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# Non symmetry of the function $\mathfrak{S}\left(\frac{\zeta(s)}{s}\right)$

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## Abstract

We prove that  $\mathfrak{S}\left(\frac{\zeta(s)}{s}\right) + \mathfrak{S}\left(\frac{\zeta(1-s)}{1-s}\right) \neq 0$  for every  $\Re(s) \in (0, \frac{1}{2})$  and  $\Im(s) \in \mathbb{R}^*$ , where  $\zeta$  is the Riemann Zeta function. In the end of the paper, we give a discussion about the Riemann hypothesis.

**Keywords:** Riemann functional equation, Zeta function.

## 1 Main result

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (1)$$

and where  $\{t\}$  is the fractional part of the real  $t$ . We prove in this paper the following Main result

**Main Result.** Consider the function  $\zeta$  defined by Equation (1). Then

$$\mathfrak{S}\left(\frac{\zeta\left(\frac{1}{2} - r + i\tau\right)}{\frac{1}{2} - r + i\tau}\right) - \mathfrak{S}\left(\frac{\zeta\left(\frac{1}{2} + r + i\tau\right)}{\frac{1}{2} + r + i\tau}\right) \neq 0, \quad \forall r \in (0, \frac{1}{2}), \quad \forall \tau \in \mathbb{R}^*.$$

## 2 Main Proposition

**Notation 1.** Let be  $m \in \mathbb{N}$  and  $q \in \mathbb{N}^*$ . We denote  $\Lambda_q^m$  as the set of  $q$ -combinations of the set  $\{0, 1, \dots, m\}$ ;

$$\Lambda_q^m := \left\{ \{l_1, \dots, l_q\} \subset \{0, 1, \dots, m\} : l_j \neq l_{j'}, \quad \forall 1 \leq j \neq j' \leq q \right\}.$$

In order to simplify the notation, we denote

$$\alpha_\tau := \frac{1}{2} + i\tau, \quad \forall \tau \in \mathbb{R}^*.$$

**Notation 2.** For every  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$  denote

$$\phi_\tau(r) := \frac{\zeta(\alpha_\tau - r)}{\alpha_\tau - r} - \frac{\zeta(\alpha_\tau + r)}{\alpha_\tau + r}.$$

We have the following Main proposition

**Proposition 3.** For every  $\tau \in \mathbb{R}^*$  and  $n \in \mathbb{N}$  we have

$$\begin{aligned} \frac{1}{(2n+1)!} \frac{d^{2n+1}}{dr^{2n+1}} \phi_\tau(0) &= -\frac{3}{2} \alpha_\tau^{-2n-1} - (-1)^n \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(n+1)}} \\ &- \sum_{k=n+2}^{+\infty} (-1)^k \left[ \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}} \right] \sum_{A \in \Lambda_{2(k-n)}^{2k+1}} \Pi_{j \in A}(\alpha_\tau + j) \end{aligned}$$

### 3 Main Lemma

**Lemma 4.** We have

$$\int_1^t (\{v\} + 1) dv = t \frac{3}{2} - \frac{1}{12} + \frac{1}{2} \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \exp(ij2\pi t), \quad \forall t \geq 0,$$

and

$$\sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(m+1)}} \leq \frac{1}{2} \frac{1}{m} \frac{1}{(2m)!}, \quad \forall m \geq 2.$$

*Proof.* The function  $v \mapsto \{v\} + 1$  is 1-periodic, then there exists a continuous 1-periodic function  $p : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\int_0^t (\{v\} + 1) dv = t \int_0^1 (\{v\} + 1) dv + p(t), \quad \forall t \in \mathbb{R}.$$

Since

$$\int_0^1 (\{v\} + 1) dv = \frac{3}{2},$$

we get

$$\int_0^t (\{v\} + 1) dv = t \frac{3}{2} + p(t), \quad \forall t \in \mathbb{R}.$$

The function  $p$  is a piecewise  $C^\infty$ , continuous on  $\mathbb{R}$  and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^n a_k \exp(ik2\pi t),$$

converge uniformly on  $\mathbb{R}_+$  to the function  $t \mapsto p(t)$ , where  $(a_k)_k \subset \mathbb{C}$  are the Fourier coefficients of the function  $p$ .

$$\int_0^t (\{v\} + 1)dv = t\frac{3}{2} + \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u) \left( \int_0^u (\{v\} + 1)dv - u\frac{3}{2} \right) du \\ &= \frac{1}{2} \int_0^1 \exp(-ij2\pi u) u(u-1) du = \frac{1}{2} \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^* \end{aligned}$$

and

$$a_0 = \frac{1}{2} \int_0^1 u(u-1) du = -\frac{1}{12}$$

Now prove the second item of the present Lemma. Let  $m \geq 2$  and consider the 1-periodic function  $g_m : \mathbb{R} \rightarrow \mathbb{R}$  defined as

$$g_m(t) := \frac{1}{m!} \frac{1}{2} (t^2 - t)^m, \quad \forall t \in [0, 1].$$

The function  $g_m$  is a piecewise  $C^\infty$ , continuous on  $\mathbb{R}$  and 1-periodic. By Dirichlet Theorem,

$$g_m(t) = \sum_{j \in \mathbb{Z}} a_{m,j} \exp(ij2\pi t), \forall t \geq 0, \quad (2)$$

where the Fourier coefficients satisfies

$$a_{m,j} = \frac{1}{m!} \frac{1}{2} \int_0^1 \exp(-ij2\pi u) (u^2 - u)^m du.$$

Use a successive integration part formula, to obtain

$$a_{m,j} = \frac{1}{(ij2\pi)^{m+1}}, \quad \forall m \geq 0,$$

and replace its in Equation (2) , we get

$$g_m(t) = \sum_{j \in \mathbb{Z}} \frac{1}{(ij2\pi)^{m+1}} \exp(ij2\pi t), \quad \forall t \geq 0,$$

Then

$$\int_0^t g_m(u) du = \sum_{j \in \mathbb{Z}} \frac{1}{(ij2\pi)^{m+2}} \left( \exp(ij2\pi t) - 1 \right), \quad \forall t \geq 0,$$

implies

$$\int_0^1 \int_0^t g_m(u) dudt = \sum_{j \in \mathbb{Z}} \frac{1}{(ij2\pi)^{m+2}}, \quad \forall t \geq 0,$$

By definition of  $g_m$  we get

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \frac{1}{(j2\pi)^{2(m+1)}} &= \left| \int_0^1 \int_0^t g_{2m}(u) dudt \right| \\ &\leq \frac{1}{2} \frac{1}{(2m)!} \int_0^1 (1-t)^m dudt \leq \frac{1}{2} \frac{1}{m} \frac{1}{(2m)!}, \quad \forall m \geq 2. \end{aligned}$$

□

**Lemma 5.** *Let be  $n \in \mathbb{N}$  and  $\tau \in \mathbb{R}^*$ . For every  $k \geq 0$  define the function  $V_{k,\tau}(n, u) : \mathbb{R} \rightarrow \mathbb{C}$  as*

$$V_{0,\tau}(n, u) := u^n \exp(-\alpha_\tau u), \quad V_{k+1,\tau}(n, u) = \exp(-u) \frac{d}{du} V_{k,\tau}(n, u), \quad \forall k \geq 0.$$

Then

$$V_{k,\tau}(n, 0) = 0, \quad \forall k \in \llbracket 0, n-1 \rrbracket, \quad V_{n,\tau}(n, 0) = n!,$$

and

$$V_{k,\tau}(n, 0) = n! (-1)^{k-n} \sum_{A \in \Lambda_{k-n}^{k-1}} \prod_{j \in A} (\alpha_\tau + j), \quad \forall k \geq n+1.$$

where we recall that  $\Lambda_{k-n}^{k-1}$  is defined in the Notation 1. In addition,

$$\int_0^{+\infty} \left| \frac{d}{du} V_{k,\tau}(n, u) \right| du \leq \frac{n}{k^{n+1}} \max\{|\alpha_\tau|, 1\} \prod_{m=1}^{k-1} |\alpha_\tau + m|, \quad \forall k \geq 2.$$

*Proof.* Let be  $n \in \mathbb{N}$  and  $\tau \in \mathbb{R}^*$ . By recurrence, for every  $k \geq 0$  there exists a polynomial  $P_k(n, u)$  such that

$$V_{k,\tau}(n, u) = \exp(-(\alpha_\tau + k)u)P_k(n, u).$$

We have

$$V_{k+1,\tau}(n, u) = \exp(-u) \frac{d}{du} \left[ \exp(-(\alpha_\tau + k)u) P_k(n, u) \right]. \quad (3)$$

Then

$$P_{k+1}(n, u) = \frac{d}{du} P_k(n, u) - (\alpha_\tau + k) P_k(n, u). \quad (4)$$

Write

$$P_k(n, u) = \sum_{j=0}^n c_{k,j}(n, \tau) \frac{d^j}{du^j} P_0(n, u), \quad (5)$$

and replace it in Equation (4),

$$\begin{aligned} \sum_{j=0}^n c_{k+1,j}(n, \tau) \frac{d^j}{du^j} P_0(n, u) &= \sum_{j=0}^n c_{k,j}(n, \tau) \frac{d^{j+1}}{du^{j+1}} P_0(n, u) \\ &\quad - (\alpha_\tau + k) \sum_{j=0}^n c_{k,j}(n, \tau) \frac{d^j}{du^j} P_0(n, u). \end{aligned}$$

Since  $P_0(n, u) = u^n$  then  $\frac{d^{n+1}}{du^{n+1}} P_0(n, u) = 0$ , we get

$$\begin{aligned} \sum_{j=0}^n c_{k+1,j}(n, \tau) \frac{d^j}{du^j} P_0(n, u) &= \sum_{j=0}^{n-1} c_{k,j}(n, \tau) \frac{d^{j+1}}{du^{j+1}} P_0(n, u) \\ &\quad - (\alpha_\tau + k) \sum_{j=0}^n c_{k,j}(n, \tau) \frac{d^j}{du^j} P_0(n, u), \end{aligned}$$

which can be written as

$$\begin{aligned} \sum_{j=0}^n c_{k+1,j} \frac{d^j}{du^j} P_0(n, u) &= \sum_{j=1}^n c_{k,j-1}(n, \tau) \frac{d^j}{du^j} P_0(n, u) \\ &\quad - (\alpha_\tau + k) \sum_{j=0}^n c_{k,j}(n, \tau) \frac{d^j}{du^j} P_0(n, u). \end{aligned}$$

Implies

$$\begin{pmatrix} c_{k+1,0}(n, \tau) \\ c_{k+1,1}(n, \tau) \\ \vdots \\ c_{k+1,n}(n, \tau) \end{pmatrix} = (C_n - (\alpha_\tau + k)Id_n) \begin{pmatrix} c_{k,0}(n, \tau) \\ c_{k,1}(n, \tau) \\ \vdots \\ c_{k,n}(n, \tau) \end{pmatrix}$$

where  $Id_n$  is the  $n \times n$  identity matrix and where  $C_n$  is the  $n \times n$  matrix defined as

$$C_n := \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} c_{k+1,0}(n, \tau) \\ c_{k+1,1}(n, \tau) \\ \vdots \\ c_{k+1,n}(n, \tau) \end{pmatrix} = \left[ \prod_{m=0}^k (C_n - (\alpha_\tau + m)Id_n) \right] \begin{pmatrix} c_{0,0}(n, \tau) \\ c_{0,1}(n, \tau) \\ \vdots \\ c_{0,n}(n, \tau) \end{pmatrix}$$

By definition of  $P_k(n, u)$  (Resp.  $V_{k+1,\tau}(n, u)$ ) in Equation (5) (Resp. Equation (3)) we have  $P_0(n, u) = u^n$ , we get

$$\begin{pmatrix} c_{k+1,0}(n, \tau) \\ c_{k+1,1}(n, \tau) \\ \vdots \\ c_{k+1,n}(n, \tau) \end{pmatrix} = \left[ \prod_{m=0}^k (C_n - (\alpha_\tau + m)Id_n) \right] \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

By Equation (3), and (5), for all  $u \geq 0$  we find

$$\begin{aligned} & \left| \frac{d}{du} V_{k+1,\tau}(n, u) \right| \\ & \leq n! \max\{|\alpha_\tau|, 1\} \left[ \prod_{m=1}^k |\alpha_\tau + m| \right] \exp\left(-\left(\frac{1}{2} + k + 1\right)u\right) u^n, \end{aligned}$$

Since

$$\int_0^{+\infty} \exp\left(-\left(\frac{1}{2} + k + 1\right)u\right) u^n du = \frac{1}{\left(\frac{1}{2} + k + 1\right)^{n+1}} \frac{n}{n!}$$

Then

$$\int_0^{+\infty} \left| \frac{d}{du} V_{k+1,\tau}(n, u) \right| du \leq n \frac{\max\{|\alpha_\tau|, 1\} \left[ \prod_{m=1}^k |\alpha_\tau + m| \right]}{\left(\frac{1}{2} + k + 1\right)^{n+1}}.$$

Use the fact  $C_n$  is a nilpotent matrix of degree  $n$ , we get  $c_{n,n}(n, \tau) = 1$  and

$$V_{k,\tau}(n, 0) = P_k(n, 0) = n!c_{k,n}(n, \tau) = n! \sum_{A \in \Lambda_{k-n}^{k-1}} (-1)^{k-n} \Pi_{j \in A}(\alpha_\tau + j).$$

□

## 4 Proof of the Main Proposition

**Lemma 6.** *Let be  $\tau \in \mathbb{R}^*$ . Consider the function  $\phi_\tau$  defined in the Notation 2. Then  $\frac{d^{2n}}{d\tau^{2n}}\phi_\tau(0) = 0$  for all  $n \geq 0$  and*

$$\frac{d^{2n+1}}{d\tau^{2n+1}}\phi_\tau(0) = 2 \int_0^{+\infty} u^{2n+1} \exp(-\alpha_\tau u) (\{\exp(u)\} + 1) du, \quad \forall n \geq 0.$$

*Proof.* Let be  $x \in (0, 1)$  and  $\tau \in \mathbb{R}^*$ . From Equation (1), we have

$$\begin{aligned} \frac{\zeta(x + i\tau)}{x + i\tau} &= - \int_0^{+\infty} \frac{\{t\}}{t^{1+x+i\tau}} dt \\ &= - \int_0^{+\infty} \exp(-i\tau \ln(t)) \frac{\{t\}}{t^{1+x}} dt \\ &= - \int_0^1 \exp(-i\tau \ln(t)) \frac{1}{t^x} dt - \int_1^{+\infty} \exp(-i\tau \ln(t)) \frac{\{t\}}{t^{1+x}} dt \end{aligned}$$

Use the change of variable,  $t \rightarrow \exp(-u)$  for the integral

$$\int_0^1 \exp(-i\tau \ln(t)) \frac{1}{t^x} dt,$$

and  $t \rightarrow \exp(u)$  for the integral

$$\int_1^{+\infty} \exp(-i\tau \ln(t)) \frac{\{t\}}{t^{1+x}} dt,$$

we obtain

$$\begin{aligned} \frac{\zeta(x + i\tau)}{x + i\tau} &= \int_0^{+\infty} \exp(-i\tau u) \exp(-(1-x)u) du \\ &\quad - \int_0^{+\infty} \exp(-i\tau u) \exp(-xu) \{\exp(u)\} du. \end{aligned}$$



Implies

$$\begin{aligned} \frac{\zeta(\alpha_\tau - r)}{\alpha_\tau - r} &= \int_0^{+\infty} \exp(-i\tau u) \exp(-(\frac{1}{2} + r)u) du \\ &\quad - \int_0^{+\infty} \exp(-i\tau u) \exp(-(\frac{1}{2} - r)u) \{\exp(u)\} du, \end{aligned}$$

and

$$\begin{aligned} \frac{\zeta(\alpha_\tau + r)}{\alpha_\tau + r} &= \int_0^{+\infty} \exp(-i\tau u) \exp(-(\frac{1}{2} - r)u) du \\ &\quad - \int_0^{+\infty} \exp(i\tau u) \exp(-(\frac{1}{2} + r)u) \{\exp(u)\} du. \end{aligned}$$

We get

$$\begin{aligned} \phi_\tau(r) &= - \int_0^{+\infty} \exp(-i\tau u) [\exp(-(\frac{1}{2} + r)u) - \exp(-(\frac{1}{2} - r)u)] du \\ &\quad + \int_0^{+\infty} \exp(-i\tau u) [\exp((\frac{1}{2} + r)u) - \exp((\frac{1}{2} - r)u)] \{\exp(u)\} du. \end{aligned}$$

Implies

$$\phi_\tau(r) = 2 \int_0^{+\infty} \exp(-\alpha_\tau u) \sinh(ru) (\{\exp(u)\} + 1) du.$$

Since  $r \in (0, \frac{1}{2})$  then

$$\frac{d^{2n}}{d^{2n}} \phi_\tau(r) = 2 \int_0^{+\infty} \exp(-\alpha_\tau u) u^{2n} \sinh(ru) (\{\exp(u)\} + 1) du.$$

implies  $\frac{d^{2n}}{d^{2n}} \phi_\tau(0) = 0$  and

$$\frac{d^{2n+1}}{d^{2n+1}} \phi_\tau(r) = 2 \int_0^{+\infty} \exp(-\alpha_\tau u) u^{2n+1} \cosh(ru) (\{\exp(u)\} + 1) du,$$

implies

$$\frac{d^{2n+1}}{d^{2n+1}} \phi_\tau(0) = 2 \int_0^{+\infty} \exp(-\alpha_\tau u) u^{2n+1} (\{\exp(u)\} + 1) du.$$

□

*Proof of the Proposition 3.* Let be  $n \geq 0$  and  $\tau \in \mathbb{R}^*$ . By Lemma 6 , we have

$$\frac{d^{2n+1}}{dr^{2n+1}}\phi_\tau(0) = 2 \int_0^{+\infty} u^{2n+1} \exp(-\alpha_\tau u) [\{\exp(u)\} + 1] du,$$

Then

$$\begin{aligned} \frac{d^{2n+1}}{dr^{2n+1}}\phi_\tau(0) &= 3 \int_0^{+\infty} u^{2n+1} \exp(-\alpha_\tau u) du \\ &\quad + 2 \int_0^{+\infty} u^{2n+1} \exp(-\alpha_\tau u) \left[ \{\exp(u)\} - \frac{1}{2} \right] du, \end{aligned}$$

we get

$$\begin{aligned} \frac{d^{2n+1}}{dr^{2n+1}}\phi_\tau(0) &= 3 \frac{(2n+1)!}{\alpha_\tau^{2n+1}} \\ &\quad + 2 \int_0^{+\infty} u^{2n+1} \exp(-\alpha_\tau u) \left[ \{\exp(u)\} - \frac{1}{2} \right] du \end{aligned}$$

In order to simplify the notation, denote for every  $m \in \mathbb{N}$ ,

$$J_{n,\tau} := 2 \int_0^{+\infty} u^{2n+1} \exp(-\alpha_\tau u) \left[ \{\exp(u)\} - \frac{1}{2} \right] du$$

Then

$$\frac{d^{2n+1}}{dr^{2n+1}}\phi_\tau(0) = -3(2n+1)! \alpha_\tau^{-2n-1} + 2J_{n,\tau}. \quad (6)$$

Use the integration part formula,

$$\begin{aligned} J_{n,\tau} &= 2 \int_0^{+\infty} u^{2n+1} \exp(-(\alpha_\tau + 1)u) \exp(u) \left[ \{\exp(u)\} - \frac{1}{2} \right] du \\ &= -2 \int_0^{+\infty} \frac{d}{du} \left[ u^{2n+1} \exp(-(\alpha_\tau + 1)u) \right] \left[ \int_0^{\exp(u)} (\{v\} - \frac{1}{2}) dv \right] du. \end{aligned}$$

By Lemma 4,

$$J_{n,\tau} = - \int_0^{+\infty} \frac{d}{du} \left[ u^{2n+1} \exp(-(\alpha_\tau + 1)u) \right] \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \exp(ij2\pi \exp(u)) du$$

Use the fact

$$\sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \exp(ij2\pi t) = 2 \sum_{j \in \mathbb{N}^*} \frac{1}{(j2\pi)^2} \cos(2j\pi t), \quad \forall t \in \mathbb{R},$$

Use a successive integration part formula, by induction we deduce that

$$J_{n,\tau} = - \sum_{k=1}^m (-1)^k \left[ \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}} \right] V_{2k,\tau}(2n+1, 0) + R_{\tau,n}(m), \quad (7)$$

where

$$\begin{aligned} V_{0,\tau}(2n+1, u) &:= u^{2n+1} \exp(-(\alpha_\tau + 1)u), \\ V_{k+1,\tau}(2n+1, u) &= \exp(-u) \frac{d}{du} V_{k,\tau}(2n+1, u), \quad \forall k \geq 0, \end{aligned} \quad (8)$$

and where

$$R_{\tau,n}(m) := \int_0^{+\infty} \left[ \frac{d}{du} V_{2(m+1),\tau}(2n+1, u) \right] p_m(u) du,$$

where

$$p_m(u) := \left[ \sum_{j \in \mathbb{Z}^*} \frac{1}{(ij2\pi)^{2(m+2)}} \exp(ij2\pi \exp(u)) \right].$$

By Lemma 4,

$$\sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(m+2)}} \leq \frac{1}{2} \frac{1}{m} \frac{1}{2m!}, \quad \forall m \geq 2.$$

and by Lemma 5,

$$\begin{aligned} \left| R_{\tau,n}(m) \right| &\leq (2n+1) \frac{1}{2} \frac{1}{m} \frac{1}{(2m)!} \frac{\max\{|\alpha_\tau|, 1\} \left[ \prod_{j=1}^{2m+1} |\alpha_\tau + j| \right]}{(2(m+1))^{2(n+1)}} \\ &= (2n+1) \frac{1}{2} \max\{|\alpha_\tau|, 1\} \frac{1}{m} \frac{|\alpha_\tau + 2m + 1|}{(2(m+1))^{2(n+1)}} \left[ \prod_{j=1}^{2m} \left| \frac{\alpha_\tau}{j} + 1 \right| \right] \end{aligned}$$

We have

$$\forall j > \theta_\tau := \left[ \frac{1}{2} + \tau^2 - 1 \right] + 1 : \quad \left| \frac{\alpha_\tau}{j} + 1 \right| < 1 + \frac{1}{j}$$

where  $\left[ \frac{1}{2} + \tau^2 - 1 \right]$  is the enter part of  $\frac{1}{2} + \tau^2 - 1$ . Then

$$\begin{aligned} \prod_{j=\theta_\tau}^{2(m+1)} \left| \frac{\alpha_\tau}{j} + 1 \right| &\leq \prod_{j=\theta_\tau}^{2m} \frac{1+j}{j} = \frac{1}{\theta_\tau} 2(m+2) \left[ \prod_{j=\theta_\tau+1}^{2m} \frac{j}{j+1} \right] \\ &\leq \frac{1}{\theta_\tau} 2(m+2) \end{aligned}$$

Then

$$\left| R_{\tau,n}(m) \right| \leq C_{n,\tau} \frac{1}{m} \frac{2(m+2) \left| \alpha_{\tau} + 2m + 1 \right|}{(2(m+1))^{2(n+1)}},$$

where

$$C_{n,\tau} := (2n+1) \frac{1}{2\theta_{\tau}} \left[ \prod_{j=1}^{\theta_{\tau}} \left| \frac{\alpha_{\tau}}{j} + 1 \right| \right] \max\{|\alpha_{\tau}|, 1\}.$$

Since  $n \geq 0$  we deduce that  $\lim_{m \rightarrow +\infty} |R_{\tau,n}(m)| = 0$ . Equation (7), can be written as

$$J_{n,\tau} = - \sum_{k=1}^{+\infty} (-1)^k \left[ \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}} \right] V_{2(k+1),\tau}(2n+1, 0)$$

In order to simplify the notation denote

$$\rho_{2(k+1)} = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}},$$

Then

$$J_{n,\tau} = - \sum_{k=0}^{+\infty} (-1)^k \rho_{2(k+1)} V_{2(k+1),\tau}(2n+1, 0)$$

By Equation (8) and Lemma 5

$$\begin{aligned} J_{n,\tau} &= -(2n+1)! (-1)^n \rho_{2(n+1)} \\ &\quad - (2n+1)! \sum_{k=n+2}^{+\infty} (-1)^k \rho_{2(k+1)} \sum_{A \in \Lambda_{2(k-n)}^{2k+1}} (-1)^{2(k-n)} \prod_{j \in A} (\alpha_{\tau} + j). \end{aligned}$$

By Equation (6)

$$\begin{aligned} \frac{1}{(2n+1)!} \frac{d^{2n+1}}{dr^{2n+1}} \phi_{\tau}(0) &= -\frac{3}{2} \alpha_{\tau}^{-2n-1} - (-1)^n \rho_{2(n+1)} \\ &\quad - \sum_{k=n+2}^{+\infty} (-1)^k \rho_{2(k+1)} \sum_{A \in \Lambda_{2(k-n)}^{2k+1}} (-1)^{2(k-n)} \prod_{j \in A} (\alpha_{\tau} + j) \end{aligned}$$

□

## 5 Proof of the Main result

*Proof of the Main result.* Let be  $\tau \in \mathbb{R}^*$  fixed. By the Proposition 3, we have

$$\begin{aligned} \frac{1}{(2n+1)!} \frac{d^{2n+1}}{dr^{2n+1}} \phi_\tau(0) &= -\frac{3}{2} \alpha_\tau^{-2n-1} - (-1)^n \left[ \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(n+1)}} \right] \\ &- \sum_{k=n+2}^{+\infty} (-1)^k \left[ \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}} \right] \sum_{A \in \Lambda_{2(k-n)}^{2k+1}} \Pi_{j \in A}(\alpha_\tau + j). \end{aligned}$$

By Lemma6 , we havet

$$\phi_\tau(r) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \frac{d^{2n+1}}{dr^{2n+1}} \phi_\tau(0) r^{2n+1},$$

Then

$$\begin{aligned} \phi_\tau(r) &= -\frac{3}{2} \sum_{n=0}^{\infty} r^{2n+1} \alpha_\tau^{-2n-1} - \sum_{n=0}^{\infty} (-1)^n r^{2n+1} \left[ \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(n+1)}} \right] \\ &- \sum_{n=0}^{\infty} r^{2n+1} \sum_{k=n+2}^{+\infty} (-1)^k \left[ \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(k+1)}} \right] \sum_{A \in \Lambda_{2(k-n)}^{2k+1}} \Pi_{j \in A}(\alpha_\tau + j), \end{aligned}$$

implies

$$\begin{aligned} \phi_\tau(r) &= -\frac{3}{2} \sum_{n=0}^{\infty} r^{2n+1} \alpha_\tau^{-2n-1} - \sum_{n=0}^{\infty} (-1)^n r^{2n+1} \left[ \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(n+1)}} \right] \\ &- \sum_{n=0}^{\infty} (-1)^n \left[ \sum_{k=0}^n r^{2k+1} \right] \left[ \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^{2(n+1)}} \right] \sum_{A \in \Lambda_2^{2n+3}} \Pi_{j \in A}(\alpha_\tau + j), \end{aligned}$$

since  $\alpha_\tau = \frac{1}{2} + i\tau$  and since

$$\sum_{A \in \Lambda_2^{2n+3}} \Pi_{j \in A}(\alpha_\tau + j) = \sum_{0 \leq j \neq j' \leq 2n+3} \left( \frac{1}{2} + j + i\tau \right) \left( \frac{1}{2} + j' + i\tau \right),$$

we get

$$\Im(\phi_\tau(r)) = \tau r \left[ \frac{3}{2} \frac{\frac{1}{4} + r^2 + \tau^2}{\left(\frac{1}{4} - r^2\right)^2 + 2\left(\frac{1}{4} - r^2\right)\tau^2 + \tau^4} - \psi(r) \right],$$

where

$$\psi(r) := \frac{1}{1-r} \sum_{j \in \mathbb{Z}^*} \sum_{n=0}^{\infty} \frac{(-1)^n (1-r^{2n})}{(j2\pi)^{2(n+1)}} \sum_{0 \leq m \neq m' \leq 2n+3} (1+m+m').$$

We have

$$\begin{aligned} \sum_{0 \leq m \neq m' \leq 2n+3} (1+m+m') &= 2(n+2) + \sum_{m=1}^{2(n+1)} (2n+3-m)m \\ &= 2n^2 + 3n - 1 \end{aligned}$$

Since the sequence

$$n \mapsto \frac{(1-r^{2n})}{(j2\pi)^{2(n+1)}} (2n^2 + 3n - 1), \quad n \geq 1,$$

is positive and decreasing for every  $r \in (0, \frac{1}{2})$ , then

$$\psi(r) = \frac{1}{1-r} \sum_{j \in \mathbb{Z}^*} \sum_{n=0}^{\infty} \frac{(-1)^n (1-r^{2n})}{(j2\pi)^{2(n+1)}} (2n^2 + 3n - 1) < 0, \quad \forall r \in (0, \frac{1}{2}).$$

We deduce that  $\tau \Im(\phi_\tau(r)) > 0$  for every  $r \in (0, \frac{1}{2})$  and  $\tau \in \mathbb{R}^*$ .

□

## 6 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function  $\zeta$  defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (9)$$

and where  $\{t\}$  is the fractional part of the real  $t$ . Define the  $\tau_*$ -non trivial zeros of the function  $\zeta$  in the following sense

**Definition 7.** Let be  $s \in \mathbb{C}$ . We say that  $s$  is a *non trivial zero of the function  $\zeta$*  if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

By the Riemann functional equation [[1], page 13 Theorem 2.1], the non trivial zeros of the function  $\zeta$  are symmetric about the line  $\Re(z) = \frac{1}{2}$ . By the Main result, the non trivial zeros  $s \in \mathbb{C}$  symmetric about the line  $\Re(z) = \frac{1}{2}$  satisfies  $\Re(s) = \frac{1}{2}$ .

## References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).