



**HAL**  
open science

## Non trivial zeros of the Zeta function

W Oukil

► **To cite this version:**

| W Oukil. Non trivial zeros of the Zeta function. 2023. hal-03581978v23

**HAL Id: hal-03581978**

**<https://hal.science/hal-03581978v23>**

Preprint submitted on 11 Oct 2023 (v23), last revised 26 Nov 2023 (v25)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Non trivial zeros of the Zeta function

W. Oukil

Faculty of Mathematics.

University of Science and Technology Houari Boumediene.

BP 32 EL ALIA 16111 Bab Ezzouar, Algiers, Algeria.

October 11, 2023

## Abstract

Using the symmetry of the non-trivial zeros of the Riemann Zeta function, we give two statements where one of them is true. The first statement asserts that the non-trivial zero of the function Zeta is of real part equal to  $\frac{1}{2}$ .

**Keywords:** Zeta function, Non trivial zeros, Riemann functional equation.

**AMS subject classifications:** 00A05

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where  $\{t\}$  is the fractional part of the real  $t$ . In order to simplify the notation, denote

$$B := \{s \in \mathbb{C} : \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*\}.$$

It is proved in [2] that the zeros of the function  $\zeta$  and the function  $f : B \rightarrow \mathbb{C}$  coincide on  $B$ , where the function  $f$  is defined as

$$f(s) := 1 - 4s(1-s) \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k}}{2k-1-s}, \quad \forall s \in B. \quad (2)$$

We prove the following theorem

**Theorem 1.** *Let be  $s \in B$ . Suppose that  $s$  is a zero of the function  $\zeta$ . Then, one of the following statements is true*

- $1 - 2\Re(s) = 0$ ,
- $h_s(t) = 0, \quad \forall t \in (0, 1)$ ,

where

$$h_s(t) := \frac{1}{s(1-\bar{s})} - \frac{1}{(1-s)\bar{s}}t + 4 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1}}{(2k-1-s)(2k-2+\bar{s})}.$$

*Proof.* Let be  $s \in B$ . In one hand, for every  $t \in (0, 1)$  we have

$$\begin{aligned} \int_t^1 u^{-2-s} \left[ \int_0^u \left( v - \frac{1}{2} \right) dv \right] du &= \frac{1}{2} \int_t^1 u^{-2-s} (u^2 - u) du \\ &= \frac{1}{2s(1-s)} - \frac{1}{2} t^{-s} \left( \frac{t}{1-s} + \frac{1}{s} \right). \end{aligned} \quad (3)$$

We recall that the function  $p : \mathbb{R} \rightarrow \mathbb{R}$ , defined as

$$p(t) := \int_0^t \left( \{u\} - \frac{1}{2} \right) du, \quad \forall t \in \mathbb{R}$$

is a continuous 1-periodic function and satisfies

$$\begin{aligned} p(t) &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \left( \cos(j2\pi t) - 1 \right) \\ &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k} t^{2k}}{(2k)!}, \quad \forall t \in \mathbb{R}. \end{aligned}$$

On the other hand, for every  $t \in (0, 1)$  we have

$$\begin{aligned} & \int_t^1 u^{-2-s} \left[ \int_0^u \left( v - \frac{1}{2} \right) dv \right] du \\ &= 2 \int_t^1 u^{-2-s} \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k} u^{2k}}{(2k)!} du \end{aligned} \quad (4)$$

$$\begin{aligned} &= 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k}}{2k-1-s} \\ &- 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1-s}}{2k-1-s} \end{aligned} \quad (5)$$

Suppose that  $s$  is a zeros of the function  $\zeta$ . Since the zeros of the function  $\zeta$  and the function  $f : B \rightarrow \mathbb{C}$  coincide on  $B$ , where the function  $f$  is given by the Equation (2). Then the Equations (3) and (5) implies that

$$\frac{1}{2} \left( \frac{t}{1-s} + \frac{1}{s} \right) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1}}{2k-1-s}, \quad \forall t \in (0, 1).$$

Thanks to the Riemann functional equation, the point  $1 - \bar{s}$  is a zero of the function  $\zeta$ . Then

$$\frac{1}{2} \left( \frac{t}{\bar{s}} + \frac{1}{1-\bar{s}} \right) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1}}{2k-2+\bar{s}}, \quad \forall t \in (0, 1).$$

The subtraction implies that

$$(1 - 2\Re(s))h_s(t) = 0, \quad \forall t \in (0, 1),$$

where

$$\begin{aligned} h_s(t) &:= \frac{1}{s(1-\bar{s})} - \frac{1}{(1-s)\bar{s}} t \\ &+ 4 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k \in \mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1}}{(2k-1-s)(2k-2+\bar{s})}. \end{aligned}$$

□

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).

- [2] W. Oukil, Non trivial zeros of the Zeta function using the differential equations, URL: <https://hal.science/hal-03849860>, HAL-VERSION: v9, working paper or preprint, (2023).