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► **To cite this version:**

| W Oukil. Non symmetry of the function $\frac{\zeta(s)}{s}$. 2022. hal-03581978v10

HAL Id: hal-03581978

<https://hal.science/hal-03581978v10>

Preprint submitted on 28 Apr 2022 (v10), last revised 26 Nov 2023 (v25)

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Non symmetry of the function $\frac{\zeta(s)}{s}$

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April 28, 2022

Abstract

We prove that $\frac{\zeta(s)}{s} \neq \frac{\zeta(1-s)}{1-s}$ for every $\Re(s) \in (0, \frac{1}{2})$ and $\Im(s) \in \mathbb{R}^*$, where ζ is the Riemann Zeta function. In the end of the paper, we give a discussion about the Riemann hypothesis.

Keywords: Riemann functional equation, Zeta function.

1 Main result

Let be $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a bounded function. We study the function $\psi[\eta] : (0, 1) \times \mathbb{R}^* \rightarrow \mathbb{C}$ defined as

$$\forall x \in (0, 1), \forall \tau \in \mathbb{R}^* : \quad \psi[\eta](x, \tau) := \int_0^{+\infty} \frac{\eta(t)}{t^{1+x+i\tau}} dt, \quad (1)$$

We focus about the function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfying the following hypotheses

$$H_1 : \quad 0 \leq \eta(t) \leq 1 \text{ for all } t \geq 0,$$

$$H_2 : \quad \text{the function } \eta \text{ is a } C^1 \text{ piecewise function.}$$

We prove in this paper the following Main result

Main Result. *Consider the function ψ defined by Equation (1). Then, for every function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ that satisfies the hypotheses H_1 and H_2 we have*

$$\forall \tau \in \mathbb{R}^*, \forall x \in (0, \frac{1}{2}) : \quad \|\psi[\eta](x, \tau) - \psi[\eta](1-x, \tau)\| \neq 0.$$

Notation 1. Let $\eta : \mathbb{R}_+ \mapsto \mathbb{R}$ be a function that satisfies H_1 and H_2 . For every $r \in (0, \frac{1}{2})$ define the functions $t \mapsto h_r(\eta, t)$ as

$$h_r(\eta, t) := 2 \exp\left(-\frac{1}{2}t\right)[\eta(\exp(t)) + 1] \sinh(rt), \quad \forall t \geq 0.$$

In order to simplify the notation we denote $h_r(t) := h_r(\eta, t)$ for all $t \geq 0$.

Proof of the Main result. Let be $r \in (0, \frac{1}{2})$ and $\tau \in \mathbb{R}^*$. Use the change of variable, $t \rightarrow \exp(u)$, in Equation (1), we obtain

$$\begin{aligned} \|\psi[\eta]\left(\frac{1}{2} - r, \tau\right) - \psi[\eta]\left(\frac{1}{2} + r, \tau\right)\| &= \left\| \int_0^{+\infty} h_r(s) \exp(i\tau s) ds \right\|, \\ \forall r \in (0, \frac{1}{2}), \forall \tau \in \mathbb{R}^*. \end{aligned}$$

Then

$$\|\psi[\eta]\left(\frac{1}{2} - r, \tau\right) - \psi[\eta]\left(\frac{1}{2} + r, \tau\right)\| = 0,$$

implies

$$\left\| \int_0^{+\infty} \sin(\tau s) h_r(s) ds \right\| = 0.$$

Use the hypothesis H_2 and the integration part formula, we get

$$\lim_{n \rightarrow +\infty} \int_0^{\frac{2n\pi}{\tau}} \cos(\tau u) \int_0^u h_r(v) dv du = 0.$$

Since $r \in (0, \frac{1}{2})$, there exists $\alpha_r > 0$ such that

$$\lim_{u \rightarrow +\infty} \int_0^u h_r(v) dv = \alpha_r.$$

Then

$$\lim_{n \rightarrow +\infty} \int_0^{\frac{2n\pi}{\tau}} \cos(\tau u) [\alpha_r - \int_0^u h_r(v) dv] du = 0.$$

In order to simplify the notation, denote

$$\tilde{h}_r(t) := \alpha_r - \int_0^t h_r(v) dv, \quad \forall t \geq 0.$$

Use again the integration part formula,

$$\lim_{n \rightarrow +\infty} \left[\int_0^{\frac{2n\pi}{\tau}} \tilde{h}_r(v) dv + \tau \int_0^{\frac{2n\pi}{\tau}} \sin(\tau u) \int_0^u \tilde{h}_r(v) dv du \right] = 0. \quad (2)$$

By hypothesis H_1 the function $v \mapsto h_r(v)$ is positive, which implies that

$$u \mapsto \int_0^u \tilde{h}_r(v) dv,$$

is positive, increasing, concave on \mathbb{R}_+ and converge. Then there exists a sequence $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that

- $t_n \in [\frac{(2n-1)\pi}{\tau}, \frac{2n\pi}{\tau}]$, $\forall n \geq 0$,
- $\lim_{n \rightarrow +\infty} \left| t_n - [\frac{2n\pi}{\tau} - \frac{\pi}{2\tau}] \right| \neq 0$,
- $\int_0^{t_n} \sin(\tau u) \int_0^u \tilde{h}_r(v) dv du = 0$.

Use the Cauchy mean value Theorem on Equation (2), for all $n \geq 0$ there exists $\eta_{\tau,r,n} \in (t_n, \frac{2n\pi}{\tau})$ such that

$$\lim_{n \rightarrow +\infty} \left[\int_0^{\frac{2n\pi}{\tau}} \tilde{h}_r(v) dv + \int_0^{\eta_{\tau,r,n}} \tilde{h}_r(v) dv \int_{t_n}^{\frac{2n\pi}{\tau}} \tau \sin(\tau u) du \right] = 0$$

which can be written as

$$\lim_{n \rightarrow +\infty} \left[1 + \int_{t_n}^{\frac{2n\pi}{\tau}} \tau \sin(\tau u) du \right] = 0.$$

Contradiction with $\lim_{n \rightarrow +\infty} \left| t_n - [\frac{2n\pi}{\tau} - \frac{\pi}{2\tau}] \right| \neq 0$. □

2 Discussion about the Riemann hypothesis

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*, \quad (3)$$

and where $\{t\}$ is the fractional part of the real t . Define the *non-trivial zeros* of the function ζ in the following sense

Definition 2. Let be $s \in \mathbb{C}$. We say that s is a *non-trivial zero* of the function ζ if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

Denote $\eta_*(t) := \{t\}$. For every $\tau \in \mathbb{R}^*$ and every $x \in (0, 1)$ we have

$$\psi[\eta_*](x, \tau) = \frac{\zeta(s)}{s} \quad \text{and} \quad \psi[\eta_*](1-x, \tau) = -\frac{\zeta(1-s)}{1-s}, \quad s := x + i\tau.$$

Since

$$0 \leq \eta_*(t) \leq 1, \quad \forall t \geq 0,$$

Then η_* satisfies the hypotheses H_1 . The function η_* is C^1 on \mathbb{R}_+/\mathbb{N} . Then η_* satisfies the hypotheses H_2 . By the Riemann functional equation [[1], page 13 Theorem 2.1], the non-trivial zeros of the function ζ are symmetric about the line $\Re(z) = \frac{1}{2}$. By the Main result, the non-trivial zeros $s \in \mathbb{C}$ symmetric about the line $\Re(z) = \frac{1}{2}$ satisfies $\Re(s) = \frac{1}{2}$.

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).