Non symmetry of the function $\Im\Big(\frac{\zeta(s)}{s}\Big)$

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July 27, 2022

Abstract

We prove that $\Im\Big(\frac{\zeta(s)}{s}\Big) + \Im\Big(\frac{\zeta(1-s)}{1-s}\Big) \neq 0$ for every $\Re(s) \in (0, \frac{1}{2})$ and $\Im(s) \in \mathbb{R}^*$, where ζ is the Riemann Zeta function. At the end of the paper, we give a discussion about the Riemann hypothesis.

Keywords: Riemann functional equation, Zeta function.

1 Main result

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s\int_0^{+\infty} \frac{\{t\}}{t^{1+s}}dt, \quad \Re(s) \in (0,1), \quad \Im(s) \in \mathbb{R}^*, \tag{1}$$

where $\{t\}$ is the fractional part of the real t. We prove in this paper the following Main result

Main Result. *Consider the function ζ defined by Equation* (1)*. Then*

$$\Im\Big(\frac{\zeta(\frac{1}{2}+r+i\tau)}{\frac{1}{2}+r+i\tau}\Big) - \Im\Big(\frac{\zeta(\frac{1}{2}-r+i\tau)}{\frac{1}{2}-r+i\tau}\Big) \neq 0, \quad \forall r \in (0, \frac{1}{2}),\ \forall \tau \in \mathbb{R}^*.$$

2 Main Proposition

Notation 1. In order to simplify the notation, we denote

$$\alpha_\tau := \frac{1}{2} + i\tau, \quad \forall \tau \in \mathbb{R}^*.$$

For every $r \in (0, \frac{1}{2})$ and $\tau \in \mathbb{R}^*$ denote

$$\phi_\tau(r) := \frac{\zeta(\alpha_\tau + r)}{\alpha_\tau + r} - \frac{\zeta(\alpha_\tau - r)}{\alpha_\tau - r}.$$

For every $n \geq 1$ we denote

$$\mathbb{Z}_n := \{j \in \mathbb{Z}^*, \quad |j| \leq n\}.$$

We have the following Main proposition

Proposition 2. *For every $\tau \in \mathbb{R}^*$ and $r \in (0, \frac{1}{2})$ we have*

$$\phi_\tau(r) = \frac{-2r}{(1-\alpha_\tau)^2 - r^2} + \frac{r}{\alpha_\tau^2 - r^2} + \lim_{n \to +\infty} \phi_\tau(r, n),$$

where

$$\phi_\tau(r, n) = \int_1^{+\infty} \Big(\frac{1}{u^{1+\alpha_\tau - r}} - \frac{1}{u^{1+\alpha_\tau + r}} \Big) \sum_{j \in \mathbb{Z}_n} \frac{i}{j2\pi} \exp(ij2\pi u) du.$$

3 Main Lemma

Lemma 3. *We have*

$$\int_1^t \Big(\{v\} - \frac{1}{2} \Big) dv = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \Big(\exp(ij2\pi t) - 1 \Big), \quad \forall t \geq 0.$$

Proof. The function $v \mapsto \{v\} - \frac{1}{2}$ is 1-periodic, then there exists a continuous 1-periodic function $p : \mathbb{R} \to \mathbb{R}$ such that

$$\int_1^t \Big(\{v\} - \frac{1}{2} \Big) dv = (t-1) \int_0^1 \Big(\{v\} - \frac{1}{2} \Big) dv + p(t), \quad \forall t \in \mathbb{R}.$$

Since

$$\int_0^1 \Big(\{v\} - \frac{1}{2} \Big) dv = 0,$$

we get

$$\int_1^t \Big(\{v\} - \frac{1}{2} \Big) dv = p(t), \quad \forall t \in \mathbb{R}.$$

The function p is a piecewise C^∞, continuous on $\mathbb{R}$ and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^{n} a_k \exp(ik2\pi t),$$

converge uniformly on $\mathbb{R}_+$ to the function $t \mapsto p(t)$, where $(a_k)_k \subset \mathbb{C}$ are the Fourier coefficients of the function p.

$$\int_0^t \Big(\{v\} - \frac{1}{2}\Big) dv = \sum_{j\in\mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u)\Big(\int_0^u \Big(\{v\} - \frac{1}{2}\Big) dv\Big) du \\ &= \frac{1}{2}\int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2}\int_0^1 u(u-1) du = -\frac{1}{12}.$$

□

4 Proof of the Main Proposition

Proof of the Proposition 2. Let be $r \in (0, \frac{1}{2})$ and $\tau \in \mathbb{R}^*$. We have

$$\begin{aligned} \phi_\tau(r) = &\int_0^1 \Big(\frac{1}{u^{\alpha_\tau - r}} - \frac{1}{u^{\alpha_\tau + r}}\Big) du + \frac{1}{2}\int_1^{+\infty} \Big(\frac{1}{u^{1+\alpha_\tau - r}} - \frac{1}{u^{1+\alpha_\tau + r}}\Big) du \\ &+ \int_1^{+\infty} \Big(\frac{1}{u^{1+\alpha_\tau - r}} - \frac{1}{u^{1+\alpha_\tau + r}}\Big)\Big(\{u\} - \frac{1}{2}\Big) du. \end{aligned}$$

Use the integration part formula,

$$\begin{aligned} \phi_\tau(r) = &\frac{-2r}{(1-\alpha_\tau)^2 - r^2} + \frac{r}{\alpha_\tau^2 - r^2} \\ &+ \int_1^{+\infty} \Big(\frac{1+\alpha_\tau - r}{u^{2+\alpha_\tau - r}} - \frac{1+\alpha_\tau + r}{u^{2+\alpha_\tau + r}}\Big)\Big[\int_1^u \Big(\{v\} - \frac{1}{2}\Big) dv\Big] du. \end{aligned}$$

By Lemma 3,

$$\begin{aligned} \phi_\tau(r) = &\frac{-2r}{(1-\alpha_\tau)^2 - r^2} + \frac{r}{\alpha_\tau^2 - r^2} \\ &+ \int_1^{+\infty} \Big(\frac{1+\alpha_\tau - r}{u^{2+\alpha_\tau} - r} - \frac{1+\alpha_\tau + r}{u^{2+\alpha_\tau + r}}\Big) \sum_{j\in\mathbb{Z}^*} \frac{\exp(ij2\pi u)}{(j2\pi)^2} du. \end{aligned}$$

Then

$$\phi_\tau(r) = \frac{-2r}{(1-\alpha_\tau)^2 - r^2} + \frac{r}{\alpha_\tau^2 - r^2} + \lim_{n\to+\infty} \phi_\tau(r,n),$$

where

$$\phi_\tau(r,n) = \int_1^{+\infty} \Big(\frac{1+\alpha_\tau + r}{u^{2+\alpha_\tau - r}} - \frac{1+\alpha_\tau - r}{u^{2+\alpha_\tau + r}} \Big) \sum_{j\in\mathbb{Z}_n} \frac{\exp(ij2\pi u)}{(j2\pi)^2} du.$$

Use the integration part formula, we get

$$\phi_\tau(r,n) = \int_1^{+\infty} \Big(\frac{1}{u^{1+\alpha_\tau - r}} - \frac{1}{u^{1+\alpha_\tau + r}} \Big) \sum_{j\in\mathbb{Z}_n} \frac{i}{j2\pi} \exp(ij2\pi u) du.$$

$\square$

5 Proof of the Main result

Proof of the Main result. Let be $\tau \in \mathbb{R}^*$ fixed. By the Proposition 2,

$$\phi_\tau(r) = \frac{-2r}{(1-\alpha_\tau)^2 - r^2} + \frac{r}{\alpha_\tau^2 - r^2} + \lim_{n\to+\infty} \phi_\tau(r,n),$$

where

$$\phi_\tau(r,n) = \int_1^{+\infty} \Big(\frac{1}{u^{1+\alpha_\tau - r}} - \frac{1}{u^{1+\alpha_\tau + r}} \Big) \sum_{j\in\mathbb{Z}_n} \frac{i}{j2\pi} \exp(ij2\pi u) du.$$

Denote for ever $T \geq 1$,

$$\phi_\tau(r,n,T) = \int_1^{T} \Big(\frac{1}{u^{1+\alpha_\tau - r}} - \frac{1}{u^{1+\alpha_\tau + r}} \Big) \sum_{j\in\mathbb{Z}_n} \frac{i}{j2\pi} \exp(ij2\pi u) du.$$

Use the Taylor formula

$$\phi_\tau(r,n,T) = \sum_{j\in\mathbb{Z}_n} \frac{i}{j2\pi} \sum_{k=1}^{\infty} \frac{1}{k!}(ij2\pi)^k \int_1^T \Big(u^{k-1-\alpha_\tau + r} - u^{k-1-\alpha_\tau - r} \Big) du.$$

implies

$$\phi_\tau(r,n,T) = \sum_{k=1}^{\infty} \frac{i^{k+1}}{k!} \Big[\sum_{j\in\mathbb{Z}_n} (j2\pi)^{k-1} \Big] \int_1^T \Big(u^{k-1-\alpha_\tau + r} - u^{k-1-\alpha_\tau - r} \Big) du$$

Since $\sum_{j\in\mathbb{Z}_n}(j2\pi)^{2k-1}=0$ for every $k\geq 1$ we get

$$\phi_\tau(r,n,T)=-\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\sum_{j\in\mathbb{Z}_n}(j2\pi)^{2k}\int_1^T\left(u^{2k-\alpha_\tau+r}-u^{2k-\alpha_\tau-r}\right)du$$

Then

$$\phi_\tau(r,n,T)=-\psi_\tau(r,n,T)+\psi_\tau(r,n,1), \tag{2}$$

where for every $T\geq 1$ the function $T\mapsto\psi_\tau(r,n,T)$ is defined as

$$\begin{aligned}\psi_\tau(r,n,T):=&\sum_{j\in\mathbb{Z}_n}\sum_{k=0}^{\infty}\frac{(-1)^k(j2\pi)^{2k}}{(2k+1)!}\frac{T^{2k+1-\alpha_\tau+r}}{2k+1-\alpha_\tau+r}\\&-\sum_{j\in\mathbb{Z}_n}\sum_{k=0}^{\infty}\frac{(-1)^k(j2\pi)^{2k}}{(2k+1)!}\frac{T^{2k+1-\alpha_\tau-r}}{2k+1-\alpha_\tau-r}.\end{aligned}$$

We have

$$\Im(\psi_\tau(r,n,T)):=\cos(\tau\ln(T))V_\tau(r,n,T)+\sin(\tau\ln(T))W_\tau(r,nT), \tag{3}$$

where

$$\begin{aligned}V_\tau(r,n,T):=&\frac{\tau}{T^{\frac{1}{2}-r}}\sum_{j\in\mathbb{Z}_n}\sum_{k=0}^{\infty}\frac{(-1)^k(j2\pi)^{2k}}{(2k+1)!}\frac{T^{2k+1}}{|2k+1-\alpha+r|^2}\\&-\frac{\tau}{T^{\frac{1}{2}+r}}\sum_{j\in\mathbb{Z}_n}\sum_{k=0}^{\infty}\frac{(-1)^k(j2\pi)^{2k}}{(2k+1)!}\frac{T^{2k+1}}{|2k+1-\alpha_\tau-r|^2},\end{aligned}$$

and where

$$\begin{aligned}W_\tau(r,n,T):=&\frac{1}{T^{\frac{1}{2}-r}}\sum_{j\in\mathbb{Z}_n}\sum_{k=0}^{\infty}\frac{(-1)^k(j2\pi)^{2k}}{(2k+1)!}T^{2k+1}\frac{2k+1-\frac{1}{2}+r}{|2k+1-\alpha+r|^2}\\&-\frac{1}{T^{\frac{1}{2}+r}}\sum_{j\in\mathbb{Z}_n}\sum_{k=0}^{\infty}\frac{(-1)^k(j2\pi)^{2k}}{(2k+1)!}T^{2k+1}\frac{2k+1-\frac{1}{2}-r}{|2k+1-\alpha-r|^2}\end{aligned}$$

Thanks to Equation (2), we obtain

$$\begin{aligned}\Im(\phi_\tau(r,n,T))=&-\cos(\tau\ln(T))V_\tau(r,n,T)-\sin(\tau\ln(T))W_\tau(r,nT)\\&+V_\tau(r,n,1).\end{aligned} \tag{4}$$

We have

$$V_\tau(r,n,T) := \frac{\tau}{T^{\frac{1}{2}-r}} \sum_{j\in\mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k}}{(4k+1)!} \frac{T^{4k+1}}{|4k+1-\alpha+r|^2}$$
$$+ \frac{\tau}{T^{\frac{1}{2}+r}} \sum_{j\in\mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k+2}}{(4k+3)!} \frac{T^{4k+3}}{|4k+3-\alpha-r|^2}$$
$$- \frac{\tau}{T^{\frac{1}{2}+r}} \sum_{j\in\mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k}}{(4k+1)!} \frac{T^{4k+1}}{|4k+1-\alpha-r|^2}$$
$$- \frac{\tau}{T^{\frac{1}{2}-r}} \sum_{j\in\mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k+2}}{(4k+3)!} \frac{T^{4k+3}}{|4k+3-\alpha+r|^2},$$

which can be written as

$$V_\tau(r,n,T) := \tau\sqrt{T} \sum_{j\in\mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k}}{(4k+1)!} \tau T^{4k} \int_{-r}^{r} \frac{d}{du} \frac{T^{-u}}{|4k+1-\alpha+u|^2} du$$
$$- \tau\sqrt{T} \sum_{j\in\mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k+2}}{(4k+3)!} T^{4k+2} \int_{-r}^{r} \frac{d}{du} \frac{T^{u}}{|4k+3-\alpha-u|^2} du.$$

Since

$$\frac{d}{du} \frac{T^{-u}}{|4k+1-\alpha+u|^2} = -\frac{1}{T^u} \frac{\ln(T)((4k+\frac{1}{2}+u)^2+\tau^2)+2(4k+\frac{1}{2}+u)}{((4k+\frac{1}{2}+u)^2+\tau^2)^2},$$

and

$$\frac{d}{du} \frac{T^{u}}{|4k+3-\alpha-u|^2} = \frac{\exp(u\ln(T))\ln(T)}{\Big((4k+2+\frac{1}{2}-u)^2+\tau^2\Big)} + \frac{2\exp(u\ln(T))(4k+2+\frac{1}{2}-u)}{\Big((4k+2+\frac{1}{2}-u)^2+\tau^2\Big)^2}.$$

Then $\tau V_\tau(r,n,T) < 0$ for every $T > 1$, by continuity $\tau V_\tau(r,n,1) \le 0$. There exists a sequence $(t_{k,n})_k \to +\infty$ such that $\lim_{k\to+\infty} t_{k,n} = +\infty$ and such that

$$\cot(\tau\ln(t_{k,n})) = -\frac{W_\tau(r,n,t_{k,n})}{V_\tau(r,n,t_{k,n})}.$$

By Equation (3), we get $\Im(\psi_\tau(r,n,t_{k,n}) = 0$ for every $k \ge 0$. By Equation (4) we get

$$\Im(\phi_\tau(r,n,t_{k,n})) = V_\tau(r,n,1), \quad \forall k \ge 0.$$

Since $\lim_{k\to+\infty} t_{k,n} = +\infty$, then

$$\tau\Im(\phi_\tau(r,n)) = \tau V_\tau(r,n,1) \leq 0.$$

Since

$$\Im(\phi_\tau(r)) = \frac{-2r\tau}{|(1-\alpha_\tau)^2 - r^2|^2} + \frac{-r\tau}{|\alpha_\tau^2 - r^2|^2} + \lim_{n\to+\infty} \Im(\phi_\tau(r,n)),$$

we get

$$\tau\Im(\phi_\tau(r)) \leq \frac{-2r\tau^2}{|(1-\alpha_\tau)^2 - r^2|^2} + \frac{-r\tau^2}{|\alpha_\tau^2 - r^2|^2} < 0.$$

□

6 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function ζ defined by Equation (1). Define the *non trivial zeros* of the function ζ in the following sense

Definition 4. Let be $s \in \mathbb{C}$. We say that s is a *non trivial zero of the function* ζ if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0,1), \quad \Im(s) \in \mathbb{R}^*.$$

By the Riemann functional equation [[1], page 13 Theorem 2.1], the non trivial zeros of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. By the Main result, the non trivial zeros $s \in \mathbb{C}$ symmetric about the line $\Re(z) = \frac{1}{2}$ satisfies $\Re(s) = \frac{1}{2}$.

References

[1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).