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Non symmetry of the function $\frac{\zeta(s)}{s}$

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Abstract

We prove that $\frac{\zeta(s)}{s} \neq \frac{\zeta(1-s)}{1-s}$ for every $\Re(s) \in (0, \frac{1}{2})$ and $\Im(s) \in \mathbb{R}^*$, where ζ is the Riemann Zeta function. In the end of the paper, we give a discussion about the Riemann hypothesis.

Keywords: Riemann functional equation, Zeta function.

1 Main result

Let be $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a bounded function. We study the function $\psi[\eta] : (0, 1) \times \mathbb{R}^* \rightarrow \mathbb{C}$ defined as

$$\forall x \in (0, 1), \forall \tau \in \mathbb{R}^* : \quad \psi[\eta](x, \tau) := \int_0^{+\infty} \frac{\eta(t)}{t^{1+x+i\tau}} dt, \quad (1)$$

We focus about the function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfying the following hypotheses

$$H_1 : \quad 0 \leq \eta(t) \leq 1 \text{ for all } t \geq 0,$$

$$H_2 : \quad \text{the function } \eta \text{ is a } C^1 \text{ piecewise function.}$$

We prove in this paper the following Main result

Main Result. *Consider the function ψ defined by Equation (1). Then, for every function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ that satisfies the hypotheses H_1 and H_2 we have*

$$\forall \tau \in \mathbb{R}^*, \forall x \in (0, \frac{1}{2}) : \quad \|\psi[\eta](x, \tau) - \psi[\eta](1-x, \tau)\| \neq 0.$$

Notation 1. Let $\eta : \mathbb{R}_+ \mapsto \mathbb{R}$ be a function that satisfies H_1 and H_2 . For every $r \in (0, \frac{1}{2})$ define the functions $t \mapsto h_r(\eta, t)$ as

$$h_r(\eta, t) := 2 \exp(-\frac{1}{2}t)[\eta(\exp(t)) + 1] \sinh(rt), \quad \forall t \geq 0.$$

In order to simplify the notation we denote $h_r(t) := h_r(\eta, t)$ for all $t \geq 0$.

Proof of the Main result. Use the change of variable, $t \rightarrow \exp(u)$, in Equation (1), we obtain

$$\begin{aligned} \|\psi[\eta](\frac{1}{2} - r, \tau) - \psi[\eta](\frac{1}{2} + r, \tau)\| &= \left\| \int_0^{+\infty} h_r(s) \exp(i\tau s) ds \right\|, \\ \forall r \in (0, \frac{1}{2}), \forall \tau \in \mathbb{R}^*. \end{aligned}$$

The strategy is to prove that there exists a continuous function $\theta : \mathbb{R}^* \times (0, \frac{1}{2}) \rightarrow \mathbb{R}_+^*$ such that for every $\tau \in \mathbb{R}^*$ and every $r \in (0, \frac{1}{2})$ we have

$$\int_0^{+\infty} \frac{d}{dr} h_r(s) \exp(i\tau s) ds \neq \theta(\tau, r) \int_0^{+\infty} h_r(s) \exp(i\tau s) ds. \quad (2)$$

Let be $\beta > 0$ that we fix later. We have

$$\int_0^{+\infty} \frac{d}{dr} h_r(s) \exp(i\tau s) ds = \beta \int_0^{+\infty} h_r(s) \exp(i\tau s) ds,$$

implies

$$\int_0^{+\infty} \sin(\tau s) \left[\frac{d}{dr} h_r(s) - \beta h_r(s) \right] ds = 0.$$

By Rolle Theorem and the hypothesis H_1 , there exists $\beta_{\tau, r} > 0$ such that

$$\int_0^{\frac{\pi}{\tau}} \sin(\tau s) \left[\frac{d}{dr} h_r(s) - \beta_{\tau, r} h_r(s) \right] ds = 0. \quad (3)$$

Then

$$\int_{\frac{\pi}{\tau}}^{+\infty} \sin(\tau s) \left[\frac{d}{dr} h_r(s) - \beta_{\tau, r} h_r(s) \right] ds = 0.$$

Use the hypothesis H_2 and the integration part formula, we get

$$\lim_{n \rightarrow +\infty} \int_{\frac{\pi}{\tau}}^{\frac{2n\pi}{\tau}} \cos(\tau u) \int_{\frac{\pi}{\tau}}^u \left[\frac{d}{dr} h_r(v) - \beta_{\tau, r} h_r(v) \right] dv du = 0.$$

Since $r \in (0, \frac{1}{2})$ and by the hypothesis H_1 , then there exists $\alpha_{\tau,r} > 0$ such that

$$\int_{\frac{\pi}{\tau}}^{+\infty} \left[\frac{d}{dr} h_r(v) - \beta_{\tau,r} h_r(v) \right] dv = \alpha_{\tau,r}.$$

Since the convergence to $\alpha_{\tau,r}$ is exponential, we get

$$\lim_{t \rightarrow +\infty} \int_{\frac{\pi}{\tau}}^t \cos(\tau u) \left[\alpha_{\tau,r} - \int_{\frac{\pi}{\tau}}^u \left[\frac{d}{dr} h_r(v) - \beta_{\tau,r} h_r(v) \right] dv \right] du = 0. \quad (4)$$

In order to simplify the notation, denote

$$\tilde{h}_{\tau,r}(t) := \alpha_{\tau,r} - \int_{\frac{\pi}{\tau}}^t \left[\frac{d}{dr} h_r(v) - \beta_{\tau,r} h_r(v) \right] dv, \quad \forall t \geq 0.$$

Use again the integration part formula on Equation (4), we obtain

$$\lim_{n \rightarrow +\infty} \left[\int_{\frac{\pi}{\tau}}^{\frac{(2n+1)\pi}{\tau}} \tilde{h}_{\tau,r}(v) dv - \tau \int_{\frac{\pi}{\tau}}^{\frac{(2n+1)\pi}{\tau}} \sin(\tau u) \int_{\frac{\pi}{\tau}}^u \tilde{h}_{\tau,r}(v) dv du \right] = 0. \quad (5)$$

The function

$$u \mapsto \int_{\frac{\pi}{\tau}}^u \tilde{h}_{\tau,r}(v) dv,$$

is positive, increasing, concave on $[\frac{\pi}{\tau}, +\infty)$ and converge. Then there exists $\epsilon_{\tau,r} \in (0, \frac{\pi}{2})$ and a sequence $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that

$$\tau t_n \in [2n\pi, (2n+1)\pi - \frac{\pi}{2} - \epsilon_{\tau,r}] \cup [(2n+1)\pi - \frac{\pi}{2} + \epsilon_{\tau,r}, (2n+1)\pi], \quad (6)$$

and

$$\int_{\frac{\pi}{\tau}}^{t_n} \sin(\tau u) \int_{\frac{\pi}{\tau}}^u \tilde{h}_{\tau,r}(v) dv du = 0, \quad \forall n \geq 0.$$

Use the Cauchy mean value Theorem on Equation (5), for all $n \geq 0$ there exists $\eta_{\tau,r,n} \in (t_n, \frac{(2n+1)\pi}{\tau})$ such that

$$\lim_{n \rightarrow +\infty} \left[\int_{\frac{\pi}{\tau}}^{\frac{(2n+1)\pi}{\tau}} \tilde{h}_{\tau,r}(v) dv - \int_{\frac{\pi}{\tau}}^{\eta_{\tau,r,n}} \tilde{h}_{\tau,r}(v) dv \int_{t_n}^{\frac{(2n+1)\pi}{\tau}} \tau \sin(\tau u) du \right] = 0$$

which can be written as

$$\lim_{n \rightarrow +\infty} \left[1 - \int_{t_n}^{\frac{(2n+1)\pi}{\tau}} \tau \sin(\tau u) du \right] = 0.$$

Contradiction with Equation (6). By definition of $\beta_{\tau,r}$ on Equation (3) the function $\theta : \mathbb{R}^* \times (0, \frac{1}{2}) \rightarrow \mathbb{R}_+^*$ defined as $\theta(\tau, r) := \beta_{\tau,r}$ is continuous. We have proved Equation (2). Then for every fixed $\tau \in \mathbb{R}^*$ we get

$$\int_0^{+\infty} \frac{d}{dr} h_r(s) \sin(\tau s) ds > \theta(\tau, r) \int_0^{+\infty} h_r(s) \sin(\tau s) ds, \quad \forall r \in (0, \frac{1}{2}),$$

or

$$\int_0^{+\infty} \frac{d}{dr} h_r(s) \sin(\tau s) ds < \theta(\tau, r) \int_0^{+\infty} h_r(s) \sin(\tau s) ds, \quad \forall r \in (0, \frac{1}{2}).$$

Integrate to deduce the Main result. \square

2 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (7)$$

and where $\{t\}$ is the fractional part of the real t . Define the *non-trivial zeros* of the function ζ in the following sense

Definition 2. Let be $s \in \mathbb{C}$. We say that s is a *non-trivial zero of the function* ζ if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

Denote $\eta_*(t) := \{t\}$. For every $\tau \in \mathbb{R}^*$ and every $x \in (0, 1)$ we have

$$\psi[\eta_*](x, \tau) = \frac{\zeta(s)}{s} \quad \text{and} \quad \psi[\eta_*](1-x, \tau) = -\frac{\zeta(1-s)}{1-s}, \quad s := x + i\tau.$$

Since

$$0 \leq \eta_*(t) \leq 1, \quad \forall t \geq 0,$$

Then η_* satisfies the hypotheses H_1 . The function η_* is C^1 on \mathbb{R}_+/\mathbb{N} . Then η_* satisfies the hypotheses H_2 . By the Riemann functional equation [[1], page 13 Theorem 2.1], the non-trivial zeros of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. By the Main result, the non-trivial zeros $s \in \mathbb{C}$ symmetric about the line $\Re(z) = \frac{1}{2}$ satisfies $\Re(s) = \frac{1}{2}$.

References

- [1] E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function* (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).