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Non symmetry of the function $\frac{\zeta(s)}{s}$

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Abstract

We prove that $|\frac{\zeta(\bar{s})}{\bar{s}} - \frac{\zeta(1-s)}{1-s}| \neq 0$ for every $\Re(s) \in (0, \frac{1}{2})$ and $\Im(s) \in \mathbb{R}^*$, where ζ is the Riemann Zeta function. At the end of the paper, we give a discussion about the Riemann hypothesis.

Keywords: Riemann functional equation, Zeta function.

AMS subject classifications: 00A05

1 Main result

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where $\{t\}$ is the fractional part of the real t . We prove in this paper the following Main result

Main Result. *Consider the function ζ defined by Equation (1). Then*

$$\left| \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau} \right| \neq 0, \quad \forall r \in (0, \frac{1}{2}), \quad \forall \tau \in \mathbb{R}^*.$$

2 Main Proposition

Notation 1. In order to simplify the notation, for every $r \in (0, \frac{1}{2})$ and $\tau \in \mathbb{R}^*$ denote

$$\phi(\tau, r) := \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau}.$$

For every $r \in (0, \frac{1}{2})$ denote

$$f_r(u) := \{u\} \frac{1}{u^{\frac{3}{2}}} (u^r - u^{-r}), \quad u > 0. \quad (2)$$

By Equation (1) we have

$$\phi(\tau, r) = \int_0^{+\infty} u^{-i\tau} f_r(u) du,$$

The aim is to study the differential equation of the solution function

$$t \mapsto t^{i\tau} \left[z + \int_0^t u^{-i\tau} f_r(u) du \right], \quad t > 0, \quad z \in \mathbb{C}.$$

We focus only on the solutions that converge to zero. More precisely, the strategy to prove the Main result, is to prove that the above function converges to zero when $t \rightarrow +\infty$ implies that $|z| \neq 0$ (the initial condition). We deduce that $\phi(\tau, r) \neq 0$.

For every $\tau \in \mathbb{R}^*$ and $r \in (0, \frac{1}{2})$ we consider the following differential equation

$$\frac{d}{dt} x = \frac{i\tau}{t} x + f_r(t), \quad t \in \mathbb{R}_+/\mathbb{N}, \quad x(0) = 0, \quad x : (0, +\infty) \rightarrow \mathbb{C}. \quad (3)$$

Lemma 2. For every $\tau \in \mathbb{R}^*$ and $r \in (0, \frac{1}{2})$ there exists a unique continuous solution $\psi_{\tau, r}(t) : [0, +\infty) \rightarrow \mathbb{C}$ of the differential equation (3). Further,

$$\psi_{\tau, r}(t) = t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du, \quad \forall t > 0,$$

and

$$\psi_{\tau, r}(t) = \frac{t^{\frac{1}{2}+r}}{-i\tau + \frac{1}{2} + r} - \frac{t^{\frac{1}{2}-r}}{-i\tau + \frac{1}{2} - r}, \quad \forall t \in (0, 1).$$

Proof. Since $0 \leq f_r(u) \leq \frac{1}{t^{\frac{1}{2}}}(u^r - u^{-r})$ for every $u > 1$ and $f_r(u) = \frac{1}{u^{\frac{1}{2}}}(u^r - t^{-r})$ for every $u \in (0, 1)$ then the function

$$t \mapsto t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du,$$

is continuous and C^1 on \mathbb{R}_+/\mathbb{N} . The Equation (3) is a non-homogeneous linear differential equation, then unique continuous solution $\psi_{\tau,r}(t) : (0, +\infty) \rightarrow \mathbb{C}$ such that $[\psi_{\tau,r}(0)] = 0$, is given by

$$\psi_{\tau,r}(t) = t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du, \quad \forall t > 0.$$

Since $f_r(u) = t^{-\frac{1}{2}}(t^r - t^{-r})$ for every $t \in (0, 1)$, then

$$\psi_{\tau,r}(t) = \frac{t^{\frac{1}{2}+r}}{-i\tau + \frac{1}{2} + r} - \frac{t^{\frac{1}{2}-r}}{-i\tau + \frac{1}{2} - r}, \quad \forall t \in (0, 1).$$

□

Proposition 3. For every $\tau \in \mathbb{R}^*$ and $r \in (0, \frac{1}{2})$ the unique continuous solution $\psi_{\tau,r}(t) : [0, +\infty) \rightarrow \mathbb{C}$ of of the differential equation (3) satisfies

$$\lim_{t \rightarrow +\infty} |\psi_{\tau,r}(t)| \neq 0.$$

Proof. Let be $r \in (0, \frac{1}{2})$ and $\tau \in \mathbb{R}^*$ fixed. By contradiction, suppose that

$$\lim_{t \rightarrow +\infty} |\psi_{\tau,r}(t)| = 0. \quad (4)$$

Equation (3) implies

$$\psi_{\tau,r}(t) = -i\tau \int_t^{+\infty} \frac{1}{u} \psi_{\tau,r}(u) du - \int_t^{+\infty} f_r(u) du, \quad \forall t > 0, \quad (5)$$

In order to simplify the notation, denote

$$\Psi_{\tau,r}(t) := \int_t^{+\infty} \frac{1}{u} \psi_{\tau,r}(u) du, \quad \forall t > 0.$$

Equation (5) implies

$$\frac{d}{dt} \Psi_{\tau,r} = i\tau \frac{1}{t} \Psi_{\tau,r} + \frac{1}{t} \int_t^{+\infty} f_r(u) du, \quad \forall t > 0,$$

which is a non-homogeneous linear differential equation. Integrate, for every $t' > t > 0$ we have

$$\Psi_{\tau,r}(t') = \left(\frac{t'}{t}\right)^{i\tau} \left[\Psi_{\tau,r}(t) + \int_t^{t'} \left(\frac{t}{u}\right)^{i\tau} \frac{1}{u} \left[\int_u^{+\infty} f_r(v) dv \right] du \right]. \quad (6)$$

By the hypothesis (4) and Lemma 2 we get

$$\int_0^{+\infty} u^{-i\tau} f_r(u) du = 0,$$

then

$$\int_0^t u^{-i\tau} f_r(u) du = - \int_t^{+\infty} u^{-i\tau} f_r(u) du, \quad \forall t > 0.$$

implies

$$|\psi_{\tau,r}(t)| \leq \int_t^{+\infty} |f_r(u)| du, \quad \forall t > 0.$$

By definition of $\Psi_{\tau,r}$ and f_r we get

$$\lim_{t' \rightarrow +\infty} |\Psi_{\tau,r}(t')| = 0.$$

Equation (6) implies that for every $t > 0$ we have

$$\Psi_{\tau,r}(t) = -t^{i\tau} \int_t^{+\infty} u^{-i\tau-1} \left[\int_u^{+\infty} f_r(v) dv \right] du.$$

In order to simplify the notation, denote

$$\omega_{\tau,r}(t) := \int_t^{+\infty} u^{-i\tau-1} \left[\int_u^{+\infty} f_r(v) dv \right] du, \quad \forall t > 0, \quad (7)$$

and

$$I_n(\tau, r) := \tau \Im \left(\omega_{\tau,r}(s_n) \right), \quad s_n := \exp \left(-\frac{2n\pi}{\tau} \right) \quad \forall n \in \mathbb{N}.$$

We have Then

$$\Psi_{\tau,r}(t) = -t^{i\tau} \omega_{\tau,r}(t), \quad \forall t > 0. \quad (8)$$

By Equation (5) we obtain

$$\begin{aligned}\Psi_{\tau,r}(t) &= -\frac{1}{i\tau}\psi_{\tau,r} - \frac{1}{i\tau} \int_t^{+\infty} f_r(u)du \\ &= -\frac{1}{i\tau}\psi_{\tau,r} + \frac{1}{i\tau} \int_0^t f_r(u)du - \frac{1}{i\tau} \int_0^{+\infty} f_r(u)du, \quad \forall t > 0,\end{aligned}$$

By Lemma 2, for every $t \in (0, 1)$ we get

$$\begin{aligned}\Psi_{\tau,r}(t) &= -\frac{1}{i\tau} \left(\frac{t^{\frac{1}{2}+r}}{-i\tau + \frac{1}{2} + r} - \frac{t^{\frac{1}{2}-r}}{-i\tau + \frac{1}{2} - r} \right) \\ &\quad + \frac{1}{i\tau} \left(\frac{t^{\frac{1}{2}+r}}{\frac{1}{2} + r} - \frac{t^{\frac{1}{2}-r}}{\frac{1}{2} - r} \right) \\ &\quad - \frac{1}{i\tau} \int_0^{+\infty} f_r(u)du, \quad \forall t \in (0, 1),\end{aligned}$$

which can be written as

$$\begin{aligned}\Psi_{\tau,r}(t) &= -\frac{t^{\frac{1}{2}+r}}{(-i\tau + \frac{1}{2} + r)(\frac{1}{2} + r)} + \frac{t^{\frac{1}{2}-r}}{(-i\tau + \frac{1}{2} - r)(\frac{1}{2} - r)} \\ &\quad - \frac{1}{i\tau} \int_0^{+\infty} f_r(u)du, \quad \forall t \in (0, 1).\end{aligned}\tag{9}$$

By Equation (8)

$$I_n(\tau, r) = -\Psi_{\tau,r}(s_n),$$

Since $r \in (0, \frac{1}{2})$, Equation (9) implies

$$\begin{aligned}&\frac{d}{dr} I_{n+1}(\tau, r) - \frac{d}{dr} I_n(\tau, r) \\ &\approx \frac{d}{dr} \left[\frac{\tau^2 \exp(-\frac{2n\pi}{\tau}(\frac{1}{2} - r))}{((\tau^2 + (\frac{1}{2} - r)^2)(\frac{1}{2} - r))} \left(1 - \exp(-\frac{\pi}{\tau}(\frac{1}{2} - r)) \right) \right], \quad n \approx +\infty.\end{aligned}$$

there exists $n_{\tau,r} > 0$ such that

$$\frac{d}{dr} I_{n+1}(\tau, r) - \frac{d}{dr} I_n(\tau, r) > 0, \quad \forall n > n_{\tau,r}.\tag{10}$$

By Equation (7), for every $n \in \mathbb{N}$ we have

$$I_n(\tau, r) = -\tau \int_{s_n}^{+\infty} \sin(\tau \ln(u)) \frac{1}{u} \left[\int_u^{+\infty} f_r(v)dv \right] du.$$

Then

$$\frac{d}{dr}I_n(\tau, r) = -\tau \int_{s_n}^{+\infty} \sin(\tau \ln(u)) \frac{1}{u} \left[\int_u^{+\infty} \frac{d}{dr}f_r(v)dv \right] du.$$

Use the change of variable $u \mapsto \exp(v)$ we get

$$\frac{d}{dr}I_{n+1}(\tau, r) - \frac{d}{dr}I_n(\tau, r) = -\tau \int_{-\frac{2(n+1)\pi}{\tau}}^{-\frac{2n\pi}{\tau}} \sin(\tau u) \left[\int_{\exp(u)}^{+\infty} \frac{d}{dr}f_r(v)dv \right] du.$$

By definition of f_r in Equation , we have

$$\frac{d}{dr}f_r(v) > 0, \quad \forall v \in \mathbb{R}_+/\mathbb{N}, \quad \forall r \in (0, \frac{1}{2}),$$

then the function

$$\int_{\exp(u)}^{+\infty} \frac{d}{dr}f_r(v)dv,$$

is positive, decreasing and converge to zero. Use the symmetry of $\sin(\tau u)$, we get

$$\frac{d}{dr}I_{n+1}(\tau, r) - \frac{d}{dr}I_n(\tau, r) < 0, \quad \forall n > 0.$$

We have obtained a contradiction with Equation (10). \square

3 Proof of the Main result

Proof of the Main result. Let be $r \in (0, \frac{1}{2})$ and $\tau \in \mathbb{R}^*$ fixed. We have

$$\phi(\tau, r) = \int_0^{+\infty} u^{-i\tau} f_r(u) du.$$

Let $\psi_{\tau, r}(t) : (0, +\infty) \rightarrow \mathbb{C}$ be the unique continuous solution of the differential equation (3). By Lemma 2 , we have

$$\lim_{t \rightarrow +\infty} |\psi_{\tau, r}(t)| = |\phi(\tau, r)|,$$

By the Main Proposition 3,

$$0 \neq \lim_{t \rightarrow +\infty} |\psi_{\tau, r}(t)| = |\phi(\tau, r)|,$$

\square

4 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function ζ defined by Equation (1). Define the *non trivial zeros* of the function ζ in the following sense

Definition 4. Let be $s \in \mathbb{C}$. We say that s is a *non trivial zero of the function* ζ if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

By the Riemann functional equation [[1], page 13 Theorem 2.1], the non trivial zeros of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. By the Main result, the non trivial zeros $s \in \mathbb{C}$ symmetric about the line $\Re(z) = \frac{1}{2}$ satisfies $\Re(s) = \frac{1}{2}$.

Theorem 5 (Riemann hypothesis). *Let $s \in \mathbb{C}$ be a non-trivial zero of the function ζ . Then $\Re(s) = \frac{1}{2}$.*

Proof. By contradiction, suppose that there exists a non-trivial zero $s_* \in \mathbb{C}$ of the function ζ such that $\Re(s_*) \neq \frac{1}{2}$. Then

$$\frac{\zeta(s_*)}{s_*} = 0.$$

The Main result implies

$$\frac{\zeta(1 - s_*)}{1 - s_*} \neq 0. \tag{11}$$

But the Riemann functional equation implies that the non-trivial zero of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. In other words,

$$\zeta(1 - s_*) = 0.$$

We obtain a contradiction with the Equation (11). We deduce that $\Re(s_*) = \frac{1}{2}$. \square

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).