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# Non symmetry of the function $\mathfrak{S}\left(\frac{\zeta(s)}{s}\right)$

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## Abstract

We prove that  $\mathfrak{S}\left(\frac{\zeta(s)}{s}\right) + \mathfrak{S}\left(\frac{\zeta(1-s)}{1-s}\right) \neq 0$  for every  $\Re(s) \in (0, \frac{1}{2})$  and  $\Im(s) \in \mathbb{R}^*$ , where  $\zeta$  is the Riemann Zeta function. At the end of the paper, we give a discussion about the Riemann hypothesis.

**Keywords:** Riemann functional equation, Zeta function.

## 1 Main result

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

where  $\{t\}$  is the fractional part of the real  $t$ . We prove in this paper the following Main result

**Main Result.** Consider the function  $\zeta$  defined by Equation (1). Then

$$\mathfrak{S}\left(\frac{\zeta\left(\frac{1}{2} + r + i\tau\right)}{\frac{1}{2} + r + i\tau}\right) - \mathfrak{S}\left(\frac{\zeta\left(\frac{1}{2} - r + i\tau\right)}{\frac{1}{2} - r + i\tau}\right) \neq 0, \quad \forall r \in (0, \frac{1}{2}), \quad \forall \tau \in \mathbb{R}^*.$$

## 2 Main Proposition

**Notation 1.** In order to simplify the notation, we denote

$$\alpha_\tau := \frac{1}{2} + i\tau, \quad \forall \tau \in \mathbb{R}^*.$$

For every  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$  denote

$$\phi_\tau(r) := \frac{\zeta(\alpha_\tau + r)}{\alpha_\tau + r} - \frac{\zeta(\alpha_\tau - r)}{\alpha_\tau - r}.$$

For every  $n \geq 1$  we denote

$$\mathbb{Z}_n := \{j \in \mathbb{Z}^*, \quad |j| \leq n\}.$$

We have the following Main proposition

**Proposition 2.** *For every  $\tau \in \mathbb{R}^*$  and  $r \in (0, \frac{1}{2})$  we have*

$$\phi_\tau(r) = \frac{-2r}{(1 - \alpha_\tau)^2 - r^2} + \frac{r}{\alpha_\tau^2 - r^2} + \lim_{n \rightarrow +\infty} \phi_\tau(r, n),$$

where

$$\phi_\tau(r, n) = \int_1^{+\infty} \left( \frac{1}{u^{1+\alpha_\tau-r}} - \frac{1}{u^{1+\alpha_\tau+r}} \right) \sum_{j \in \mathbb{Z}_n} \frac{i}{j2\pi} \exp(ij2\pi u) du.$$

### 3 Main Lemma

**Lemma 3.** *We have*

$$\int_1^t \left( \{v\} - \frac{1}{2} \right) dv = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left( \exp(ij2\pi t) - 1 \right), \quad \forall t \geq 0.$$

*Proof.* The function  $v \mapsto \{v\} - \frac{1}{2}$  is 1-periodic, then there exists a continuous 1-periodic function  $p : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\int_1^t \left( \{v\} - \frac{1}{2} \right) dv = (t - 1) \int_0^1 \left( \{v\} - \frac{1}{2} \right) dv + p(t), \quad \forall t \in \mathbb{R}.$$

Since

$$\int_0^1 \left( \{v\} - \frac{1}{2} \right) dv = 0,$$

we get

$$\int_1^t \left( \{v\} - \frac{1}{2} \right) dv = p(t), \quad \forall t \in \mathbb{R}.$$

The function  $p$  is a piecewise  $C^\infty$ , continuous on  $\mathbb{R}$  and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on  $\mathbb{R}_+$  to the function  $t \mapsto p(t)$ , where  $(a_k)_k \subset \mathbb{C}$  are the Fourier coefficients of the function  $p$ .

$$\int_0^t \left( \{v\} - \frac{1}{2} \right) dv = \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left( \int_0^u \left( \{v\} - \frac{1}{2} \right) dv \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12}.$$

□

## 4 Proof of the Main Proposition

*Proof of the Proposition 2.* Let be  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$ . We have

$$\begin{aligned} \phi_\tau(r) &= \int_0^1 \left( \frac{1}{u^{\alpha_\tau - r}} - \frac{1}{u^{\alpha_\tau + r}} \right) du + \frac{1}{2} \int_1^{+\infty} \left( \frac{1}{u^{1+\alpha_\tau - r}} - \frac{1}{u^{1+\alpha_\tau + r}} \right) du \\ &\quad + \int_1^{+\infty} \left( \frac{1}{u^{1+\alpha_\tau - r}} - \frac{1}{u^{1+\alpha_\tau + r}} \right) \left( \{u\} - \frac{1}{2} \right) du. \end{aligned}$$

Use the integration part formula,

$$\begin{aligned} \phi_\tau(r) &= \frac{-2r}{(1 - \alpha_\tau)^2 - r^2} + \frac{r}{\alpha_\tau^2 - r^2} \\ &\quad + \int_1^{+\infty} \left( \frac{1 + \alpha_\tau - r}{u^{2+\alpha_\tau - r}} - \frac{1 + \alpha_\tau + r}{u^{2+\alpha_\tau + r}} \right) \left[ \int_1^u \left( \{v\} - \frac{1}{2} \right) dv \right] du. \end{aligned}$$

By Lemma 3,

$$\begin{aligned} \phi_\tau(r) &= \frac{-2r}{(1 - \alpha_\tau)^2 - r^2} + \frac{r}{\alpha_\tau^2 - r^2} \\ &\quad + \int_1^{+\infty} \left( \frac{1 + \alpha_\tau - r}{u^{2+\alpha_\tau - r}} - \frac{1 + \alpha_\tau + r}{u^{2+\alpha_\tau + r}} \right) \sum_{j \in \mathbb{Z}^*} \frac{\exp(ij2\pi u)}{(j2\pi)^2} du. \end{aligned}$$

Then

$$\phi_\tau(r) = \frac{-2r}{(1 - \alpha_\tau)^2 - r^2} + \frac{r}{\alpha_\tau^2 - r^2} + \lim_{n \rightarrow +\infty} \phi_\tau(r, n),$$

where

$$\phi_\tau(r, n) = \int_1^{+\infty} \left( \frac{1 + \alpha_\tau + r}{u^{2+\alpha_\tau-r}} - \frac{1 + \alpha_\tau - r}{u^{2+\alpha_\tau+r}} \right) \sum_{j \in \mathbb{Z}_n} \frac{\exp(ij2\pi u)}{(j2\pi)^2} du.$$

Use the integration part formula, we get

$$\phi_\tau(r, n) = \int_1^{+\infty} \left( \frac{1}{u^{1+\alpha_\tau-r}} - \frac{1}{u^{1+\alpha_\tau+r}} \right) \sum_{j \in \mathbb{Z}_n} \frac{i}{j2\pi} \exp(ij2\pi u) du.$$

□

## 5 Proof of the Main result

*Proof of the Main result.* Let be  $\tau \in \mathbb{R}^*$  fixed. By the Proposition 2,

$$\phi_\tau(r) = \frac{-2r}{(1 - \alpha_\tau)^2 - r^2} + \frac{r}{\alpha_\tau^2 - r^2} + \lim_{n \rightarrow +\infty} \phi_\tau(r, n),$$

where

$$\phi_\tau(r, n) = \int_1^{+\infty} \left( \frac{1}{u^{1+\alpha_\tau-r}} - \frac{1}{u^{1+\alpha_\tau+r}} \right) \sum_{j \in \mathbb{Z}_n} \frac{i}{j2\pi} \exp(ij2\pi u) du.$$

Denote for ever  $T \geq 1$ ,

$$\phi_\tau(r, n, T) = \int_1^T \left( \frac{1}{u^{1+\alpha_\tau-r}} - \frac{1}{u^{1+\alpha_\tau+r}} \right) \sum_{j \in \mathbb{Z}_n} \frac{i}{j2\pi} \exp(ij2\pi u) du.$$

Use the Taylor formula

$$\phi_\tau(r, n, T) = \sum_{j \in \mathbb{Z}_n} \frac{i}{j2\pi} \sum_{k=1}^{\infty} \frac{1}{k!} (ij2\pi)^k \int_1^T \left( u^{k-1-\alpha_\tau+r} - u^{k-1-\alpha_\tau-r} \right) du.$$

implies

$$\phi_\tau(r, n, T) = \sum_{k=1}^{\infty} \frac{i^{k+1}}{k!} \left[ \sum_{j \in \mathbb{Z}_n} (j2\pi)^{k-1} \right] \int_1^T \left( u^{k-1-\alpha_\tau+r} - u^{k-1-\alpha_\tau-r} \right) du$$

Since  $\sum_{j \in \mathbb{Z}_n} (j2\pi)^{2k-1} = 0$  for every  $k \geq 1$  we get

$$\phi_\tau(r, n, T) = - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \sum_{j \in \mathbb{Z}_n} (j2\pi)^{2k} \int_1^T \left( u^{2k-\alpha_\tau+r} - u^{2k-\alpha_\tau-r} \right) du$$

Then

$$\phi_\tau(r, n, T) = -\psi_\tau(r, n, T) + \psi_\tau(r, n, 1), \quad (2)$$

where for every  $T \geq 1$  the function  $T \mapsto \psi_\tau(r, n, T)$  is defined as

$$\begin{aligned} \psi_\tau(r, n, T) &:= \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(-1)^k (j2\pi)^{2k}}{(2k+1)!} \frac{T^{2k+1-\alpha_\tau+r}}{2k+1-\alpha_\tau+r} \\ &\quad - \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(-1)^k (j2\pi)^{2k}}{(2k+1)!} \frac{T^{2k+1-\alpha_\tau-r}}{2k+1-\alpha_\tau-r}. \end{aligned}$$

We have

$$\Im(\psi_\tau(r, n, T)) := \cos(\tau \ln(T))V_\tau(r, n, T) + \sin(\tau \ln(T))W_\tau(r, nT), \quad (3)$$

where

$$\begin{aligned} V_\tau(r, n, T) &:= \frac{\tau}{T^{\frac{1}{2}-r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(-1)^k (j2\pi)^{2k}}{(2k+1)!} \frac{T^{2k+1}}{|2k+1-\alpha+r|^2} \\ &\quad - \frac{\tau}{T^{\frac{1}{2}+r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(-1)^k (j2\pi)^{2k}}{(2k+1)!} \frac{T^{2k+1}}{|2k+1-\alpha-r|^2}, \end{aligned}$$

and where

$$\begin{aligned} W_\tau(r, n, T) &:= \frac{1}{T^{\frac{1}{2}-r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(-1)^k (j2\pi)^{2k}}{(2k+1)!} T^{2k+1} \frac{2k+1-\frac{1}{2}+r}{|2k+1-\alpha+r|^2} \\ &\quad - \frac{1}{T^{\frac{1}{2}+r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(-1)^k (j2\pi)^{2k}}{(2k+1)!} T^{2k+1} \frac{2k+1-\frac{1}{2}-r}{|2k+1-\alpha-r|^2} \end{aligned}$$

Thanks to Equation (2), we obtain

$$\begin{aligned} \Im(\phi_\tau(r, n, T)) &= -\cos(\tau \ln(T))V_\tau(r, n, T) - \sin(\tau \ln(T))W_\tau(r, nT) \\ &\quad + V_\tau(r, n, 1). \end{aligned} \quad (4)$$

We have

$$\begin{aligned}
V_\tau(r, n, T) &:= \frac{\tau}{T^{\frac{1}{2}-r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k}}{(4k+1)!} \frac{T^{4k+1}}{|4k+1-\alpha+r|^2} \\
&+ \frac{\tau}{T^{\frac{1}{2}+r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k+2}}{(4k+3)!} \frac{T^{4k+3}}{|4k+3-\alpha-r|^2} \\
&- \frac{\tau}{T^{\frac{1}{2}+r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k}}{(4k+1)!} \frac{T^{4k+1}}{|4k+1-\alpha-r|^2} \\
&- \frac{\tau}{T^{\frac{1}{2}-r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k+2}}{(4k+3)!} \frac{T^{4k+3}}{|4k+3-\alpha+r|^2},
\end{aligned}$$

which can be written as

$$\begin{aligned}
V_\tau(r, n, T) &:= \tau\sqrt{T} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k}}{(4k+1)!} \tau T^{4k} \int_{-r}^r \frac{d}{du} \frac{T^{-u}}{|4k+1-\alpha+u|^2} du \\
&- \tau\sqrt{T} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k+2}}{(4k+3)!} T^{4k+2} \int_{-r}^r \frac{d}{du} \frac{T^u}{|4k+3-\alpha-u|^2} du.
\end{aligned}$$

Since

$$\frac{d}{du} \frac{T^{-u}}{|4k+1-\alpha+u|^2} = -\frac{1}{T^u} \frac{\ln(T)((4k+\frac{1}{2}+u)^2 + \tau^2) + 2(4k+\frac{1}{2}+u)}{((4k+\frac{1}{2}+u)^2 + \tau^2)^2},$$

and

$$\begin{aligned}
\frac{d}{du} \frac{T^u}{|4k+3-\alpha-u|^2} &= \frac{\exp(u \ln(T)) \ln(T)}{\left((4k+2+\frac{1}{2}-u)^2 + \tau^2\right)} \\
&+ \frac{2 \exp(u \ln(T))(4k+2+\frac{1}{2}-u)}{\left((4k+2+\frac{1}{2}-u)^2 + \tau^2\right)^2}.
\end{aligned}$$

Then  $\tau V_\tau(r, n, T) < 0$  for every  $T > 1$ , by continuity  $\tau V_\tau(r, n, 1) \leq 0$ . There exists a sequence  $(t_{k,n})_k \rightarrow +\infty$  such that  $\lim_{k \rightarrow +\infty} t_{k,n} = +\infty$  and such that

$$\cot(\tau \ln(t_{k,n})) = -\frac{W_\tau(r, n, t_{k,n})}{V_\tau(r, n, t_{k,n})}.$$

By Equation (3), we get  $\Im(\psi_\tau(r, n, t_{k,n})) = 0$  for every  $k \geq 0$ . By Equation (4) we get

$$\Im(\phi_\tau(r, n, t_{k,n})) = V_\tau(r, n, 1), \quad \forall k \geq 0.$$

Since  $\lim_{k \rightarrow +\infty} t_{k,n} = +\infty$ , then

$$\tau \mathfrak{S}(\phi_\tau(r, n)) = \tau V_\tau(r, n, 1) \leq 0.$$

Since

$$\mathfrak{S}(\phi_\tau(r)) = \frac{-2r\tau}{|(1 - \alpha_\tau)^2 - r^2|^2} + \frac{-r\tau}{|\alpha_\tau^2 - r^2|^2} + \lim_{n \rightarrow +\infty} \mathfrak{S}(\phi_\tau(r, n)),$$

we get

$$\tau \mathfrak{S}(\phi_\tau(r)) \leq \frac{-2r\tau^2}{|(1 - \alpha_\tau)^2 - r^2|^2} + \frac{-r\tau^2}{|\alpha_\tau^2 - r^2|^2} < 0.$$

□

## 6 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function  $\zeta$  defined by Equation (1). Define the *non trivial zeros* of the function  $\zeta$  in the following sense

**Definition 4.** Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non trivial zero of the function*  $\zeta$  if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

By the Riemann functional equation [[1], page 13 Theorem 2.1], the non trivial zeros of the function  $\zeta$  are symmetric about the line  $\Re(z) = \frac{1}{2}$ . By the Main result, the non trivial zeros  $s \in \mathbb{C}$  symmetric about the line  $\Re(z) = \frac{1}{2}$  satisfies  $\Re(s) = \frac{1}{2}$ .

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).