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Non symmetry of the function $\frac{\zeta(s)}{s}$

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Abstract

We prove that $|\frac{\zeta(\bar{s})}{\bar{s}} - \frac{\zeta(1-s)}{1-s}| \neq 0$ for every $\Re(s) \in (0, \frac{1}{2})$ and $\Im(s) \in \mathbb{R}^*$, where ζ is the Riemann Zeta function. At the end of the paper, we give a discussion about the Riemann hypothesis.

Keywords: Riemann functional equation, Zeta function.

AMS subject classifications: 00A05

1 Main result

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where $\{t\}$ is the fractional part of the real t . We prove in this paper the following Main result

Main Result. *Consider the function ζ defined by Equation (1). Then*

$$\left| \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau} \right| \neq 0, \quad \forall r \in (0, \frac{1}{2}), \quad \forall \tau \in \mathbb{R}^*.$$

2 Main Proposition

Notation 1. In order to simplify the notation, for every $r \in (0, \frac{1}{2})$ and $\tau \in \mathbb{R}^*$ denote

$$\phi(\tau, r) := \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau}.$$

For every $r \in (0, \frac{1}{2})$ denote

$$f_r(u) := \{u\} \frac{1}{u^{\frac{3}{2}}} (u^r - u^{-r}), \quad u > 0. \quad (2)$$

By Equation (1) we have

$$\phi(\tau, r) = \int_0^{+\infty} u^{-i\tau} f_r(u) du,$$

The aim is to study the differential equation of the solution function

$$t \mapsto t^{i\tau} \left[z + \int_0^t u^{-i\tau} f_r(u) du \right], \quad t > 0, \quad z \in \mathbb{C}.$$

We focus only on the solutions that converge to zero. More precisely, the strategy to prove the Main result, is to prove that the above function converges to zero when $t \rightarrow +\infty$ implies that $|z| \neq 0$ (the initial condition). We deduce that $\phi(\tau, r) \neq 0$.

For every $\tau \in \mathbb{R}^*$ and $r \in (0, \frac{1}{2})$ we consider the following differential equation

$$\frac{d}{dt} x = \frac{i\tau}{t} x + f_r(t), \quad t \in \mathbb{R}_+/\mathbb{N}, \quad x(0) = 0, \quad x : (0, +\infty) \rightarrow \mathbb{C}. \quad (3)$$

Lemma 2. For every $\tau \in \mathbb{R}^*$ and $r \in (0, \frac{1}{2})$ there exists a unique continuous solution $\psi_{\tau, r}(t) : [0, +\infty) \rightarrow \mathbb{C}$ of the differential equation (3). Further,

$$\psi_{\tau, r}(t) = t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du, \quad \forall t > 0,$$

and

$$\psi_{\tau, r}(t) = \frac{t^{\frac{1}{2}+r}}{-i\tau + \frac{1}{2} + r} - \frac{t^{\frac{1}{2}-r}}{-i\tau + \frac{1}{2} - r}, \quad \forall t \in (0, 1).$$

Proof. Since $0 \leq f_r(u) \leq \frac{1}{t^{\frac{1}{2}}}(u^r - u^{-r})$ for every $u > 1$ and $f_r(u) = \frac{1}{u^{\frac{1}{2}}}(u^r - t^{-r})$ for every $u \in (0, 1)$ then the function

$$t \mapsto t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du,$$

is continuous and C^1 on \mathbb{R}_+/\mathbb{N} . The Equation (3) is a non-homogeneous linear differential equation, then unique continuous solution $\psi_{\tau,r}(t) : (0, +\infty) \rightarrow \mathbb{C}$ such that $[\psi_{\tau,r}(0)] = 0$, is given by

$$\psi_{\tau,r}(t) = t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du, \quad \forall t > 0.$$

Since $f_r(u) = t^{-\frac{1}{2}}(t^r - t^{-r})$ for every $t \in (0, 1)$, then

$$\psi_{\tau,r}(t) = \frac{t^{\frac{1}{2}+r}}{-i\tau + \frac{1}{2} + r} - \frac{t^{\frac{1}{2}-r}}{-i\tau + \frac{1}{2} - r}, \quad \forall t \in (0, 1).$$

□

Proposition 3. For every $\tau \in \mathbb{R}^*$ and $r \in (0, \frac{1}{2})$ the unique continuous solution $\psi_{\tau,r}(t) : [0, +\infty) \rightarrow \mathbb{C}$ of of the differential equation (3) satisfies

$$\lim_{t \rightarrow +\infty} |\psi_{\tau,r}(t)| \neq 0.$$

Lemma 4. For every $r \in (0, \frac{1}{2})$ we have

$$\int_0^{+\infty} f_r(v) dv < 0.$$

Proof. We have

$$\int_0^{+\infty} f_r(u) du = \int_0^1 f_r(s) ds + \int_1^{+\infty} f_r(s) ds.$$

Use the change of variable $s \mapsto \frac{1}{u}$ for the first integral, we get

$$\int_0^{+\infty} f_r(u) du = \int_1^{+\infty} \frac{1}{u^2} f_r\left(\frac{1}{u}\right) du + \int_1^{+\infty} f_r(s) ds.$$

By definition of f_r we have

$$f_r(u) = \{u\} u^{-\frac{3}{2}} (u^r - u^{-r}).$$

In particular

$$f_r(u) = u^{-\frac{1}{2}}(u^r - u^{-r}), \quad \forall u \in (0, 1).$$

Then

$$\begin{aligned} \int_0^{+\infty} f_r(u) du &= - \int_1^{+\infty} u^{-\frac{3}{2}}(u^r - u^{-r}) ds + \int_1^{+\infty} f_r(u) du \\ &= - \int_1^{+\infty} (1 - \{u\}) u^{-\frac{3}{2}}(u^r - u^{-r}) ds < 0, \quad \forall r \in (0, \frac{1}{2}). \end{aligned}$$

□

Proof of the Main Proposition 3. Let be $r \in (0, \frac{1}{2})$ and $\tau \in \mathbb{R}^*$ fixed. By contradiction, suppose that

$$\lim_{t \rightarrow +\infty} |\psi_{\tau, r}(t)| = 0. \quad (4)$$

Equation (3) implies

$$\psi_{\tau, r}(t) = -i\tau \int_t^{+\infty} \frac{1}{u} \psi_{\tau, r}(u) du - \int_t^{+\infty} f_r(u) du, \quad \forall t > 0, \quad (5)$$

In order to simplify the notation, denote

$$\Psi_{\tau, r}(t) := \int_t^{+\infty} \frac{1}{u} \psi_{\tau, r}(u) du, \quad \forall t > 0.$$

Equation (5) implies

$$\frac{d}{dt} \Psi_{\tau, r} = i\tau \frac{1}{t} \Psi_{\tau, r} + \frac{1}{t} \int_t^{+\infty} f_r(u) du, \quad \forall t > 0,$$

which is a non-homogeneous linear differential equation. Integrate, for every $t' > t > 0$ we have

$$\Psi_{\tau, r}(t') = \left(\frac{t'}{t}\right)^{i\tau} \left[\Psi_{\tau, r}(t) + \int_t^{t'} \left(\frac{t}{u}\right)^{i\tau} \frac{1}{u} \left[\int_u^{+\infty} f_r(v) dv \right] du \right]. \quad (6)$$

By the hypothesis (4) and Lemma 2 we get

$$\int_0^{+\infty} u^{-i\tau} f_r(u) du = 0,$$

then

$$\int_0^t u^{-i\tau} f_r(u) du = - \int_t^{+\infty} u^{-i\tau} f_r(u) du, \quad \forall t > 0.$$

implies

$$|\psi_{\tau,r}(t)| \leq \int_t^{+\infty} |f_r(u)| du, \quad \forall t > 0.$$

By definition of $\Psi_{\tau,r}$ and f_r we get

$$\lim_{t' \rightarrow +\infty} |\Psi_{\tau,r}(t')| = 0.$$

Equation (6) implies that for every $t > 0$ we have

$$\Psi_{\tau,r}(t) = -t^{i\tau} \int_t^{+\infty} u^{-i\tau-1} \left[\int_u^{+\infty} f_r(v) dv \right] du.$$

In order to simplify the notation, denote

$$\omega_{\tau,r}(t) := \int_t^{+\infty} u^{-i\tau-1} \left[\int_u^{+\infty} f_r(v) dv \right] du, \quad \forall t > 0. \quad (7)$$

Then

$$\Psi_{\tau,r}(t) = -t^{i\tau} \omega_{\tau,r}(t), \quad \forall t > 0. \quad (8)$$

By Equation (5)

$$\Psi_{\tau,r}(t) = -\frac{1}{i\tau} \psi_{\tau,r}(t) - \frac{1}{i\tau} \int_t^{+\infty} f_r(u) du, \quad \forall t > 0,$$

By Lemma 2

$$\begin{aligned} \Psi_{\tau,r}(t) &= -\frac{1}{i\tau} \left[\frac{t^{\frac{1}{2}+r}}{-i\tau + \frac{1}{2} + r} - \frac{t^{\frac{1}{2}-r}}{-i\tau + \frac{1}{2} - r} \right] \\ &\quad - \frac{1}{i\tau} \int_t^{+\infty} f_r(u) du, \quad \forall t \in (0, 1). \end{aligned}$$

Use the fact $f_r(u) = u^{-\frac{1}{2}+r} - u^{-\frac{1}{2}-r}$ for all $u \in (0, 1)$, we obtain

$$\begin{aligned} \Psi_{\tau,r}(t) &= -\frac{1}{i\tau} \left[\frac{t^{\frac{1}{2}+r}}{-i\tau + \frac{1}{2} + r} - \frac{t^{\frac{1}{2}-r}}{-i\tau + \frac{1}{2} - r} \right] \\ &\quad - \frac{1}{i\tau} \left[\frac{t^{-\frac{1}{2}+r}}{-\frac{1}{2} + r} - \frac{t^{-\frac{1}{2}-r}}{-\frac{1}{2} - r} \right], \quad \forall t \in (0, 1). \end{aligned}$$

By Equation (8),

$$\begin{aligned}\Re\left(\omega_{\tau,r}(t)\right) &= t^{\frac{1}{2}+r} \frac{(\frac{1}{2}+r) \cos(\tau \ln(t)) + \tau \sin(\tau \ln(t))}{(\tau^2 + (\frac{1}{2}+r)^2)(\frac{1}{2}+r)} \\ &\quad - t^{\frac{1}{2}-r} \frac{(\frac{1}{2}-r) \cos(\tau \ln(t)) + \tau \sin(\tau \ln(t))}{(\tau^2 + (\frac{1}{2}-r)^2)(\frac{1}{2}-r)} \\ &\quad - \frac{1}{\tau} \sin(\tau \ln(t)) \int_0^{+\infty} f_r(v) dv \quad \forall t \in (0, 1),\end{aligned}$$

and

$$\begin{aligned}\Im\left(\omega_{\tau,r}(t)\right) &= -t^{\frac{1}{2}+r} \frac{(\frac{1}{2}+r) \sin(\tau \ln(t)) - \tau \cos(\tau \ln(t))}{(\tau^2 + (\frac{1}{2}+r)^2)(\frac{1}{2}+r)} \\ &\quad + t^{\frac{1}{2}-r} \frac{(\frac{1}{2}-r) \sin(\tau \ln(t)) - \tau \cos(\tau \ln(t))}{(\tau^2 + (\frac{1}{2}-r)^2)(\frac{1}{2}-r)} \\ &\quad - \frac{1}{\tau} \cos(\tau \ln(t)) \int_0^{+\infty} f_r(v) dv, \quad \forall t \in (0, 1).\end{aligned}$$

In order to simplify the notation, denote

$$\begin{aligned}R_n &:= \Re\left(\omega_{\tau,r}(t_n)\right), & t_n &:= \exp\left(-\frac{2n\pi}{\tau}\right), \\ I_n &:= \Im\left(\omega_{\tau,r}(s_n)\right), & s_n &:= \exp\left(-\frac{2n\pi}{\tau} - \frac{\pi}{2\tau}\right) \quad \forall n \in \mathbb{N}.\end{aligned}$$

Then for every $n \in \mathbb{N}$ we have

$$\begin{aligned}R_n &= \exp\left(-\left(\frac{1}{2}+r\right)\frac{2n\pi}{\tau}\right) \frac{1}{\tau^2 + (\frac{1}{2}+r)^2} \\ &\quad - \exp\left(-\left(\frac{1}{2}-r\right)\frac{2n\pi}{\tau}\right) \frac{1}{\tau^2 + (\frac{1}{2}-r)^2},\end{aligned}$$

and

$$\begin{aligned}I_n &= \exp\left(-\left(\frac{1}{2}+r\right)\left(\frac{2n\pi}{\tau} + \frac{\pi}{2\tau}\right)\right) \frac{1}{\tau^2 + (\frac{1}{2}+r)^2} \\ &\quad - \exp\left(-\left(\frac{1}{2}-r\right)\left(\frac{2n\pi}{\tau} + \frac{\pi}{2\tau}\right)\right) \frac{1}{\tau^2 + (\frac{1}{2}-r)^2}.\end{aligned}$$

Implies

$$\begin{aligned}R_{n+1} - R_n + I_{n+1} - I_n \\ = -\exp\left(-\left(\frac{1}{2}+r\right)\frac{2n\pi}{\tau}\right) \alpha_{\tau,r} + \exp\left(-\left(\frac{1}{2}-r\right)\frac{2n\pi}{\tau}\right) \beta_{\tau,r}.\end{aligned}$$

where

$$\alpha_{\tau,r} := \left(1 - \exp\left(-\left(\frac{1}{2} + r\right)\frac{2\pi}{\tau}\right)\right) \left(\exp\left(-\left(\frac{1}{2} + r\right)\frac{\pi}{2\tau}\right) + 1\right) \frac{1}{\tau^2 + \left(\frac{1}{2} + r\right)^2},$$

$$\beta_{\tau,r} := \left(1 - \exp\left(-\left(\frac{1}{2} - r\right)\frac{2\pi}{\tau}\right)\right) \left(\exp\left(-\left(\frac{1}{2} - r\right)\frac{\pi}{2\tau}\right) + 1\right) \frac{1}{\tau^2 + \left(\frac{1}{2} - r\right)^2}.$$

Since $r \in (0, \frac{1}{2})$, there exists $n_{\tau,r} > 0$ such that

$$R_{n+1} - R_n + I_{n+1} - I_n > 0, \quad \forall n > n_{\tau,r}. \quad (9)$$

By Equation (7), for every $n \in \mathbb{N}$ we have

$$\begin{aligned} R_{n+1} - R_n + I_{n+1} - I_n &= \int_{t_{n+1}}^{t_n} \cos(\tau \ln(u)) \frac{1}{u} \left[\int_u^{+\infty} f_r(v) dv \right] du \\ &\quad - \int_{s_{n+1}}^{s_n} \sin(\tau \ln(u)) \frac{1}{u} \left[\int_u^{+\infty} f_r(v) dv \right] du \\ &= \int_{t_{n+1}}^{t_n} \cos(\tau \ln(u)) \frac{1}{u} h_{\tau,r}(u) du \end{aligned} \quad (10)$$

where

$$h_{\tau,r}(u) := \int_u^{+\infty} f_r(v) dv + \int_{u \exp(-\frac{\pi}{2\tau})}^{+\infty} f_r(v) dv.$$

Since $f_r(u) := u^{-\frac{1}{2}+r} - u^{-\frac{1}{2}-r}$ for all $u \in (0, 1)$ then

$$\frac{d}{du} h_{\tau,r}(u) = -f_r(u) - \exp\left(-\frac{\pi}{2\tau}\right) f_r\left(u \exp\left(-\frac{\pi}{2\tau}\right)\right) > 0, \quad \forall u \in (0, 1),$$

and

$$\frac{d^2}{du^2} h_{\tau,r}(u) = -\frac{d}{du} f_r(u) - \exp\left(-2\frac{\pi}{2\tau}\right) \frac{d}{du} f_r\left(u \exp\left(-\frac{\pi}{2\tau}\right)\right) < 0, \quad \forall u \in (0, 1),$$

By Lemma 4, there exists $t_{\tau,r} \in (0, 1)$ such that

$$h_{\tau,r}(u) < 0, \quad \forall (0, t_{\tau,r}),$$

Since $t_n = \exp\left(-\frac{2n\pi}{\tau}\right) \rightarrow 0$ then there exists $\tilde{n}_{\tau,r} > 0$ such that the Equation (10) implies

$$R_{n+1} - R_n + I_{n+1} - I_n < 0, \quad \forall n > \tilde{n}_{\tau,r}.$$

We have obtained a contradiction with Equation (9). \square

3 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function ζ defined by Equation (1). Define the *non trivial zeros* of the function ζ in the following sense

Definition 5. Let be $s \in \mathbb{C}$. We say that s is a *non trivial zero of the function* ζ if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

By the Riemann functional equation [[1], page 13 Theorem 2.1], the non trivial zeros of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. By the Main result, the non trivial zeros $s \in \mathbb{C}$ symmetric about the line $\Re(z) = \frac{1}{2}$ satisfies $\Re(s) = \frac{1}{2}$.

Theorem 6 (Riemann hypothesis). *Let $s \in \mathbb{C}$ be a non-trivial zero of the function ζ . Then $\Re(s) = \frac{1}{2}$.*

Proof. By contradiction, suppose that there exists a non-trivial zero $s_* \in \mathbb{C}$ of the function ζ such that $\Re(s_*) \neq \frac{1}{2}$. Then

$$\frac{\zeta(s_*)}{s_*} = 0.$$

The Main result implies

$$\frac{\zeta(1 - s_*)}{1 - s_*} \neq 0. \tag{11}$$

But the Riemann functional equation implies that the non-trivial zero of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. In other words,

$$\zeta(1 - s_*) = 0.$$

We obtain a contradiction with the Equation (11). We deduce that $\Re(s_*) = \frac{1}{2}$. \square

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).