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Differential equations properties and the Riemann hypothesis

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Abstract

Using the differential equations properties, we prove that any *non-trivial zero* of Riemann's zeta function is of real part equal to $\frac{1}{2}$. In other words, we prove that the Riemann hypothesis is true.

Keywords: Riemann hypothesis, Zeta function, Abel summation formula.

1 Introduction

We consider the representation of Riemann's zeta function [[1], page 14 Equation 2.1.5] defined for all $s \in \mathbb{C}$ such that $\Re(s) > 0$ by the Abel summation formula as

$$\zeta(s) := -s \int_0^{+\infty} \frac{\{t\}}{t^{1+s}} dt, \quad (1)$$

where $\{t\}$ is the fractional part of the real t . In this paper, we focus on the *non-trivial zeros* of the function ζ , defined by Equation (1), in the following sense

Definition 1. Let be $s \in \mathbb{C}$. We say that s is a *non-trivial zero* of the function ζ if

$$\zeta(s) = 0 \quad \text{and} \quad \Re(s) \in (0, 1), \quad \Im(s) \in \mathbb{R}^*.$$

The following theorem affirms the Riemann hypothesis.

Main Result. Consider the function ζ defined by Equation (1). Suppose that $s \in \mathbb{C}$ is a non-trivial zero of the function ζ then $\Re(s) = \frac{1}{2}$.

2 Main proposition

The Equation (1) can be written as

$$\zeta(s) = -s \int_1^{+\infty} \frac{1}{t^{2-s}} + \frac{\{t\}}{t^{1+s}} dt, \quad \forall s \in \mathbb{C}, \quad \Re(s) \in (0, 1).$$

By a change of variable, $t \rightarrow \exp(u)$ we get

$$\begin{aligned} \forall s := x + i\tau \in \mathbb{C}, \quad x \in (0, 1), \quad \tau \in \mathbb{R}^* : \\ \zeta(s) := -s \int_0^{+\infty} \exp(-(1-s)u) + \exp(-su)\{\exp(u)\} du. \end{aligned}$$

For every $\tau \in \mathbb{R}^*$, define the function $\phi_\tau : (0, 1) \times \mathbb{R}_+ \rightarrow \mathbb{R}$ as

$$\phi_\tau(x, t) := - \int_0^t \left[\exp(-(1-x)u) - \exp(-xu)\{\exp(u)\} \right] \sin(\tau u) du. \quad (2)$$

Denote

$$\forall x \in (0, 1) : \quad \phi_\tau(x) = \lim_{t \rightarrow +\infty} \phi_\tau(x, t).$$

By identification, for every $\tau \in \mathbb{R}^*$ and every $x \in (0, 1)$, we have $\phi_\tau(x) = \Im\left(\frac{\zeta(s)}{s}\right)$ where $s := x + i\tau$ and we have

$$\zeta(x + i\tau) = 0 \implies \phi_\tau(x) = 0. \quad (3)$$

Proposition 2. *Let be $\tau \in \mathbb{R}^*$. Then,*

$$\forall r \in (0, \frac{1}{2}) : \quad \phi_\tau\left(\frac{1}{2} - r\right) \neq \phi_\tau\left(\frac{1}{2} + r\right).$$

Lemma 3. *Let be $\tau \in \mathbb{R}^*$ and $z \geq 0$. Let be $s > v \geq 0$, then*

$$\left| \int_v^s \exp(-zt)\{\exp(t)\} \sin(\tau \ln(t)) dt - [w_\tau(z, s) - w_\tau(z, v)] \right| \leq h_\tau(z, v, s),$$

where $h_\tau(0, v, s) = s - v$ and $h_\tau(z, v, s) = 1 + \frac{1}{z}$ for every $z > 0$ and where

$$\begin{aligned} w_\tau(z, t) = & -\frac{1}{\tau} \left[\frac{1-z}{(1-z)^2 + \tau^2} - \frac{z}{z^2 + \tau^2} \right] \exp((1-z)t) \cos(\tau t) \\ & \left[\frac{1}{(1-z)^2 + \tau^2} - \frac{1}{z^2 + \tau^2} \right] \exp((1-z)t) \sin(\tau t), \quad \forall t \geq 0. \end{aligned}$$

Proof. Without loss of generality, we prove the present Lemma for $s := \ln(n)$ and $v := \ln(m)$ where $n, m \in \mathbb{N}^*$ and $n \geq m$. By Abel summation formula,

$$\begin{aligned} \int_{\ln(m)}^{\ln(n)} \exp(-zt) \{ \exp(t) \} \exp(i\tau t) dt &= \frac{1}{i\tau} \sum_{k=m}^{n-1} \exp((-z + i\tau) \ln(k+1)) \\ &\quad - \frac{1}{i\tau(1-z+i\tau)} [n^{1-z} \exp(i\tau \ln(n)) - m^{1-z} \exp(i\tau \ln(m))]. \end{aligned}$$

Then

$$\begin{aligned} \int_{\ln(m)}^{\ln(n)} \exp(-zt) \{ \exp(t) \} \sin(\tau \ln(t)) dt &= -\frac{1}{\tau} \sum_{k=m}^{n-1} (k+1)^{-z} \cos(\tau \ln(k+1)) \\ &\quad + \frac{1-z}{\tau((1-z)^2 + \tau^2)} [n^{1-z} \cos(\tau \ln(n)) - m^{1-z} \cos(\tau \ln(m))] \\ &\quad + \frac{1}{(1-z)^2 + \tau^2} [n^{1-z} \sin(\tau \ln(n)) - m^{1-z} \sin(\tau \ln(m))]. \end{aligned}$$

We have

$$\begin{aligned} &\left| \sum_{k=m}^{n-1} (k+1)^{-z} \cos(\tau \ln(k+1)) - \int_m^n t^{-z} \cos(\tau \ln(t)) dt \right| \\ &= \left| \sum_{k=m}^{n-1} \int_k^{k+1} [(k+1)^{-z} \cos(\tau \ln(k+1)) - t^{-z} \cos(\tau \ln(t))] dt \right| \\ &= \left| \sum_{k=m}^{n-1} \int_k^{k+1} \int_t^{k+1} s^{-z-1} [-z \cos(\tau \ln(s)) + \sin(\tau \ln(s))] ds dt \right| \\ &\leq (1+z) \sum_{k=m}^{n-1} \int_k^{k+1} t^{-z-1} dt = (1+z) \int_m^n t^{-z-1} dt. \end{aligned}$$

Then if $z = 0$ we get

$$\left| \sum_{k=m}^{n-1} \cos(\tau \ln(k+1)) - \int_m^n \cos(\tau \ln(t)) dt \right| \leq \ln\left(\frac{n}{m}\right),$$

if $z > 0$ we get

$$\left| \sum_{k=m}^{n-1} (k+1)^{-z} \cos(\tau \ln(k+1)) - \int_m^n t^{-z} \cos(\tau \ln(t)) dt \right| \leq \frac{1+z}{z}.$$

Use the fact

$$\begin{aligned} \int_m^n t^{-z} \cos(\tau \ln(t)) dt &= \Re \left[\int_m^n t^{-z+i\tau} t \right] \\ &= \frac{-z}{z^2 + \tau^2} \left[n^{1-z} \cos(\tau \ln(n)) - m^{1-z} \cos(\tau \ln(m)) \right] \\ &\quad - \frac{\tau}{z^2 + \tau^2} \left[n^{1-z} \sin(\tau \ln(n)) - m^{1-z} \sin(\tau \ln(m)) \right], \end{aligned}$$

to deduce the Lemma. \square

In order to simplify the notation, denote

$$A := \{t \geq 0, t \neq \ln(n), \quad \forall n \geq 2.\}.$$

Lemma 4. *Let be $\tau \in \mathbb{R}^*$. There exists a squence $(t_n)_n \subset \mathbb{R}_+$ such that*

$$\lim_{n \rightarrow +\infty} t_n = +\infty,$$

and such that

$$\lim_{n \rightarrow +\infty} \int_0^{t_{2n}} \{\exp(t)\} \sin(\tau t) dt = - \lim_{n \rightarrow +\infty} \int_0^{t_{2n+1}} \{\exp(t)\} \sin(\tau t) dt = +\infty.$$

Proof. By Lemma 3,

$$n \approx +\infty \implies \frac{1}{n} \int_0^{\ln(n)} \{\exp(t)\} \sin(\tau t) dt \approx \frac{1}{n} w_\tau(0, \ln(n)).$$

Since

$$\frac{1}{n} w_\tau(0, \ln(n)) \approx -\frac{1}{\tau} \frac{1}{1 + \tau^2} \cos(\tau \ln(n)) + \left[\frac{1}{1 + \tau^2} - \frac{1}{\tau^2} \right] \sin(\tau \ln(n)),$$

using the fact $\tau \neq 0$ and the fact $\lim_{n \rightarrow +\infty} \ln(n+1) - \ln(n) = 0$, there exists a sequence $(n_m)_m \subset \mathbb{N}^*$ such that

$$\lim_{m \rightarrow +\infty} n_m = +\infty \quad \text{and} \quad \lim_{m \rightarrow +\infty} \sin(\tau n_m) = 0,$$

and such that

$$\begin{aligned} \lim_{m \rightarrow +\infty} \frac{1}{n_{2m}} w_\tau(0, \ln(n_{2m})) &= - \lim_{m \rightarrow +\infty} \frac{1}{n_{2m+1}} w_\tau(0, \ln(n_{2m+1})) \\ &= \frac{1}{\tau} \frac{1}{1 + \tau^2}. \end{aligned}$$

\square

Lemma 5. *Let be $\tau \in \mathbb{R}^*$ and $r \in (0, \frac{1}{2})$. Let $(v_n)_n \subset \mathbb{R}_+$ be a sequence. For every $n \in \mathbb{N}$, let $t \mapsto \tilde{\theta}_\tau(r, n, t)$ be a continuous function satisfying the following differential equation,*

$$\begin{aligned} \frac{d}{dt} \tilde{\theta}_\tau(r, n, t) &= \left(\frac{1}{2} - r\right) \tilde{\theta}_\tau(r, n, t) - [1 + \{\exp(t)\}] \sin(\tau t), \\ \tilde{\theta}_\tau(r, n, v_n) &= 0, \quad t \in A, \quad t \geq v_n. \end{aligned} \quad (4)$$

For every sequence $(s_n)_n \subset \mathbb{R}_+$ such that

$$\lim_{n \rightarrow +\infty} s_n = +\infty, \quad \lim_{n \rightarrow +\infty} s_n - v_n = +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} \sin(\tau s_n) = 0,$$

we have,

$$\lim_{n \rightarrow +\infty} |\tilde{\theta}_\tau(r, n, s_n)| = +\infty.$$

Proof. In order to simplify the notation, denote $x_r := \frac{1}{2} - r$. Integrate Equation (4) and use the fact $\tilde{\theta}_\tau(r, n, v_n) = 0$, we get

$$\tilde{\theta}_\tau(r, n, s_n) = \int_{v_n}^{s_n} \exp(x_r(s_n - t)) [1 + \{\exp(t)\}] \sin(\tau t) dt.$$

Then

$$\begin{aligned} \tilde{\theta}_\tau(r, n, s_n) &= \frac{x_r}{x_r^2 + \tau^2} \left[\exp(x_r(s_n - v_n)) \sin(\tau v_n) - \sin(\tau s_n) \right] \\ &\quad - \frac{\tau}{x_r^2 + \tau^2} \left[\exp(x_r(s_n - v_n)) \cos(\tau v_n) - \cos(\tau s_n) \right] \\ &\quad + \int_{v_n}^{s_n} \exp(x_r(s_n - t)) \{\exp(t)\} \sin(\tau t) dt. \end{aligned}$$

By Lemma 3,

$$\begin{aligned} & \left| \int_{v_n}^{s_n} \exp(x_r(s_n - t)) \{\exp(t)\} \sin(\tau t) dt - \exp(x_r s_n) [w_\tau(x_r, s_n) - w_\tau(x_r, v_n)] \right| \\ & \leq \exp(x_r s_n) \frac{3 - 2r}{1 - 2r}. \end{aligned}$$

Since

$$\lim_{n \rightarrow +\infty} s_n = +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} s_n - v_n = +\infty,$$

then

$$n \approx +\infty \implies \exp(-s_n) \tilde{\theta}_\tau(r, n, s_n) \approx \tilde{w}_\tau(x_r, s_n),$$

where

$$\begin{aligned}\tilde{w}_\tau(x_r, s_n) &:= \left[\frac{1-x_r}{(1-x_r)^2 + \tau^2} - \frac{x_r}{x_r^2 + \tau^2} \right] \cos(\tau s_n) \\ &\quad - \left[\frac{1}{(1-x_r)^2 + \tau^2} - \frac{1}{x_r^2 + \tau^2} \right] \sin(\tau s_n).\end{aligned}$$

By hypothesis of the present Lemma we have

$$\lim_{n \rightarrow +\infty} \sin(\tau s_n) = 0,$$

since $x_r \in (0, \frac{1}{2})$, then

$$\lim_{n \rightarrow +\infty} |\tilde{\theta}_\tau(r, n, s_n)| = +\infty.$$

□

Lemma 6. *Let be $r \in (0, \frac{1}{2})$ and $t \mapsto \theta_\tau(r, t)$ be a continuous function satisfying the following differential equation,*

$$\begin{aligned}\frac{d}{dt}\theta_\tau(r, t) &= \left(\frac{1}{2} - r\right)\theta_\tau(r, t) - [1 - \exp(-2rt)][1 + \{\exp(t)\}] \sin(\tau t), \\ \theta_\tau(r, 0) &= 0, \quad t \in A.\end{aligned}\tag{5}$$

Then,

$$\sup_{t \geq 0} |\theta_\tau(r, t)| = +\infty.$$

Proof. By contradiction, suppose that there exists $r \in (0, \frac{1}{2})$ and $\omega_r > 0$ such that

$$\sup_{t \geq 0} |\theta_\tau(r, t)| \leq \omega_r.\tag{6}$$

Since $t \mapsto \theta_\tau(r, t)$ is a continuous function, integrate Equation (5) to obtain

$$\begin{aligned}\theta_\tau(r, t) &= \left(\frac{1}{2} - r\right) \int_0^t \theta_\tau(r, s) ds \\ &\quad - \int_0^t [1 - \exp(-2rs)][1 + \{\exp(s)\}] \sin(\tau s) ds.\end{aligned}\tag{7}$$

By Lemma 4, there exists $(t_n)_n \subset \mathbb{R}_+$ such that

$$\lim_{n \rightarrow +\infty} t_n = +\infty,$$

and such that

$$\lim_{n \rightarrow +\infty} \int_0^{t_{2n}} \{\exp(s)\} \sin(\tau s) dt = - \lim_{n \rightarrow +\infty} \int_0^{t_{2n+1}} \{\exp(s)\} \sin(\tau s) dt = +\infty.$$

Then

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_0^{t_{2n}} [1 - \exp(-2rs)][1 + \{\exp(s)\}] \sin(\tau s) ds \\ &= - \lim_{n \rightarrow +\infty} \int_0^{t_{2n+1}} [1 - \exp(-2rs)][1 + \{\exp(s)\}] \sin(\tau s) ds = +\infty. \end{aligned}$$

By hypothesis (6) and Equation (7), we obtain

$$\lim_{n \rightarrow +\infty} \int_0^{t_{2n}} \theta_\tau(r, s) ds = - \lim_{n \rightarrow +\infty} \int_0^{t_{2n+1}} \theta_\tau(r, s) ds = +\infty.$$

Implies, there exists a sequence $(v_n)_n \subset \mathbb{R}_+$ such that

$$\lim_{n \rightarrow +\infty} v_n = +\infty, \quad (8)$$

and such that

$$\theta_\tau(r, v_n) = 0, \quad \forall n \geq 0.$$

For every $n \in \mathbb{N}$, let $t \mapsto \tilde{\theta}_\tau(r, n, t)$ be a continuous function satisfying the following differential equation,

$$\begin{aligned} \frac{d}{dt} \tilde{\theta}_\tau(r, n, t) &= \left(\frac{1}{2} - r\right) \tilde{\theta}_\tau(r, n, t) - [1 + \{\exp(t)\}] \sin(\tau t), \\ \tilde{\theta}_\tau(r, n, v_n) &= 0, \quad t \in A, \quad t \geq v_n. \end{aligned} \quad (9)$$

By Lemma 5, for every sequence $(s_n)_n$ such that

$$\lim_{n \rightarrow +\infty} s_n = +\infty, \quad \lim_{n \rightarrow +\infty} s_n - v_n = +\infty \text{ and } \lim_{n \rightarrow +\infty} \sin(\tau s_n) = 0.$$

we have

$$\lim_{n \rightarrow +\infty} |\tilde{\theta}_\tau(r, n, s_n)| = +\infty. \quad (10)$$

Denote $\eta_\tau(x, n, t) := \theta_\tau(r, t) - \tilde{\theta}_\tau(r, n, t)$. By Equations (5) and (9),

$$\frac{d}{dt} \eta_\tau(x, n, t) = \left(\frac{1}{2} - r\right) \eta_\tau(x, n, t) + \exp(-2rt)[1 + \{\exp(t)\}] \sin(\tau t),$$

where $t \in A$ and $t \geq v_n$. Integrate,

$$\begin{aligned} |\eta_\tau(x, n, t)| &\leq 2 \exp(-2v_n) \exp\left(\left(\frac{1}{2} - r\right)t\right) \int_{v_n}^t \exp\left(-\left(\frac{1}{2} - r\right)s\right) ds \\ &= \frac{2}{\frac{1}{2} - r} \exp(-2v_n) \left[\exp\left(\left(\frac{1}{2} - r\right)(t - v_n)\right) - 1 \right]. \end{aligned}$$

$$|\eta_\tau(x, n, t)| \leq \frac{4}{1 - 2r} \left[\exp\left(\left(\frac{1}{2} - r\right)(t - v_n) - 2v_n\right) - \exp(-2v_n) \right].$$

For

$$s_n \in \left[v_n, v_n + \frac{1}{\frac{1}{2} - r} 2v_n \right],$$

we get

$$\sup_{n \geq 0} |\eta_\tau(x, n, s_n)| \leq \frac{2}{\frac{1}{2} - r}.$$

By Equation (8), we have $\lim_{n \rightarrow +\infty} v_n = +\infty$. Then we can choose $(s_n)_n$ such that

$$\lim_{n \rightarrow +\infty} s_n = +\infty, \quad \lim_{n \rightarrow +\infty} s_n - v_n = +\infty \text{ and } \lim_{n \rightarrow +\infty} \sin(\tau s_n) = 0.$$

and since $\eta_\tau(x, n, t) := \theta_\tau(r, t) - \tilde{\theta}_\tau(r, n, t)$, use Equation (10) to deduce that

$$\lim_{n \rightarrow +\infty} |\theta_\tau(r, t_n)| = +\infty.$$

Contradiction with the hypothesis (6). \square

Proof of Proposition 2. Equation (2), implies that $\forall t \geq 0$ we have

$$\phi_\tau\left(\frac{1}{2} + r, t\right) = - \int_0^t \exp\left(-\frac{1}{2}u\right) [\exp(ru) - \exp(-ru) \{\exp(u)\}] \sin(\tau u) du,$$

$$\phi_\tau\left(\frac{1}{2} - r, t\right) = - \int_0^t \exp\left(-\frac{1}{2}u\right) [\exp(-ru) - \exp(ru) \{\exp(u)\}] \sin(\tau u) du.$$

Denote

$$y_\tau(r, t) := \exp\left(\frac{1}{2}t\right) \left[\phi_\tau\left(\frac{1}{2} + r, t\right) - \phi_\tau\left(\frac{1}{2} - r, t\right) \right], \quad \forall r \in \left(0, \frac{1}{2}\right), \quad \forall t \geq 0.$$

Then

$$y_\tau(r, t) = -2 \exp\left(\frac{1}{2}t\right) \int_0^t \exp\left(-\frac{1}{2}u\right) \sinh(ru) [1 + \{\exp(u)\}] \sin(\tau u) du. \quad (11)$$

The function $t \mapsto y_\tau(r, t)$ is a continuous function and satisfies the following differential equation

$$\frac{d}{dt}y_\tau(r, t) = \frac{1}{2}y_\tau(r, t) - 2\sinh(rt)[1 + \{\exp(t)\}] \sin(\tau t), \quad \forall t \in A. \quad (12)$$

By contradiction, suppose that there exists $r \in (0, \frac{1}{2})$ such that

$$\phi_\tau(\frac{1}{2} - r) = \phi_\tau(\frac{1}{2} + r).$$

Then

$$\int_0^{+\infty} \exp(-\frac{1}{2}u) \sinh(ru)[1 + \{\exp(u)\}] \sin(\tau u) du = 0,$$

implies

$$\begin{aligned} & \int_0^t \exp(-\frac{1}{2}u) \sinh(ru)[1 + \{\exp(u)\}] \sin(\tau u) du \\ &= - \int_t^{+\infty} \exp(-\frac{1}{2}u) \sinh(ru)[1 + \{\exp(u)\}] \sin(\tau u) du. \end{aligned}$$

Then

$$\begin{aligned} & 2 \left| \int_0^t \exp(-\frac{1}{2}u) \sinh(ru)[1 + \{\exp(u)\}] \sin(\tau u) du \right| \\ & \leq \frac{2}{\frac{1}{2} - r} \exp(-(\frac{1}{2} - r)t) + \frac{2}{\frac{1}{2} + r} \exp(-(\frac{1}{2} + r)t), \quad \forall t \geq 0. \end{aligned}$$

Denote $\theta_\tau(r, t) := \exp(-rt)y_\tau(r, t)$. By the last Inequality and Equation (11), we get

$$\sup_{t \geq 0} |\theta_\tau(r, t)| = \sup_{t \geq 0} |\exp(-rt)y_\tau(r, t)| \leq \frac{2}{\frac{1}{4} - r^2} \quad (13)$$

By Equation (12), The function $t \mapsto \theta_\tau(r, t)$ is a continuous function and satisfies the following differential equation

$$\frac{d}{dt}\theta_\tau(r, t) = (\frac{1}{2} - r)\theta_\tau(r, t) - [1 - \exp(-2rt)][1 + \{\exp(t)\}] \sin(\tau t), \quad t \in A.$$

By Lemma 6 we have $\sup_{t \geq 0} |\theta_\tau(r, t)| = +\infty$. Contradiction with Equation (13). Then

$$\forall r \in (0, \frac{1}{2}) : \quad \phi_\tau(\frac{1}{2} - r) \neq \phi_\tau(\frac{1}{2} + r).$$

□

3 Proof of the Main result

Proof of the Main result. Suppose that there exist $\tau \in \mathbb{R}^*$ and $r \in [0, \frac{1}{2})$ such that

$$\zeta\left(\frac{1}{2} - r + i\tau\right) = 0,$$

by Equation (3), we get

$$\phi_\tau\left(\frac{1}{2} - r\right) = 0.$$

Using the Riemann's functional equation [[1], page 13 Theorem 2.1], the non-trivial zeros of the function ζ are symmetric about the line $\Re(s) = \frac{1}{2}$. We get

$$\zeta\left(\frac{1}{2} + r + i\tau\right) = 0.$$

By Equation (3) we obtain

$$\phi_\tau\left(\frac{1}{2} + r\right) = 0.$$

By Proposition 2, we obtain $r = \frac{1}{2}$. □

References

- [1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).