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# Non symmetry of the function $\frac{\zeta(s)}{s}$

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## Abstract

We prove that  $|\frac{\zeta(\bar{s})}{\bar{s}} - \frac{\zeta(1-s)}{1-s}| \neq 0$  for every  $\Re(s) \in (0, \frac{1}{2})$  and  $|\Im(s)| > \frac{1}{4}$ , where  $\zeta$  is the Riemann Zeta function. At the end of the paper, we give a discussion about the Riemann hypothesis.

**Keywords:** Riemann functional equation, Zeta function.

**AMS subject classifications:** 00A05

## 1 Main result

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where  $\{t\}$  is the fractional part of the real  $t$ . We prove in this paper the following Main result

**Main Result.** Consider the function  $\zeta$  defined by Equation (1). Then

$$\left| \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau} \right| \neq 0, \quad \forall r \in (0, \frac{1}{2}), \quad \forall |\tau| > \frac{1}{4}.$$

**Notation 1.** In order to simplify the notation, for every  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$  denote

$$\phi(\tau, r) := \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau}.$$

For every  $r \in (0, \frac{1}{2})$  denote

$$f_r(u) := \{u\}u^{-\frac{3}{2}}(u^r - u^{-r}), \quad u > 0. \quad (2)$$

**Lemma 2.** *Let be  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a positive, increasing and convergent function. Let be  $(\alpha, \beta) \in \mathbb{R}^2$  and denote*

$$\omega := -\arctan\left(\frac{\beta}{\alpha}\right).$$

Suppose that

$$\beta < 0, \quad \text{and} \quad \alpha < 0.$$

Then,

$$\lim_{n \rightarrow +\infty} \int_0^{2n\pi + \omega} (\beta \sin(u) - \alpha \cos(u))g(u)du \neq 0.$$

*Proof.* In order to simplify the notation, denote

$$h(t) := \int_0^{\frac{2n\pi}{\tau} + \omega} (\beta \sin(u) - \alpha \cos(u)) \left[ \int_0^{\exp(u)} f_r(v)dv \right] du, \quad t \geq 0.$$

Let be  $(s_n)_{n \neq 0}$  be the zeros of the function  $h$ . We suppose the order

$$s_0 \leq s_1 \leq \dots, s_2 \leq \dots$$

By hypothesis of the present Lemma the function  $g$  is positive and increasing. Since

$$\beta < 0, \quad \text{and} \quad \alpha < 0$$

By symmetry of the function

$$u \mapsto \beta \sin(u) - \alpha \cos(u),$$

the second zero  $s_1$  of  $h$  satisfies  $-\omega + 2\pi < s_1 < -\omega + 2\pi + \frac{\pi}{2}$ . Since  $g$  is increasing then

$$0 \leq s_{n+4} - s_{n+2} < s_{n+2} - s_n, \quad \forall n \geq 0.$$

Then there exists  $\gamma \in (0, \frac{\pi}{2})$  such that

$$-\omega \leq s_n - 2\pi n < -\omega + \frac{\pi}{2} - \gamma, \quad \forall n \geq 0.$$

We deduce that

$$\lim_{n \rightarrow +\infty} \left( \frac{2n\pi}{\tau} + \omega - s_n \right) \neq 0.$$

□

*Proof of the Main result.* Let be  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$  fixed. By Equation (1) we have

$$\phi(\tau, r) = \int_0^{+\infty} u^{-i\tau} f_r(u) du,$$

which can be written as

$$\begin{aligned} \phi(\tau, r) &= \int_0^1 (u^{-i\tau} - 1) f_r(u) du + \int_0^1 f_r(u) du \\ &\quad + \int_1^{+\infty} u^{-i\tau} f_r(u) du. \end{aligned}$$

Since  $f_r(u) = u^{-\frac{1}{2}}(u^r - u^r)$  for every  $u \in (0, 1)$  then

$$\begin{aligned} \int_0^1 (u^{-i\tau} - 1) f_r(u) du &= i\tau \int_0^1 u^{-i\tau-1} \int_0^u f_r(v) dv du \\ &= i\tau \frac{1}{(-i\tau + \frac{1}{2} + r)(\frac{1}{2} + r)} - i\tau \frac{1}{(-i\tau + \frac{1}{2} - r)(\frac{1}{2} - r)} \\ &= \frac{\tau}{c_{\tau, r}} (\alpha_{\tau, r} + i\beta_{\tau, r}), \end{aligned}$$

where

$$\begin{aligned} c_{\tau, r} &:= \left( \tau^2 + \left( \frac{1}{2} + r \right)^2 \right) \left( \tau^2 + \left( \frac{1}{2} - r \right)^2 \right) \left( \frac{1}{4} - r^2 \right) \quad (3) \\ \alpha_{\tau, r} &:= \tau \left( \tau^2 + \left( \frac{1}{2} - r \right)^2 \right) \left( \frac{1}{2} - r \right) - \tau \left( \tau^2 + \left( \frac{1}{2} + r \right)^2 \right) \left( \frac{1}{2} + r \right) \\ \beta_{\tau, r} &:= -2r \left( \tau^2 - \left( \frac{1}{4} - r^2 \right) \right) \left( \frac{1}{4} - r^2 \right). \\ &\quad \tau \left( 2\tau^2 r + \left( \frac{1}{2} + r \right)^2 \right) \left( \frac{1}{2} + r \right) - \tau \left( \left( \frac{1}{2} - r \right)^2 \right) \left( \frac{1}{2} - r \right) \end{aligned}$$

Since  $r \in (0, \frac{1}{2})$  and  $|\tau| > \frac{1}{4}$  then

$$\beta_{\tau, r} < 0 \quad \text{and} \quad \alpha_{\tau, r} < 0. \quad (4)$$

Then

$$\begin{aligned} \beta_{\tau, r} \Re(\phi(\tau, r)) - \alpha_{\tau, r} \Im(\phi(\tau, r)) &= \beta_{\tau, r} \int_0^1 f_r(u) du \\ &\quad + \int_1^{+\infty} \left( \beta_{\tau, r} \cos(\tau \ln(u)) + \alpha_{\tau, r} \sin(\tau \ln(u)) \right) f_r(u) du. \end{aligned}$$

Suppose that,

$$\Im(\phi(\tau, r)) = 0, \quad (5)$$

The Main result is proved if we prove that  $\Re(\phi(\tau, r)) \neq 0$ . The hypothesis (5), implies

$$\begin{aligned} \beta_{\tau,r} \Re(\phi(\tau, r)) &= \beta_{\tau,r} \int_0^1 f_r(u) du \\ &+ \int_1^{+\infty} (\beta_{\tau,r} \cos(\tau \ln(u)) + \alpha_{\tau,r} \sin(\tau \ln(u))) f_r(u) du. \end{aligned}$$

Use the integration part formula and use the change of variable  $u = \exp(v)$  we get,

$$\beta_{\tau,r} \Re(\phi(\tau, r)) = \tau \lim_{n \rightarrow +\infty} h_{\tau,r}(t_n), \quad (6)$$

where

$$\begin{aligned} t_n &:= -\frac{1}{\tau} \arctan\left(\frac{\beta_{\tau,r}}{\alpha_{\tau,r}}\right) + \frac{n\pi}{\tau}, \quad n \geq 0, \\ h_{\tau,r}(t) &:= \int_0^t (\beta_{\tau,r} \sin(\tau u) - \alpha_{\tau,r} \cos(\tau u)) \left[ \int_0^{\exp(u)} f_r(v) dv \right] du, \quad t > 0. \end{aligned}$$

By definition of  $f_r$  the function

$$u \mapsto \int_0^{\exp(u)} f_r(v) dv, \quad u > 0,$$

is positive, increasing and converge. By Lemma 2,

$$\lim_{n \rightarrow +\infty} h_{\tau,r}(t_n) \neq 0.$$

Equation (6) implies

$$\beta_{\tau,r} \Re(\phi(\tau, r)) \neq 0.$$

□

## 2 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function  $\zeta$  defined by Equation (1). Define the *non trivial zeros* of the function  $\zeta$  in the following sense

**Definition 3.** Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non trivial zero of the function*  $\zeta$  if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

By the Riemann functional equation [[1], page 13 Theorem 2.1], the non trivial zeros of the function  $\zeta$  are symmetric about the line  $\Re(z) = \frac{1}{2}$ . By the Main result, the non trivial zeros  $s \in \mathbb{C}$  such that  $|\Im(s)| > \frac{1}{4}$  and symmetric about the line  $\Re(z) = \frac{1}{2}$  satisfies  $\Re(s) = \frac{1}{2}$ .

**Theorem 4** (Partial Riemann hypothesis). *Let  $s \in \mathbb{C}$  be a non-trivial zero of the function  $\zeta$  such that  $|\Im(s)| > \frac{1}{4}$ . Then  $\Re(s) = \frac{1}{2}$ .*

*Proof.* By contradiction, suppose that there exists a non-trivial zero  $s_* \in \mathbb{C}$  of the function  $\zeta$  such that  $|\Im(s_*)| > \frac{1}{4}$  and such that  $\Re(s_*) \neq \frac{1}{2}$ . Then

$$\frac{\zeta(s_*)}{s_*} = 0.$$

The Main result implies

$$\frac{\zeta(1 - s_*)}{1 - s_*} \neq 0. \tag{7}$$

But the Riemann functional equation implies that the non-trivial zero of the function  $\zeta$  are symmetric about the line  $\Re(z) = \frac{1}{2}$ . In other words,

$$\zeta(1 - s_*) = 0.$$

We obtain a contradiction with the Equation (7). We deduce that  $\Re(s_*) = \frac{1}{2}$ .  $\square$

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).