Non trivial zeros of the Zeta function

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Abstract

In [2] a more flexible function is found whose zeros coincide with the zeros of the Zeta function in the critical strip. We prove that any zero, of this new function, is of real part equal to $\frac{1}{2}$.

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Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s\int_1^{+\infty} \frac{\{t\}}{t^{1+s}}dt, \quad \Re(s) \in (0,1), \quad \Im(s) \in \mathbb{R}^*, \tag{1}$$

where $\{t\}$ is the fractional part of the real t. In order to simplify the notation, denote

$$B := \{s \in \mathbb{C} : \quad \Re(s) \in (0,1), \quad \Im(s) \in \mathbb{R}^*\}.$$

It is proved in [2] that the zeros of the function ζ and the function $f : B \to \mathbb{C}$ coincide on B, where the function f is defined as

$$f(s) := 1 - 4s(1-s)\sum_{j\in\mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k\in\mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k}}{2k-1-s}, \quad \forall s \in B. \tag{2}$$

We prove the following theorem

Theorem 1. *Consider the function f given by the Equation (2). Any zero of the function f in B is of real part equal to $\frac{1}{2}$.*

In order to prove the previous Theorem, we need the following Lemma.

Lemma 2. *Consider the function f given by the Equation (2). Suppose that $s \in B$ is a zero of the function f. Then, one of the following statements is true*

- $1 - 2\Re(s) = 0,$
- $|h_s(t)| = 0, \quad \forall t \in (0,1),$

where

$$\begin{aligned} h_s(t) := &\frac{1}{s(1-\overline{s})} - \frac{1}{(1-s)\overline{s}}t \\ &+ 4\sum_{j\in\mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k\in\mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1}}{(2k-1-s)(2k-2+\overline{s})}. \end{aligned}$$

Proof. Let be $s \in B$. In one hand, for every $t \in (0,1)$ we have

$$\begin{aligned} \int_t^1 u^{-2-s}\Big[\int_0^u (v-\frac{1}{2})dv\Big]du &= \frac{1}{2}\int_t^1 u^{-2-s}\Big(u^2-u\Big)du \\ &= \frac{1}{2s(1-s)} - \frac{1}{2}t^{-s}\Big(\frac{t}{1-s}+\frac{1}{s}\Big). \end{aligned} \tag{3}$$

We recall that the function $p:\mathbb{R}\to\mathbb{R}$, defined as

$$p(t) := \int_0^t \Big(\{u\} - \frac{1}{2}\Big)du, \quad \forall t \in \mathbb{R}$$

is a continuous 1-periodic function and satisfies

$$\begin{aligned} p(t) &= 2\sum_{j\in\mathbb{N}^*} \frac{1}{(j2\pi)^2}\Big(\cos(j2\pi t) - 1\Big) \\ &= 2\sum_{j\in\mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k\in\mathbb{N}^*} (-1)^k \frac{(j2\pi)^{2k} t^{2k}}{(2k)!}, \quad \forall t \in \mathbb{R}. \end{aligned}$$

On the other hand, for every $t \in (0,1)$ we have

$$\begin{aligned}
&\int_t^1 u^{-2-s}\Big[\int_0^u (v-\frac{1}{2})dv\Big]du \\
&\qquad = 2\int_t^1 u^{-2-s}\sum_{j\in\mathbb{N}^*}\frac{1}{(j2\pi)^2}\sum_{k\in\mathbb{N}^*}(-1)^k\frac{(j2\pi)^{2k}u^{2k}}{(2k)!}du \\
&\qquad = 2\sum_{j\in\mathbb{N}^*}\frac{1}{(j2\pi)^2}\sum_{k\in\mathbb{N}^*}\frac{1}{(2k)!}\frac{(-1)^k(j2\pi)^{2k}}{2k-1-s} \\
&\qquad - 2\sum_{j\in\mathbb{N}^*}\frac{1}{(j2\pi)^2}\sum_{k\in\mathbb{N}^*}\frac{1}{(2k)!}\frac{(-1)^k(j2\pi)^{2k}t^{2k-1-s}}{2k-1-s}. \qquad (4)
\end{aligned}$$

Suppose that s is a zero of the function f. Then the Equations (3) and (4) implies that

$$\frac{1}{2}\Big(\frac{t}{1-s}+\frac{1}{s}\Big) = 2\sum_{j\in\mathbb{N}^*}\frac{1}{(j2\pi)^2}\sum_{k\in\mathbb{N}^*}\frac{1}{(2k)!}\frac{(-1)^k(j2\pi)^{2k}t^{2k-1}}{2k-1-s},\ \forall t\in(0,1).$$

It is proved in [2], that the zeros of the function ζ and the function $f : B \to \mathbb{C}$ coincide on B. Thanks to the Riemann functional equation, the point $1-\overline{s}$ is a zero of the function ζ and f. Then

$$\frac{1}{2}\Big(\frac{t}{\overline{s}}+\frac{1}{1-\overline{s}}\Big) = 2\sum_{j\in\mathbb{N}^*}\frac{1}{(j2\pi)^2}\sum_{k\in\mathbb{N}^*}\frac{1}{(2k)!}\frac{(-1)^k(j2\pi)^{2k}t^{2k-1}}{2k-2+\overline{s}},\ \forall t\in(0,1).$$

The subtraction implies that

$$|(1-2\Re(s))h_s(t)| = 0, \quad \forall t\in(0,1),$$

where

$$\begin{aligned}
h_s(t) :=& \frac{1}{s(1-\overline{s})} - \frac{1}{(1-s)\overline{s}}t \\
&+ 4\sum_{j\in\mathbb{N}^*}\frac{1}{(j2\pi)^2}\sum_{k\in\mathbb{N}^*}\frac{1}{(2k)!}\frac{(-1)^k(j2\pi)^{2k}t^{2k-1}}{(2k-1-s)(2k-2+\overline{s})}.
\end{aligned}$$

□

Proof of the Theorem 1. Let be $s \in B$ and suppose that s is a zero of the function f. Prove the present Theorem by contradiction. Suppose that $1-2\Re(s) \neq 0$. Thanks to the Lemma 2, it sufficient to prove that there exists

$t \in (0,1)$ such that $|h_s(t)| \neq 0$. By contradiction, suppose that $|h_s(t)| = 0$ for all $t \in (0,1)$. Then

$$\begin{aligned} 4|s(1-s)|^2 \sum_{j\in\mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k\in\mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k-1}}{(2k-1-s)(2k-2+\overline{s})}, \\ = s(1-\overline{s})t - \overline{s}(1-s), \quad \forall t \in (0,1). \end{aligned} \tag{5}$$

In order to simplify the notation, denote for every $t \in (0,1)$ the function Ψ_s as follow

$$\Psi_s(t) := 4|s(1-s)|^2 \sum_{j\in\mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k\in\mathbb{N}^*} \frac{1}{(2k)!} \frac{(-1)^k (j2\pi)^{2k} t^{2k}}{|(2k-1-s)(2k-2+\overline{s})|^2}.$$

The imaginary part of the equation (5) implies

$$2\frac{d}{dt}\Psi_s(t) - 3t^{-1}\Psi_s(t) = t+1, \quad \forall t \in (0,1).$$

Then there exists a constant $c_s \in \mathbb{R}$ such that

$$\Psi_s(t) = c_s t^{\frac{3}{2}} + t^2 - t, \quad \forall t \in (0,1). \tag{6}$$

The real part of the equation (5) implies

$$\begin{aligned} &4\sum_{j\in\mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k\in\mathbb{N}^*} \frac{(2k-1)(2k-2)}{(2k)!} \frac{|s(1-s)|^2 (-1)^k (j2\pi)^{2k} t^{2k-1}}{|(2k-1-s)(2k-2+\overline{s})|^2} \\ &+(\Re(s) - |s|^2)t^{-1}\Psi_s(t) \\ &\quad = (\Re(s) - |s|^2)(t-1), \quad \forall t \in (0,1). \end{aligned}$$

Then for all $t \in (0,1)$ we have

$$\begin{aligned} &4\sum_{j\in\mathbb{N}^*} \frac{1}{(j2\pi)^2} \sum_{k\in\mathbb{N}^*} \frac{(2k-1)(2k-2)}{(2k)!} \frac{s(1-s)|^2 (-1)^k (j2\pi)^{2k} t^{2k-1}}{|(2k-1-s)(2k-2+\overline{s})|^2} = \tilde{c}_s t^{\frac{1}{2}}, \\ &\tilde{c}_s := (\Re(s) - |s|^2)c_s. \end{aligned} \tag{7}$$

Integrate, for all $[t_1, t_2] \subset (0,1)$ we obtain

$$\left[\frac{d}{dt}\Psi_s(t_2) - 2t_2^{-1}\Psi_s(t_2)\right] - \left[\frac{d}{dt}\Psi_s(t_1) - 2t_1^{-1}\Psi_s(t_1)\right] = 2\tilde{c}_s\left(t_2^{\frac{1}{2}} - t_1^{\frac{1}{2}}\right).$$

Using the Equation (6) and $t_1 \approx 0$, we obtain a contradiction. $\square$

References

[1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).

[2] W. Oukil, Non trivial zeros of the Zeta function using the differential equations, eprint: 2112.05521, archivePrefix: arXiv, primaryClass: math.GM. (2023)