Non symmetry of the function $\frac{\zeta(s)}{s}$

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Abstract

We prove that $|\frac{\zeta(\overline{s})}{\overline{s}} - \frac{\zeta(1-s)}{1-s}| \neq 0$ for every $\Re(s) \in (0, \frac{1}{2})$ and $\Im(s) \in \mathbb{R}^*$, where ζ is the Riemann Zeta function. At the end of the paper, we give a discussion about the Riemann hypothesis.

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1 Main result

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0,1), \quad \Im(s) \in \mathbb{R}^*, \tag{1}$$

where $\{t\}$ is the fractional part of the real t. We prove in this paper the following Main result

Main Result. *Consider the function ζ defined by Equation* (1)*. Then*

$$\Big| \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau} \Big| \neq 0, \quad \forall r \in (0, \frac{1}{2}), \ \forall \tau \in \mathbb{R}^*.$$

2 Main Proposition

Notation 1. In order to simplify the notation, for every $r \in (0, \frac{1}{2})$ and $\tau \in \mathbb{R}^*$ denote

$$\phi(\tau, r) := \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau}.$$

For every $r \in (0, \frac{1}{2})$ denote

$$f_r(u) := \{u\}u^{-\frac{3}{2}}(u^r - u^{-r}), \quad u > 0. \tag{2}$$

By Equation (1) we have

$$\phi(\tau, r) = \int_0^{+\infty} u^{-i\tau} f_r(u)du,$$

The aims is to studies the differential equation of solution the function

$$t \mapsto t^{i\tau}\Big[z + \int_0^t u^{-i\tau} f_r(u)du\Big], \quad t > 0, \quad z \in \mathbb{C}.$$

We focus only on the solutions that converge to zero. More precisely, the strategy to prove the Main result, is to prove that the above function converge to zero when $t \to +\infty$ implies that $|z| \neq 0$ (the initial condition). We deduce that $\phi(\tau, r) \neq 0$.

For every $\tau \in \mathbb{R}^*$ and $r \in (0, \frac{1}{2})$ we consider the following differential equation

$$\frac{d}{dt}x = \frac{i\tau}{t}x + f_r(t), \quad t \in \mathbb{R}_+/\mathbb{N}, \quad x(0) = 0, \quad x : (0, +\infty) \to \mathbb{C}. \tag{3}$$

Lemma 2. *For every $\tau \in \mathbb{R}^*$ and $r \in (0, \frac{1}{2})$ there exists a unique continuous solution $\psi_{\tau,r}(t) : [0, +\infty) \to \mathbb{C}$ of the differential equation (3). Further,*

$$\psi_{\tau,r}(t) = t^{i\tau} \int_0^t u^{-i\tau} f_r(u)du, \quad \forall t > 0,$$

and

$$\psi_{\tau,r}(t) = \frac{t^{\frac{1}{2}+r}}{-i\tau + \frac{1}{2} + r} - \frac{t^{\frac{1}{2}-r}}{-i\tau + \frac{1}{2} - r}, \quad \forall t \in (0, 1).$$

Proof. Since $0 \leq f_r(u) \leq \frac{1}{t^{\frac{3}{2}}}(u^r - u^{-r})$ for every $u > 1$ and $f_r(u) = \frac{1}{u^{\frac{1}{2}}}(u^r - t^{-r})$ for every $u \in (0,1)$ then the function

$$t \mapsto t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du,$$

is continuous and C^1 on $\mathbb{R}_+/\mathbb{N}$. The Equation (3) is a non-homogeneous linear differential equation, then unique continuous solution $\psi_{\tau,r}(t) : (0,+\infty) \to \mathbb{C}$ such that $[\psi_{\tau,r}(0)| = 0$, is given by

$$\psi_{\tau,r}(t) = t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du, \ \forall t > 0.$$

Since $f_r(u) = t^{-\frac{1}{2}}(t^r - t^{-r})$ for every $t \in (0,1)$, then

$$\psi_{\tau,r}(t) = \frac{t^{\frac{1}{2}+r}}{-i\tau + \frac{1}{2} + r} - \frac{t^{\frac{1}{2}-r}}{-i\tau + \frac{1}{2} - r}, \quad \forall t \in (0,1).$$

□

Proposition 3. *For every $\tau \in \mathbb{R}^*$ and $r \in (0, \frac{1}{2})$ the unique continuous solution $\psi_{\tau,r}(t) : [0,+\infty) \to \mathbb{C}$ of of the differential equation (3) satisfies*

$$\lim_{t \to +\infty} |\psi_{\tau,r}(t)| \neq 0.$$

Notation 4. Denote

$$p(t) = \int_0^t \Big(\{u\} - \frac{1}{2} \Big) du, \quad \forall t \geq 0,$$

where we recall that $\{u\}$ is the fractional part of the real u. Denote

$$\omega := 2 \max_{t \geq [0,1]} |p(t)|.$$

Lemma 5. *The function p is a continuous 1-periodic function.*

Proof. The function $u \mapsto \{u\}$ is 1-periodic, then there exists a continuous 1-periodic function $p : \mathbb{R} \to \mathbb{R}$ such that

$$\int_0^t \{u\} du = t \int_0^1 \{u\} du + p(t), \quad \forall t \geq 0.$$

Since

$$\int_0^1 \{u\}du = \frac{1}{2},$$

we get

$$\int_0^t \{u\}du = t\frac{1}{2} + p(t), \quad \forall t \geq 0.$$

□

Lemma 6. *Let $\tau \in \mathbb{R}^*$ and $r \in (0, \frac{1}{2})$. Let $\psi_{\tau,r}(t) : [0, +\infty) \to \mathbb{C}$ be the unique continuous solution of the differential equation* (3)*. Suppose that*

$$\lim_{t\to+\infty} |\psi_{\tau,r}(t)| = 0.$$

Then

$$\lim_{t\to+\infty} t^{\frac{1}{2}-r}\psi_{\tau,r}(t) = -\frac{1}{2}\frac{\frac{1}{2}-r}{(\frac{1}{2}-r)^2+\tau^2} + i\frac{1}{2}\frac{\tau}{(\frac{1}{2}-r)^2+\tau^2}.$$

Proof. By Lemma 2

$$\psi_{\tau,r}(t) = t^{i\tau}\int_0^t u^{-i\tau}f_r(u)du, \ \forall t > 0.$$

Suppose that

$$\lim_{t\to+\infty} |\psi_{\tau,r}(t)| = 0.$$

Then

$$\int_0^{+\infty} u^{-i\tau}f_r(u)du = 0,$$

implies

$$\int_0^t u^{-i\tau}f_r(u)du = -\int_t^{+\infty} u^{-i\tau}f_r(u)du, \quad \forall t > 0.$$

We get

$$\psi_{\tau,r}(t) = -t^{i\tau}\int_t^{+\infty} u^{-i\tau}f_r(u)du, \ \forall t > 0. \tag{4}$$

Consider the function p given in the Notation 4. Use the integration part formula in Equation (4),

$$\begin{aligned}\psi_{\tau,r}(t) = &-\frac{1}{2}t^{i\tau}\int_t^{+\infty} u^{-i\tau-\frac{3}{2}}(u^r - u^{-r})du \\ &+ t^{i\tau}\int_t^{+\infty} h_{\tau,r}(u)\Big(p(u) - p(t)\Big)du,\end{aligned}$$

where

$$h_{\tau,r}(u) := \frac{d}{du}\Big[u^{-i\tau-\frac{3}{2}}(u^r - u^{-r})\Big] \\ = (-i\tau - \frac{3}{2} + r)u^{-i\tau-\frac{5}{2}+r} - (-i\tau - \frac{3}{2} - r)u^{-i\tau-\frac{5}{2}-r}.$$

Since

$$|p(u) - p(t)| < \omega, \quad \forall u \geq 0,$$

where ω is given by the Notation 4, we obtain

$$\Big|\psi_{\tau,r}(t) + \frac{1}{2}t^{i\tau}\int_t^{+\infty}[u^{-i\tau-\frac{3}{2}+r} - u^{-i\tau-\frac{3}{2}-r}]du\Big| \\ < 2\omega(|\tau| + \frac{3}{2} + r)\int_t^{+\infty} u^{-\frac{5}{2}+r}du = \frac{2\omega(|\tau| + \frac{3}{2} + r)}{\frac{3}{2} - r}t^{-\frac{3}{2}+r}.$$

We have

$$t^{i\tau}\int_t^{+\infty}[u^{-i\tau-\frac{3}{2}+r} - u^{-i\tau-\frac{3}{2}-r}]dv = \frac{1}{-\frac{1}{2} - r - i\tau}t^{-\frac{1}{2}-r} \\ - \frac{1}{-\frac{1}{2} + r - i\tau}t^{-\frac{1}{2}+r},$$

Then,

$$\lim_{t\to+\infty} t^{\frac{1}{2}-r}\psi_{\tau,r}(t) = \frac{1}{2}\frac{1}{-\frac{1}{2} + r - i\tau},$$

in other words

$$\lim_{t\to+\infty} t^{\frac{1}{2}-r}\psi_{\tau,r}(t) = -\frac{1}{2}\frac{\frac{1}{2} - r}{(\frac{1}{2} - r)^2 + \tau^2} + i\frac{1}{2}\frac{\tau}{(\frac{1}{2} - r)^2 + \tau^2}.$$

□

Lemma 7. *For every $r \in (0, \frac{1}{2})$ there exists $t_r > 0$ such that*

$$\frac{d}{dt}\Big[\frac{1}{t}\int_{\frac{1}{t}}^{+\infty} f_r(u)du\Big] < 0, \quad \forall t \in (0, t_r).$$

Proof. We have

$$\frac{d}{dt}\Big[\frac{1}{t}\int_{\frac{1}{t}}^{+\infty} f_r(u)du\Big] = -\frac{1}{t^2}\Big[\int_{\frac{1}{t}}^{+\infty} f_r(u)du - \frac{1}{t}f_r(\frac{1}{t})\Big]$$

As in the Proof of the Lemma 6, using the integration part formula, we have

$$\int_{\frac{1}{t}}^{+\infty} f_r(u)du = \frac{1}{2}\Big[\frac{t^{\frac{1}{2}-r}}{\frac{1}{2}-r} - \frac{t^{\frac{1}{2}+r}}{\frac{1}{2}+r}\Big] - \int_{\frac{1}{t}}^{+\infty} h_r(u)\Big(p(u)-p(\frac{1}{t})\Big)du,$$

where p given in the Notation 4 and where

$$h_r(u) := \frac{d}{du}\Big[u^{-\frac{3}{2}}(u^r-u^{-r})\Big] = (-\frac{3}{2}+r)u^{-\frac{5}{2}+r} - (-\frac{3}{2}-r)u^{-\frac{5}{2}-r}.$$

Since

$$|p(u)-p(\frac{1}{t})| < \omega, \quad \forall u \geq 0,$$

Then

$$\lim_{t\to 0} t^{-\frac{1}{2}+r}\int_{\frac{1}{t}}^{+\infty} f_r(u)du = \frac{1}{2}\frac{1}{\frac{1}{2}-r} > 1.$$

Since

$$\lim_{t\to 0} t^{-\frac{3}{2}+r} f_r(\frac{1}{t}) < 1,$$

Then there exists $\epsilon_r > 0$ such that

$$\frac{d}{dt}\Big[\frac{1}{t}\int_{\frac{1}{t}}^{+\infty} f_r(u)du\Big] < 0, \quad \forall t \in (0,\epsilon_r).$$

□

Proof of the Proposition 3. Let be $r \in (0,\frac{1}{2})$ and $\tau \in \mathbb{R}^*$ fixed. By contradiction, suppose that

$$\lim_{t\to+\infty} |\psi_{\tau,r}(t)| = 0. \tag{5}$$

In order to simplify the notation, denote

$$\Psi_{\tau,r}(t) := \int_0^{\frac{1}{t}} \frac{1}{u}\psi_{\tau,r}(u)du, \quad \forall t \in (0,+\infty).$$

Equation (3), implies

$$\frac{d}{dt}\Psi_{\tau,r}(t) = -\frac{1}{t}\psi_{\tau,r}(\frac{1}{t})du = -\frac{i\tau}{t}\Psi_{\tau,r}(t) - \frac{1}{t}\int_0^{\frac{1}{t}} f_r(u)du, \tag{6}$$

In order to simplify the notation, denote

$$w(t) := \Re\Big(\Psi_{\tau,r}(t)\Big) \quad \text{and} \quad v(t) := \Re\Big(\Psi_{\tau,r}(t)\Big),\ \forall t > 0.$$

By Hypothesis (5) and by Equation (3), we have

$$w(0) = 0 \quad \text{and} \quad v(0) = \frac{1}{\tau}\int_0^{+\infty} f_r(u)du. \tag{7}$$

Equation (6) can be written as

$$\begin{aligned} &\frac{d}{dt}w(t) = \frac{\tau}{t}v(t) - \frac{1}{t}\int_0^{\frac{1}{t}} f_r(u)du, && v(0) = \frac{1}{\tau}\int_0^{+\infty} f_r(u)du, \quad t \geq 0, \\ &\frac{d}{dt}v(t) = -\frac{\tau}{t}w(t), && w(0) = 0, \quad t \geq 0. \end{aligned}$$

By Lemma 7 there exists $t_r > 0$

$$\frac{d}{dt}\Big[\frac{1}{t}\int_{\frac{1}{t}}^{+\infty} f_r(u)du\Big] < 0, \quad \forall t \in (0, t_r).$$

Then there exists $\epsilon_{\tau,r} > 0$ such that

$$\frac{d}{dt}w(t) < 0, \quad \forall t \in (0, \epsilon_{\tau,r}),$$

in other words

$$\Re\Big(\psi_{\tau,r}(\frac{1}{t})\Big) > 0, \quad \forall t \in (0, \epsilon_{\tau,r}),$$

But by the hypothesis (5) an Lemma 6, we have

$$\lim_{t\to+\infty} t^{\frac{1}{2}-r}\psi_{\tau,r}(t) = -\frac{1}{2}\frac{\frac{1}{2}-r}{(\frac{1}{2}-r)^2+\tau^2} + i\frac{1}{2}\frac{\tau}{(\frac{1}{2}-r)^2+\tau^2}.$$

We obtain a contradiction. □

3 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function ζ defined by Equation (1). Define the *non trivial zeros* of the function ζ in the following sense

Definition 8. Let be $s \in \mathbb{C}$. We say that s is a *non trivial zero of the function* ζ if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0,1), \quad \Im(s) \in \mathbb{R}^*.$$

By the Riemann functional equation [[1], page 13 Theorem 2.1], the non trivial zeros of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. By the Main result, the non trivial zeros $s \in \mathbb{C}$ symmetric about the line $\Re(z) = \frac{1}{2}$ satisfies $\Re(s) = \frac{1}{2}$.

Theorem 9 (Riemann hypothesis)**.** *Let $s \in \mathbb{C}$ be a non-trivial zero of the function ζ. Then $\Re(s) = \frac{1}{2}$.*

Proof. By contradiction, suppose that there exists a non-trivial zero $s_* \in \mathbb{C}$ of the function ζ such that $\Re(s_*) \neq \frac{1}{2}$. Then

$$\frac{\zeta(s_*)}{s_*} = 0.$$

The Main result implies

$$\frac{\zeta(1-s_*)}{1-s_*} \neq 0. \tag{8}$$

But the Riemann functional equation implies that the non-trivial zero of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. In other words,

$$\zeta(1-s_*) = 0.$$

We obtain a contradiction with the Equation (8). We deduce that $\Re(s_*) = \frac{1}{2}$. □

References

[1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).