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► **To cite this version:**

| W Oukil. Differential equations properties and the Riemann hypothesis. 2022. hal-03581978v2

**HAL Id: hal-03581978**

**<https://hal.science/hal-03581978v2>**

Preprint submitted on 23 Feb 2022 (v2), last revised 26 Nov 2023 (v25)

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# Differential equations properties and the Riemann hypothesis

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February 23, 2022

## Abstract

Using the differential equations properties, we prove that any *non-trivial zero* of Riemann's zeta function is of real part equal to  $\frac{1}{2}$ . In other words, we prove that the Riemann hypothesis is true.

**Keywords:** Riemann hypothesis, Zeta function, Abel summation formula.

## 1 Introduction

We consider the representation of Riemann's zeta function [[1], page 14 Equation 2.1.5] defined for all  $s \in \mathbb{C}$  such that  $\Re(s) > 0$  by the Abel summation formula as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad (1)$$

where  $\{t\}$  is the fractional part of the real  $t$ . In this paper, we focus on the *non-trivial zeros* of the function  $\zeta$ , defined by Equation (1), in the following sense

**Definition 1.** Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non-trivial zero* of the function  $\zeta$  if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

The following theorem affirms the Riemann hypothesis.

**Main Result.** Consider the function  $\zeta$  defined by Equation (1). Suppose that  $s \in \mathbb{C}$  is a non-trivial zero of the function  $\zeta$  then  $\Re(s) = \frac{1}{2}$ .

## 2 Main proposition

The Equation (1) can be written as

$$\zeta(s) = s \int_1^{+\infty} \frac{1}{t^{2-s}} - \frac{\{t\}}{t^{1+s}} dt, \quad \forall s \in \mathbb{C}, \quad \Re(s) \in (0, 1).$$

By a change of variable,  $t \rightarrow \exp(u)$  we get

$$\begin{aligned} \forall s := x + i\tau \in \mathbb{C}, \quad x \in (0, 1), \quad \tau \in \mathbb{R}^* : \\ \zeta(s) := -s \int_0^{+\infty} \exp(-(1-s)u) + \exp(-su)\{\exp(u)\} du. \end{aligned}$$

For every  $\tau \in \mathbb{R}^*$ , define the function  $\phi_\tau : (0, 1) \times \mathbb{R}_+ \rightarrow \mathbb{R}$  as

$$\phi_\tau(x, t) := - \int_0^t \left[ \exp(-(1-x)u) - \exp(-xu)\{\exp(u)\} \right] \sin(\tau u) du. \quad (2)$$

Denote

$$\forall x \in (0, 1) : \quad \phi_\tau(x) = \lim_{t \rightarrow +\infty} \phi_\tau(x, t).$$

By identification, for every  $\tau \in \mathbb{R}^*$  and every  $x \in (0, 1)$ , we have  $\phi_\tau(x) = \Im\left(\frac{\zeta(s)}{s}\right)$  where  $s := x + i\tau$  and we have

$$\zeta(x + i\tau) = 0 \implies \phi_\tau(x) = 0. \quad (3)$$

**Proposition 2.** *Let be  $\tau \in \mathbb{R}^*$ . Then,*

$$\forall r \in (0, \frac{1}{2}) : \quad \phi_\tau(\frac{1}{2} - r) \neq \phi_\tau(\frac{1}{2} + r).$$

**Lemma 3.** *Let be  $\tau \in \mathbb{R}^*$  and  $z > 0$ . Then there exists  $\beta_{z,\tau} > 0$  such that for all  $s > v \geq 0$  we have*

$$\left| \int_v^s \exp(-zt)\{\exp(t)\} \sin(\tau t) dt - [w_\tau(z, s) - w_\tau(z, v)] \right| \leq h_\tau(z, \beta, v, s),$$

where

$$\begin{aligned} w_\tau(z, t) &:= -\frac{1}{2} \frac{1}{z^2 + \tau^2} \exp(-zt) [z \sin(\tau t) + \tau \cos(\tau t)], \quad \forall t \geq 0, \\ h_\tau(z, \beta, v, s) &:= \beta_{z,\tau} \left[ \exp(-(1+z)s) + \exp(-(1+z)v) \right], \quad \forall s > v. \end{aligned}$$

*Proof.* In order to simplify the notation, denote

$$I_\tau(z, s, v) := \int_v^s \exp(-zt) \{\exp(t)\} \sin(\tau t) dt.$$

Use the change of variable  $\exp(t) = u$  we get

$$I_\tau(z, s, v) := \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \{t\} \sin(\tau \ln(t)) dt.$$

Since the function  $u \mapsto \{u\}$  is 1-periodic function and since

$$\int_0^1 t dt = \frac{1}{2},$$

Then there exists a 1-periodic function  $t \mapsto p(t)$  such that

$$\int_s^t \{u\} du = \frac{1}{2}(t - s) + p(t) - p(s), \quad \forall t \geq s \geq 0.$$

Since the function  $u \mapsto \{u\}$  is continuous on  $\mathbb{R}/\mathbb{Z}$  then the function  $t \mapsto p(t)$  is continuous on  $\mathbb{R}_+$  and  $C^1$  on  $\mathbb{R}/\mathbb{Z}$ . Use the integration part formula,

$$\begin{aligned} I_\tau(z, s, v) &= \frac{1}{2} \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \sin(\tau \ln(t)) dt \\ &\quad + \exp(-(1+z)s) \sin(\tau s) (p(\exp(s)) - p(\exp(v))) \\ &\quad + \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{2+z}} f_\tau(z, t) (p(\exp(t)) - p(\exp(v))) dt, \\ f_\tau(z, t) &:= (1+z) \sin(\tau \ln(t)) + \tau \cos(\tau \ln(t)). \end{aligned} \tag{4}$$

Since the function  $t \mapsto p(t)$  is 1-periodic, there exists  $\alpha > 0$  such that

$$\sup_{t \geq s \geq 0} |p(t) - p(s)| < \alpha.$$

By Equation (4), we get

$$\begin{aligned} &\left| I_\tau(z, s, v) - \frac{1}{2} \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \sin(\tau \ln(t)) dt \right| \\ &\leq \alpha \left[ \exp(-(1+z)s) + [1+z + |\tau|] \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{2+z}} dt \right] \\ &\leq \beta_{\tau, \tau} \left[ \exp(-(1+z)s) + \exp(-(1+z)v) \right], \end{aligned}$$

where

$$\beta_{r,\tau} := \alpha \left[ 1 + \frac{|\tau|}{1+z} \right].$$

Use the fact

$$\begin{aligned} \int_{\exp(v)}^{\exp(s)} \frac{1}{t^{1+z}} \sin(\tau \ln(t)) dt &= \Im \left[ \int_{\exp(v)}^{\exp(s)} t^{1-z+i\tau} dt \right] \\ &= \frac{1}{z^2 + \tau^2} \exp(-zs) [z \sin(\tau s) + \tau \cos(\tau s)] \\ &\quad + \frac{1}{z^2 + \tau^2} \exp(-zv) [z \sin(\tau v) + \tau \cos(\tau v)], \end{aligned}$$

to deduce the Lemma.  $\square$

In order to simplify the notation, denote

$$A := \{t \geq 0, t \neq \ln(n), \quad \forall n \geq 2.\}.$$

In order to simplify the notation we denote on all this Section,

$$x_r := \frac{1}{2} - r.$$

**Lemma 4.** *Let be  $\tau \in \mathbb{R}^*$  and  $r \in (0, \frac{1}{2})$ . Let  $(\epsilon_n)_n, (v_n)_n, (s_n)_n \subset \mathbb{R}_+$  be a sequences such that*

- $\lim_{n \rightarrow +\infty} [\epsilon_n - \frac{1}{2} \Im [\frac{1}{x_r + i\tau} \exp(-i\tau v_n)]] \neq 0,$
- $\lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} v_n = \lim_{n \rightarrow +\infty} s_n - v_n = +\infty,$
- $s_n \leq 2v_n,$

For every  $n \in \mathbb{N}$ , let  $t \mapsto \Delta_\tau(r, n, t)$  be the continuous function satisfying the following differential equation,

$$\begin{aligned} \frac{d}{dt} \Delta_\tau(r, n, t) &= x_r \Delta_\tau(r, n, t) - [1 + \{\exp(t)\}] \sin(\tau t), \\ \Delta_\tau(r, n, v_n) &= \epsilon_n, \quad t \in A, t \geq v_n. \end{aligned} \tag{5}$$

Then,

$$\lim_{n \rightarrow +\infty} |\Delta_\tau(r, n, s_n)| = +\infty.$$

*Proof.* Integrate Equation (5) and use the fact  $\Delta_\tau(r, n, v_n) = \epsilon_n$ , we get

$$\begin{aligned}
\Delta_\tau(r, n, s_n) &= \exp\left(x_r(s_n - v_n)\right)\epsilon_n \\
&\quad + \int_{v_n}^{s_n} \exp\left(x_r(s_n - t)\right)[1 + \{\exp(t)\}] \sin(\tau t) dt \\
&= \exp\left(x_r(s_n - v_n)\right)\epsilon_n \\
&\quad + \int_{v_n}^{s_n} \exp\left(x_r(s_n - t)\right) \sin(\tau t) dt \\
&\quad + \int_{v_n}^{s_n} \exp\left(x_r(s_n - t)\right) \{\exp(t)\} \sin(\tau t) dt.
\end{aligned} \tag{6}$$

By Lemma 3,

$$\begin{aligned}
\left| \int_{v_n}^{s_n} \exp(-x_r t) \{\exp(t)\} \sin(\tau t) dt - [w_\tau(x_r, s_n) - w_\tau(x_r, v_n)] \right| \\
\leq h_\tau(z, \beta, v_n, s_n),
\end{aligned}$$

where

$$\begin{aligned}
w_\tau(z, t) &:= -\frac{1}{2} \frac{1}{z^2 + \tau^2} \exp(-zs) [z \sin(\tau s) + \tau \cos(\tau s)], \quad \forall t \geq 0, \\
h_\tau(z, \beta, v, s) &:= \beta_{\tau, r} \left[ \exp(-(1+z)s) + \left[1 + \frac{|\tau|}{1+z}\right] \exp(-(1+z)v) \right].
\end{aligned}$$

By hypothesis of the present Lemma we have

$$x_r s_n \leq 2x_r v_n \leq (2x_r + 1)v_n, \quad \text{and} \quad \lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} v_n = +\infty,$$

then

$$\lim_{n \rightarrow +\infty} \exp(x_r(s_n - v_n)) h_\tau(z, \beta, v_n, s_n) = 0.$$

Use the fact

$$\int_{v_n}^{s_n} \exp(-x_r t) \sin(\tau t) dt = 2[w_\tau(x_r, s_n) - w_\tau(x_r, v_n)]$$

We have

$$\lim_{n \rightarrow +\infty} \exp(-x_r(s_n - v_n)) \left[ \Delta_\tau(r, n, s_n) + \exp(x_r s_n) [w_\tau(x_r, s_n) - w_\tau(x_r, v_n)] \right] = 0.$$

We have

$$\begin{aligned} & \exp(x_r v_n)[w_\tau(x_r, s_n) - w_\tau(x_r, v_n)] \\ &= -\frac{1}{2} \frac{1}{x_r^2 + \tau^2} \exp(-x_r(s_n - v_n))[x_r \sin(\tau s_n) + \tau \cos(\tau s_n)] \\ &+ \frac{1}{2} \frac{1}{x_r^2 + \tau^2} [x_r \sin(\tau v_n) + \tau \cos(\tau v_n)]. \end{aligned}$$

By hypothesis of the present Lemma we have  $\lim_{n \rightarrow +\infty} s_n - v_n = +\infty$  and

$$\alpha(\tau, r) := \lim_{n \rightarrow +\infty} [\epsilon_n - \frac{1}{2} \Im[\frac{1}{x_r + i\tau} \exp(-i\tau v_n)]] \neq 0,$$

thanks to Equation (6), we get

$$\lim_{n \rightarrow +\infty} \exp(-x_r(s_n - v_n)) \Delta_\tau(r, n, s_n) = \alpha(\tau, r) \neq 0,$$

since  $\lim_{n \rightarrow +\infty} s_n - v_n = +\infty$ , then

$$\lim_{n \rightarrow +\infty} |\Delta_\tau(r, n, s_n)| = +\infty.$$

□

**Lemma 5.** *Let be  $r \in (0, \frac{1}{2})$  and  $t \mapsto \theta_\tau(r, t)$  be the continuous function satisfying the following differential equation,*

$$\begin{aligned} \frac{d}{dt} \theta_\tau(r, t) &= x_r \theta_\tau(r, t) - [1 - \exp(-2rt)][1 + \{\exp(t)\} \sin(\tau t)], \\ \theta_\tau(r, 0) &= 0, \quad t \in A. \end{aligned} \tag{7}$$

Then,

$$\sup_{t \geq 0} |\theta_\tau(r, t)| = +\infty.$$

*Proof.* By contradiction, suppose that there exists  $r \in (0, \frac{1}{2})$  and  $\omega_r > 0$  such that

$$\sup_{t \geq 0} |\theta_\tau(r, t)| \leq \omega_r. \tag{8}$$

Prove that there exists a sequence  $(v_n)_n \subset \mathbb{R}_+$  such that

$$\lim_{n \rightarrow +\infty} v_n = +\infty, \tag{9}$$

and such that

$$\lim_{n \rightarrow +\infty} \left[ \theta_\tau(r, v_n) - \frac{1}{2} \Im\left[\frac{1}{x_r + i\tau} \exp(-i\tau v_n)\right] \right] \neq 0. \tag{10}$$

By contradiction, suppose that

$$\lim_{t \rightarrow +\infty} \left[ \theta_\tau(r, t) - \frac{1}{2} \Im \left[ \frac{1}{x_r + i\tau} \exp(-i\tau t) \right] \right] = 0. \quad (11)$$

Since  $t \mapsto \theta_\tau(r, t)$  is a continuous function, integrate Equation (7), we obtain for all  $s \geq t \geq 0$  the following equality

$$\begin{aligned} \theta_\tau(r, t) - \theta_\tau(r, s) &= x_r \int_s^t \theta_\tau(r, u) du \\ &\quad - \int_s^t [1 - \exp(-2ru)][1 + \{\exp(u)\}] \sin(\tau u) du. \end{aligned} \quad (12)$$

By the hypothesis (11), for every sequences  $(u_n)_n, (\tilde{u}_n)_n \subset \mathbb{R}_+$  such that

$$\sup_{n \geq 0} |u_n - \tilde{u}_n| < +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} u_n = +\infty,$$

we find

$$\begin{aligned} &\lim_{n \rightarrow +\infty} [g_\tau(r, u_n) - g_\tau(r, \tilde{u}_n)] \\ &= \lim_{n \rightarrow +\infty} \int_{\tilde{u}_n}^{u_n} [1 - \exp(-2ru)][1 + \{\exp(u)\}] \sin(\tau u) du. \end{aligned}$$

where the  $C^\infty$  and  $\frac{2\pi}{\tau}$ -periodic function  $s \mapsto g_\tau(r, s)$  is defined as

$$g_\tau(r, s) := \frac{1}{2} \Im \left[ \frac{\tau - ix_r}{\tau(x_r + i\tau)} \exp(-i\tau s) \right], \quad \forall s \geq 0.$$

Contradiction. We deduce then Equation (9) and (10).

Now, for every  $n \in \mathbb{N}$ , let  $t \mapsto \Delta_\tau(r, n, t)$  be the continuous function satisfying the following differential equation,

$$\begin{aligned} \frac{d}{dt} \Delta_\tau(r, n, t) &= x_r \Delta_\tau(r, n, t) - [1 + \{\exp(t)\}] \sin(\tau t), \\ \Delta_\tau(r, n, v_n) &= \theta_\tau(r, v_n), \quad t \in A, \quad t \geq v_n. \end{aligned} \quad (13)$$

Denote  $\eta_\tau(x, n, t) := \theta_\tau(r, t) - \Delta_\tau(r, n, t)$ . By Equations (7) and (13),

$$\frac{d}{dt} \eta_\tau(x, n, t) = x_r \eta_\tau(x, n, t) + \exp(-2rt)[1 + \{\exp(t)\}] \sin(\tau t), \quad \eta_\tau(x, n, v_n) = 0.$$



where  $t \in A$  and  $t \geq v_n$ . Since  $t \mapsto \eta_\tau(x, n, t)$  is a continuous function, integrate,

$$\begin{aligned} |\eta_\tau(x, n, t)| &\leq 2 \exp(-2v_n) \exp(x_r t) \int_{v_n}^t \exp(-x_r s) ds \\ &= \frac{2}{x_r} \exp(-2v_n) \left[ \exp(x_r(t - v_n)) - 1 \right]. \\ |\eta_\tau(x, n, t)| &\leq \frac{4}{1 - 2r} [\exp(x_r(t - v_n) - 2v_n) - \exp(-2v_n)]. \end{aligned}$$

For

$$s_n \in [v_n, v_n + \frac{1}{x_r} 2v_n],$$

we get

$$\sup_{n \geq 0} |\eta_\tau(x, n, s_n)| \leq \frac{2}{x_r}. \quad (14)$$

By Equation (9), we have  $\lim_{n \rightarrow +\infty} v_n = +\infty$ . Then we can choose  $(s_n)_n$  such that

- $\lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} s_n - v_n = +\infty$ ,
- $s_n \leq 2v_n$ .

By Equations (10), (13) and by Lemma 4, we obtain

$$\lim_{n \rightarrow +\infty} |\Delta_\tau(r, n, s_n)| = +\infty.$$

Since  $\eta_\tau(x, n, t) := \theta_\tau(r, t) - \Delta_\tau(r, n, t)$ , thanks to Equation (14) we find,

$$\lim_{n \rightarrow +\infty} |\theta_\tau(r, t_n)| = +\infty.$$

Contradiction with the hypothesis (8). □

*Proof of Proposition 2 .* Equation (2), implies that  $\forall t \geq 0$  we have

$$\begin{aligned} \phi_\tau\left(\frac{1}{2} + r, t\right) &= - \int_0^t \exp\left(-\frac{1}{2}u\right) [\exp(ru) - \exp(-ru) \{\exp(u)\}] \sin(\tau u) du, \\ \phi_\tau\left(\frac{1}{2} - r, t\right) &= - \int_0^t \exp\left(-\frac{1}{2}u\right) [\exp(-ru) - \exp(ru) \{\exp(u)\}] \sin(\tau u) du. \end{aligned}$$

Denote

$$y_\tau(r, t) := \exp\left(\frac{1}{2}t\right) \left[ \phi_\tau\left(\frac{1}{2} + r, t\right) - \phi_\tau\left(\frac{1}{2} - r, t\right) \right], \quad \forall r \in \left(0, \frac{1}{2}\right), \quad \forall t \geq 0.$$

Then

$$y_\tau(r, t) = -2 \int_0^t \exp\left(\frac{1}{2}(t-u)\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du. \quad (15)$$

The function  $t \mapsto y_\tau(r, t)$  is a continuous function and satisfies the following differential equation

$$\begin{aligned} \frac{d}{dt} y_\tau(r, t) &= \frac{1}{2} y_\tau(r, t) - 2 \sinh(rt) [1 + \{\exp(t)\}] \sin(\tau t), \\ y_\tau(r, 0) &= 0, \quad \forall t \in A. \end{aligned} \quad (16)$$

By contradiction, suppose that there exists  $r \in (0, \frac{1}{2})$  such that

$$\phi_\tau\left(\frac{1}{2} - r\right) = \phi_\tau\left(\frac{1}{2} + r\right).$$

Then

$$\int_0^{+\infty} \exp\left(-\frac{1}{2}u\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du = 0,$$

implies

$$\begin{aligned} &\int_0^t \exp\left(-\frac{1}{2}u\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du \\ &= - \int_t^{+\infty} \exp\left(-\frac{1}{2}u\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du. \end{aligned}$$

Then

$$\begin{aligned} &2 \left| \int_0^t \exp\left(-\frac{1}{2}u\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du \right| \\ &\leq \frac{2}{\frac{1}{2} - r} \exp\left(-\left(\frac{1}{2} - r\right)t\right) + \frac{2}{\frac{1}{2} + r} \exp\left(-\left(\frac{1}{2} + r\right)t\right), \quad \forall t \geq 0. \end{aligned}$$

Denote  $\theta_\tau(r, t) := \exp(-rt)y_\tau(r, t)$ . By the last Inequality and Equation (15), we get

$$\sup_{t \geq 0} |\theta_\tau(r, t)| = \sup_{t \geq 0} |\exp(-rt)y_\tau(r, t)| \leq \frac{2}{\frac{1}{4} - r^2} \quad (17)$$

By Equation (16), The function  $t \mapsto \theta_\tau(r, t)$  is a continuous function satisfies  $\theta_\tau(r, 0) = 0$  and satisfies the following differential equation

$$\frac{d}{dt} \theta_\tau(r, t) = \left(\frac{1}{2} - r\right) \theta_\tau(r, t) - [1 - \exp(-2rt)] [1 + \{\exp(t)\}] \sin(\tau t), \quad t \in A.$$

By Lemma 5 we have  $\sup_{t \geq 0} |\theta_\tau(r, t)| = +\infty$ . Contradiction with Equation (17). Then

$$\forall r \in (0, \frac{1}{2}) : \quad \phi_\tau(\frac{1}{2} - r) \neq \phi_\tau(\frac{1}{2} + r).$$

□

### 3 Proof of the Main result

*Proof of the Main result.* Suppose that there exist  $\tau \in \mathbb{R}^*$  and  $r \in [0, \frac{1}{2})$  such that

$$\zeta(\frac{1}{2} - r + i\tau) = 0,$$

by Equation (3), we get

$$\phi_\tau(\frac{1}{2} - r) = 0.$$

Using the Riemann's functional equation [[1], page 13 Theorem 2.1], the non-trivial zeros of the function  $\zeta$  are symmetric about the line  $\Re(s) = \frac{1}{2}$ . We get

$$\zeta(\frac{1}{2} + r + i\tau) = 0.$$

By Equation (3) we obtain

$$\phi_\tau(\frac{1}{2} + r) = 0.$$

By Proposition 2, we obtain  $r = \frac{1}{2}$ . □

### References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).