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# Non symmetry of the function $\mathfrak{S}\left(\frac{\zeta(s)}{s}\right)$

W. Oukil

Laboratory of Dynamical Systems, Faculty of Mathematics.  
University of Science and Technology Houari Boumediene,  
BP 32 EL ALIA 16111 Bab Ezzouar, Algiers, Algeria.

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## Abstract

We prove that  $\mathfrak{S}\left(\frac{\zeta(s)}{s}\right) + \mathfrak{S}\left(\frac{\zeta(1-s)}{1-s}\right) \neq 0$  for every  $\Re(s) \in (0, \frac{1}{2})$  and  $\Im(s) \in \mathbb{R}^*$ , where  $\zeta$  is the Riemann Zeta function. In the end of the paper, we give a discussion about the Riemann hypothesis.

**Keywords:** Riemann functional equation, Zeta function.

## 1 Main result

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

and where  $\{t\}$  is the fractional part of the real  $t$ . We prove in this paper the following Main result

**Main Result.** *Consider the function  $\zeta$  defined by Equation (1). Then*

$$\mathfrak{S}\left(\frac{\zeta\left(\frac{1}{2} - r + i\tau\right)}{\frac{1}{2} - r + i\tau}\right) - \mathfrak{S}\left(\frac{\zeta\left(\frac{1}{2} + r + i\tau\right)}{\frac{1}{2} + r + i\tau}\right) \neq 0, \quad \forall r \in (0, \frac{1}{2}), \quad \forall \tau \in \mathbb{R}^*.$$

*More precisely, for every  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$ , we have*

$$\frac{\zeta\left(\frac{1}{2} - r + i\tau\right)}{\frac{1}{2} - r + i\tau} + \frac{\zeta\left(\frac{1}{2} + r + i\tau\right)}{\frac{1}{2} + r + i\tau} = \left(3 - \frac{1}{6}\right) \frac{r}{\left(\frac{1}{2} + i\tau\right)^2 - r^2}.$$

## 2 Main Proposition

**Notation 1.** In order to simplify the notation, we denote

$$\rho_0 := 3, \quad \text{and} \quad \rho_k := \sum_{j \in \mathbb{Z}^*} \frac{1}{(ij2\pi)^{k+1}}, \quad \forall k \geq 1.$$

and we denote

$$\alpha_\tau := \frac{1}{2} + i\tau, \quad \forall \tau \in \mathbb{R}^*.$$

For every  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$  denote

$$\phi_\tau(r) := \frac{\zeta(\alpha_\tau - r)}{\alpha_\tau - r} - \frac{\zeta(\alpha_\tau + r)}{\alpha_\tau + r}.$$

We have the following Main proposition

**Proposition 2.** *For every  $\tau \in \mathbb{R}^*$  and  $r \in (0, \frac{1}{2})$  we have*

$$\frac{d^{2n+1}}{dr^{2n+1}} \phi_\tau(0) = (2n+1)!(3+2\rho_1) \frac{1}{\alpha_\tau^{2(n+1)}}. \quad (2)$$

where  $(\rho_k)_k$  are given in the Notation 1.

## 3 Main Lemma

**Lemma 3.** *We have*

$$\int_0^t (\{v\} + 1) dv = t \frac{3}{2} - \frac{1}{12} + \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \exp(ij2\pi t), \quad \forall t \geq 0,$$

*Proof.* The function  $v \mapsto \{v\} + 1$  is 1-periodic, then there exists a continuous 1-periodic function  $p : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\int_0^t (\{v\} + 1) dv = t \int_0^1 (\{v\} + 1) dv + p(t), \quad \forall t \in \mathbb{R}.$$

Since

$$\int_0^1 (\{v\} + 1) dv = \frac{3}{2},$$

we get

$$\int_0^t (\{v\} + 1) dv = t \frac{3}{2} + p(t), \quad \forall t \in \mathbb{R}.$$

The function  $p$  is a piecewise  $C^\infty$ , continuous on  $\mathbb{R}$  and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on  $\mathbb{R}_+$  to the function  $t \mapsto p(t)$ , where  $(a_k)_k \subset \mathbb{C}$  are the Fourier coefficients of the function  $p$ .

$$\int_0^t (\{v\} + 1)dv = t\frac{3}{2} + \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left( \int_0^u (\{v\} + 1)dv - u\frac{3}{2} \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^* \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12}.$$

□

**Lemma 4.** For every  $k \geq 0$  define the function  $W_k(u) : \mathbb{R}_+ \rightarrow \mathbb{R}$  as

$$W_0(u) := \int_1^u (\{v\} - \frac{1}{2})dv, \quad W_{k+1}(u) = \int_1^u W_k(v)dv, \quad \forall k \geq 0.$$

Then for all  $k \geq 1$  and  $u \geq 1$  we have

$$\left| W_k(u) + \frac{\rho_1}{k!} (u-1)^k \right| \leq \frac{\rho_1}{k!} u^k.$$

*Proof.* By Lemma 3

$$\int_1^u (\{v\} - \frac{1}{2})dv = \sum_{j \in \mathbb{Z}^*} \frac{1}{(2j\pi)^2} (\exp(ij2\pi t) - 1),$$

where we recall that

$$\frac{1}{12} = - \sum_{j \in \mathbb{Z}^*} \frac{1}{(ij2\pi)^2},$$

Let be  $k \geq 1$  we have

$$\begin{aligned} W_k(u) &= \int_1^u \int_1^{u_{k-1}} \cdots \int_1^{u_1} \sum_{j \in \mathbb{Z}^*} \frac{1}{(ij2\pi)^2} \left( \exp(ij2\pi u) - 1 \right) du du_1, \dots, du_{k-1}. \end{aligned}$$

Implies

$$\begin{aligned} W_k(u) &= -\rho_1 \int_1^u \int_1^{u_{k-1}} \cdots \int_1^{u_1} 1 du du_1, \dots, du_{k-1} \\ &+ \int_1^u \int_1^{u_{k-1}} \cdots \int_1^{u_2} \sum_{j \in \mathbb{Z}^*} \frac{1}{(ij2\pi)^2} \exp(ij2\pi u_1) du_1, \dots, du_{k-1} \\ &= -\rho_1 \frac{1}{k!} (u-1)^k \\ &+ \int_1^u \int_1^{u_{k-1}} \cdots \int_1^{u_2} \sum_{j \in \mathbb{Z}^*} \frac{1}{(ij2\pi)^3} \left( \exp(ij2\pi u_2) - 1 \right) du_2, \dots, du_{k-1}. \end{aligned}$$

By induction

$$W_k(u) = - \sum_{m=0}^{k-1} \frac{\rho_{m+1}}{(k-m)!} (u-1)^{k-m} + f_k(u), \quad \forall u \geq 1, \quad (3)$$

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$$f_k(u) := \sum_{j \in \mathbb{Z}^*} \frac{1}{(ij2\pi)^{2+k}} \left( \exp(ij2\pi u) - 1 \right), \quad \forall u \geq 1.$$

Use the fact

$$f_k(u) := \sum_{j \in \mathbb{Z}^*} \frac{1}{(ij2\pi)^{2+k}} \left( \exp(ij2\pi u) - \exp(ij2\pi) \right),$$

an use the Taylor formula, for every  $u \geq 1$  we get

$$\left| f_k(u) - \sum_{j \in \mathbb{Z}^*} \frac{1}{(ij2\pi)^{2+k}} \sum_{m=1}^{k-1} \frac{(ij2\pi)^m}{m!} (u-1)^m \right| \leq \frac{u^k}{k!} \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2},$$

which can be written as

$$\left| f_k(u) - \sum_{m=1}^{k-1} \frac{1}{m!} (u-1)^m \sum_{j \in \mathbb{Z}^*} \frac{1}{(ij2\pi)^{k-m+2}} \right| \leq \frac{u^k}{k!} \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2}, \quad \forall u \geq 1,$$

by the notation of  $\rho_k$  is the Notation 1, that is equivalent to

$$\left| f_k(u) - \sum_{m=1}^{k-1} \frac{1}{m!} (u-1)^m \rho_{k+1-m} \right| \leq \frac{\rho_1}{k!} u^k, \quad \forall u \geq 1,$$

in other word

$$\left| f_k(u) - \sum_{m=1}^{k-1} \frac{1}{(k-m)!} (u-1)^{k-m} \rho_{m+1} \right| \leq \frac{\rho_1}{k!} u^k, \quad \forall u \geq 1.$$

From Equation (3), we deduce that

$$\left| W_k(u) + \frac{1}{k!} (u-1)^k \rho_1 \right| \leq \frac{\rho_1}{k!} u^k, \quad \forall u \geq 1.$$

□

**Lemma 5.** *Let be  $n \in \mathbb{N}^*$  and  $\tau \in \mathbb{R}^*$ . For every  $k \geq 0$  define the function  $V_{k,\tau}(n, u) : \mathbb{R} \rightarrow \mathbb{C}$  as*

$$\begin{aligned} V_{0,\tau}(n, u) &:= u^n \exp(-(\alpha_\tau + 1)u), \\ V_{k+1,\tau}(n, u) &= \exp(-u) \frac{d}{du} V_{k,\tau}(n, u), \quad \forall k \geq 0. \end{aligned}$$

There exists  $\gamma_{\tau,n} > 0$  such that

$$\int_0^{+\infty} \left| \frac{d}{du} V_{k,\tau}(n, u) \right| \exp(ku) du \leq \gamma_{\tau,n} \Pi_{m=1}^{k-2} |\alpha_\tau + m|, \quad \forall k \geq 4.$$

*Proof.* Let be  $n \in \mathbb{N}^*$  and  $\tau \in \mathbb{R}^*$ . By recurrence, for every  $k \geq 0$  there exists a polynomial  $P_k(n, u)$  such that

$$\frac{d}{du} V_{k,\tau}(n, u) = \exp(-(\alpha_\tau + k + 1)u) P_k(n, u). \quad (4)$$

We have

$$\frac{d}{du} V_{k,\tau}(n, u) = \exp(-u) \left( \frac{d^2}{du^2} V_{k-1,\tau}(n, u) - \frac{d}{du} V_{k-1,\tau}(n, u) \right).$$

Then

$$\begin{aligned} \frac{d}{du} V_{k,\tau}(n, u) &= \exp(-u) \frac{d}{du} \left[ \exp(-(\alpha_\tau + k)u) P_{k-1}(n, u) \right] \\ &\quad - \exp(-(\alpha_\tau + k + 1)u) P_{k-1}(n, u). \end{aligned} \quad (5)$$

Implies

$$P_k(n, u) = \frac{d}{du} P_{k-1}(n, u) - (\alpha_\tau + k + 1) P_{k-1}(n, u). \quad (6)$$

Write

$$P_k(n, u) = \sum_{j=0}^n c_{k,j}(n, \tau) \frac{d^j}{du^j} P_0(n, u), \quad (7)$$

and replace it in Equation (6),

$$\begin{aligned} \sum_{j=0}^n c_{k,j}(n, \tau) \frac{d^j}{du^j} P_0(n, u) &= \sum_{j=0}^n c_{k-1,j}(n, \tau) \frac{d^{j+1}}{du^{j+1}} P_0(n, u) \\ &\quad - (\alpha_\tau + k + 1) \sum_{j=0}^n c_{k-1,j}(n, \tau) \frac{d^j}{du^j} P_0(n, u). \end{aligned}$$

Since  $P_0(n, u) = nu^{n-1} - (\alpha + 1)u^n$  then  $\frac{d^{n+1}}{du^{n+1}} P_0(n, u) = 0$ , we get

$$\begin{aligned} \sum_{j=0}^n c_{k,j}(n, \tau) \frac{d^j}{du^j} P_0(n, u) &= \sum_{j=0}^{n-1} c_{k-1,j}(n, \tau) \frac{d^{j+1}}{du^{j+1}} P_0(n, u) \\ &\quad - (\alpha_\tau + k + 1) \sum_{j=0}^n c_{k-1,j}(n, \tau) \frac{d^j}{du^j} P_0(n, u), \end{aligned}$$

which can be written as

$$\begin{aligned} \sum_{j=0}^n c_{k,j} \frac{d^j}{du^j} P_0(n, u) &= \sum_{j=1}^n c_{k-1,j-1}(n, \tau) \frac{d^j}{du^j} P_0(n, u) \\ &\quad - (\alpha_\tau + k + 1) \sum_{j=0}^n c_{k-1,j}(n, \tau) \frac{d^j}{du^j} P_0(n, u). \end{aligned}$$

Implies

$$\begin{pmatrix} c_{k,0}(n, \tau) \\ c_{k,1}(n, \tau) \\ \vdots \\ c_{k,n}(n, \tau) \end{pmatrix} = \left( C_n - (\alpha_\tau + k + 1) Id_n \right) \begin{pmatrix} c_{k-1,0}(n, \tau) \\ c_{k-1,1}(n, \tau) \\ \vdots \\ c_{k-1,n}(n, \tau) \end{pmatrix}$$

where  $Id_n$  is the  $n \times n$  identity matrix and where  $C_n$  is the nilpotent  $n \times n$  matrix defined as

$$C_n := \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} c_{k,0}(n, \tau) \\ c_{k,1}(n, \tau) \\ \vdots \\ c_{k,n}(n, \tau) \end{pmatrix} = \left[ \prod_{m=1}^k \left( C_n - (\alpha_\tau + m + 1) Id_n \right) \right] \begin{pmatrix} c_{0,0}(n, \tau) \\ c_{0,1}(n, \tau) \\ \vdots \\ c_{0,n}(n, \tau) \end{pmatrix}$$

By definition of  $P_k(n, u)$  (Resp.  $V_{k+1, \tau}(n, u)$ ) in Equation (7) (Resp. Equation (5)) we have  $P_0(n, u) = nu^{n-1} - (\alpha + 1)u^n$ , we get

$$\begin{pmatrix} c_{k,0}(n, \tau) \\ c_{k,1}(n, \tau) \\ \vdots \\ c_{k,n}(n, \tau) \end{pmatrix} = \left[ \prod_{m=0}^k \left( C_n - (\alpha_\tau + m + 1) Id_n \right) \right] \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

By Equation (4), for all  $u \geq 0$  we find

$$\left| \frac{d}{du} V_{k, \tau}(n, u) \right| \leq n! \left[ \prod_{m=1}^{k+1} |\alpha_\tau + m| \right] \exp\left(-\left(\frac{1}{2} + k + 2\right)u\right) u^n, \quad (8)$$

Since

$$V_{k, \tau}(n, u) = \exp(-u) \frac{d}{du} V_{k, \tau}(n, u),$$

then

$$\frac{d}{du} V_{k, \tau}(n, u) = \exp(-u) \left( \frac{d^2}{du^2} V_{k-1, \tau}(n, u) - \frac{d}{du} V_{k-1, \tau}(n, u) \right), \quad (9)$$

implies

$$\frac{d}{du} V_{k-1, \tau}(n, u) = \exp(-u) \left( \frac{d^2}{du^2} V_{k-2, \tau}(n, u) - \frac{d}{du} V_{k-2, \tau}(n, u) \right),$$

replace in Equation (9), we get

$$\begin{aligned} & \frac{d}{du} V_{k, \tau}(n, u) \\ &= \exp(-2u) \left( \frac{d^3}{du^3} V_{k-2, \tau}(n, u) - 3 \frac{d^2}{du^2} V_{k-2, \tau}(n, u) + 2 \frac{d}{du} V_{k-2, \tau}(n, u) \right). \end{aligned}$$

Since the last Equation is satisfied for every  $k \geq 2$ , we obtain for every  $k \geq 4$  the following equality

$$\begin{aligned} & \frac{d}{du} V_{k-2, \tau}(n, u) \\ &= \exp(-2u) \left( \frac{d^3}{du^3} V_{k-4, \tau}(n, u) - 3 \frac{d^2}{du^2} V_{k-4, \tau}(n, u) + 2 \frac{d}{du} V_{k-4, \tau}(n, u) \right). \end{aligned}$$



By Equation (8) there exists  $\omega_{\tau,n} > 0$  such that

$$\left| \frac{d}{du} V_{k,\tau}(n, u) \right| \leq n! \omega_{\tau,n} \left[ \prod_{m=1}^{k-2} |\alpha_{\tau} + m| \right] \exp\left(-\left(\frac{1}{2} + k + 2\right)u\right) u^n,$$

Since

$$\int_0^{+\infty} \exp\left(-\frac{3}{2}u\right) u^n du = n! \left(\frac{2}{3}\right)^{n+1},$$

we deduce that

$$\int_0^{+\infty} \left| \frac{d}{du} V_{k,\tau}(n, u) \right| \exp(ku) du \leq \gamma_{\tau,n} \prod_{m=1}^{k-2} |\alpha_{\tau} + m|,$$

where

$$\gamma_{\tau,n} := n_{\tau} (n!)^2 \left(\frac{2}{3}\right)^{n+1}.$$

□

## 4 Proof of the Main Proposition

**Lemma 6.** *Let be  $\tau \in \mathbb{R}^*$ . Consider the function  $\phi_{\tau}$  defined in the Notation*

1. *Then  $\frac{d^{2n}}{d\tau^{2n}} \phi_{\tau}(0) = 0$  for all  $n \geq 0$  and*

$$\frac{d^{2n+1}}{d\tau^{2n+1}} \phi_{\tau}(0) = 2 \int_0^{+\infty} u^{2n+1} \exp(-\alpha_{\tau} u) (\{\exp(u)\} + 1) du, \quad \forall n \geq 0.$$

*Proof.* Let be  $x \in (0, 1)$  and  $\tau \in \mathbb{R}^*$ . From Equation (1), we have

$$\begin{aligned} \frac{\zeta(x + i\tau)}{x + i\tau} &= - \int_0^{+\infty} \frac{\{t\}}{t^{1+x+i\tau}} dt \\ &= - \int_0^{+\infty} \exp(-i\tau \ln(t)) \frac{\{t\}}{t^{1+x}} dt \\ &= - \int_0^1 \exp(-i\tau \ln(t)) \frac{1}{t^x} dt - \int_1^{+\infty} \exp(-i\tau \ln(t)) \frac{\{t\}}{t^{1+x}} dt \end{aligned}$$

Use the change of variable,  $t \rightarrow \exp(-u)$  for the integral

$$\int_0^1 \exp(-i\tau \ln(t)) \frac{1}{t^x} dt,$$

and  $t \rightarrow \exp(u)$  for the integral

$$\int_1^{+\infty} \exp(-i\tau \ln(t)) \frac{\{t\}}{t^{1+x}} dt,$$

we obtain

$$\begin{aligned} \frac{\zeta(x+i\tau)}{x+i\tau} &= \int_0^{+\infty} \exp(-i\tau u) \exp(-(1-x)u) du \\ &\quad - \int_0^{+\infty} \exp(-i\tau u) \exp(-xu) \{\exp(u)\} du. \end{aligned}$$

We get

$$\begin{aligned} \phi_\tau(r) &= - \int_0^{+\infty} \exp(-i\tau u) [\exp(-(\frac{1}{2}+r)u) - \exp(-(\frac{1}{2}-r)u)] du \\ &\quad + \int_0^{+\infty} \exp(-i\tau u) [\exp((\frac{1}{2}+r)u) - \exp((\frac{1}{2}-r)u)] \{\exp(u)\} du. \end{aligned}$$

Implies

$$\phi_\tau(r) = 2 \int_0^{+\infty} \exp(-\alpha_\tau u) \sinh(ru) \left( \{\exp(u)\} + 1 \right) du.$$

Since  $r \in (0, \frac{1}{2})$  then

$$\frac{d^{2n}}{d^{2n}} \phi_\tau(r) = 2 \int_0^{+\infty} \exp(-\alpha_\tau u) u^{2n} \sinh(ru) \left( \{\exp(u)\} + 1 \right) du.$$

implies  $\frac{d^{2n}}{d^{2n}} \phi_\tau(0) = 0$  and

$$\frac{d^{2n+1}}{d^{2n+1}} \phi_\tau(r) = 2 \int_0^{+\infty} \exp(-\alpha_\tau u) u^{2n+1} \cosh(ru) \left( \{\exp(u)\} + 1 \right) du,$$

implies

$$\frac{d^{2n+1}}{d^{2n+1}} \phi_\tau(0) = 2 \int_0^{+\infty} \exp(-\alpha_\tau u) u^{2n+1} \left( \{\exp(u)\} + 1 \right) du.$$

□

*Proof of the Proposition 2.* Let be  $n \geq 0$  and  $\tau \in \mathbb{R}^*$ . By Lemma 6 , we have

$$\frac{d^{2n+1}}{dr^{2n+1}} \phi_\tau(0) = 2 \int_0^{+\infty} u^{2n+1} \exp(-\alpha_\tau u) \left( \{\exp(u)\} + 1 \right) du.$$

Then

$$\begin{aligned} \frac{d^{2n+1}}{dr^{2n+1}} \phi_\tau(0) &= 3 \int_0^{+\infty} u^{2n+1} \exp(-\alpha_\tau u) du \\ &\quad + 2 \int_0^{+\infty} u^{2n+1} \exp(-\alpha_\tau u) \left( \{\exp(u)\} - \frac{1}{2} \right) du, \end{aligned}$$

we get

$$\begin{aligned} \frac{d^{2n+1}}{dr^{2n+1}}\phi_\tau(0) &= 3\frac{(2n+1)!}{\alpha_\tau^{2(n+1)}} \\ &\quad + 2\int_0^{+\infty} u^{2n+1} \exp(-\alpha_\tau u) \left(\{\exp(u)\} - \frac{1}{2}\right) du. \end{aligned}$$

In order to simplify the notation, denote for every  $n \in \mathbb{N}$ ,

$$J_{n,\tau} := \int_0^{+\infty} u^{2n+1} \exp(-\alpha_\tau u) \left(\{\exp(u)\} - \frac{1}{2}\right) du.$$

Then

$$\frac{d^{2n+1}}{dr^{2n+1}}\phi_\tau(0) = 3(2n+1)!\alpha_\tau^{-2n-1} + 2J_{n,\tau}. \quad (10)$$

Use the integration part formula,

$$\begin{aligned} J_{n,\tau} &= \int_0^{+\infty} u^{2n+1} \exp(-(\alpha_\tau + 1)u) \exp(u) \left(\{\exp(u)\} - \frac{1}{2}\right) du \\ &= -\int_0^{+\infty} \left[\frac{d}{du} V_{0,\tau}(n, u)\right] \left[\int_1^{\exp(u)} \left(\{v\} - \frac{1}{2}\right) dv\right] du, \end{aligned}$$

where

$$V_{0,\tau}(n, u) := u^{2n+1} \exp(-(\alpha_\tau + 1)u),$$

Use a successive integration part formula, we get

$$\begin{aligned} J_{n,\tau} &= -\int_0^{+\infty} \left[\exp(-u) \frac{d}{du} V_{0,\tau}(n, u)\right] \left[\exp(u) \left[\int_1^{\exp(u)} \left(\{v\} - \frac{1}{2}\right) dv\right]\right] du \\ &= \int_0^{+\infty} \left[\frac{d}{du} V_{1,\tau}(n, u)\right] \left[\int_1^{\exp(u)} \int_1^{u_1} \left(\{v\} - \frac{1}{2}\right) dv u_1\right] du, \end{aligned}$$

where

$$V_{1,\tau}(n, u) := \exp(-u) \frac{d}{du} V_{0,\tau}(n, u)$$

by induction we deduce that

$$J_{n,\tau} = (-1)^{k+1} \int_0^{+\infty} \left[\frac{d}{du} V_{k,\tau}(n, u)\right] W_k(\exp(u)) du \quad (11)$$

where

$$\begin{aligned} V_{0,\tau}(n, u) &:= u^{2n+1} \exp(-(\alpha_\tau + 1)u), \\ V_{k+1,\tau}(n, u) &= \exp(-u) \frac{d}{du} V_{k,\tau}(n, u), \quad \forall k \geq 0, \end{aligned}$$

and where

$$W_0(u) := \int_1^u (\{v\} - \frac{1}{2}) dv, \quad W_{k+1}(u) = \int_1^u W_k(v) dv, \quad \forall k \geq 0. \quad (12)$$

By Lemma 4. Equation (11), can be written as

$$J_{n,\tau} = (-1)^k \frac{\rho_1}{k!} \int_0^{+\infty} \left[ \frac{d}{du} V_{k,\tau}(n, u) \right] (\exp(u) - 1)^k du + R_{\tau,n}(k), \quad (13)$$

where

$$\left| R_{\tau,n}(k) \right| \leq \frac{\rho_1}{k!} \int_0^{+\infty} \left| \frac{d}{du} V_{k,\tau}(n, u) \right| \exp((k+1)u) du.$$

Using the backward chaining technique, a successive integration part gives in Equation (13),

$$\begin{aligned} J_{n,\tau} &= \rho_1 \int_0^{+\infty} V_{0,\tau}(n, u) \exp(u) du + R_{\tau,n}(k) \\ &= \rho_1 \int_0^{+\infty} u^{2n+1} \exp(-\alpha_\tau u) du + R_{\tau,n}(k) \\ &= (2n+1)! \rho_1 \alpha_\tau^{-2(n+1)} + R_{\tau,n}(k) \end{aligned} \quad (14)$$

By Lemma 5,

$$\left| R_{\tau,n}(k) \right| \leq \frac{\rho_1 \gamma_{\tau,n}}{k!} \prod_{m=1}^{k-2} \left| \alpha_\tau + m \right| = \frac{\rho_1 \gamma_{\tau,n}}{k(k-1)} \prod_{j=1}^{k-2} \left| \frac{\alpha_\tau}{j} + 1 \right| \quad (15)$$

We have

$$\forall j > \theta_\tau := \lfloor \tau^2 - \frac{1}{2} \rfloor + 2 : \quad \left| \frac{\alpha_\tau}{j} + 1 \right| < 1 + \frac{1}{j}$$

where  $\lfloor \tau^2 - \frac{1}{2} \rfloor$  is the enter part of  $\tau^2 - \frac{1}{2}$ . Then

$$\prod_{j=\theta_\tau}^{k-2} \left| \frac{\alpha_\tau}{j} + 1 \right| \leq \prod_{j=\theta_\tau}^{k-2} \frac{j+1}{j} = \frac{k-1}{\theta_\tau}.$$

Then Equation (15), implies

$$\left| R_{\tau,n}(k) \right| \leq C_{n,\tau} \frac{1}{k},$$

where

$$C_{n,\tau} := \frac{\rho_1 \gamma_{\tau,n}}{\theta_\tau} \left( \prod_{j=0}^{\theta_\tau-1} \left| \frac{\alpha_\tau}{j} + 1 \right| \right).$$

Since  $n \geq 0$  is fixed, we deduce that  $\lim_{k \rightarrow +\infty} |R_{\tau,n}(k)| = 0$ . Equation (14) can be written as

$$J_{n,\tau} = (2n+1)! \rho_1 \frac{1}{\alpha_\tau^{2(n+1)}}.$$

Thanks to Equation (10), we get

$$\frac{d^{2n+1}}{dr^{2n+1}} \phi_\tau(0) = (2n+1)! (3+2\rho_1) \frac{1}{\alpha_\tau^{2(n+1)}}. \quad (16)$$

□

## 5 Proof of the Main result

*Proof of the Main result.* Let be  $\tau \in \mathbb{R}^*$  fixed. By the Proposition 2, we have

$$\frac{d^{2n+1}}{dr^{2n+1}} \phi_\tau(0) = (2n+1)! (3+2\rho_1) \frac{1}{\alpha_\tau^{2(n+1)}}$$

By Lemma 6, we have

$$\phi_\tau(r) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \frac{d^{2n+1}}{dr^{2n+1}} \phi_\tau(0) r^{2n+1},$$

Then

$$\phi_\tau(r) = (3+2\rho_1) \frac{r}{\alpha_\tau^2} \sum_{n=0}^{\infty} \left( \frac{r}{\alpha_\tau} \right)^{2n} = (3+2\rho_1) \frac{r}{\alpha_\tau^2 - r^2} \neq 0.$$

□

## 6 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (17)$$

and where  $\{t\}$  is the fractional part of the real  $t$ . Define the  $\tau_*$ -non trivial zeros of the function  $\zeta$  in the following sense

**Definition 7.** Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non trivial zero of the function*  $\zeta$  if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

By the Riemann functional equation [[1], page 13 Theorem 2.1], the non trivial zeros of the function  $\zeta$  are symmetric about the line  $\Re(z) = \frac{1}{2}$ . By the Main result, the non trivial zeros  $s \in \mathbb{C}$  symmetric about the line  $\Re(z) = \frac{1}{2}$  satisfies  $\Re(s) = \frac{1}{2}$ .

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).