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# Non symmetry of the function $\frac{\zeta(s)}{s}$

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## Abstract

We prove that  $|\frac{\zeta(\bar{s})}{\bar{s}} - \frac{\zeta(1-s)}{1-s}| \neq 0$  for every  $\Re(s) \in (0, \frac{1}{2})$  and  $\Im(s) \in \mathbb{R}^*$ , where  $\zeta$  is the Riemann Zeta function. At the end of the paper, we give a discussion about the Riemann hypothesis.

**Keywords:** Riemann functional equation, Zeta function.

**AMS subject classifications:** 00A05

## 1 Main result

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where  $\{t\}$  is the fractional part of the real  $t$ . We prove in this paper the following Main result

**Main Result.** *Consider the function  $\zeta$  defined by Equation (1). Then*

$$\left| \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau} \right| \neq 0, \quad \forall r \in (0, \frac{1}{2}), \quad \forall \tau \in \mathbb{R}^*.$$

**Notation 1.** For every  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$  denote

$$\phi(\tau, r) := \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau}.$$

For every  $r \in (0, \frac{1}{2})$  denote

$$f_r(u) := \{u\} \frac{1}{u^{\frac{3}{2}}} (u^r - u^{-r}), \quad u > 0.$$

By Equation (1) we have

$$\phi(\tau, r) = \int_0^{+\infty} u^{-i\tau} f_r(u) du,$$

The aim is to study the differential equation of the solution function

$$t \mapsto t^{i\tau} \left[ z + \int_0^t u^{-i\tau} f_r(u) du \right], \quad t > 0, \quad z \in \mathbb{C}.$$

We focus only on the solutions that converge to zero. More precisely, the strategy to prove the Main result, is to prove that the above function converges to zero when  $t \rightarrow +\infty$  implies that  $|z| \neq 0$  (the initial condition). We deduce that  $\phi(\tau, r) \neq 0$ .

**Lemma 2.** *For every  $\tau > 0$  and  $r \in (0, \frac{1}{2})$ , the unique continuous solution  $\psi_{\tau, r}(t) : [0, +\infty) \rightarrow \mathbb{C}$  of the following differential equation*

$$\frac{d}{dt} \psi_{\tau, r}(t) = \frac{i\tau}{t} \psi_{\tau, r}(t) + f_r(t), \quad t \in \mathbb{R}_+/\mathbb{N}, \quad \psi_{\tau, r}(0) = 0, \quad (2)$$

is defined by

$$\psi_{\tau, r}(t) = t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du, \quad \forall t > 0.$$

*Proof.* Since  $0 \leq f_r(u) \leq \frac{1}{t^{\frac{3}{2}}} (t^r - t^{-r})$  for every  $t > 1$  and  $0 \leq f_r(u) \leq \frac{1}{t^{\frac{3}{2}}} (t^r - t^{-r})$  for every  $t \in (0, 1)$  then the function

$$t \mapsto t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du,$$

is continuous and  $C^1$  on  $\mathbb{R}_+/\mathbb{N}$ . The Equation (2) is a non-homogeneous linear differential equation, the unique continuous solution is given by

$$\psi_{\tau, r}(t) = t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du.$$

□

**Lemma 3.** For every  $r \in (0, \frac{1}{2})$  there exists a continuous and pies-wise  $C^1$  function  $\epsilon_r : (1, +\infty) \rightarrow \mathbb{R}$  such that

$$\int_{\epsilon_r(t)}^t f_r(u) du = 0,$$

In addition,

$$\epsilon_r(t) \in (0, 1) \quad \text{and} \quad \frac{d}{dt} \epsilon_r(t) < 0, \quad \forall t > 1.$$

*Proof.* We have

$$\int_0^{+\infty} f_r(s) ds = \int_0^1 f_r(s) ds + \int_1^{+\infty} f_r(s) ds.$$

Use the change of variable  $s \mapsto \frac{1}{u}$  for the first integral, we get

$$\int_0^{+\infty} f_r(s) ds = \int_1^{+\infty} \frac{1}{u^2} f_r\left(\frac{1}{u}\right) du + \int_1^{+\infty} f_r(s) ds.$$

By definition of  $f_r$  we have

$$f_r(u) = \frac{1}{u^{\frac{3}{2}}} \{u\} (u^r - u^{-r}).$$

In particular

$$f_r(u) = \frac{1}{u^{\frac{3}{2}}} (u^r - u^{-r}), \quad \forall u \in (0, 1).$$

Then

$$\begin{aligned} \int_0^{+\infty} f_r(s) ds &= - \int_1^{+\infty} \frac{1}{u^{\frac{3}{2}}} (u^r - u^{-r}) ds + \int_1^{+\infty} f_r(u) du \\ &= - \int_1^{+\infty} (1 - \{u\}) \frac{1}{u^{\frac{3}{2}}} (u^r - u^{-r}) ds < 0, \quad \forall r \in (0, \frac{1}{2}). \end{aligned}$$

□

*Proof of the Main result.* Let be  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$  fixed. The Main result is proved if we prove that  $\phi(\tau, r) \neq 0$ . We have

$$\phi(\tau, r) = \int_0^{+\infty} u^{-i\tau} f_r(u) du.$$

By Lemma 2 , we have

$$\lim_{t \rightarrow +\infty} |\psi_{\tau, r}(t)| = \phi(\tau, r),$$

then main result is proved if we prove that  $\lim_{t \rightarrow +\infty} |\psi_{\tau,r}(t)| \neq 0$ . Let be  $\epsilon_r$  the function defined in Lemma 3. Equation (2) implies

$$\psi_{\tau,r}(t) - \psi_{\tau,r}(\epsilon_r(t)) = i\tau \int_{\epsilon_r(t)}^t \frac{1}{u} \psi_{\tau,r}(u) du, \quad \forall t \geq 1.$$

By contradiction, suppose that

$$\lim_{t \rightarrow +\infty} |\psi_{\tau,r}(t)| = 0.$$

Then

$$\psi_{\tau,r}(\epsilon_r(t)) = -i\tau \int_{\epsilon_r(t)}^{+\infty} \frac{1}{u} \psi_{\tau,r}(u) du, \quad \forall t \geq 1,$$

implies

$$\frac{d}{dt} [\psi_{\tau,r}(\epsilon_r(t))] = i\tau \frac{\frac{d}{dt} \epsilon_r(t)}{\epsilon_r(t)} \psi_{\tau,r}(\epsilon_r(t)), \quad \forall t \geq 1. \quad (3)$$

which is a linear differential equation. By Lemma 3 we have  $\frac{d}{dt} \epsilon_r(t) < 0$  and  $\epsilon_r(t) \in (0, 1)$  for every  $t > 1$ . Integrate the Equation (3), for every  $t > t' > 1$  we have

$$\psi_{\tau,r}(\epsilon_r(t)) = \left( \frac{\epsilon_r(t)}{\epsilon_r(t')} \right)^{i\tau} \psi_{\tau,r}(\epsilon_r(t')). \quad (4)$$

Since  $f_r(u) = t^{-\frac{1}{2}}(t^r - t^{-r})$  for every  $t \in (0, 1)$  by Lemma 2, we have

$$\psi_{\tau,r}(u) = \frac{u^{\frac{1}{2}+r}}{-i\tau + \frac{1}{2} + r} - \frac{u^{\frac{1}{2}-r}}{-i\tau + \frac{1}{2} - r}, \quad \forall u \in (0, 1). \quad (5)$$

Then  $|\psi_{\tau,r}(\epsilon_r(t))| \neq 0$  for every  $t \geq 1$ . By Equation (4), there exists  $t_* \in (0, 1)$  such that

$$\Im(\psi_{\tau,r}(\epsilon_r(t_*))) = 0$$

Contradiction with Equation (5). We deduce that  $\phi(\tau, r) \neq 0$  for every  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$ .  $\square$

## 2 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function  $\zeta$  defined by Equation (1). Define the *non trivial zeros* of the function  $\zeta$  in the following sense

**Definition 4.** Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non trivial zero of the function  $\zeta$*  if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

By the Riemann functional equation [[1], page 13 Theorem 2.1], the non trivial zeros of the function  $\zeta$  are symmetric about the line  $\Re(z) = \frac{1}{2}$ . By the Main result, the non trivial zeros  $s \in \mathbb{C}$  symmetric about the line  $\Re(z) = \frac{1}{2}$  satisfies  $\Re(s) = \frac{1}{2}$ .

**Theorem 5** (Riemann hypothesis). *Let  $s \in \mathbb{C}$  be a non-trivial zero of the function  $\zeta$ . Then  $\Re(s) = \frac{1}{2}$ .*

*Proof.* By contradiction, suppose that there exists a non-trivial zero  $s_* \in \mathbb{C}$  of the function  $\zeta$  such that  $\Re(s_*) \neq \frac{1}{2}$ . Then

$$\frac{\zeta(s_*)}{s_*} = 0.$$

The Main result implies

$$\frac{\zeta(1 - s_*)}{1 - s_*} \neq 0. \tag{6}$$

But the Riemann functional equation implies that the non-trivial zero of the function  $\zeta$  are symmetric about the line  $\Re(z) = \frac{1}{2}$ . In other words,

$$\zeta(1 - s_*) = 0.$$

We obtain a contradiction with the Equation (6). We deduce that  $\Re(s_*) = \frac{1}{2}$ .  $\square$

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).