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Non symmetry of the function $\mathfrak{S}\left(\frac{\zeta(s)}{s}\right)$

W. Oukil

Laboratory of Dynamical Systems, Faculty of Mathematics.
University of Science and Technology Houari Boumediene,
BP 32 EL ALIA 16111 Bab Ezzouar, Algiers, Algeria.

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Abstract

We prove that $\mathfrak{S}\left(\frac{\zeta(s)}{s}\right) + \mathfrak{S}\left(\frac{\zeta(1-s)}{1-s}\right) \neq 0$ for every $\Re(s) \in (0, \frac{1}{2})$ and $\Im(s) \in \mathbb{R}^*$, where ζ is the Riemann Zeta function. At the end of the paper, we give a discussion about the Riemann hypothesis.

Keywords: Riemann functional equation, Zeta function.

1 Main result

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

and where $\{t\}$ is the fractional part of the real t . We prove in this paper the following Main result

Main Result. Consider the function ζ defined by Equation (1). Then

$$\mathfrak{S}\left(\frac{\zeta\left(\frac{1}{2} + r + i\tau\right)}{\frac{1}{2} + r + i\tau}\right) - \mathfrak{S}\left(\frac{\zeta\left(\frac{1}{2} - r + i\tau\right)}{\frac{1}{2} - r + i\tau}\right) \neq 0, \quad \forall r \in (0, \frac{1}{2}), \quad \forall \tau \in \mathbb{R}^*.$$

2 Main Proposition

Notation 1. In order to simplify the notation, we denote

$$\alpha_\tau := \frac{1}{2} + i\tau, \quad \forall \tau \in \mathbb{R}^*.$$

For every $r \in (0, \frac{1}{2})$ and $\tau \in \mathbb{R}^*$ denote

$$\phi_\tau(r) := \frac{\zeta(\alpha_\tau + r)}{\alpha_\tau + r} - \frac{\zeta(\alpha_\tau - r)}{\alpha_\tau - r}.$$

For every $n \geq 1$ we denote

$$\mathbb{Z}_n := \{j \in \mathbb{Z}^*, \quad |j| \leq n\}.$$

We have the following Main proposition

Proposition 2. *For every $\tau \in \mathbb{R}^*$ and $r \in (0, \frac{1}{2})$ we have*

$$\phi_\tau(r) = \lim_{n \rightarrow +\infty} \phi_\tau(r, n),$$

where

$$\begin{aligned} \phi_\tau(r, n) &= \frac{-3r}{(1 - \alpha_\tau)^2 - r^2} \\ &+ \int_1^{+\infty} \left(\frac{1}{u^{1+\alpha_\tau-r}} - \frac{1}{u^{1+\alpha_\tau+r}} \right) \sum_{j \in \mathbb{Z}_n} \frac{i}{j2\pi} \exp(ij2\pi u) du. \end{aligned}$$

3 Main Lemma

Lemma 3. *We have*

$$\int_0^t (\{v\} + 1) dv = t \frac{3}{2} + \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} (\exp(ij2\pi t) - 1), \quad \forall t \geq 0.$$

Proof. The function $v \mapsto \{v\} + 1$ is 1-periodic, then there exists a continuous 1-periodic function $p : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\int_0^t (\{v\} + 1) dv = t \int_0^1 (\{v\} + 1) dv + p(t), \quad \forall t \in \mathbb{R}.$$

Since

$$\int_0^1 (\{v\} + 1) dv = \frac{3}{2},$$

we get

$$\int_0^t (\{v\} + 1) dv = t \frac{3}{2} + p(t), \quad \forall t \in \mathbb{R}.$$

The function p is a piecewise C^∞ , continuous on \mathbb{R} and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on \mathbb{R}_+ to the function $t \mapsto p(t)$, where $(a_k)_k \subset \mathbb{C}$ are the Fourier coefficients of the function p .

$$\int_0^t (\{v\} + 1)dv = t\frac{3}{2} + \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left(\int_0^u (\{v\} + 1)dv - u\frac{3}{2} \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12}.$$

□

4 Proof of the Main Proposition

Proof of the Proposition 2. Let be $r \in (0, \frac{1}{2})$ and $\tau \in \mathbb{R}^*$. We have

$$\begin{aligned} \phi_\tau(r) &= \frac{-3r}{(1 - \alpha_\tau)^2 - r^2} \\ &\quad + \int_1^{+\infty} \left(\frac{1}{u^{1+\alpha_\tau-r}} - \frac{1}{u^{1+\alpha_\tau+r}} \right) \left(\{u\} - \frac{1}{2} \right) du. \end{aligned}$$

Use the integration part formula,

$$\begin{aligned} \phi_\tau(r) &= \frac{-3r}{(1 - \alpha_\tau)^2 - r^2} \\ &\quad + \int_1^{+\infty} \left(\frac{1 + \alpha_\tau - r}{u^{2+\alpha_\tau-r}} - \frac{1 + \alpha_\tau + r}{u^{2+\alpha_\tau+r}} \right) \left[\int_0^u \left(\{v\} - \frac{1}{2} \right) dv \right] du. \end{aligned}$$

By Lemma 3,

$$\begin{aligned}\phi_\tau(r) &= \frac{-3r}{(1-\alpha_\tau)^2 - r^2} \\ &+ \int_1^{+\infty} \left(\frac{1+\alpha_\tau-r}{u^{2+\alpha_\tau-r}} - \frac{1+\alpha_\tau+r}{u^{2+\alpha_\tau+r}} \right) \sum_{j \in \mathbb{Z}^*} \frac{\exp(ij2\pi u)}{(j2\pi)^2} du.\end{aligned}$$

Then

$$\phi_\tau(r) = \lim_{n \rightarrow +\infty} \phi_\tau(r, n),$$

where

$$\begin{aligned}\phi_\tau(r, n) &= \frac{-3r}{(1-\alpha_\tau)^2 - r^2} \\ &+ \int_1^{+\infty} \left(\frac{1+\alpha_\tau+r}{u^{2+\alpha_\tau-r}} - \frac{1+\alpha_\tau-r}{u^{2+\alpha_\tau+r}} \right) \sum_{j \in \mathbb{Z}_n} \frac{\exp(ij2\pi u)}{(j2\pi)^2} du.\end{aligned}$$

Use the integration part formula, we get

$$\begin{aligned}\phi_\tau(r, n) &= \frac{-3r}{(1-\alpha_\tau)^2 - r^2} \\ &+ \int_1^{+\infty} \left(\frac{1}{u^{1+\alpha_\tau-r}} - \frac{1}{u^{1+\alpha_\tau+r}} \right) \sum_{j \in \mathbb{Z}_n} \frac{i}{j2\pi} \exp(ij2\pi u) du.\end{aligned}$$

□

5 Proof of the Main result

Proof of the Main result. Let be $\tau \in \mathbb{R}^*$ fixed. By the Proposition 2,

$$\phi_\tau(r) = \lim_{n \rightarrow +\infty} \phi_\tau(r, n),$$

where

$$\begin{aligned}\phi_\tau(r, n) &= \frac{-3r}{(1-\alpha_\tau)^2 - r^2} \\ &+ \int_1^{+\infty} \left(\frac{1}{u^{1+\alpha_\tau-r}} - \frac{1}{u^{1+\alpha_\tau+r}} \right) \sum_{j \in \mathbb{Z}_n} \frac{i}{j2\pi} \exp(ij2\pi u) du.\end{aligned}$$

Denote for ever $T \geq 1$,

$$\begin{aligned} \phi_\tau(r, n, T) &= \frac{-3r}{(1 - \alpha_\tau)^2 - r^2} \\ &+ \int_1^T \left(\frac{1}{u^{1+\alpha_\tau-r}} - \frac{1}{u^{1+\alpha_\tau+r}} \right) \sum_{j \in \mathbb{Z}_n} \frac{i}{j2\pi} \exp(ij2\pi u) du. \end{aligned}$$

Use the Taylor formula

$$\begin{aligned} \phi_\tau(r, n, T) &= \frac{-3r}{(1 - \alpha_\tau)^2 - r^2} \\ &+ \sum_{j \in \mathbb{Z}_n} \frac{i}{j2\pi} \sum_{k=1}^{\infty} \frac{1}{k!} (ij2\pi)^k \int_1^T \left(u^{k-1-\alpha_\tau+r} - u^{k-1-\alpha_\tau-r} \right) du. \end{aligned}$$

implies

$$\begin{aligned} \phi_\tau(r, n, T) &= \frac{-3r}{(1 - \alpha_\tau)^2 - r^2} \\ &+ \sum_{k=1}^{\infty} \frac{i^{k+1}}{k!} \left[\sum_{j \in \mathbb{Z}_n} (j2\pi)^{k-1} \right] \int_1^T \left(u^{k-1-\alpha_\tau+r} - u^{k-1-\alpha_\tau-r} \right) du \end{aligned}$$

Since $\sum_{j \in \mathbb{Z}_n} (j2\pi)^{2k-1} = 0$ for every $k \geq 1$ we get

$$\begin{aligned} \phi_\tau(r, n, T) &= \frac{-3r}{(1 - \alpha_\tau)^2 - r^2} \\ &- \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \sum_{j \in \mathbb{Z}_n} (j2\pi)^{2k} \int_1^T \left(u^{2k-\alpha_\tau+r} - u^{2k-\alpha_\tau-r} \right) du \end{aligned}$$

Then

$$\phi_\tau(r, n, T) = \frac{-3r}{(1 - \alpha_\tau)^2 - r^2} - \psi_\tau(r, n, T) + \psi_\tau(r, n, 1), \quad (2)$$

and where for every $T \geq 1$ the function $T \mapsto \psi_\tau(r, n, T)$ is defined as

$$\begin{aligned} \psi_\tau(r, n, T) &:= \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(-1)^k (j2\pi)^{2k}}{(2k+1)!} \frac{T^{2k+1-\alpha_\tau+r}}{2k+1-\alpha_\tau+r} \\ &- \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(-1)^k (j2\pi)^{2k}}{(2k+1)!} \frac{T^{2k+1-\alpha_\tau-r}}{2k+1-\alpha_\tau-r}. \end{aligned}$$

We have

$$\Im(\psi_\tau(r, n, T)) := \cos(\tau \ln(T))V_\tau(r, n, T) + \sin(\tau \ln(T))W_\tau(r, n, T), \quad (3)$$

where

$$\begin{aligned} V_\tau(r, n, T) &:= \frac{\tau}{T^{\frac{1}{2}-r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(-1)^k (j2\pi)^{2k}}{(2k+1)!} \frac{T^{2k+1}}{|2k+1-\alpha+r|^2} \\ &\quad - \frac{\tau}{T^{\frac{1}{2}+r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(-1)^k (j2\pi)^{2k}}{(2k+1)!} \frac{T^{2k+1}}{|2k+1-\alpha_\tau-r|^2}, \end{aligned}$$

and where

$$\begin{aligned} W_\tau(r, n, T) &:= \frac{1}{T^{\frac{1}{2}-r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(-1)^k (j2\pi)^{2k}}{(2k+1)!} T^{2k+1} \frac{2k+1-\frac{1}{2}+r}{|2k+1-\alpha+r|^2} \\ &\quad - \frac{1}{T^{\frac{1}{2}+r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(-1)^k (j2\pi)^{2k}}{(2k+1)!} T^{2k+1} \frac{2k+1-\frac{1}{2}-r}{|2k+1-\alpha-r|^2} \end{aligned}$$

Thanks to Equation (2), we obtain

$$\begin{aligned} \Im(\phi_\tau(r, n, T)) &= \frac{-3r\tau}{|(1-\alpha_\tau)^2-r^2|^2} + V_\tau(r, n, 1) \\ &\quad - \cos(\tau \ln(T))V_\tau(r, n, T) - \sin(\tau \ln(T))W_\tau(r, n, T). \quad (4) \end{aligned}$$

We have

$$\begin{aligned} V_\tau(r, n, T) &:= \frac{\tau}{T^{\frac{1}{2}-r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k}}{(4k+1)!} \frac{T^{4k+1}}{|4k+1-\alpha+r|^2} \\ &\quad + \frac{\tau}{T^{\frac{1}{2}+r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k+3}}{(4k+3)!} \frac{T^{4k+3}}{|4k+3-\alpha-r|^2} \\ &\quad - \frac{\tau}{T^{\frac{1}{2}+r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k}}{(4k+1)!} \frac{T^{4k+1}}{|4k+1-\alpha-r|^2} \\ &\quad - \frac{\tau}{T^{\frac{1}{2}-r}} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k+3}}{(4k+3)!} \frac{T^{4k+3}}{|4k+3-\alpha+r|^2}, \end{aligned}$$

which can be written as

$$V_\tau(r, n, T) := \tau\sqrt{T} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k}}{(4k+1)!} \tau T^{4k} \int_{-r}^r \frac{d}{du} \frac{T^{-u}}{|4k+1-\alpha+u|^2} du \\ - \tau\sqrt{T} \sum_{j \in \mathbb{Z}_n} \sum_{k=0}^{\infty} \frac{(j2\pi)^{4k+3}}{(4k+3)!} T^{4k+2} \int_{-r}^r \frac{d}{du} \frac{T^u}{|4k+3-\alpha-u|^2} du.$$

Since

$$\frac{d}{du} \frac{T^{-u}}{|4k+1-\alpha+u|^2} = -\frac{1}{T^u} \frac{\ln(T)((4k+\frac{1}{2}+u)^2 + \tau^2) + 2(4k+\frac{1}{2}+u)}{((4k+\frac{1}{2}+u)^2 + \tau^2)^2},$$

and

$$\frac{d}{du} \frac{T^u}{|4k+3-\alpha-u|^2} = \frac{\exp(u \ln(T)) \ln(T)}{\left((4k+2+\frac{1}{2})^2 + \tau^2\right)} \\ + \frac{2 \exp(u \ln(T))(4k+2+\frac{1}{2}-u)}{\left((4k+2+\frac{1}{2})^2 + \tau^2\right)^2}.$$

Then $V_\tau(r, n, T) < 0$ for every $T > 1$, by continuity $V_\tau(r, n, 1) \leq 0$. There exists a sequence $(t_{k,n})_k \rightarrow +\infty$ such that $\lim_{k \rightarrow +\infty} t_{k,n} = +\infty$ and such that

$$\cot(\tau \ln(t_{k,n})) = -\frac{W_\tau(r, n, t_{k,n})}{V_\tau(r, n, t_{k,n})}.$$

By Equation (3), we get $\Im(\psi_\tau(r, n, t_{k,n})) = 0$ for every $k \geq 0$. By Equation (4) we get

$$\Im(\phi_\tau(r, n, t_{k,n})) = \frac{-3r\tau}{|(1-\alpha_\tau)^2 - r^2|^2} + V_\tau(r, n, 1), \quad \forall k \geq 0.$$

Since $\lim_{k \rightarrow +\infty} t_{k,n} = +\infty$, then

$$\Im(\phi_\tau(r, n)) = \frac{-3r\tau}{|(1-\alpha_\tau)^2 - r^2|^2} + V_\tau(r, n, 1).$$

Since $\Im(\phi_\tau(r)) = \lim_{n \rightarrow +\infty} \Im(\phi_\tau(r, n))$ we get

$$\Im(\phi_\tau(r)) \leq \frac{-3r\tau}{|(1-\alpha_\tau)^2 - r^2|^2} < 0.$$

□

6 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function ζ defined by Equation (1). Define the *non trivial zeros* of the function ζ in the following sense

Definition 4. Let be $s \in \mathbb{C}$. We say that s is a *non trivial zero of the function ζ* if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

By the Riemann functional equation [[1], page 13 Theorem 2.1], the non trivial zeros of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. By the Main result, the non trivial zeros $s \in \mathbb{C}$ symmetric about the line $\Re(z) = \frac{1}{2}$ satisfies $\Re(s) = \frac{1}{2}$.

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).