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Non trivial zeros of the Zeta function

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Abstract

Using the symmetry of the non-trivial zeros of the Riemann Zeta function, we give two statements where one of them is true. The first statement asserts that the non-trivial zero of the function Zeta is of real part equal to $\frac{1}{2}$.

Keywords: Zeta function, Non trivial zeros, Riemann functional equation.

AMS subject classifications: 00A05

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where $\{t\}$ is the fractional part of the real t . In order to simplify the notation, denote

$$B := \{s \in \mathbb{C} : \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*\}.$$

It is proved in [2] that the zeros of the function ζ and the function $f : B \rightarrow \mathbb{C}$ coincide on B , where the function f is defined as

$$f(s) := 1 - 4s(1-s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k}}{2k-1-s}, \quad \forall s \in B. \quad (2)$$

We prove the following theorem

Theorem 1. *Let be $s \in B$. Suppose that s is a zero of the function ζ . Then, one of the following statements is true*

- $1 - 2\Re(s) = 0$,
- $h_s(t) = 0, \quad \forall t \in (0, 1)$,

where

$$h_s(t) := \frac{1}{s(1-\bar{s})} - \frac{1}{(1-s)\bar{s}}t + 4 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1}}{(2k-1-s)(2k-2+\bar{s})}.$$

Proof. Let be $s \in B$. In one hand, for every $t \in (0, 1)$ we have

$$\begin{aligned} \int_t^1 u^{-2-s} \left[\int_0^u \left(v - \frac{1}{2} \right) dv \right] du &= \frac{1}{2} \int_t^1 u^{-2-s} (u^2 - u) du \\ &= \frac{1}{2s(1-s)} - \frac{1}{2} t^{-s} \left(\frac{t}{1-s} + \frac{1}{s} \right). \end{aligned} \quad (3)$$

We recall that the function $p : \mathbb{R} \rightarrow \mathbb{R}$, defined as

$$p(t) := \int_0^t \left(\{u\} - \frac{1}{2} \right) du, \quad \forall t \in \mathbb{R}$$

is a continuous 1-periodic function and satisfies

$$\begin{aligned} p(t) &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \left(\cos(j2\pi t) - 1 \right) \\ &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k} t^{2k}}{(2k)!}, \quad \forall t \in \mathbb{R}. \end{aligned}$$

On the other hand, for every $t \in (0, 1)$ we have

$$\begin{aligned} & \int_t^1 u^{-2-s} \left[\int_0^u \left(v - \frac{1}{2} \right) dv \right] du \\ &= 2 \int_t^1 u^{-2-s} \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k} u^{2k}}{(2k)!} du \end{aligned} \quad (4)$$

$$\begin{aligned} &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k}}{2k-1-s} \\ &- 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1-s}}{2k-1-s} \end{aligned} \quad (5)$$

Suppose that s is a zeros of the function ζ . Since the zeros of the function ζ and the function $f : B \rightarrow \mathbb{C}$ coincide on B , where the function f is given by the Equation (2). Then the Equations (3) and (5) implies that

$$\frac{1}{2} \left(\frac{t}{1-s} + \frac{1}{s} \right) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1}}{2k-1-s}, \quad \forall t \in (0, 1).$$

Thanks to the Riemann functional equation, the point $1 - \bar{s}$ is a zero of the function ζ . Then

$$\frac{1}{2} \left(\frac{t}{\bar{s}} + \frac{1}{1-\bar{s}} \right) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1}}{2k-2+\bar{s}}, \quad \forall t \in (0, 1).$$

The subtraction implies that

$$(1 - 2\Re(s))h_s(t) = 0, \quad \forall t \in (0, 1),$$

where

$$\begin{aligned} h_s(t) &:= \frac{1}{s(1-\bar{s})} - \frac{1}{(1-s)\bar{s}} t \\ &+ 4 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1}}{(2k-1-s)(2k-2+\bar{s})}. \end{aligned}$$

□

Proposition 2. Consider the function h_s given by the Theorem 1. Then

$$t|s||1-s|\Im(h_s(t)) = -4 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} f_s(j2\pi t), \quad \forall t \in (0, 1),$$

where $f_s : (0, 1) \rightarrow \mathbb{R}$ is defined as

$$f_s(z) := \sum_{k \in \mathbb{N}^*} (-1)^k \frac{z^{2k}}{(2k)!} (4k - 3) \beta_{s,k},$$

and where $(\beta_{s,k})_{k \in \mathbb{N}^*} \subset (0, 1)$ are defined as

$$\beta_{s,k} := 1 - \left[\frac{|s||1-s|}{|2k-1-s||2k-2+s|} \right]^2$$

Proof. By definition we have

$$\begin{aligned} h_s(t) &:= \frac{1}{s(1-\bar{s})} - \frac{1}{(1-s)\bar{s}} t \\ &+ 4 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1}}{(2k-1-s)(2k-2+\bar{s})}. \end{aligned}$$

Take the imaginary part, we obtain

$$\begin{aligned} \Im(h_s(t)) &:= -\frac{1}{|s||1-s|} (1+t) \\ &+ 4 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1} (4k-3)}{|2k-1-s|^2 |2k-2+s|^2}. \end{aligned}$$

Use the fact

$$\frac{1}{2}(t+t^2) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} (-1)^k (j2\pi)^{2k} t^{2k} (4k-3), \quad \forall t \in (0, 1).$$

□

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).
- [2] W. Oukil, Non trivial zeros of the Zeta function using the differential equations, url: <https://hal.science/hal-03849860>, Hal-version: v9, (2023).