



# Non symmetry of the function $\frac{\zeta(s)}{s}$

W Oukil

## ► To cite this version:

| W Oukil. Non symmetry of the function  $\frac{\zeta(s)}{s}$ . 2022. hal-03581978v18

**HAL Id: hal-03581978**

**<https://hal.science/hal-03581978v18>**

Preprint submitted on 10 Oct 2022 (v18), last revised 26 Nov 2023 (v25)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Non symmetry of the function $\frac{\zeta(s)}{s}$

W. Oukil

Laboratory of Dynamical Systems, Faculty of Mathematics.  
University of Science and Technology Houari Boumediene,  
BP 32 EL ALIA 16111 Bab Ezzouar, Algiers, Algeria.

October 10, 2022

## Abstract

We prove that  $|\frac{\zeta(\overline{s})}{\overline{s}} - \frac{\zeta(1-s)}{1-s}| \neq 0$  for every  $\Re(s) \in (0, \frac{1}{2})$  and  $\Im(s) \in \mathbb{R}^*$ , where  $\zeta$  is the Riemann Zeta function. At the end of the paper, we give a discussion about the Riemann hypothesis.

**Keywords:** Riemann functional equation, Zeta function.

**AMS subject classifications:** 00A05

## 1 Main result

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where  $\{t\}$  is the fractional part of the real  $t$ . We prove in this paper the following Main result

**Main Result.** *Consider the function  $\zeta$  defined by Equation (1). Then*

$$\left| \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau} \right| \neq 0, \quad \forall r \in (0, \frac{1}{2}), \quad \forall \tau \in \mathbb{R}^*.$$

**Notation 1.** For every  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$  denote

$$\phi(\tau, r) := \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau}.$$

For every  $r \in (0, \frac{1}{2})$  denote

$$f_r(u) := \{u\} \frac{1}{u^{\frac{3}{2}}}(u^r - u^{-r}), \quad u > 0.$$

By Equation (1) we have

$$\phi(\tau, r) = \int_0^{+\infty} u^{-i\tau} f_r(u) du,$$

The aim is to study the differential equation of the solution function

$$t \mapsto t^{i\tau} \left[ z + \int_0^t u^{-i\tau} f_r(u) du \right], \quad t > 0, \quad z \in \mathbb{C}.$$

We focus only on the solutions that converge to zero. More precisely, the strategy to prove the Main result, is to prove that the above function converges to zero when  $t \rightarrow +\infty$  implies that  $|z| \neq 0$  (the initial condition). We deduce that  $\phi(\tau, r) \neq 0$ .

**Lemma 2.** *For every  $\tau > 0$  and  $r \in (0, \frac{1}{2})$ , the unique continuous solution  $\psi_{\tau, r}(t) : [0, +\infty) \rightarrow \mathbb{C}$  of the following differential equation*

$$\frac{d}{dt} \psi_{\tau, r}(t) = \frac{i\tau}{t} \psi_{\tau, r}(t) + f_r(t), \quad t \in \mathbb{R}_+/\mathbb{N}, \quad \psi_{\tau, r}(0) = 0, \quad (2)$$

is defined by

$$\psi_{\tau, r}(t) = t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du, \quad \forall t > 0.$$

*Proof.* Since  $0 \leq f_r(u) \leq \frac{1}{t^{\frac{3}{2}}}(t^r - t^{-r})$  for every  $t > 1$  and  $0 \leq f_r(u) \leq \frac{1}{t^{\frac{3}{2}}}(t^r - t^{-r})$  for every  $t \in (0, 1)$  then the function

$$t \mapsto t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du,$$

is continuous and  $C^1$  on  $\mathbb{R}_+/\mathbb{N}$ . The Equation (2) is a non-homogeneous linear differential equation, the unique continuous solution is given by

$$\psi_{\tau, r}(t) = t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du.$$

□

**Lemma 3.** For every  $r \in (0, \frac{1}{2})$  there exists a continuous and pies-wise  $C^1$  function  $\epsilon_r : (1, +\infty) \rightarrow \mathbb{R}$  such that

$$\int_{\epsilon_r(t)}^t f_r(u) du = 0,$$

In addition,

$$\epsilon_r(t) \in (0, 1) \quad \text{and} \quad \frac{d}{dt} \epsilon_r(t) < 0, \quad \forall t > 1.$$

*Proof.* We have

$$\int_0^{+\infty} f_r(s) ds = \int_0^1 f_r(s) ds + \int_1^{+\infty} f_r(s) ds.$$

Use the change of variable  $s \mapsto \frac{1}{u}$  for the first integral, we get

$$\int_0^{+\infty} f_r(s) ds = \int_1^{+\infty} \frac{1}{u^2} f_r\left(\frac{1}{u}\right) du + \int_1^{+\infty} f_r(s) ds.$$

By definition of  $f_r$  we have

$$f_r(u) = \frac{1}{u^{\frac{3}{2}}} \{u\} (u^r - u^{-r}).$$

In particular

$$f_r(u) = \frac{1}{u^{\frac{1}{2}}} (u^r - u^{-r}), \quad \forall u \in (0, 1).$$

Then

$$\begin{aligned} \int_0^{+\infty} f_r(s) ds &= - \int_1^{+\infty} \frac{1}{u^{\frac{3}{2}}} (u^r - u^{-r}) ds + \int_1^{+\infty} f_r(u) du \\ &= - \int_1^{+\infty} (1 - \{u\}) \frac{1}{u^{\frac{3}{2}}} (u^r - u^{-r}) ds < 0, \quad \forall r \in (0, \frac{1}{2}). \end{aligned}$$

□

*Proof of the Main result.* Let be  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$  fixed. The Main result is proved if we prove that  $\phi(\tau, r) \neq 0$ . We have

$$\phi(\tau, r) = \int_0^{+\infty} u^{-i\tau} f_r(u) du.$$

By Lemma 2 , we have

$$\lim_{t \rightarrow +\infty} |\psi_{\tau, r}(t)| = \phi(\tau, r),$$

then main result is proved if we prove that  $\lim_{t \rightarrow +\infty} |\psi_{\tau,r}(t)| \neq 0$ . Let be  $\epsilon_r$  the function defined in Lemma 3. Equation (2) implies

$$\psi_{\tau,r}(t) - \psi_{\tau,r}(\epsilon_r(t)) = i\tau \int_{\epsilon_r(t)}^t \frac{1}{u} \psi_{\tau,r}(u) du, \quad \forall t \geq 1.$$

By contradiction, suppose that

$$\lim_{t \rightarrow +\infty} |\psi_{\tau,r}(t)| = 0.$$

Then

$$\psi_{\tau,r}(\epsilon_r(t)) = -i\tau \int_{\epsilon_r(t)}^{+\infty} \frac{1}{u} \psi_{\tau,r}(u) du, \quad \forall t \geq 1,$$

implies

$$\frac{d}{dt} [\psi_{\tau,r}(\epsilon_r(t))] = i\tau \frac{\frac{d}{dt} \epsilon_r(t)}{\epsilon_r(t)} \psi_{\tau,r}(\epsilon_r(t)), \quad \forall t \geq 1. \quad (3)$$

which is a linear differential equation. By Lemma 3 we have  $\frac{d}{dt} \epsilon_r(t) < 0$  and  $\epsilon_r(t) \in (0, 1)$  for every  $t > 1$ . Integrate the Equation (3), for every  $t > t' > 1$  we have

$$\psi_{\tau,r}(\epsilon_r(t)) = \left( \frac{\epsilon_r(t)}{\epsilon_r(t')} \right)^{i\tau} \psi_{\tau,r}(\epsilon_r(t')). \quad (4)$$

Since  $f_r(u) = t^{-\frac{1}{2}}(t^r - t^{-r})$  for every  $t \in (0, 1)$  by Lemma 2, we have

$$\psi_{\tau,r}(u) = \frac{u^{\frac{1}{2}+r}}{-i\tau + \frac{1}{2} + r} - \frac{u^{\frac{1}{2}-r}}{-i\tau + \frac{1}{2} - r}, \quad \forall u \in (0, 1). \quad (5)$$

Then  $|\psi_{\tau,r}(\epsilon_r(t))| \neq 0$  for every  $t \geq 1$ . By Equation (4), there exists  $t_* \in (0, 1)$  such that

$$\Im(\psi_{\tau,r}(\epsilon_r(t_*))) = 0$$

Contradiction with Equation (5). We deduce that  $\phi(\tau, r) \neq 0$  for every  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$ .  $\square$

## 2 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function  $\zeta$  defined by Equation (1). Define the *non trivial zeros* of the function  $\zeta$  in the following sense

**Definition 4.** Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non trivial zero of the function  $\zeta$*  if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

By the Riemann functional equation [[1], page 13 Theorem 2.1], the non trivial zeros of the function  $\zeta$  are symmetric about the line  $\Re(z) = \frac{1}{2}$ . By the Main result, the non trivial zeros  $s \in \mathbb{C}$  symmetric about the line  $\Re(z) = \frac{1}{2}$  satisfies  $\Re(s) = \frac{1}{2}$ .

**Theorem 5** (Riemann hypothesis). *Let  $s \in \mathbb{C}$  be a non-trivial zero of the function  $\zeta$ . Then  $\Re(s) = \frac{1}{2}$ .*

*Proof.* By contradiction, suppose that there exists a non-trivial zero  $s_* \in \mathbb{C}$  of the function  $\zeta$  such that  $\Re(s_*) \neq \frac{1}{2}$ . Then

$$\frac{\zeta(s_*)}{s_*} = 0.$$

The Main result implies

$$\frac{\zeta(1 - s_*)}{1 - s_*} \neq 0. \quad (6)$$

But the Riemann functional equation implies that the non-trivial zero of the function  $\zeta$  are symmetric about the line  $\Re(z) = \frac{1}{2}$ . In other words,

$$\zeta(1 - s_*) = 0.$$

We obtain a contradiction with the Equation (6). We deduce that  $\Re(s_*) = \frac{1}{2}$ .  $\square$

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).