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# Non symmetry of the function $\frac{\zeta(s)}{s}$

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## Abstract

We prove that  $|\frac{\zeta(\bar{s})}{\bar{s}} - \frac{\zeta(1-s)}{1-s}| \neq 0$  for every  $\Re(s) \in (0, \frac{1}{2})$  and  $\Im(s) \in \mathbb{R}^*$ , where  $\zeta$  is the Riemann Zeta function. At the end of the paper, we give a discussion about the Riemann hypothesis.

**Keywords:** Riemann functional equation, Zeta function.

**AMS subject classifications:** 00A05

## 1 Main result

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where  $\{t\}$  is the fractional part of the real  $t$ . We prove in this paper the following Main result

**Main Result.** *Consider the function  $\zeta$  defined by Equation (1). Then*

$$\left| \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau} \right| \neq 0, \quad \forall r \in (0, \frac{1}{2}), \quad \forall \tau \in \mathbb{R}^*.$$

**Notation 1.** For every  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$  denote

$$\phi(\tau, r) := \frac{\zeta(\frac{1}{2} + r + i\tau)}{\frac{1}{2} + r + i\tau} - \frac{\zeta(\frac{1}{2} - r + i\tau)}{\frac{1}{2} - r + i\tau}.$$

For every  $r \in (0, \frac{1}{2})$  denote

$$f_r(u) := \{u\} \frac{1}{u^{\frac{3}{2}}}(u^r - u^{-r}), \quad u > 0.$$

By Equation (1) we have

$$\phi(\tau, r) = \int_0^{+\infty} u^{-i\tau} f_r(u) du,$$

The aim is to study the differential equation of the solution function

$$t \mapsto t^{i\tau} \left[ z + \int_0^t u^{-i\tau} f_r(u) du \right], \quad t > 0, \quad z \in \mathbb{C}.$$

We focus only on the solutions that converge to zero. More precisely, the strategy to prove the Main result, is to prove that the above function converges to zero when  $t \rightarrow +\infty$  implies that  $|z| \neq 0$  (the initial condition). We deduce that  $\phi(\tau, r) \neq 0$ .

**Lemma 2.** *For every  $\tau > 0$  and  $r \in (0, \frac{1}{2})$ , the unique continuous solution  $\psi_{\tau, r}(t) : [0, +\infty) \rightarrow \mathbb{C}$  of the following differential equation*

$$\frac{d}{dt} \psi_{\tau, r}(t) = \frac{i\tau}{t} \psi_{\tau, r}(t) + f_r(t), \quad t \in \mathbb{R}_+/\mathbb{N}, \quad \psi_{\tau, r}(0) = 0, \quad (2)$$

is defined by

$$\psi_{\tau, r}(t) = t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du, \quad \forall t > 0.$$

*Proof.* Since  $0 \leq f_r(u) \leq \frac{1}{t^{\frac{3}{2}}}(t^r - t^{-r})$  for every  $t > 1$  and  $0 \leq f_r(u) \leq \frac{1}{t^{\frac{3}{2}}}(t^r - t^{-r})$  for every  $t \in (0, 1)$  then the function

$$t \mapsto t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du,$$

is continuous and  $C^1$  on  $\mathbb{R}_+/\mathbb{N}$ . The Equation (2) is a non-homogeneous linear differential equation, the unique continuous solution is given by

$$\psi_{\tau, r}(t) = t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du.$$

□

**Lemma 3.** Let be  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$ . Let  $\psi_{\tau,r}(t) : [0, +\infty) \rightarrow \mathbb{C}$  the unique continuous solution of Equation (2). Suppose that

$$\lim_{t \rightarrow +\infty} |\psi_{\tau,r}(t)| = 0.$$

Then

$$\lim_{t \rightarrow +\infty} t^{\frac{1}{2}-r} \psi_{\tau,r}(t) = -\frac{1}{2} \frac{\frac{1}{2}-r}{(\frac{1}{2}-r)^2 + \tau^2} + i \frac{1}{2} \frac{\tau}{(\frac{1}{2}-r)^2 + \tau^2}.$$

*Proof.* By Lemma 2

$$\psi_{\tau,r}(t) = t^{i\tau} \int_0^t u^{-i\tau} f_r(u) du, \quad \forall t > 0.$$

Suppose that

$$\lim_{t \rightarrow +\infty} |\psi_{\tau,r}(t)| = 0.$$

Then

$$\int_0^{+\infty} u^{-i\tau} f_r(u) du = 0,$$

implies

$$\int_0^t u^{-i\tau} f_r(u) du = - \int_t^{+\infty} u^{-i\tau} f_r(u) du, \quad \forall t > 0.$$

We get

$$\psi_{\tau,r}(t) = -t^{i\tau} \int_t^{+\infty} u^{-i\tau} f_r(u) du, \quad \forall t > 0. \quad (3)$$

The function  $v \mapsto \{v\}$  is 1-periodic, then there exists a continuous 1-periodic function  $p : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\int_0^t \{v\} dv = t \int_0^1 \{v\} dv + p(t), \quad \forall t \geq 0.$$

Since

$$\int_0^1 \{v\} dv = \frac{1}{2},$$

we get

$$\int_0^t \{v\} dv = t \frac{1}{2} + p(t), \quad \forall t \geq 0.$$

Use the integration part formula in Equation (3),

$$\begin{aligned}\psi_{\tau,r}(t) &= -\frac{1}{2}t^{i\tau} \int_t^{+\infty} v^{-i\tau} \frac{1}{v^{\frac{3}{2}}} (v^r - v^{-r}) dv \\ &\quad + t^{i\tau} \int_t^{+\infty} h_{\tau,r}(v) (p(v) - p(t)) dv,\end{aligned}$$

where

$$\begin{aligned}h_{\tau,r}(v) &:= \frac{d}{dv} \left[ v^{-i\tau} \frac{1}{v^{\frac{3}{2}}} (v^r - v^{-r}) \right] \\ &= -i\tau v^{-i\tau} \left( \frac{1}{v^{\frac{5}{2}-r}} - \frac{1}{v^{\frac{5}{2}+r}} \right) + v^{-i\tau} \left( \frac{\frac{3}{2}+r}{v^{\frac{5}{2}+r}} - \frac{\frac{3}{2}-r}{v^{\frac{5}{2}-r}} \right).\end{aligned}$$

Since  $p$  is a periodic function then it is bounded:

$$\exists \alpha > 0, \quad |p(v) - p(t)| < \alpha, \quad \forall v \geq 0.$$

We obtain

$$\begin{aligned}\left| \psi_{\tau,r}(t) + \frac{1}{2}t^{i\tau} \int_t^{+\infty} \left[ \frac{1}{v^{\frac{3}{2}-r+i\tau}} - \frac{1}{v^{\frac{3}{2}+r+i\tau}} \right] dv \right| \\ < 2\alpha \left( \tau + \frac{3}{2} + r \right) \int_{t_n}^{+\infty} \frac{1}{v^{\frac{5}{2}-r}} dv = \frac{2\alpha \left( \tau + \frac{3}{2} + r \right)}{\frac{3}{2} - r} \frac{1}{t_n^{\frac{3}{2}-r}}.\end{aligned}$$

Since

$$\begin{aligned}t^{i\tau} \int_t^{+\infty} \left[ \frac{1}{v^{\frac{3}{2}-r+i\tau}} - \frac{1}{v^{\frac{3}{2}+r+i\tau}} \right] dv \\ = \frac{1}{-\frac{1}{2} - r - i\tau} \frac{1}{t^{\frac{1}{2}+r}} - \frac{1}{-\frac{1}{2} + r - i\tau} \frac{1}{t^{\frac{1}{2}-r}},\end{aligned}$$

Then,

$$\lim_{t \rightarrow +\infty} t^{\frac{1}{2}-r} \psi_{\tau,r}(t) = \frac{1}{2} \frac{1}{-\frac{1}{2} + r - i\tau},$$

equivalent to

$$\lim_{t \rightarrow +\infty} t^{\frac{1}{2}-r} \psi_{\tau,r}(t) = -\frac{1}{2} \frac{\frac{1}{2} - r}{\left(\frac{1}{2} - r\right)^2 + \tau^2} + i \frac{1}{2} \frac{\tau}{\left(\frac{1}{2} - r\right)^2 + \tau^2}.$$

□

**Definition 4.** Let  $h : \mathbb{R}_+ \rightarrow \mathbb{R}$  be a continuous and piecewise  $C^1$  function. Let be  $x \in \mathbb{R}_+$ . We say that  $x$  is a *regular zero* of the function  $h$  if

- $x \neq 0$ ,
- $h(x) = 0$ ,
- $\exists \epsilon > 0 : h(t)h(s) < 0, \forall t \in (x - \epsilon, x), \forall s \in (x, x + \epsilon)$ .

We denote  $\Lambda(h)$  the cardinal of the set of all regular zeros of the function  $h$ .

*Proof of the Main result.* Let be  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$  fixed. The Main result is proved if we prove that  $\phi(\tau, r) \neq 0$ . We have

$$\phi(\tau, r) = \int_0^{+\infty} u^{-i\tau} f_r(u) du.$$

By Lemma 2 , we have

$$\lim_{t \rightarrow +\infty} |\psi_{\tau, r}(t)| = \phi(\tau, r),$$

then main result is proved if we prove that  $\lim_{t \rightarrow +\infty} |\psi_{\tau, r}(t)| \neq 0$ . In order to simplify the notation, denote

$$w(t) := \Re(\psi_{\tau, r}(t)) \quad \text{and} \quad v(t) := \Im(\psi_{\tau, r}(t)), \quad \forall t > 0.$$

By Lemma 3, we have

$$\lim_{t \rightarrow +\infty} v(t) > 0.$$

We have

$$\forall t \in [0, 1] : v(t) = \frac{\tau}{(\frac{1}{2} + r)^2 + \tau^2} t^{\frac{1}{2} + r} - \frac{\tau}{(\frac{1}{2} - r)^2 + \tau^2} t^{\frac{1}{2} - r}.$$

implies

$$v(t) < 0, \quad \forall t \in (0, 1]. \quad (4)$$

Then there exists a finite odd number of regular zeros (in the sense of the Definition 4) of the function  $v$  in  $(0, +\infty)$ ;

$$\exists n \in \mathbb{N} : \Lambda(v) = 2n + 1.$$

By Lemma 2, the function  $v$  is given by

$$\begin{aligned} v(t) &= \sin(\tau \ln(t)) \int_0^t \cos(\tau \ln(u)) f_r(u) du \\ &\quad - \cos(\tau \ln(t)) \int_0^t \sin(\tau \ln(u)) f_r(u) du, \quad \forall t \geq 0. \end{aligned} \quad (5)$$

Since  $\lim_{t \rightarrow +\infty} v(t) > 0$ , then there exists  $k_* \in \mathbb{N}$  such that

$$\int_0^{\exp(\frac{2k\pi}{\tau})} \sin(\tau \ln(u)) f_r(u) du < 0, \quad \forall k \geq k_*, \quad k \in \mathbb{N}.$$

and

$$\int_0^{\exp(\frac{(2k+1)\pi}{\tau})} \sin(\tau \ln(u)) f_r(u) du > 0, \quad \forall k \geq k_*, \quad k \in \mathbb{N}.$$

By Equation (4) and (5), we have

$$\int_0^1 \sin(\tau \ln(u)) f_r(u) du > 0.$$

We have  $f_r(u) > 0$  for every  $u \in (1, +\infty)$ . Using the variation, for every  $k \in \mathbb{N}^*$  the function

$$t \mapsto \int_0^t \sin(\tau \ln(u)) f_r(u) du,$$

admits a even number of regular zeros on  $[\exp(\frac{\pi}{\tau}), \exp(\frac{(2k+1)\pi}{\tau})]$ . Then there exists a even number of solution  $t \in (1, \exp(\frac{2k\pi}{\tau})]$  or there exists a even number of solution  $t \in (1, \exp(\frac{(2k+1)\pi}{\tau})]$  that satisfies

$$\cot(\tau \ln(t)) = \frac{\int_0^t \cos(\tau \ln(u)) f_r(u) du}{\int_0^t \sin(\tau \ln(u)) f_r(u) du}.$$

By the explicit Equation (5), there exists a even number of regular zeros of the function  $v$  on  $(0, \exp(\frac{2k\pi}{\tau})]$  or there exists a even number of regular zeros of the function  $v$  on  $(0, \exp(\frac{(2k+1)\pi}{\tau})]$ . By the hypothesis  $\lim_{t \rightarrow +\infty} v(t) \neq 0$ , we deduce that there exists a finite even number of regular zeros of the function  $v$ ,

$$\exists m \in \mathbb{N} : \quad \Lambda(v) = 2m.$$

We obtain a contradiction. We deduce that  $\phi(\tau, r) \neq 0$  for every  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$ .  $\square$

## 2 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function  $\zeta$  defined by Equation (1). Define the *non trivial zeros* of the function  $\zeta$  in the following sense

**Definition 5.** Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non trivial zero of the function*  $\zeta$  if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

By the Riemann functional equation [[1], page 13 Theorem 2.1], the non trivial zeros of the function  $\zeta$  are symmetric about the line  $\Re(z) = \frac{1}{2}$ . By the Main result, the non trivial zeros  $s \in \mathbb{C}$  symmetric about the line  $\Re(z) = \frac{1}{2}$  satisfies  $\Re(s) = \frac{1}{2}$ .

**Theorem 6** (Riemann hypothesis). *Let  $s \in \mathbb{C}$  be a non-trivial zero of the function  $\zeta$ . Then  $\Re(s) = \frac{1}{2}$ .*

*Proof.* By contradiction, suppose that there exists a non-trivial zero  $s_* \in \mathbb{C}$  of the function  $\zeta$  such that  $\Re(s_*) \neq \frac{1}{2}$ . Then

$$\frac{\zeta(s_*)}{s_*} = 0.$$

The Main result implies

$$\frac{\zeta(1 - s_*)}{1 - s_*} \neq 0. \tag{6}$$

But the Riemann functional equation implies that the non-trivial zero of the function  $\zeta$  are symmetric about the line  $\Re(z) = \frac{1}{2}$ . In other words,

$$\zeta(1 - s_*) = 0.$$

We obtain a contradiction with the Equation (6). We deduce that  $\Re(s_*) = \frac{1}{2}$ .  $\square$

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).