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Non symmetry of the function $\frac{\zeta(s)}{s}$

W. Oukil

Laboratory of Dynamical Systems, Faculty of Mathematics.
University of Science and Technology Houari Boumediene,
BP 32 EL ALIA 16111 Bab Ezzouar, Algiers, Algeria.

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Abstract

We prove that $\frac{\zeta(s)}{s} \neq \frac{\zeta(1-s)}{1-s}$ for every $\Re(s) \in (0, \frac{1}{2})$ and $\Im(s) > 0$, where ζ is the Riemann Zeta function. In the end of the paper, we give a discussion about the Riemann hypothesis.

Keywords: Riemann functional equation, Zeta function.

1 Main result

Let be $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a bounded function. We study the function $\psi[\eta] : (0, 1) \times \mathbb{R}_+^* \rightarrow \mathbb{R}$ defined as

$$\forall x \in (0, 1), \forall \tau > 0 : \quad \psi[\eta](x, \tau) := \int_0^{+\infty} \frac{\eta(t)}{t^{1+x}} \sin(\tau \ln(t)) dt, \quad (1)$$

We focus about the function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfying the following hypotheses

H₁ : $0 \leq \eta(t) \leq 1$ for all $t \geq 0$,

H₂ : the function η is a C^1 piecewise function.

We prove in this paper the following Main result

Main Result. Consider the function ψ defined by Equation (1). Then, for every function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ that satisfies the hypotheses H₁ and H₂ we have

$$\psi[\eta](x, \tau) \neq \psi[\eta](1-x, \tau), \quad \forall \tau > 0, \forall x \in (0, \frac{1}{2}).$$

2 Main Proposition

Notation 1. Let $\eta : \mathbb{R}_+ \mapsto \mathbb{R}$ be a function that satisfies H_1 and H_2 . For every $r \in (0, \frac{1}{2})$ and $\tau > 0$ denote

$$\phi_\tau(r) := \psi[\eta](\frac{1}{2} + r, \tau) - \psi[\eta](\frac{1}{2} - r, \tau).$$

Proposition 2. *There exists a continuous function $\theta : \mathbb{R}_+^* \times (0, \frac{1}{2}) \rightarrow \mathbb{R}_+^*$ such that*

$$\frac{d}{dr}\phi_\tau(r) > \theta(\tau, r)\phi_\tau(r), \quad \forall r \in (0, \frac{1}{2}), \quad \forall \tau > 0.$$

Proof. For every $r \in (0, \frac{1}{2})$, define the functions $t \mapsto h_r(\eta, t)$ as

$$h_r(\eta, t) := 2 \exp(-\frac{1}{2}t)[\eta(\exp(t)) + 1] \sinh(rt), \quad \forall t \geq 0.$$

In order to simplify the notation we denote $h_r(t) := h_r(\eta, t)$ for all $t \geq 0$. Use the change of variable, $t \rightarrow \exp(u)$, in Equation (1), we obtain

$$\phi_\tau(r) = - \int_0^{+\infty} h_r(s) \sin(\tau s) ds, \quad \forall r \in (0, \frac{1}{2}), \quad \forall \tau > 0.$$

Let be $\beta > 0$ that we fix later. We have

$$\frac{d}{dr}\phi_\tau(r) - \beta\phi_\tau(r) = - \int_0^{+\infty} \sin(\tau s) [\frac{d}{dr}h_r(s) - \beta h_r(s)] ds.$$

By Rolle Theorem and the hypothesis H_1 , there exists $\beta_{\tau, r} > 0$ such that

$$\int_0^{\frac{\pi}{\tau}} \sin(\tau s) [\frac{d}{dr}h_r(s) - \beta_{\tau, r} h_r(s)] ds = 0. \quad (2)$$

Then

$$\frac{d}{dr}\phi_\tau(r) - \beta_{\tau, r}\phi_\tau(r) = - \int_{\frac{\pi}{\tau}}^{+\infty} \sin(\tau s) [\frac{d}{dr}h_r(s) - \beta_{\tau, r} h_r(s)] ds.$$

Use the hypothesis H_2 and the integration part formula, we get

$$\frac{d}{dr}\phi_\tau(r) - \beta_{\tau, r}\phi_\tau(r) = \tau \lim_{n \rightarrow +\infty} \int_{\frac{\pi}{\tau}}^{\frac{2n\pi}{\tau}} \cos(\tau u) \int_{\frac{\pi}{\tau}}^u [\frac{d}{dr}h_r(v) - \beta_{\tau, r} h_r(v)] dv du.$$

Since $r \in (0, \frac{1}{2})$ and by the hypothesis H_1 , then there exists $\alpha_{\tau,r} > 0$ such that

$$\int_{\frac{\pi}{\tau}}^{+\infty} \left[\frac{d}{dr} h_r(v) - \beta_{\tau,r} h_r(v) \right] dv = \alpha_{\tau,r}.$$

In order to simplify the notation, denote

$$\tilde{h}_{\tau,r}(t) := \alpha_{\tau,r} - \int_{\frac{\pi}{\tau}}^t \left[\frac{d}{dr} h_r(v) - \beta_{\tau,r} h_r(v) \right] dv, \quad \forall t \geq 0.$$

Since the convergence to $\alpha_{\tau,r}$ is exponential, we get

$$\frac{d}{dr} \phi_\tau(r) - \beta_{\tau,r} \phi_\tau(r) = -\tau \lim_{t \rightarrow +\infty} \int_{\frac{\pi}{\tau}}^t \cos(\tau u) \tilde{h}_{\tau,r}(u) du.$$

Use the integration part formula, we obtain

$$\begin{aligned} & \frac{d}{dr} \phi_\tau(r) - \beta_{\tau,r} \phi_\tau(r) \\ = & \tau \lim_{n \rightarrow +\infty} \left[\int_{\frac{\pi}{\tau}}^{\frac{(2n+1)\pi}{\tau}} \tilde{h}_{\tau,r}(v) dv - \tau \int_{\frac{\pi}{\tau}}^{\frac{(2n+1)\pi}{\tau}} \sin(\tau u) \int_{\frac{\pi}{\tau}}^u \tilde{h}_{\tau,r}(v) dv du \right]. \end{aligned} \quad (3)$$

The function

$$u \mapsto \int_{\frac{\pi}{\tau}}^u \tilde{h}_{\tau,r}(v) dv,$$

is positive, increasing, concave on $[\frac{\pi}{\tau}, +\infty)$ and converge. Then there exists $\epsilon_{\tau,r} \in (0, \frac{\pi}{2})$ and a sequence $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that

$$\tau t_n \in [2n\pi + \frac{\pi}{2} + \epsilon_{\tau,r}, (2n+1)\pi], \quad (4)$$

and

$$\int_{\frac{\pi}{\tau}}^{t_n} \sin(\tau u) \int_{\frac{\pi}{\tau}}^u \tilde{h}_{\tau,r}(v) dv du = 0, \quad \forall n \geq 0.$$

Use the Cauchy mean value Theorem on Equation (3), for all $n \geq 0$ there exists $\eta_{\tau,r,n} \in (t_n, \frac{(2n+1)\pi}{\tau})$ such that

$$\begin{aligned} & \frac{d}{dr} \phi_\tau(r) - \beta_{\tau,r} \phi_\tau(r) \\ = & \tau \lim_{n \rightarrow +\infty} \left[\int_{\frac{\pi}{\tau}}^{\frac{(2n+1)\pi}{\tau}} \tilde{h}_{\tau,r}(v) dv - \int_{\frac{\pi}{\tau}}^{\eta_{\tau,r,n}} \tilde{h}_{\tau,r}(v) dv \int_{t_n}^{\frac{(2n+1)\pi}{\tau}} \tau \sin(\tau u) du \right], \end{aligned}$$

which can be written as

$$\frac{d}{dr}\phi_\tau(r) - \beta_{\tau,r}\phi_\tau(r) = \tau\tilde{\alpha}_{\tau,r} \lim_{n \rightarrow +\infty} [1 - \int_{t_n}^{\frac{(2n+1)\pi}{\tau}} \tau \sin(\tau u) du],$$

where

$$\tilde{\alpha}_{\tau,r} = \lim_{t \rightarrow +\infty} \int_{\frac{\pi}{\tau}}^t \tilde{h}_{\tau,r}(v) dv.$$

Thanks to Equation (4) we deduce that

$$\frac{d}{dr}\phi_\tau(r) - \beta_{\tau,r}\phi_\tau(r) > 0.$$

By definition of $\beta_{\tau,r}$ on Equation (2) the function $\theta : \mathbb{R}^* \times (0, \frac{1}{2}) \rightarrow \mathbb{R}_+$ defined as $\theta(\tau, r) := \beta_{\tau,r}$ is continuous and we have

$$\frac{d}{dr}\phi_\tau(r) > \theta(\tau, r)\phi_\tau(r), \quad \forall r \in (0, \frac{1}{2}), \quad \forall \tau > 0.$$

□

3 Proof of the Main result

Proof of the Main result. Let be $\tau > 0$ fixed. By the Proposition 2 we have

$$\frac{d}{dr}\phi_\tau(r) > \theta(\tau, r)\phi_\tau(r), \quad \forall r \in (0, \frac{1}{2}). \quad (5)$$

Then there exists $r \in (0, \frac{1}{2})$ such that $\phi_\tau(r) \neq 0$. Suppose that there exists a point $r_* \in (0, \frac{1}{2})$ such that $\phi_\tau(r_*) = 0$, then $\frac{d}{dr}\phi_\tau(r_*) > 0$. There exists $z \in (0, \frac{1}{2})$ such that $\phi_\tau(z) > 0$. Integrate Equation (5) we get

$$\phi_\tau(r) > \exp\left(\int_z^r \theta(\tau, v) dv\right)\phi_\tau(z), \quad \forall r \in (0, \frac{1}{2}).$$

Contradiction with the fact $\phi_\tau(r_*) = 0$. By the Notation 1 we deduce that

$$\psi[\eta](\frac{1}{2} - r, \tau) - \psi[\eta](\frac{1}{2} + r, \tau) \neq 0, \quad \forall r \in (0, \frac{1}{2}).$$

□

4 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (6)$$

and where $\{t\}$ is the fractional part of the real t . Define the *non-trivial zeros* of the function ζ in the following sense

Definition 3. Let be $s \in \mathbb{C}$. We say that s is a *non-trivial zero of the function* ζ if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

Denote $\eta_*(t) := \{t\}$. For every $\tau > 0$ and every $x \in (0, 1)$ we have

$$\psi[\eta_*](x, \tau) = \Im\left[\frac{\zeta(s)}{s}\right] \quad \text{and} \quad \psi[\eta_*](1-x, \tau) = -\Im\left[\frac{\zeta(1-s)}{1-s}\right], \quad s := x + i\tau.$$

Since

$$0 \leq \eta_*(t) \leq 1, \quad \forall t \geq 0,$$

Then η_* satisfies the hypotheses H_1 . The function η_* is C^1 on \mathbb{R}_+/\mathbb{N} . Then η_* satisfies the hypotheses H_2 . By the Riemann functional equation [[1], page 13 Theorem 2.1], the non-trivial zeros of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. By the Main result, the non-trivial zeros $s \in \mathbb{C}$ symmetric about the line $\Re(z) = \frac{1}{2}$ satisfies $\Re(s) = \frac{1}{2}$.

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).