



HAL
open science

FOLIATIONS AND STABLE MAPS

Bertrand Toen, Gabriele Vezzosi

► **To cite this version:**

Bertrand Toen, Gabriele Vezzosi. FOLIATIONS AND STABLE MAPS. Current developments in Hodge theory. Proceedings of Hodge theory at IMSA., Mar 2021, Miami, United States. <hal-03581642v2>

HAL Id: hal-03581642

<https://hal.science/hal-03581642v2>

Submitted on 26 Jul 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

FOLIATIONS AND STABLE MAPS

BERTRAND TOËN AND GABRIELE VEZZOSI

ABSTRACT. This paper is part of an ongoing series of works on the study of foliations on algebraic varieties via derived algebraic geometry. We focus here on the specific case of globally defined vector fields and the global behaviour of their algebraic integral curves. For a smooth and proper variety X with a global vector field ν , we consider the induced vector field $\nu_{g,n}$ on the derived stack of stable maps, of genus g with n marked points, to X . When (g, n) is either $(0, 2)$ or $(1, 0)$, the derived stack of zeros of $\nu_{g,n}$ defines a proper *moduli of algebraic trajectories* of ν . When $(g, n) = (0, 2)$ algebraic trajectories behave very much like rational algebraic paths from one zero of ν to another, and in particular they can be composed. This composition is represented by the usual gluing maps in Gromov-Witten theory, and we use it to give three categorical constructions, of different categorical levels, related, in a certain sense, by decategorification. In order to do this, in particular, we have to deal with virtual fundamental classes of non-quasi-smooth derived stacks. When $(g, n) = (1, 0)$, zeros of $\nu_{1,0}$ might be thought as algebraic analogues of periodic orbits of vector fields on smooth real manifolds. In particular, we propose a Zeta function counting the zeros of $\nu_{1,0}$, that we like to think of as an algebraic version of Ruelle's dynamical Zeta function. We conclude the paper with a brief indication on how to extend these results to the case of general one dimensional foliation F , by considering the derived stack of F -equivariant stable maps.

CONTENTS

Introduction	2
1. Vector fields and stable maps	4
1.1. Recollections on the derived moduli of stable maps	4
1.2. Induced vector fields	6
1.3. Genus 0 and 1	8
2. Localized Euler class in the derived setting	10
2.1. The refined Euler class	11
2.2. The virtual Euler series and Euler numbers	14
2.3. Categories of shifted singularities	19
3. Applications	24

3.1. Categories associated to a vector field	24
3.2. An algebraic dynamical Zeta function	30
3.3. Higher genus considerations	30
4. Appendix: K-theory and G-theory of quasi-smooth derived stacks	32
References	34

INTRODUCTION

This paper is part of an ongoing series of works on the study of foliations on algebraic varieties via derived techniques ([TVc, TVb, TVa]). We focus here on the specific case of globally defined vector fields (and more generally one dimensional foliations, see our section §4) and the global behaviour of their algebraic integral curves. The results presented in this paper are preliminary, and we will limit ourselves to survey the main constructions and results, more details as well as more general results will appear elsewhere.

For a smooth and proper variety X and a global vector field $\nu \in H^0(X, \mathbb{T}_X)$, we consider curves in X which are globally invariant by ν , with a particular emphasis on their moduli and on the various numerical and homological invariants they give rise to.

To start with, we will show how ν defines an induced vector field $\nu_{g,n}$ on $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$, the derived moduli stack of stable maps of genus g with n marked points in X (see Section 1.1). An important first observation is that when (g, n) is either $(0, 2)$ or $(1, 0)$, the zeros of the vector field $\nu_{g,n}$ on $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$ are precisely given by stable maps that are globally invariant by ν , or, equivalently, by *algebraic trajectories* of ν in X . The derived stack of zeros of $\nu_{g,n}$ therefore defines a proper moduli space of algebraic trajectories of ν , which is the central object of this work.

When $(g, n) = (0, 2)$, algebraic trajectories appear as *broken rational curves* in X along ν whose marked points are sent to zeros of the original vector field ν on X . Therefore, they behave very much like algebraic paths from one zero of ν to another, and in particular they can be composed. This composition is represented by gluing maps on the level of derived moduli stacks, induced by the usual gluing construction of stable maps in Gromov-Witten theory. Out of this, we extract three constructions, of different categorical levels, that can be thought as the starting point of an *enumerative geometry of algebraic trajectories*. We begin by considering derived categories attached to the derived moduli Z of zeros of $\nu_{0,2}$, namely the derived category of coherent complexes on

Z which are relatively perfect along the evaluation map $Z \rightarrow Z_0 \times Z_0$ where Z_0 is the derived zero locus of ν (see Definition 3.1.1). As base change always holds in derived algebraic geometry (see [Toë14, 3.1]), the gluing maps induce a composition on these derived categories, and we get this way a linear 2-category $\widetilde{\text{Traj}}(X, \nu)$ whose set of objects is the set of zeros of ν in X , and whose categories of maps are derived categories of the derived stack of zeros of $\nu_{0,2}$ (see Definition 3.1.1). This structure can be viewed as a generalization of the categorical Gromov-Witten invariants of [MR18] suitably localized around the zeros of the vector field ν . It is also a vector field analogue of the usual localization techniques in Gromov-Witten theory for \mathbb{G}_m -actions (see for instance [GP99]).

The 2-category $\widetilde{\text{Traj}}(X, \nu)$ can then be decategorified twice. As a first decategorification, we construct (Definition 3.1.2) a 1-category, whose set of objects is again the zeros of ν , and the space of morphisms are given by Borel-Moore homology of the derived zero loci of $\nu_{0,2}$ (again relative to the evaluation map). A second decategorification is obtained by considering virtual fundamental classes of the derived zero loci of $\nu_{0,2}$. This is now a bivector in the homology of the zero locus of ν : a vector field version of the usual Gromov-Witten invariants in cohomology. On a technical level, the derived moduli stacks involved in this construction are not quasi-smooth in general, and thus their virtual fundamental classes do not exist in the classical sense of the word. We show however that one can always define them as *rational functions* in an indeterminate t , and that the bivector mentioned above has only $\mathbb{Q}(t)$ coefficients. More precisely, if Z_0 is the zero locus of ν in X , our construction provides an element in $H_*(Z_0)^{\otimes 2} \otimes \mathbb{Q}(t)$. When further assumptions on X are made (for instance when X is convex), the derived moduli spaces are indeed quasi-smooth, and the above rational functions are polynomials which can then be evaluated at $t = -1$. Under these conditions, we obtain a linear endomorphism of the vector space spanned by zeros of ν (assuming these zeros are simple), which we like to think of an algebraic version of the *Morse differential*. Finally, we prove (Proposition 3.1.3) that this endomorphism, even if it does not square to zero, satisfies a remarkable formula close to the splitting/associativity formula in usual Gromov-Witten theory (see e.g. [BM96, Def. 7.1-(3)]).

In Section 3.2 we share some thoughts on the case $(g, n) = (1, 0)$, where zeros of $\nu_{1,0}$ can be thought as algebraic analogues of periodic orbits of vector fields on smooth real manifolds. In particular, we use our numerical invariants with coefficients in $\mathbb{Q}(t)$ in order to propose

a formula for a *Zeta function* counting zeros of $\nu_{1,0}$, which we like to think of as an algebraic version of the dynamical Zeta function of a flow (see e.g. [Rue92, §1.4-(B)]). We hope to come back to this aspect in future work. We conclude the paper with a short section (Section 3.3) explaining how to extend our results to the case of general one dimensional foliations instead of just global vector fields, which will certainly provide a wider range of applicability of our results.

We recall in the Appendix the basic facts we need about the K -theory and G -theory of quasi-smooth derived stacks.

Conventions. We will work over base field k of characteristic 0. However, all our results and construction remains correct over an arbitrary excellent base ring k (but some algebraic operations such as symmetric dg-algebras have to be replaced with their simplicial versions).

We will work in the context of derived algebraic geometry of [TV08, Toë14]

1. VECTOR FIELDS AND STABLE MAPS

In this first section we show how a global vector field on an algebraic variety X induces a vector field on $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$, the derived moduli stack of stable maps to X . When $(g, n) = (0, 2)$ or $(g, n) = (1, 0)$, we explain how the zeros of the induced vector field correspond to *algebraic trajectories* of the original vector field, namely stable maps which are globally invariant.

1.1. Recollections on the derived moduli of stable maps. The reader can find details concerning the derived moduli stack of stable maps in [STV15]. We remind below some notations and constructions.

We let $\bar{\mathcal{M}}_{g,n}^{pr}$ be the Artin stack of prestable curves of genus g with n marked points. We denote by $\pi : \bar{\mathcal{C}}_{g,n} \rightarrow \bar{\mathcal{M}}_{g,n}^{pr}$ be the universal prestable curve. The morphism π induces an adjunction on the ∞ -categories of relative derived stacks

$$\pi^* : \mathbf{dSt}/\bar{\mathcal{M}}_{g,n}^{pr} \rightleftarrows \mathbf{dSt}/\bar{\mathcal{C}}_{g,n} : \pi_*$$

where π^* is the usual base change ∞ -functor. The right adjoint π_* is also known as the Weil restriction along π .

Because π is a proper and flat morphism, it is known that π_* preserves derived Artin stacks locally of finite presentation. In particular, if X is any smooth and proper scheme over k , we can consider

$$\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X) := \pi_*(X \times \bar{\mathcal{C}}_{g,n}),$$

which is a derived Artin stack endowed with a natural projection to $\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}$.

Definition 1.1.1. *The derived stack of pre-stable maps on X (of genus g with n marked points) is the derived Artin stack $\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)$ defined above.*

The projection $p : \mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X) \rightarrow \bar{\mathcal{M}}_{g,n}^{pr}$ is a morphism of derived Artin stacks, whose fibers are the derived mapping schemes of pre-stable curves to X . In more precise terms, for any derived scheme S and morphism $S \rightarrow \bar{\mathcal{M}}_{g,n}^{pr}$, the pull-back $S \times_{\bar{\mathcal{M}}_{g,n}^{pr}} \mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)$ is canonically identified with $\mathbf{Map}_S(\bar{\mathcal{C}}_{g,n} \times_{\bar{\mathcal{M}}_{g,n}^{pr}} S, X \times S)$, the derived mapping stack over S from the curve $\bar{\mathcal{C}}_{g,n} \times_{\bar{\mathcal{M}}_{g,n}^{pr}} S$ to $X \times S$. We see in particular that the morphism $p : \mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X) \rightarrow \bar{\mathcal{M}}_{g,n}^{pr}$ is representable and its fibers are derived schemes locally of finite type.

As usual, the derived stack of stable maps is defined as the open substack $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X) \subset \mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)$ whose geometric points consists of stable maps to X . The derived stack $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$ is Deligne-Mumford, and each component $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X, \beta)$, of stable maps $f : C \rightarrow X$ with $f_*([C]) = \beta$, is proper over $\mathbf{Spec} k$ (see [BM, Thm. 3.14]). Here β is an element in $H_2(\bar{X}, \mathbb{Q}_\ell(-1))$, is a class in \mathbb{Q}_ℓ -adic homology of $\bar{X} = X \otimes_k \bar{k}$?

We then define the open (and closed) substacks $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X, \beta) \subset \mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$ by considering geometric points $f : \mathbf{Spec} K \rightarrow \mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$ corresponding to stable maps $C \rightarrow X \otimes_k \bar{k}$, with C a prestable curve over K , such that $f_*([C]) = \beta \in H_2(X \otimes_k K, \mathbb{Q}_\ell(-1))$, where $[C] \in H_2(C, \mathbb{Q}_\ell(-1))$ is the fundamental class of C .

Definition 1.1.2. *The derived stack of stable maps on X (of genus g with n marked points) is the derived Deligne-Mumford stack $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$ defined above. The open and closed substack of stable maps of class $\beta \in H_2(X, \mathbb{Q}_\ell(1))$ is $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X, \beta)$ defined above.*

An important fact that we will use in a crucial way is the explicit computation of the tangent complex of $\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)$. For this, we consider the evaluation morphism

$$ev : \mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X) \times_{\bar{\mathcal{M}}_{g,n}^{pr}} \bar{\mathcal{C}}_{g,n} \longrightarrow X,$$

as well as the projection to the first factor

$$q : \mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X) \times_{\bar{\mathcal{M}}_{g,n}^{pr}} \bar{\mathcal{C}}_{g,n} \longrightarrow \mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X).$$

The tangent complex of $p : \mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X) \rightarrow \bar{\mathcal{M}}_{g,n}^{pr}$ is then given by (see [Toë14, §3.4])

$$\mathbb{T}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)/\bar{\mathcal{M}}_{g,n}^{pr}} \simeq q_*(ev^*(\mathbb{T}_X)),$$

where T_X is the tangent vector bundle of X relative to k . As q is a flat and proper curve, we see immediately that $\mathbb{T}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)/\bar{\mathcal{M}}_{g,n}^{pr}}$ is a perfect complex whose amplitude is contained in $[0, 1]$. To obtain the full tangent complex of $\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)$, relative to k , we use the exact triangle

$$\mathbb{T}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)/\bar{\mathcal{M}}_{g,n}^{pr}} \longrightarrow \mathbb{T}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)} \longrightarrow p^*(\mathbb{T}_{\bar{\mathcal{M}}_{g,n}^{pr}}).$$

As $\bar{\mathcal{M}}_{g,n}^{pr}$ is a smooth Artin stack over k , $p^*(\mathbb{T}_{\bar{\mathcal{M}}_{g,n}^{pr}})$ is itself perfect and of amplitude in $[-1, 0]$. As a result, the tangent complex $\mathbb{T}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)}$ is perfect and of amplitude in $[-1, 1]$. The derived Artin stack $\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)$ is thus quasi-smooth over k (see Appendix), and therefore $\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)$ is itself a quasi-smooth and proper derived Deligne-Mumford stack over k (its tangent complex is a complex and of Tor-amplitude $[0, 1]$).

1.2. Induced vector fields. We keep working with X a smooth and proper scheme over k . As before \mathbb{T}_X denotes the tangent bundle of X relative to k .

Proposition 1.2.1. *Any vector field $\nu \in H^0(X, \mathbb{T}_X)$ determines a natural vector field*

$$\nu_{g,n} \in H^0(\mathbb{R}\bar{\mathcal{M}}_{g,n}(X), \mathbb{T}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}).$$

Proof. Start with $\nu : \mathcal{O}_X \rightarrow \mathbb{T}_X$ on X . By pull-back along the evaluation map $ev : \mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X) \times_{\bar{\mathcal{M}}_{g,n}^{pr}} \bar{\mathcal{C}}_{g,n} \rightarrow X$, it provides a new section $ev^*(\nu) : \mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)} \rightarrow ev^*(\mathbb{T}_X)$. Its push-forward by $q : \mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X) \rightarrow \bar{\mathcal{M}}_{g,n}^{pr}$ produces a morphism

$$\mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}} \longrightarrow q_*q^*(\mathcal{O}) \longrightarrow q_*ev^*(\mathbb{T}_X).$$

We now use that $q_*ev^*(\mathbb{T}_X) \simeq \mathbb{T}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)/\bar{\mathcal{M}}_{g,n}^{pr}}$, as well as the natural morphism $\mathbb{T}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)/\bar{\mathcal{M}}_{g,n}^{pr}} \rightarrow \mathbb{T}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)}$ in order to get a canonical element $\nu_{g,n} \in H^0(\mathbb{R}\bar{\mathcal{M}}_{g,n}(X), \mathbb{T}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)})$. \square

Remark 1.2.2. One can understand the above proposition in various ways. For instance, it can be seen as the fact that an infinitesimal automorphism of X induces an infinitesimal automorphism of $\mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)$, simply by using the functoriality of the construction $X \mapsto \mathbb{R}\bar{\mathcal{M}}_{g,n}^{pr}(X)$. In another direction, and more generally, a global vector field defines a derived foliation of dimension 1 in the sense of [TVc], and the above proposition is a very particular case of a more general result stating the existence of direct images of derived foliations (see [TVa]).

It will be important to remind (see the proof of Proposition 1.2.1) that the vector field $\nu_{g,n}$ is in fact the image of a natural relative vector field $\nu'_{g,n} \in H^0(\mathbb{R}\mathcal{M}_{g,n}^{pr}(X), \mathbb{T}_{\mathbb{R}\mathcal{M}_{g,n}^{pr}(X)/\mathcal{M}_{g,n}^{pr}})$ by the canonical map $\mathbb{T}_{\mathbb{R}\mathcal{M}_{g,n}^{pr}(X)/\mathcal{M}_{g,n}^{pr}} \rightarrow \mathbb{T}_{\mathbb{R}\mathcal{M}_{g,n}^{pr}(X)}$, and thus is vertical relative to this morphism. Our main object of interest is $\nu_{g,n}$, but it will be sometimes important to use $\nu'_{g,n}$ in some of the constructions. We thus gather these two objects $\nu_{g,n}$ and $\nu'_{g,n}$ in the definition below.

Definition 1.2.3. *For $\nu \in H^0(X, \mathbb{T}_X)$ a global vector field on X (relative to k). The induced vector field (resp. relative induced vector field) of type (g, n) is $\nu_{g,n}$ (resp. $\nu'_{g,n}$) constructed above.*

Pointwise, the vector field $\nu_{g,n}$ can be described as follows. Let $f : C \rightarrow X$ by a stable map to X (say define over k), and $\Sigma \subset C$ be the divisor defining the marked points on C . We consider $\mathbb{T}_X(\Sigma)$, the tangent complex of C twisted by $\mathcal{O}(-\Sigma)$, and $\mathbb{T}_C(-\Sigma) \rightarrow f^*(\mathbb{T}_X)$ the induced by the derivative of f . The cone of this morphism is a perfect complex E on C of amplitude contained in $[-1, 0]$ on C . The tangent complex of $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$ at the point C is given by the complex of k -vector spaces $H^*(C, E)$.

We have a canonical morphism $f^*(\mathbb{T}_X) \rightarrow E$, as well as a global section $\mathcal{O}_C \rightarrow f^*(\mathbb{T}_X)$ given by the restriction of ν on C . The vector field $\nu_{g,n}$ evaluated at the point C , is the given by the composition

$$k \rightarrow H^*(C, \mathcal{O}_C) \rightarrow H^*(C, f^*(\mathbb{T}_X)) \rightarrow H^*(C, E).$$

Let X be a smooth and proper scheme over k and $\nu \in H^*(X, \mathbb{T}_X)$. The zeros of the induced vector $\nu_{g,n}$ on $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$ defines a closed derived substack $Z(\nu_{g,n}) \subset \mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$. More precisely, we can form the sheaf of commutative simplicial rings $Sym_{\mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}}(\mathbb{L}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)})$, whose spectrum (relative to $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$) is the total tangent stack

$$\mathrm{TR}\bar{\mathcal{M}}_{g,n}(X) = \mathbf{Spec}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)} Sym_{\mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}}(\mathbb{L}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}).$$

The augmentation $Sym_{\mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}}(\mathbb{L}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}) \rightarrow \mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}$ defines a natural projection $\mathrm{TR}\bar{\mathcal{M}}_{g,n}(X) \rightarrow \mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$. The vector field $\nu_{g,n}$ provides a natural section $s : \mathbb{R}\bar{\mathcal{M}}_{g,n}(X) \rightarrow \mathrm{TR}\bar{\mathcal{M}}_{g,n}(X)$. Similarly, the zero vector field provides a section $0 : \mathbb{R}\bar{\mathcal{M}}_{g,n}(X) \rightarrow \mathrm{TR}\bar{\mathcal{M}}_{g,n}(X)$. The derived stack of zeros of $\nu_{g,n}$ is naturally defined as the derived intersections of these two sections.

Definition 1.2.4. *With the above notations, the derived stack of zeros of $\nu_{g,n}$ is defined by*

$$Z(\nu_{g,n}) := \mathbb{R}\bar{\mathcal{M}}_{g,n}(X) \times_{s, \mathrm{TR}\bar{\mathcal{M}}_{g,n}(X), 0} \mathbb{R}\bar{\mathcal{M}}_{g,n}(X).$$

The derived stack $Z(\nu_{g,n})$ can be seen as a closed substack of $\bar{\mathcal{M}}_{g,n}(X)$ by means of the second projection

$$j : Z(\nu_{g,n}) := \mathbb{R}\bar{\mathcal{M}}_{g,n}(X) \times_{s, \mathbb{T}\mathbb{R}\bar{\mathcal{M}}_{g,n}(X), 0} \mathbb{R}\bar{\mathcal{M}}_{g,n}(X) \longrightarrow \mathbb{R}\bar{\mathcal{M}}_{g,n}(X).$$

The derived substack $Z(\nu_{g,n})$ can also be described in terms of a relative spectrum of an explicit sheaf of commutative simplicial rings, as follows. The section $\nu_{g,n}$ defines a morphism of perfect complexes on $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$

$$\mathbb{L}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)} \longrightarrow \mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}.$$

By the universal properties of symmetric algebras, this morphism defines a morphism of sheaves of commutative simplicial rings

$$\mathit{Sym}_{\mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}}(\mathbb{L}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}) \longrightarrow \mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}.$$

In the same manner we have the canonical augmentation morphism

$$\mathit{Sym}_{\mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}}(\mathbb{L}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}) \longrightarrow \mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}.$$

Then, the derived substack $Z(\nu_{g,n})$ can also be described as

$$Z(\nu_{g,n}) \simeq \mathbf{Spec} \left(\mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)} \otimes_{\mathit{Sym}_{\mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}}(\mathbb{L}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)})} \mathcal{O}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)} \right).$$

A first important comment is the fact that, even though $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$ is quasi-smooth, the derived stack $Z(\nu_{g,n})$ is no more quasi-smooth in general. Indeed, its tangent complex is in general concentrated in amplitude $[0, 2]$. As an example, when $\nu = 0$ is the zero vector field on X , then $Z(\nu_{g,n}) \simeq \mathbb{T}\bar{\mathcal{M}}_{g,n}(X)[-1]$ is the shifted total tangent space, whose tangent complex taken at a geometric point x is $\mathbb{T}_{\bar{\mathcal{M}}_{g,n}(X), x} \oplus \mathbb{T}_{\bar{\mathcal{M}}_{g,n}(X), x}[-1]$. As a consequence, the derived stack $Z(\nu_{g,n})$ does not carry a virtual class (see Appendix) as its structure sheaf is in general not bounded. We will see however in the next sections that it is possible to overcome this problem and to associate to $Z(\nu_{g,n})$ a homology class that deserves the name of *virtual class* and that can be used in order to make several interesting constructions.

1.3. Genus 0 and 1. Let $f : \mathbf{Spec} L \rightarrow \mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$ be a morphism with L a separably closed field. By replacing k by L and X by $X \otimes_k L$, we can furthermore assume that $k = L$. The morphism f is then described as a stable map

$$f : C \longrightarrow X,$$

with C a prestable curve of type $(0, 2)$ or $(1, 0)$.

Case $(g, n) = (0, 2)$. Let us first assume that $(g, n) = (0, 2)$ and that C is smooth to make things simpler, so it consists of a \mathbb{P}^1 marked at 0

and ∞ . We can describe the vector field $\nu_{g,n}$ at the point f as follows. The tangent complex of $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$ at f is given by (see [STV15])

$$\mathbb{T}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X),f} \simeq \mathbb{H}(C, \mathbb{T}_C(-0 - \infty) \rightarrow f^*(\mathbb{T}_X)).$$

The vector field $\nu : \mathcal{O}_X \rightarrow \mathbb{T}_X$ provides a morphism $\mathbb{H}(C, \mathcal{O}_C) \rightarrow \mathbb{H}(C, f^*(\mathbb{T}_X)) \rightarrow \mathbb{H}(C, \mathbb{T}_C(-0 - \infty) \rightarrow f^*(\mathbb{T}_X))$. Since $C = \mathbb{P}^1$, we have $\mathbb{H}(C, \mathcal{O}_C) \simeq k$, and the corresponding morphism of complexes of k -vector spaces

$$k \longrightarrow \mathbb{H}(C, \mathbb{T}_C(-0 - \infty) \rightarrow f^*(\mathbb{T}_X))$$

is the vector field $\nu_{0,2}$ evaluated at the point f . We also note that $k \rightarrow \mathbb{H}(C, f^*(\mathbb{T}_X))$ is $\nu'_{2,0}$ evaluated at f , where $\nu'_{0,2}$ is the relative induced vector field of Definition 1.2.3.

We now consider the commutative diagram

$$\begin{array}{ccc} \mathbb{H}(C, f^*(\mathbb{T}_X)) & \longrightarrow & \mathbb{H}(C, \mathbb{T}_C(-0 - \infty) \rightarrow f^*(\mathbb{T}_X)) \\ & \swarrow \nu'_{0,2} & \searrow \nu_{0,2} \\ & k & \end{array}$$

A homotopy to zero of $\nu_{0,2}$ then consists of a factorization of $\nu'_{0,2}$ via the canonical morphism induced by the tangent map of f .

$$Tf : \mathbb{H}(C, \mathbb{T}_C(-0 - \infty)) \longrightarrow \mathbb{H}(C, f^*(\mathbb{T}_X)).$$

Again as $C = \mathbb{P}^1$, we have $\mathbb{H}(C, \mathbb{T}_C(-0 - \infty)) \simeq k$, and a canonical generator is given by the vector field $z \cdot \partial_z$ on \mathbb{P}^1 which vanishes at 0 and ∞ . We thus arrive at the conclusion that the geometric point f lies in the substack $Z(\nu_{0,2})$ if and only if there exists $\lambda \in k$ such that the image of $\lambda \cdot z \partial_z$ by Tf is ν restricted to C . In other words, f lies in $Z(\nu_{0,2})$ if and only if f exhibits C as an algebraic trajectory of ν in X .

We have seen that a stable map $f : (\mathbb{P}^1, 0, \infty) \rightarrow X$, which is a zero of the induced vector field $\nu_{0,2}$ must be an algebraic trajectory of ν on X . In particular, the images of 0 and ∞ in X must be a zero of ν (and moreover these zeros must be non-degenerate along C itself). In general, when C possesses several irreducible components, and thus is a concatenation of \mathbb{P}^1 glued along 0 and ∞ , $f : C \rightarrow X$ lies in $Z(\nu_{0,2})$ if and only if each component C_i of C is an algebraic trajectory of ν in X . Therefore, f becomes a trajectories of broken \mathbb{P}^1 in X for the vector field ν , each irreducible components of C sitting between two zeros of ν .

Case $(g, n) = (1, 0)$. Let us now assume that $(g, n) = (1, 0)$. The same reasoning shows that the geometric points of $Z(\nu_{1,0})$ correspond to stable maps $f : E \rightarrow X$, with E a curve of genus 1, such that there is a global vector field $\alpha \in H^0(E, \mathbb{T}_E) \simeq k$ whose image by Tf is the

restriction of ν to E . These are again the trajectories of ν in X .

As a conclusion, we have defined for any vector field ν on X an induced vector field $\nu_{g,n}$ on the derived stack of stable maps $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$. When (g, n) is either equal to $(0, 2)$, or to $(1, 0)$, the zeros of $\nu_{g,n}$ on $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$ precisely are the algebraic trajectories of ν on X . Therefore, the derived stack of zeros $Z(\nu_{g,n})$ is a Deligne-Mumford derived stack over k , which can be written as a disjoint union of proper Deligne-Mumford derived stack $Z(\nu_{g,n}) \simeq \coprod_{\beta} Z(\nu_{g,n})(\beta)$, and is starting point to study enumerative properties of algebraic trajectories. However, the derived stack $Z(\nu_{g,n})$ is not quasi-smooth in general, and extracting numerical or cohomological invariants out of it requires more work. We will see in the next section that $Z(\nu_{g,n})$ still possesses a nice cohomology theory and an associated virtual class, which can be used in order to consider enumerative questions concerning algebraic trajectories of the original vector field ν . Also, when X is convex $Z(\nu_{g,n})$ is automatically quasi-smooth and things simplify a lot (see our section §3.2).

2. LOCALIZED EULER CLASS IN THE DERIVED SETTING

In this section we study the zero locus of a section s of a perfect complex E , of amplitude contained in $[0, 1]$, over a quasi-smooth derived Deligne-Mumford stack X . We will see how we can associate several cohomological, numerical and categorical invariants to such data. In the next section, it will be applied to the special case where $E = \mathbb{T}_{\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)}$ and $s = \nu_{g,n}$ is the induced vector field of Definition 1.2.3.

We fix once for all a quasi-smooth derived Deligne-Mumford stack X over k , $E \in \mathbf{Perf}(X)$ of amplitude contained in $[0, 1]$ and $s : \mathcal{O}_X \rightarrow E$ a morphism in $\mathbf{Perf}(X)$. We denote by $Z(s) \subset X$ the derived substack of zeros of s , which is defined to be $\mathbf{Spec}(\mathcal{O}_X \otimes_{\mathit{Sym}_{\mathcal{O}_X}(E^\vee)} \mathcal{O}_X)$, where the first map $\mathit{Sym}_{\mathcal{O}_X}(E^\vee) \rightarrow \mathcal{O}_X$ is induced by the section s , and the second map $\mathit{Sym}_{\mathcal{O}_X}(E^\vee) \rightarrow \mathcal{O}_X$ is the natural augmentation (i.e. the previous map, with $s = 0$).

When the perfect complex E is of amplitude 0 (i.e. is a vector bundle), $Z(s)$ is quasi-smooth and thus gives rise to a virtual fundamental class $[Z(s)]^{vir} \in G_0(Z(s))$, in the Grothendieck group of coherent sheaves (or in some Borel-Moore homology theory, see Appendix). However, when E is no more a vector bundle, then $Z(s)$ is not quasi-smooth in general, as it is the derived intersection of quasi-smooth derived stacks inside an ambient quasi-smooth derived stack. The virtual class $[Z(s)]^{vir}$ thus no more makes sense in this generality. However, we will see that the Euler class do exists as a \mathbb{G}_m -equivariant

class (with support on $Z(s)$) on $\mathbb{V}(E[1]) = \mathbf{Spec}(Sym_{\mathcal{O}_X}(E[1]))$, the linear stack associated to the shifted complex $E[1]$. The purpose of this section is to explain the construction of this class, as well as of the numerical and cohomological invariants that can be extracted from it. We will also provide a categorical interpretation in terms of certain categories of singularities associated to functions of degree -1 (Section 2.3).

2.1. The refined Euler class. We consider the linear derived stack $\mathbb{V}(E^\vee) = \mathbf{Spec} Sym_{\mathcal{O}_X}(E^\vee)$, associated to the dual complex E^\vee (see [Mon21, Toë14] for more on linear stacks). It comes equipped with a natural projection $\pi : Z \rightarrow X$ making it into an affine derived scheme relative to X . The derived stack $\mathbb{V}(E^\vee)$ is thus itself a derived Deligne-Mumford stack which is affine over X . Moreover, as E^\vee has its amplitude contained in $[-1, 0]$, Z is also quasi-smooth and so is the natural projection $\pi : \mathbb{V}(E^\vee) \rightarrow X$.

The projection π possesses two sections $e : X \rightarrow \mathbb{V}(E^\vee)$ and $s : X \rightarrow \mathbb{V}(E^\vee)$. The section e is the zero section and corresponds to the natural augmentation $Sym_{\mathcal{O}_X}(E^\vee) \rightarrow \mathcal{O}_X$. The section s is induced by $s : \mathcal{O}_X \rightarrow E$ and by the induced morphism of commutative \mathcal{O}_X -algebras $Sym_{\mathcal{O}_X}(E^\vee) \rightarrow \mathcal{O}_X$. We have an ∞ -functor

$$\underline{Hom}_{\mathcal{O}_X}(s^*e_*(\mathcal{O}_X), -) : \mathbf{QCoh}(X) \longrightarrow \mathbf{QCoh}(X).$$

We let $\mathcal{A} := \underline{Hom}_{\mathcal{O}_X}(e_*(\mathcal{O}_X), e_*(\mathcal{O}_X))$, considered as a quasi-coherent associative \mathcal{O}_X -algebra. By construction, the above ∞ -functor factors as

$$\mathbf{QCoh}(X) \longrightarrow \mathcal{A} - \mathbf{Mod}^{qcoh} \longrightarrow \mathbf{QCoh}(X),$$

where $\mathcal{A} - \mathbf{Mod}^{qcoh}$ is the ∞ -category of quasi-coherent \mathcal{A} -modules, and the second ∞ -functor is the forgetful ∞ -functor. The first ∞ -functor will be denoted by $Kos_s : \mathbf{QCoh}(X) \rightarrow \mathcal{A} - \mathbf{Mod}^{qcoh}$.

By construction, the underlying quasi-coherent complex of \mathcal{A} is naturally equivalent to $\bigoplus_{n \geq 0} (Sym_{\mathcal{O}_X}^n(E^\vee[1]))^\vee$. Therefore, there exists a canonical inclusion morphism $E[-1] \rightarrow \mathcal{A}$ corresponding to the $n = 1$ factor. This in turn induces a morphism on cohomology sheaves $H^1(E)[-2] \rightarrow H^*(\mathcal{A})$. The graded algebra $H^*(\mathcal{A})$ is moreover graded commutative, as \mathcal{A} underlies an E_∞ -structure, and therefore the above morphism induces a natural morphism of (underived) graded $H^0(\mathcal{O}_X)$ -algebras

$$\mathcal{B} := Sym_{H^0(\mathcal{O}_X)}(H^1(E)) \longrightarrow H^*(\mathcal{A}),$$

where $H^1(E)$ sits in weight 2 in the graded algebra \mathcal{B} . We warn the reader here that the $Sym_{H^0(\mathcal{O}_X)}$ must be understood in the non-derived setting. In particular, any quasi-coherent \mathcal{A} -module $M \in \mathcal{A} - \text{Mod}^{qcoh}$ gives rise to a sheaf of graded \mathcal{B} -modules by considering $H^*(M)$ with its graded $H^*(\mathcal{A})$ -module structure and induced graded \mathcal{B} -module structure from the above morphism.

Lemma 2.1.1. *For any perfect complex $M \in \text{Perf}(X)$, the sheaf of graded \mathcal{B} -modules $H^*(Kos_s(M))$ is locally of finite type.*

Proof. The statement is local for the étale topology on X and we can thus assume that $X = \mathbf{Spec} A$ is affine. Moreover, as \mathcal{O}_X generates perfect complexes (because X is assumed to be affine), it is enough to prove the statement for $M = \mathcal{O}_X$. We can present E as the fiber of a morphism d of vector bundles $E \longrightarrow E_0 \xrightarrow{d} E_1$. This defines a closed embedding of total spaces $j : \mathbb{V}(E^\vee) \hookrightarrow \mathbb{V}(E_0^\vee)$ as well as an epimorphism of graded algebras $H^0(Sym_A(E_1)) \simeq Sym_{H^0(\mathcal{O}_X)}(H^0(E_1)) \rightarrow \mathcal{B}$.

The derived scheme $\mathbb{V}(E^\vee)$ sits inside $\mathbb{V}(E_0^\vee)$ as the fiber at 0 of the morphism $\mathbb{V}(E_0^\vee) \rightarrow \mathbb{V}(E_1^\vee)$ induced by d . This implies that for two bounded coherent complexes \mathcal{F} and \mathcal{F}' on $\mathbb{V}(E^\vee)$, the quasi-coherent complex $\underline{Hom}(j_*(\mathcal{F}), j_*(\mathcal{F}'))$ comes equipped with a canonical structure of a module over $Sym_A(E_1^\vee[1])$. This structure is moreover such that we have

$$\underline{Hom}(\mathcal{F}, \mathcal{F}') \simeq \underline{Hom}_{Sym_A(E_1^\vee[1])}(A, \underline{Hom}(j_*(\mathcal{F}), j_*(\mathcal{F}'))).$$

In our situation, we apply this to $\mathcal{F} = e_*(\mathcal{O}_X)$ and $\mathcal{F}' = s_*(M)$, and we therefore see that because M is perfect, $j_*(\mathcal{F})$ and $j_*(\mathcal{F}')$ are again perfect over $\mathbb{V}(E_0^\vee)$. Therefore, $\underline{Hom}(j_*(\mathcal{F}), j_*(\mathcal{F}'))$ is perfect A -module, and thus it is bounded with coherent cohomologies. As an $Sym_A(E^\vee[1])$ -module, it thus lies in the thick triangulated subcategory of generated by $H^0(A)$ -module of finite type (considered as $Sym_A(E^\vee[1])$ -modules via the augmentation $Sym_A(E^\vee[1]) \rightarrow A \rightarrow H^0(A)$). By the formula above, this implies that $\underline{Hom}(\mathcal{F}, \mathcal{F}')$ is, as a module over $\underline{End}_{Sym_A(E_1^\vee[1])}(A)$, in the thick triangulated subcategory generated by $\underline{End}_{Sym_A(E_1^\vee[1])}(A) \otimes_A P$ with P an $H^0(A)$ -module of finite type. In particular, $H^*(\underline{Hom}(\mathcal{F}, \mathcal{F}'))$ is a finitely generated graded module over $H^*(\underline{End}_{Sym_A(E_1^\vee[1])}(A)) \simeq Sym_{H^0(A)}(H^0(E_1)) \otimes_{H^0(A)} H^*(A)$, and thus over $Sym_{H^0(A)}(H^0(E_1))$ because $H^*(A)$ is a finite $H^0(A)$ -module. This graded module is the image of $H^*(Kos_s(M))$ by the

canonical epimorphism $Sym_{H^0(\mathcal{O}_X)}(H^0(E_1)) \rightarrow \mathcal{B}$, and thus we conclude that $H^*(Kos_s(M))$ is also finitely generated as a graded \mathcal{B} -module. \square

We now consider the determinant line bundle $\omega_E = \det(E)$ on X and consider $Kos_s(\omega_E^{-1}) \in \mathcal{A} - \mathbf{Mod}^{qcoh}$. By the above lemma, we have a graded \mathcal{B} -module $H^*(Kos_s(\omega_E^{-1}))$. By construction, \mathcal{B} can be canonically identified with $H^0(\mathcal{O}_{\mathbb{V}(E[1])})$, the H^0 -sheaf of rings of functions on $\mathbb{V}(E^\vee[1]) \rightarrow X$, where the graduation is now induced by the canonical \mathbb{G}_m -action on $\mathbb{V}(E[1])$ given by weight 2 dilation along the fibers. In order to make this weight 2-action specific, and to distinguish it from the natural weight 1-action we will consider later, we will use the notation $\mathbb{V}^{(2)}(E[1])$.

The graded \mathcal{B} -module $H^*(Kos_s(\omega_E^{-1}))$ thus defines a class in the Grothendieck group of graded modules of finite type, which by definition is the \mathbb{G}_m -equivariant Grothendieck group of coherent sheaves on the truncation $\tau_0\mathbb{V}^{(2)}(E[1])$. As the pushforward along the inclusion $\tau_0\mathbb{V}^{(2)}(E[1]) \hookrightarrow \mathbb{V}^{(2)}(E[1])$ induces an isomorphism on Grothendieck group of coherent sheaves (by dévissage, see Appendix), we get this way a well defined class

$$[H^*(Kos_s(\omega_E^{-1}))] \in G_0^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1])).$$

Finally, we note that $H^i(Kos_s(\omega_E^{-1})) = Ext^i(s^*e_*(\mathcal{O}_X), \omega_E^{-1})$ is zero outside of $Z = Z(s) \subset X$ the zero locus of s , and thus the class $[H^*(Kos_s(\omega_E^{-1}))]$ above in fact lies in the G -theory with support on Z , or, again by dévissage, in $G_0^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1])_Z)$, where $\mathbb{V}^{(2)}(E[1])_Z = \mathbb{V}^{(2)}(E[1]) \times_X Z$ is the restriction of $\mathbb{V}^{(2)}(E[1])$ to Z .

Definition 2.1.2. *The class $[H^*(Kos_s(\omega_E^{-1}))] \in G_0^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1])_Z)$ defined above is called the refined Euler class of the section s . It is denoted by*

$$\widetilde{Eu}_E(s) := [H^*(Kos_s(\omega_E^{-1}))] \in G_0^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1])_Z).$$

We conclude this section with a couple of comments.

The bundle case. When E is a vector bundle, then the graded ring \mathcal{B} is simply \mathcal{O}_X , and $H^*(Kos_s(\omega_E^{-1}))$ is a graded $H^0(\mathcal{O}_X)$ -module of finite type. In fact, $Kos_s(\omega_E^{-1})$ is a perfect complex on X , which by Grothendieck duality is $s^*e_*(\mathcal{O}_X)[-r]$, where r is the rank of E (note that $\omega_E[r]$ is the dualizing complex of the closed embedding $s : X \rightarrow \mathbb{V}(E^\vee)$). Therefore, the class $[\widetilde{Eu}(s)] \in G_0^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1])_Z) \simeq G_0^{\mathbb{G}_m}(Z) \simeq$

$G_0(Z)[t, t^{-1}]$ is nothing else than

$$\widetilde{Eu}_E(s) = (-1)^r \sum_{i \in \mathbb{Z}} [\mathrm{Tor}_i^{\mathcal{O}_X}(e_*(\mathcal{O}_X) \otimes_{\mathcal{O}_X} s_*(\mathcal{O}_X))] t^i.$$

The value at $t = -1$ of this Laurent polynomial is, up to a sign, thus the usual Euler class of s

$$\widetilde{Eu}_E(s)(t = -1) = (-1)^r [s^* e_*(\mathcal{O}_X)] \in G_0(Z)$$

where $Z = Z(s)$ is the zero locus of s . Using base change for the cartesian diagram of derived stacks

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow s \\ X & \xrightarrow{e} & \mathbb{V}(E^\vee), \end{array}$$

we see that $\widetilde{Eu}(s)(t = -1)$ is the usual (virtual) fundamental class $[Z]^{vir} := [\mathcal{O}_Z] \in G_0(Z)$ of the derived scheme Z .

Interpretation in terms of singular supports. The class $\widetilde{Eu}_E(s)$ of Definition 2.1.2 is inspired, and very closely related to, the singular support of coherent sheaves of [AG15]. Indeed, we can apply [AG15, Thm. 4.18] to the bounded coherent complexes $e_*(\mathcal{O}_X)$ and $s_*(\omega_E^{-1})$ on the quasi-smooth derived stack $\mathbb{V}^{(2)}(E^\vee)$, and get a graded coherent sheaf on the singular schemes of $\mathbb{V}^{(2)}(E^\vee)$. In our setting, we consider the relative version over X , as both $e_*(\mathcal{O}_X)$ and $s_*(\omega_E^{-1})$ are relatively perfect over X , and thus this support lies on the relative singular scheme of $\mathbb{V}(E^\vee) \rightarrow X$. This relative singular scheme is nothing else than $\mathbb{V}^{(2)}(E[1])$, and our class $\widetilde{Eu}_E(s)$ is thus an incarnation of the relative singular support of $\mathrm{Hom}(e_*(\mathcal{O}_X), s_*(\omega_E^{-1}))$, or rather its relative characteristic cycle (i.e. the singular support but counted with its natural multiplicities).

2.2. The virtual Euler series and Euler numbers. In the previous paragraph we have constructed the refined Euler class $\widetilde{Eu}(s)$. We now explain how this class can be somehow pushed down to $Z \subset X$ itself, under the form of a rational function with coefficients in Borel-Moore homology of Z . When Z is proper (e.g. when X is so), this can be used in order to define Euler numbers which morally count the number of zeros of the section s .

We start by consider the projection $p : \mathbb{V}^{(2)}(E[1])_Z \rightarrow Z$, pull-back from the projection $\mathbb{V}^{(2)}(E[1]) \rightarrow X$ along the closed embedding $Z = Z(s) \hookrightarrow X$. The projection p is an affine morphism equipped with a natural \mathbb{G}_m -action of dilations along the fiber (with weight 2 in our setting). Algebraically, this corresponds to the graded simplicial commutative \mathcal{O}_Z -algebra $Sym_{\mathcal{O}_Z}(E[1])$, where E is given weight 2. We thus get a direct image on quasi-coherent complexes

$$p_* : \mathrm{QCoh}([\mathbb{V}^{(2)}(E[1])_Z/\mathbb{G}_m]) \rightarrow \mathrm{QCoh}(Z \times B\mathbb{G}_m).$$

As usual, we will identify the ∞ -category $\mathrm{QCoh}(Z \times B\mathbb{G}_m)$ with the ∞ -category of graded quasi-coherent complexes on Z . Using that E is perfect, it is easy to see that this push-forward sends coherent complexes to graded quasi-coherent complexes M on Z satisfying the following two conditions.

- (1) For each n , the graded piece $M(n)$ is bounded coherent on Z .
- (2) For $n \ll 0$ we have $M(n) = 0$.

We let $\widehat{\mathrm{Coh}}(Z \times B\mathbb{G}_m) \subset \mathrm{QCoh}(Z \times B\mathbb{G}_m)$ be this full sub- ∞ -category. Note that it contains $\mathrm{Coh}(Z \times B\mathbb{G}_m)$, the graded coherent complexes on Z , but is strictly bigger as it also contains objects with infinitely many non-zero positive weight pieces. Symbolically, objects in $\widehat{\mathrm{Coh}}(Z \times B\mathbb{G}_m)$ will be denoted by $\sum_{i \in \mathbb{Z}} M(i) \cdot t^i$, and thus will be considered as Laurent formal power series with coefficients in $\mathrm{Coh}(Z)$. Another natural notation for $\widehat{\mathrm{Coh}}(Z \times B\mathbb{G}_m)$ is thus $\mathrm{Coh}(Z)[[t]][t^{-1}]$.

The pushforward ∞ -functor

$$p_* : \mathrm{Coh}([\mathbb{V}^{(2)}(E[1])_Z/\mathbb{G}_m]) \rightarrow \mathrm{Coh}(Z)[[t]][t^{-1}]$$

then induces a morphism on the corresponding Grothendieck groups

$$p_* : G_0^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1])_Z) \longrightarrow G_0(Z)[[t]][t^{-1}].$$

Finally, we can compose with the Grothendieck-Riemann-Roch transformation (see Appendix)

$$\tau_Z : G_0(Z)[[t]][t^{-1}] \longrightarrow H_{2*}^{BM}(Z, \mathbb{Q}_\ell(-*))[[t]][t^{-1}],$$

where the right hand side is formal power series with coefficients in Borel-Moore homology. The composed morphism will be denoted by

$$\psi_Z : G_0^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1])_Z) \longrightarrow H_{2*}^{BM}(Z, \mathbb{Q}_\ell(-*))[[t]][t^{-1}].$$

The following result is a version of the existence of Hilbert polynomials and is well known.

Propositon 2.2.1. *The morphism*

$$\psi_Z : G_0^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1])_Z) \longrightarrow H_{2*}^{BM}(Z, \mathbb{Q}_\ell(-*))[[t]][t^{-1}]$$

factors through the subgroup consisting of formal Laurent series $P(t) \in H_{2*}^{BM}(Z, \mathbb{Q}_\ell(-*))[[t]][t^{-1}]$ such that there exist integers n and m with $(1-t^2)^n \cdot t^m \cdot P(t) \in H_{2*}^{BM}(Z, \mathbb{Q}_\ell(-*))[[t]]$. In other words $P(t)$ is always a rational function, possibly with poles at $t = -1, 0, 1$.

Proof. We start by assuming that X has the resolution property: any coherent sheaf of $H^0(\mathcal{O}_X)$ -modules \mathcal{F} is a quotient of a vector bundle $V \rightarrow \mathcal{F}$ on X . Under this assumption, the perfect complex E can be presented as the fiber of a morphism of vector bundles $d : E_0 \rightarrow E_1$. In this case, we have a \mathbb{G}_m -equivariant closed embedding

$$j : \mathbb{V}^{(2)}(E[1])_Z \hookrightarrow \mathbb{V}^{(2)}(E_1)_Z,$$

setting over Z . The morphism p_* can then be written as $q_* j_*$ where $q : \mathbb{V}^{(2)}(E_1)_Z \rightarrow Z$ is the natural projection, and as j_* preserves bounded coherent complexes, the proposition reduces to the case where $E = E_1[-1]$ is a shift of a vector bundle. By homotopy invariance, the pull-back now induces an isomorphism

$$p^* : G_0^{\mathbb{G}_m}(Z) \simeq G_0^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E_1)_Z).$$

Moreover, by the projection formula (see Appendix) we have $p_* p^*(x) = p_*(1) \cdot x$, where we consider $G_0(Z)[[t]][t^{-1}]$ as a module over the ring $K_0(Z)[[t]][t^{-1}]$ and 1 is the class of the structure sheaf in $K_0^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E_1)_Z)$. Therefore, any element $p_*(y) \in G_0(Z)[[t]][t^{-1}]$ is of the form $p_*(1) \cdot x$ with $x \in G_0(Z)[t, t^{-1}]$. As the GRR transformation τ is a morphism of modules over the Chern character on K -groups, we see that it is enough to prove there is an integer n such that $(1-t^2)^n \cdot Ch(p_*(1))$ lies in $H^{2*}(Z, \mathbb{Q}_\ell(*))[[t]]$, where $Ch : K_0(Z) \rightarrow H^{2*}(Z, \mathbb{Q}_\ell(*))$ is the Chern character map to ℓ -adic cohomology.

The element $p_*(1)$ is explicitly given by the formal power series

$$S_{t^2}([E_1^\vee]) = \sum_{i \geq 0} [Sym_{\mathcal{O}_Z}^i(E_1^\vee)] \cdot t^i$$

To prove that $(1-t^2)^n \cdot Ch(p_*(1))$ is a polynomial for n large enough, we can use the splitting principle to reduce to the case where $E_1 = \bigoplus_\alpha L_\alpha$ is a direct sum of line bundles. In this case, we have $S_{t^2}([E_1^\vee]) = \prod_\alpha S_{t^2}(L_\alpha)$, and we are reduced to the case where $E_1 = L$ is a line bundle. Let $c = Ch(1 - [L]) \in H^{2*}(Z, \mathbb{Q}_\ell(*))$. The element c is nilpotent, say $c^d = 0$. We thus have

$$(1-t^2)^{d+1} \cdot Ch(p_*(1)) = (1-t^2)^{d+1} \cdot \left(\sum_{i \geq 0} (1-c)^i \cdot t^{2i} \right) = \sum_i (-1)^i c^i \cdot (1-t^2)^{d-i} \cdot t^{2i}$$

which is indeed a polynomial.

Finally, for the general case, we can use the Chow envelope techniques of [Toë99] in order to produce a proper morphism $\pi : X' \rightarrow X$ such that, on one hand $\pi_* : G_0(X' \times_X Z) \rightarrow G_0(Z)$ is surjective, and moreover X' is smooth and possesses the resolution property (in fact X' can be taken to be a quotient stack of a smooth quasi-projective variety by a finite group action). \square

Remark 2.2.2. A slightly weaker conclusion of Proposition 2.2.1 almost holds in $G_0(Z)[[t]][t^{-1}]$, before applying the GRR transformation to Borel-Moore homology. When X is a derived algebraic space, it holds as is. However, when X is Deligne-Mumford the statement can not be true as stacky phenomena imply that the formal power series can also have poles at roots of unity.

One consequence of proposition 2.2.1 is that it is possible to evaluate the series $\widetilde{Eu}_E(s)$ at $t = -1$, at least after removing the possible poles. For this, we remind the notion of critical value of a rational function $P \in K(t)$ with coefficients in some field K . By definition, the critical value of P at $a \in K$ is the first non-zero Taylor coefficients in a Laurent series expansion of P at a . It is denoted by $P^*(a)$. In formula, it is given by

$$P^*(a) := ((t - a)^{-n} P(t))(a),$$

where n is the valuation of P at a (by convention $P^*(a) = 0$ if $P = 0$). If P is a polynomial, and $P(a) \neq 0$, then $P^*(a) = P(a)$, but when a is a root of P the critical value $P^*(a)$ is always non-zero (except when $P = 0$). One nice feature of critical values is that they define a morphism of multiplicative monoids $K(t) \rightarrow K$, i.e. $(PQ)^*(a) = P^*(a) \cdot Q^*(a)$. However, the critical value map is of course not additive in general.

With these notions and definitions, we define Euler series, global Euler series and reduced Euler numbers as follows.

Definition 2.2.3. (1) *The Euler series of s is defined to be $\widetilde{Eu}_E(s; t) = (-1)^d \psi(Td(E) \cdot Td(X)^{-1} \cap \tau_X(\widetilde{Eu}_E(s))) \in H_{2*}^{BM}(Z, \mathbb{Q}_\ell(-*))[[t]][t^{-1}]$, where $Td(E) \in H^{2*}(Z, \mathbb{Q}_\ell(*))$ is the Todd class of the perfect complex E , $Td(X)^{-1}$ is the inverse Todd class of \mathbb{T}_X and d is the rank of E .*

(2) *When $Z = Z(s) \subset X$ is proper over k , the global Euler series of s is defined to be*

$$Eu_E(s; t) := f_*(\widetilde{Eu}_E(s; t)) \in \mathbb{Q}_\ell[[t]][t^{-1}],$$

where $f_* : H_{2*}^{BM}(Z, \mathbb{Q}_\ell(-*)) \rightarrow \mathbb{Q}_\ell$ is the pushforward in Borel-Moore homology along the structure map $f : Z \rightarrow \mathbf{Spec} k$.

- (3) When Z is proper over k , the reduced Euler number of s is defined to be

$$Eu_E^r(s) := Eu_E(s; t)^*(-1) \in \mathbb{Q}_\ell.$$

Remark 2.2.4. (1) It is important to note that in point (3) above, the number $Eu_E(s; t)^*(-1)$ is well defined as Proposition 2.2.1 implies that $Eu_E(s; t) \in \mathbb{Q}_\ell(t)$ and thus its critical value at -1 is defined.

- (2) By construction $Eu_E(s; t)$ has coefficients in \mathbb{Q}_ℓ but it is easy to see that it has rational coefficients. Moreover, when X is a derived algebraic space instead of a derived DM stack, $Eu_E(s; t)$ has in fact integral coefficients. However, in general, denominators appear in the push-forward $f_* : H_{2*}^{BM}(Z, \mathbb{Q}_\ell(-*)) \rightarrow \mathbb{Q}_\ell$, as f is not a representable morphism.
- (3) The Todd classes in Definition 2.2.3 (1) are only there in order to have a comparison with the usual Euler class of vector bundles via the Grothendieck-Riemann-Roch theorem (see below), but are not truly essential.
- (4) Finally, we should also mention that it is also possible to define a non-reduced Euler number by considering the value of $(1 - t^2)^n \cdot Eu_E(s; t)$ at $t = -1$, where n is the least integer such that $(1 - t^2)^n \cdot Eu_E(s; t) \in \mathbb{Q}_\ell[t, t^{-1}]$.

The bundle case. Let us assume that E is a vector bundle on X , and that $s = 0$ to make things simpler. Then, $G_0^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1])) \simeq G_0(X)[t, t^{-1}]$ as the projection $\mathbb{V}^{(2)}(E[1]) \rightarrow X$ induces an isomorphism on the underlying truncated DM-stacks. The refined Euler class is explicitly given by the series

$$\widetilde{Eu}_E(0; t) = \sum_{i \in \mathbb{Z}} [Ext_{Sym_{\mathcal{O}_X}(E^\vee)}^i(\mathcal{O}_X, \omega_E^{-1})] \cdot t^i$$

This also be written a product of two Laurent polynomials

$$\widetilde{Eu}_E(0; t) = \left(\sum_i [H^i(\mathcal{O}_X)] \cdot t^i \right) \left(\sum_j [\wedge_{H^0(\mathcal{O}_X)}^{d-j}(H^0(E^\vee))] \cdot t^j \right),$$

where d is the rank of E . Thereforore, setting $t = -1$ we get

$$Eu_E(0; -1) := (-1)^d \cdot \lambda_{-1}(E^\vee) \cap [X]_G^{vir},$$

where $\lambda_{-1}(E^\vee) = \sum_i (-1)^i [\wedge^i E^\vee] \in K_0(X)$ and $[X]_G^{vir} = \sum_i [H^i(\mathcal{O}_X)] \in G_0(X)$ is the virtual class in G -theory of X . As the virtual class in

homology is given by the so-called Kontsevich's formula (see [Kha22, Thm. 6.12])

$$[X]^{vir} = Td(X)^{-1} \cap \tau_X([X]_G^{vir}),$$

and as we have

$$Ch(\lambda_{-1}(E^\vee)) = C_{top}(E).Td(E)^{-1},$$

we see that our Definition 2.2.3 gives that the reduced Euler number $Eu_E(0)$ equals $f_*(C_{top}(E) \cap [X]^{vir})$ for $f : X \rightarrow \mathbf{Spec} k$ the structure morphism, if this number is non-zero. In other words, the reduced Euler number coincide with the usual Euler number of E when it does not vanish. When the Euler number $f_*(C_{top}(E) \cap [X]^{vir})$ is zero, this means that $Eu_E(0; t)$ vanishes at $t = -1$, and the reduced Euler number is then equal to the first non-zero derivative of the function $Eu_E(s; t)$ evaluated at $t = -1$. Note that, however, the non-reduced Euler number always coincides with $f_*(C_{top}(E) \cap [X]^{vir})$.

2.3. Categories of shifted singularities. In this section we introduce a categorification of what we have seen so far. Namely we introduce certain categories of singularities for shifted potentials, that is for functions of cohomological degree -1 . In the same manner that categories of singularities for a usual potential f reflects the singularities of f , these categories of shifted singularities reflects singularities of shifted functions in the sense that they see the locus where the shifted function is not quasi-smooth.

Let Y be a quasi-smooth derived Deligne-Mumford stack and $f : Y \rightarrow \mathbb{A}^1[-1] := \mathbf{Spec} k[\epsilon_{-1}]$ be a function of degree -1 (here ϵ_{-1} sits in cohomological degree -1). The function f can also be considered as a morphism of \mathcal{O}_Y -algebras

$$\mathcal{O}_Y[\epsilon_{-1}] \rightarrow \mathcal{O}_Y,$$

that sends ϵ_{-1} to f . Therefore, any perfect complex M on Y is also a coherent $\mathcal{O}_Y[\epsilon_{-1}]$ -module. We will say that $M \in \mathbf{Perf}(Y)$ is f -perfect if the complex of sheaves

$$\mathbb{R}\mathcal{H}om_{\mathcal{O}_Y[\epsilon_{-1}]}(\mathcal{O}_Y, E)$$

is locally cohomologically bounded on Y . We denote by $\mathbf{Perf}(Y, f) \subset \mathbf{Perf}(Y)$ the full sub- ∞ -category of f -perfect complexes.

We then set

$$\mathbf{Sing}(Y, f) := \mathbf{Perf}(Y)/\mathbf{Perf}(Y, f),$$

where the quotient is taken in k -linear dg-categories up to Morita equivalences. In the same manner, when Y comes equipped with a

\mathbb{G}_m -action, and $f : Y \rightarrow \mathbb{A}^1[-1]$ is \mathbb{G}_m -equivariant, for the standard \mathbb{G}_m -action on $\mathbb{A}^1[-1]$ of weight 1, we have an equivariant version

$$\mathbf{Sing}^{\mathbb{G}_m}(Y, f) := \mathbf{Perf}^{\mathbb{G}_m}(Y) / \mathbf{Perf}^{\mathbb{G}_m}(Y, f).$$

We note that $\mathbf{Sing}(Y, f)$ is naturally enriched over $\mathbf{Perf}(Y)$, and thus can be considered as a quasi-coherent stack of dg-categories over Y (see [Toë12] for the general notion of quasi-coherent stacks of dg-categories). As such, it is important to notice the following fact.

Proposition 2.3.1. *As a quasi-coherent stack of dg-categories on Y , $\mathbf{Sing}(Y, f)$ (resp. $\mathbf{Sing}^{\mathbb{G}_m}(Y, f)$ in the equivariant case) is supported on the (closed) locus where $f : Y \rightarrow \mathbb{A}^1[-1]$ is not quasi-smooth.*

Proof. We have to show that if $f : Y \rightarrow \mathbb{A}^1[-1]$ is quasi-smooth then $\mathbf{Sing}(Y, f) \simeq 0$, or equivalently that all perfect complexes on Y are f -perfect. As f is quasi-smooth \mathcal{O}_Y , locally on Y , admits a finite resolution by free $k[\epsilon]_- - 1$ -dg-modules. This implies that for any perfect complex E on Y , locally on Y , E admits a finite resolution by free $k[\epsilon_-]$ -dg-modules. As $\mathbb{R}\underline{Hom}_{k[\epsilon_-]}(k, k[\epsilon]) \simeq k$, the complex of sheaves $\mathbb{R}\underline{Hom}_{\mathcal{O}_Y[\epsilon_-]}(\mathcal{O}_Y, E)$ is indeed locally cohomologically bounded. This is precisely the statement that $\mathbf{Perf}(Y, f) = \mathbf{Perf}(Y)$. \square

In our specific situation we apply this to $Y = \mathbb{V}(E[1])$ and $f_s : \mathbb{V}(E[1]) \rightarrow \mathbb{A}^1[-1]$ is the function induced from the section s . Indeed, $s : \mathcal{O}_X \rightarrow E$ induces a morphism $s[1] : \mathcal{O}_X[1] \rightarrow E[1]$, and thus a morphism on the corresponding linear stacks

$$\mathbb{V}(E[1]) \rightarrow \mathbb{V}(\mathcal{O}_X[1]) \simeq X \times \mathbb{A}^1[-1],$$

whose second projection is, by definition, our function f_s . This function is moreover compatible with the \mathbb{G}_m -action by weight 1. We thus have a dg-category

$$\mathbf{Sing}^{\mathbb{G}_m}(\mathbb{V}(E[1]), f_s) = \mathbf{Perf}^{\mathbb{G}_m}(\mathbb{V}(E[1])) / \mathbf{Perf}^{\mathbb{G}_m}(\mathbb{V}(E[1]), f_s).$$

We want to see the dg-category $\mathbf{Sing}^{\mathbb{G}_m}(\mathbb{V}(E[1]), f_s)$, together with its natural object $\omega_E^{-1} \in \mathbf{Sing}^{\mathbb{G}_m}(\mathbb{V}(E[1]), f_s)$, as some sort of categorification of the pair $(G_0^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1]))_Z, \widetilde{Eu}_E(s))$ where $\widetilde{Eu}_E(s)$ is our refined Euler class. Indeed, we will exhibit below a natural ∞ -functor

$$| - |^t : \mathbf{Sing}^{\mathbb{G}_m}(\mathbb{V}(E[1]), f_s) \longrightarrow \mathbf{Coh}_{\heartsuit}^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1]))_Z,$$

where $\mathbf{Coh}_{\heartsuit}^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1]))_Z$ is the abelian category of \mathbb{G}_m -equivariant sheaves on $\mathbb{V}^{(2)}(E[1])$ set theoretically supported on Z , such that $[|\omega_E^{-1}|^t] =$

$\widetilde{E}u_E(s)$ in $G_0^{\mathbb{G}^m}(\mathbb{V}^{(2)}(E[1])_Z)$.

Construction of the functor $|-|$. In order to define the functor $|-|$, first we notice that we have to distinguish two objects in $\mathcal{O}_{\mathbb{V}(E[1])}[\epsilon_{-1}]$ -modules. We have $\mathcal{O}_{\mathbb{V}(E[1])}$, endowed with its $\mathcal{O}_{\mathbb{V}(E[1])}[\epsilon_{-1}]$ -module structure coming from the augmentation $\epsilon_{-1} \rightarrow 0$. We also have $\mathcal{O}_{\mathbb{V}(E[1])}$ with its natural \mathcal{O}_X -algebra morphism $\mathcal{O}_{\mathbb{V}(E[1])}[\epsilon_{-1}] \rightarrow \mathcal{O}_{\mathbb{V}(E[1])}$ sending ϵ_{-1} to f_s . These are two different $\mathcal{O}_{\mathbb{V}(E[1])}[\epsilon_{-1}]$ -module structures on the same object $\mathcal{O}_{\mathbb{V}(E[1])}$ -module $\mathcal{O}_{\mathbb{V}(E[1])}$ itself. More generally, for any $M \in \mathbf{Perf}^{\mathbb{G}^m}(\mathbb{V}(E[1]))$ we have two different $\mathcal{O}_{\mathbb{V}(E[1])}[\epsilon_{-1}]$ -module structures on M , denoted by $M\{0\}$ and $M\{f_s\}$. For $M\{0\}$, ϵ_{-1} acts by 0, whereas for $M\{f_s\}$ the element ϵ_{-1} acts by multiplication by f_s . For the sake of simplicity, $M\{0\}$ will simply be denoted by M , but we will keep the notation $M\{f_s\}$ for the second one.

We start by constructing a dg-functor

$$|-| : \mathbf{Perf}^{\mathbb{G}^m}(\mathbb{V}(E[1])) \longrightarrow \mathbf{Perf}^{gr}(Sym_{\mathcal{O}_X}(E[1])[u]),$$

where u is a variable in cohomological degree 2 and weight -1 (note here that E has weight 1), and $\mathbf{Perf}^{gr}(Sym_{\mathcal{O}_X}(E[1])[u])$ are sheaves of perfect graded $Sym_{\mathcal{O}_X}(E[1])[u]$ -dg-modules. This dg-functor is essentially given by negative cyclic homology, and the weight n piece of $|M|$ is by definition

$$|M|(n) := \underline{Hom}_{\mathcal{O}_{\mathbb{V}(E[1])}[\epsilon_{-1}]}(\mathcal{O}_{\mathbb{V}(E[1])}(n), M\{f_s\}),$$

where $\mathcal{O}_{\mathbb{V}(E[1])}(n)$ is $\mathcal{O}_{\mathbb{V}(E[1])}$ with weight shifted by n . The action of u is given by the usual identification

$$\oplus_n |\mathcal{O}_{\mathbb{V}(E[1])}(n)| \simeq \mathcal{O}_{\mathbb{V}(E[1])}[u].$$

The full sub-dg-category $\mathbf{Perf}^{\mathbb{G}^m}(\mathbb{V}(E[1]), f_s)$ is sent by $|-|$ to u -torsion objects, and thus we get a well defined dg-functor after inverting u

$$|-| : \mathbf{Sing}^{\mathbb{G}^m}(\mathbb{V}(E[1]), f_s) \longrightarrow \mathbf{Perf}^{gr}(Sym_{\mathcal{O}_X}(E[1])[u, u^{-1}]).$$

Now, we recall that there exists a natural tensor autoequivalence of graded complexes, called *red shift*, which sends a complex E pure of weight n to $E[2n]$ with the same weight n . This autoequivalence, implies that $\mathbf{Perf}^{gr}(Sym_{\mathcal{O}_X}(E[1])[u, u^{-1}])$ is equivalent to $\mathbf{Perf}^{gr}(Sym_{\mathcal{O}_X}(E[-1])[t, t^{-1}])$ where now t sits in cohomological degree 0 and weight -1 . This last dg-category is naturally equivalent to $\mathbf{Perf}(Sym_{\mathcal{O}_X}(E[-1]))$, non-graded dg-modules over $Sym_{\mathcal{O}_X}(E[-1])$. Clearly, the composed dg-functor

$$|-| : \mathbf{Sing}^{\mathbb{G}^m}(\mathbb{V}(E[1]), f_s) \longrightarrow \mathbf{Perf}(Sym_{\mathcal{O}_X}(E[-1]))$$

takes its values in the full sub-dg-category $\mathrm{Perf}(Sym_{\mathcal{O}_X}(E[-1]))_Z$ of objects supported on $Z = Z(s) \subset X$. We can then compose with the total cohomology functor

$$H^* : \mathrm{Perf}(Sym_{\mathcal{O}_X}(E[-1]))_Z \longrightarrow \mathrm{Coh}_{\heartsuit}^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1]))_Z,$$

that sends M to $H^*(M)$ considered as graded $Sym_{\mathcal{O}_X}(H^1(\mathbb{T}_X))$ -module. We get this way the desired ∞ -functor

$$|-|^t : \mathrm{Sing}^{\mathbb{G}_m}(\mathbb{V}(E[1]), f_s) \longrightarrow \mathrm{Coh}_{\heartsuit}^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1]))_Z.$$

Note that $|-|^t$ is not a stable functor nor a dg-functor, and thus does not preserve exact triangles. In particular, this ∞ -functor does not induce a well defined map on K -groups. We consider the determinant line bundle $\omega_E^{-1} = \det(E)^{-1}$ on X , and its inverse image on $\mathbb{V}(E[1])$ which sits in $\mathrm{Perf}^{\mathbb{G}_m}(\mathbb{V}(E[1]))$. The image of this object by the quotient map $\mathrm{Perf}^{\mathbb{G}_m}(\mathbb{V}(E[1])) \rightarrow \mathrm{Sing}^{\mathbb{G}_m}(\mathbb{V}(E[1]))$ will still be denoted by ω_E^{-1} .

Propositon 2.3.2. *With the above notations, we have*

$$[|\omega_E^{-1}|^t] = \widetilde{Eu}_E(s)$$

in $G_0^{\mathbb{G}_m}(\mathbb{V}^{(2)}(E[1]))_Z$.

Proof. This follows from the explicit description of the dg-functor

$$|-| : \mathrm{Perf}^{\mathbb{G}_m}(\mathbb{V}(E[1])) \longrightarrow \mathrm{Perf}^{gr}(Sym_{\mathcal{O}_X}(E[1])[u]).$$

An object $M \in \mathrm{Perf}^{\mathbb{G}_m}(\mathbb{V}(E[1]))$ is given by a graded $Sym_{\mathcal{O}_X}(E[1])$ -module. The dg-functor $|-|$ sends such an M to the graded $Sym_{\mathcal{O}_X}(E[1])[u]$ whose weight p pieces are given by

$$|M|(p) = \prod_{i \geq p} M(i)[2p - 2i],$$

where the differential is given by the total differential, sum of the cohomological differential and multiplication by the degree -1 function f_s . The action of u is itself given by the canonical embedding of sub-products

$$|M|(p) = \prod_{i \geq p} M(i)[2p - 2i] \hookrightarrow \prod_{i \geq p-1} M(i)[2p - 2i] = |M|(p-1)[2].$$

In particular, $|M|^t$ is simply given by the Tate realization (see [CPT⁺17]) of the graded mixed object $M\{f_s\}$

$$|M|^t := \mathrm{colim}_p H^*\left(\prod_{i \geq p} M(i)[-2i]\right),$$

considered now as a graded- $H^*(Sym_{\mathcal{O}_X}(E[-1]))$ -module. When applied to $M = \omega_E^{-1}$, we get that $|\omega_E^{-1}|^t$ is represented by $Sym_{\mathcal{O}_X}(E[-1]) \otimes_{\mathcal{O}_X} \omega_E^{-1}$ where the differential on $Sym_{\mathcal{O}_X}(E[-1])$ is the sum of the cohomological differential and the multiplication by f_s . This clearly is $\underline{Hom}_{\mathcal{O}_{\mathbb{V}(E^\vee)}}(e_*\mathcal{O}_X, s_*\omega_E^{-1})$ as a module over

$$Sym_{\mathcal{O}_X}(E[-1]) \simeq \underline{End}_{\mathcal{O}_{\mathbb{V}(E^\vee)}}(e_*\mathcal{O}_X),$$

hence the statement. \square

Remark 2.3.3. (1) The above Proposition 2.3.2 should be related to the fact that the usual virtual structure sheaf of $Z(s)$, when E is a bundle, can be recovered from the theory of vanishing cycles of the function $\mathbb{V}(E^\vee) \rightarrow \mathbb{A}^1$ induced by s (see for instance [Dav17, Appendix A]). Here, the role of vanishing cycles has been replaced by the category of shifted singularities, which is relevant by the well known relations between vanishing cycles and categories of singularities ([BRTV18]). However, we do not try here to establish a precise relation between $\text{Sing}^{\mathbb{G}^m}(\mathbb{V}(E[1]), f_s)$ and the vanishing cycles of $\mathbb{V}(E^\vee) \rightarrow \mathbb{A}^1$, even though we think that some precise statement could be written down.

- (2) Proposition 2.3.2 is not only another way to understand the class $\widetilde{Eu}_E(s)$, it also provides a nice representative of this class as a nice object $\mathcal{O}_{\mathbb{V}(E[1])} \in \text{Sing}^{\mathbb{G}^m}(\mathbb{V}(E[1]), f_s)$, the image of the structure sheaf of $\mathbb{V}(E[1])$ by the canonical projection. One advantage of this object is that it has nice functorial properties in terms of X , E and s . Such properties are not shared by the corresponding sheaf of graded $Sym_{\mathcal{O}_X}(H^1(\mathbb{T}_X))$ -modules as this arises as the total cohomology and thus will not be stable under various base changes.
- (3) Finally, our $\text{Sing}(Y, f)$ is also closely related to the localisation of the virtual class along a cosection of [KL13]. Indeed, the non-equivariant version of our ∞ -functor $|-|^t$ is an ∞ -functor

$$|-|^t : \text{Sing}(Y, f) \longrightarrow \text{Coh}(Y)_Z$$

where Z is the zero locus of f in Y . This ∞ -functor $|-|^t$ simply sends M to its periodic cyclic homology $\underline{Hom}_{\mathcal{O}_Y[\epsilon_{-1}]}(\mathcal{O}_Y, Mf)[u^{-1}]$, which is a 2-periodic complex with coherent cohomologies on Y , supported on Z . We thus have a canonical element

$$[|M|^t] = [H^0(\underline{Hom}_{\mathcal{O}_Y[\epsilon_{-1}]}(\mathcal{O}_Y, Mf)[u^{-1}]) - [H^1(\underline{Hom}_{\mathcal{O}_Y[\epsilon_{-1}]}(\mathcal{O}_Y, Mf)[u^{-1}])]]$$

inside $G_0(Z)$. Applied to $M = \mathcal{O}_Y$ the resulting class $[[\mathcal{O}_Y|t]$ seems very closely related to the localized virtual class of [KL13] (though we do not investigate a precise formula in this work).

3. APPLICATIONS

In this final section, we will apply the results and constructions of the previous section to the specific case of the derived stack of stable maps endowed with an induced vector field as in Definition 1.2.3.

3.1. Categories associated to a vector field. Let X be a smooth and proper algebraic variety, and $\nu \in H^0(X, \mathbb{T}_X)$ a global vector field on X . We will construct three invariants of ν , based on the induced vector field $\nu_{0,2}$ on $\mathbb{R}\bar{\mathcal{M}}_{0,2}(X)$. These are of three different levels, 2-categorical, categorical and numerical/homological, and are somehow related by some form of decategorification. They are all incarnations of a possible algebraic version of Morse homology, or more precisely the specific point of view based on *moduli broken lines* exposed in [LT18].

We first set $\mathcal{M} := \coprod_{\beta} \mathbb{R}\bar{\mathcal{M}}_{0,2}(X, \beta)$, where β runs over all curve classes in $H_2(\bar{X}, \mathbb{Q}_{\ell}(-1))$. The additive monoid of curve classes will be denoted by $N \subset H_2(\bar{X}, \mathbb{Q}_{\ell}(-1))$. As a side comment, \mathcal{M} is not quasi-compact, and when considering non-quasi-compact derived stacks we will only consider coherent complexes supported on quasi-compact substacks. In particular, our notation for $\mathrm{Coh}(\mathcal{M})$ will mean $\bigoplus_{\beta} \mathrm{Coh}(\mathcal{M})$ rather than the infinite product $\prod_{\beta} \mathrm{Coh}(\mathcal{M})$. The same convention will hold each time we consider Coh of some derived stack which is an infinite disjoint union of quasi-compact derived stacks.

The two marked points of prestable curves will be denoted by 0 and ∞ , and we think of a stable map $f : C \rightarrow X$ as a path from $f(0)$ to $f(\infty)$. The gluing construction provides a composition morphism

$$\mu : \mathcal{M} \times_X \mathcal{M} \longrightarrow \mathcal{M},$$

where the two maps $\mathcal{M} \rightarrow X$ defining the fibered product on the left hand side are the two evaluation morphisms. This composition makes \mathcal{M} into an associative monoid (without units) in the ∞ -category $\mathbf{dSt}/(X \times X)$ of derived stacks over $X \times X$ endowed with the monoidal structure given by convolution. Another possible terminology here could be that \mathcal{M} is a categorical object in derived stacks whose object of objects is X . As such, \mathcal{M} is very similar to the path groupoid of a topological space.

The vector field ν induces a vector field $\nu_{0,2}$ on \mathcal{M} as explained in Proposition 1.2.1. Moreover, ν and $\nu_{0,2}$ are compatible with the

evaluation $\mathcal{M} \rightarrow X \times X$ as well as with the composition morphism μ in an obvious sense. As a result, the derived scheme of zeros $\nu_{0,2}$, denoted by $Z \subset \mathcal{M}$, comes equipped with an induced composition map

$$\mu : Z \times_{Z_0} Z \longrightarrow Z$$

making it again into an associative monoid (without units) in $\mathbf{dSt}/(Z_0 \times Z_0)$, where $Z_0 \subset X$ is the derived subscheme of zeros of ν .

The 2-category of algebraic trajectories. Let $ev : Z \rightarrow Z_0 \times Z_0$ be the evaluation morphism. We consider $\mathbf{Coh}(Z/ev) \subset \mathbf{Coh}(Z)$, the full sub-dg-category of coherent complexes which are of finite Tor-dimension over $Z_0 \times Z_0$ (i.e. that are relatively perfect with respect to ev). This is a $\mathbf{Perf}(Z_0 \times Z_0)$ linear dg-category, and will be considered as a quasi-coherent sheaf of dg-categories over $Z_0 \times Z_0$ (see [Toë12]). We note that the monoid structure on Z induces a monoid structure on $\mathbf{Coh}(Z/ev)$, as an object in quasi-coherent sheaves of dg-categories over $Z_0 \times Z_0$ endowed with the convolution monoidal structure. In concrete terms, this monoid structure is given by

$$\mathbf{Coh}(Z/ev) \otimes_k \mathbf{Coh}(Z/ev) \longrightarrow \mathbf{Coh}(Z/ev)$$

sending an object (E, E') to $\mu_*(d^*(E \boxtimes E'))$, where $d : Z \times_{Z_0} Z \rightarrow Z \times Z$ is the natural embedding. Note that d^* does preserve coherent complexes which are relatively perfect over $Z_0 \times Z_0$ even though it does not preserve coherent complexes in general. The dg-functor $\mathbf{Coh}(Z/ev) \otimes_k \mathbf{Coh}(Z/ev) \rightarrow \mathbf{Coh}(Z/ev)$ is linear over the multiplication (tensor product) $\mathbf{Perf}(Z_0) \otimes \mathbf{Perf}(Z_0) \rightarrow \mathbf{Perf}(Z_0)$ and thus descends to the mentioned monoid structure

$$\mathbf{Coh}(Z/ev) \otimes_{\mathbf{Perf}(Z_0)} \mathbf{Coh}(Z/ev) \rightarrow \mathbf{Coh}(Z/ev).$$

We therefore have defined a "linear 2-category" (without units), whose "set" of object is Z_0 , whose dg-category of morphisms is given by $\mathbf{Coh}(Z/ev)$.

Definition 3.1.1. *The 2-category of algebraic trajectories of ν on X is $\widetilde{\mathbf{Coh}}(Z/ev)$, considered as a monoid in $Dg^c(Z_0 \times Z_0)$. It is denoted by $\widetilde{\mathbf{Traj}}(X, \nu)$.*

A very important special case is when Z_0 is of dimension zero, and thus is of the form $\coprod_{i \in I} \mathbf{Spec} A_i$ where the A_i 's are local artinian k -dg-algebras A_i , and I is the set of zeros of ν in X . The derived stack Z then splits as $\coprod_{(i,j)} Z_{i,j}$, where $Z_{i,j}$ lives over $\mathbf{Spec} A_i \times \mathbf{Spec} A_j$. The 2-category $\widetilde{\mathbf{Traj}}(X, \nu)$ is then given by a concrete 2-category whose set

of objects is I . The dg-category of morphisms from i to j is given by

$$\widetilde{\mathrm{Traj}}(X, \nu)(i, j) = \mathrm{Coh}(Z_{i,j}/ev),$$

the dg-category of coherent complexes on $Z_{i,j}$ which are relatively perfect over $A_i \otimes A_j$. Composition of morphisms is given by the gluing maps

$$\mathrm{Coh}(Z_{i,j}/ev) \otimes_{A_j} \mathrm{Coh}(Z_{j,k}/ev) \longrightarrow \mathrm{Coh}(Z_{i,k}/ev).$$

By our convention, Z is itself the union of $Z(\beta)$, where $\beta \in N$ runs over curve classes. This decomposition provides an N -graduation on the dg-categories of morphisms $\mathrm{Coh}(Z_{i,j}/ev)$, in a way that the composition above is N -graded. To be more precise, we have $\mathrm{Coh}(Z_{i,j}/ev) \simeq \bigoplus_{\beta \in N} \mathrm{Coh}(Z_{i,j}(\beta)/ev)$, and the composition sends the component

$$\mathrm{Coh}(Z_{i,j}(\beta)/ev) \otimes_{A_j} \mathrm{Coh}(Z_{j,k}(\beta')/ev)$$

to $\mathrm{Coh}(Z_{i,k}(\beta + \beta')/ev)$.

Finally, when Z_0 is moreover reduced and each zero is rational over k , we have $A_i \simeq k$ for all i , and $\widetilde{\mathrm{Traj}}(X, \nu)$ becomes a genuine k -linear 2-category (but still without units).

The category of algebraic trajectories. We now construct a (dg) 1-categorical analogue of the 2-category $\widetilde{\mathrm{Traj}}(X, \nu)$. It can probably be obtained by considering the ℓ -adic realization of $\widetilde{\mathrm{Traj}}(X, \nu)$ by applying the techniques of [BRTV18] levelwise on dg-categories of morphisms (or rather on $\mathrm{Coh}(Z/ev)$ as a dg-category over $Z_0 \times Z_0$). We will however construct it directly using the ℓ -adic formalism. We come back to the evaluation map $ev : Z \rightarrow Z_0 \times Z_0$. We consider $\mathrm{Traj}(X, \nu) := ev_* ev^!(\mathbb{Q}_\ell)$, the relative Borel-Moore homology of Z over $Z_0 \times Z_0$. This is a constructible ℓ -adic complex on $Z_0 \times Z_0$. As ev is proper, the fiber of $\mathrm{Traj}(X, \nu)$ at a geometric point $(x, y) : \mathbf{Spec} K \in Z_0 \times Z_0$ is the Borel-Moore homology complex of $Z_{(x,y)} := Z \times_{Z_0 \times Z_0} \mathbf{Spec} K$.

As in the previous 2-categorical setting, we endow the ∞ -category $\mathcal{D}(Z_0 \times Z_0, \mathbb{Q}_\ell)$ of constructible ℓ -adic complexes with the natural convolution tensor product (using $*$ -pullbacks). We claim that $\mathrm{Traj}(X, \nu)$ is then endowed with a natural structure of an associative (non-unital) monoid in $\mathcal{D}(Z_0 \times Z_0, \mathbb{Q}_\ell)$. The multiplication map for this monoid structure is induced by the composition map $\mu : Z \times_{Z_0} Z \rightarrow Z$ and its natural direct image in relative homology

$$\mu_* : p_*(p^!(\mathbb{Q}_\ell)) \rightarrow ev_* ev^!(\mathbb{Q}_\ell)$$

where $p : Z \times_{Z_0} Z \rightarrow Z_0 \times Z_0$ is the natural structure map (so that μ is a morphism over $Z_0 \times Z_0$).

We get this way a monoid in $\mathcal{D}(Z_0 \times Z_0, \mathbb{Q}_\ell)$, without units, whose underlying object is $\text{Traj}(X, \nu)$. By definition, this is a \mathbb{Q}_ℓ -linear dg-category whose "set of objects" is Z_0 .

Definition 3.1.2. *The dg-category of algebraic trajectories of ν on X is $\text{Traj}(X, \nu)$, considered as a monoid in $\mathcal{D}(Z_0 \times Z_0, \mathbb{Q}_\ell)$.*

Again, as in the 2-categorical case, when Z_0 is of the form $\coprod_{\text{Spec } k} k$, $\text{Traj}(X, \nu)$ is a genuine dg-category whose set of objects is Z_0 , (possibly endowed with a Galois action when k is not algebraically closed).

Numerical and homological invariants. We now go one step further in the decategorification and define numerical and homological invariants. For each curve class $\beta \in N$ we have $ev_\beta : Z_\beta \rightarrow Z_0 \times Z_0$, the restriction of the evaluation map to the β component. The direct image by ev_β of the Euler series $\widetilde{Eu}_E(s, t) \in H_{2*}(Z_\beta, \mathbb{Q}_\ell(-*))((t))$ of Definition 2.2.3, where $E = \mathbb{T}_X$ and $s = \nu_{0,2}$, provides a natural element

$$\widetilde{d}_\beta \in (H_{2*}(Z_0, \mathbb{Q}_\ell(-*)) \otimes H_{2*}(Z_0, \mathbb{Q}_\ell(-*)))((t)).$$

This element should be thought as some form of Morse differential and as being the algebraic analogue of the usual differential in Morse homology (see [Bot88]). For instance, when Z_0 is smooth, \widetilde{d}_β can be considered, via Poincaré duality, as a $\mathbb{Q}_\ell((t))$ -linear endomorphism of $H_{2*}(Z_0, \mathbb{Q}_\ell(-*))((t))$. It is not true in general that $\widetilde{d}_\beta^2 = 0$ so this endomorphism is not quite a differential.

The situation seems slightly better when X is convex, so that $\mathbb{R}\bar{\mathcal{M}}_{0,2}(X)$ is smooth, and the Laurent series \widetilde{d}_β is in fact a polynomial in t . In this case, we can define an endomorphism by specializing $t = -1$

$$d_\beta := \widetilde{d}_\beta(t = -1) \in H_{2*}(Z_0, \mathbb{Q}_\ell(-*)) \otimes H_{2*}(Z_0, \mathbb{Q}_\ell(-*)).$$

This element is the direct image by the evaluation map

$$ev : Z_\beta \times_{Z_0} Z_\beta \longrightarrow Z_0 \times Z_0,$$

of the localized top Chern class of $\mathbb{R}\bar{\mathcal{M}}_{0,2}(X, \beta)$ along the section $\nu_{0,2}$. In particular, this element is of homological degree 0. When Z_0 is smooth and 0-dimensional (i.e. Z_0 is smooth as a derived scheme of zeros of ν), so ν has only simple and isolated zeros on X , we get this way an endomorphism of the vector space spanned by the set of zeros I

$$d_\beta : I \otimes \mathbb{Q}_\ell \longrightarrow I \otimes \mathbb{Q}_\ell.$$

To be more precise, we can set I to be the zero of geometric zeros (i.e. geometric points of Z_0), so that it comes equipped with an action

of the absolute Galois group G of k . The endomorphism d_β is then G -equivariant.

Again, d_β is not quite a differential, but there is an explicit formula for its square, very closely related to the splitting axiom in GW theory of [BM96]. To get this formula, we introduce another derived Deligne-Mumford stack W_β , together with a natural morphism $W_\beta \rightarrow Z_0 \times Z_0$. It is defined as follows. We let $\mathcal{M}_\beta := \mathbb{R}\bar{\mathcal{M}}_{0,2}(X, \beta)$, and consider the projection to the stack of prestable curves of genus zero with two marked points $\mathcal{M}_{0,2}^{pre}$.

$$p : \mathcal{M}_\beta \rightarrow \bar{\mathcal{M}}_{0,2}^{pre}.$$

On the level of prestable curves we have the gluing morphism

$$\mu : \bar{\mathcal{M}}_{0,1}^{pre} \times \bar{\mathcal{M}}_{0,3}^{pre} \amalg \bar{\mathcal{M}}_{0,2}^{pre} \times \bar{\mathcal{M}}_{0,2}^{pre} \amalg \bar{\mathcal{M}}_{0,3}^{pre} \times \bar{\mathcal{M}}_{0,1}^{pre} \rightarrow \bar{\mathcal{M}}_{0,2}^{pre}.$$

The left hand side will be denoted by $\bar{\mathcal{M}}_{0,2}^\infty$, and will be called the moduli of *decomposed prestable curves* (of genus 0 with two marked points). We then form the cartesian square

$$\begin{array}{ccc} \mathcal{M}_\beta^\infty := \mathcal{M}_\beta \times_{\bar{\mathcal{M}}_{0,2}^{pre}} \bar{\mathcal{M}}_{0,2}^\infty & \longrightarrow & \mathcal{M}_\beta \\ \downarrow & & \downarrow \\ \bar{\mathcal{M}}_{0,2}^\infty & \longrightarrow & \bar{\mathcal{M}}_{0,2}^{pre}. \end{array}$$

The derived stack \mathcal{M}_β^∞ is by definition the boundary at ∞ of \mathcal{M}_β , also called the *derived stack of decomposed stable maps*. Note that μ is a finite, lci and unramified morphism, so that \mathcal{M}_β^∞ is again a quasi-smooth proper derived Deligne-Mumford stack. Its image in \mathcal{M}_β is the canonical divisor of \mathcal{M}_β . The vector field $\nu_{0,2}$ being vertical with respect to the projection p , it induces a natural vector field $\nu_{0,2} \oplus 0$ on \mathcal{M}_β^∞ . Its derived stack of zeros is denoted by $W_\beta \subset \mathcal{M}_\beta^\infty$, and fits in a cartesian square

$$\begin{array}{ccc} W_\beta & \longrightarrow & Z_\beta \\ \downarrow & & \downarrow \\ \mathbb{T}\bar{\mathcal{M}}_{0,2}^\infty[-1] & \longrightarrow & \mathbb{T}\bar{\mathcal{M}}_{0,2}^{pre}[-1], \end{array}$$

where $\mathbb{T}Y[-1] = \mathbf{Spec} \operatorname{Sym}(\mathbb{L}_Y[1])$ is the derived stack of zeros of the zero vector field on a given derived stack Y . Note that, as X is convex, the morphism $p : \mathcal{M}_\beta \rightarrow \bar{\mathcal{M}}_{0,2}^{pre}$ is smooth. This implies that the induced morphism on zeros of vector fields $Z_\beta \rightarrow \mathbb{T}\bar{\mathcal{M}}_{0,2}^{pre}[-1]$ is itself quasi-smooth. As a result, W_β is a quasi-smooth and closed substack of \mathcal{M}_β . It comes equipped with the induced projection $ev : W_\beta \rightarrow Z_0 \times Z_0$.

The direct image of the virtual class of W_β by ev produces another endomorphism of $I \otimes \mathbb{Q}_\ell$ denoted by

$$d_\beta^\infty := ev_*([W_\beta]^{vir}) : I \otimes \mathbb{Q}_\ell \longrightarrow I \otimes \mathbb{Q}_\ell.$$

Finally, we will also need to define two other objects coming from stable maps with 1 and 3 marked points. For $\beta \in N$, we have $\mathbb{R}\bar{\mathcal{M}}_{0,1}(X, \beta)$, together with its evaluation map $\mathbb{R}\bar{\mathcal{M}}_{0,1}(X, \beta) \rightarrow X$. This induces a morphism of on the level of derived stacks of zeros of $\nu_{0,1}$

$$\mathcal{E}_\beta \rightarrow Z_0,$$

which is a proper and quasi-smooth morphism. The direct image of the virtual fundamental class $[\mathcal{E}_\beta]^{vir} \in H_0(\mathcal{E}_\beta, \mathbb{Q}_\ell)$ defines an element

$$e_\beta \in H_0(Z_0, \mathbb{Q}_\ell) \simeq I \otimes \mathbb{Q}_\ell.$$

In the same manner, we consider $\mathbb{R}\bar{\mathcal{M}}_{0,3}(X, \beta)$ with its evaluation to X^3 . On the level of derived stacks of zeros of $\nu_{0,3}$, we get another proper and quasi-smooth morphism

$$\mathcal{F}_\beta \rightarrow Z_0^3.$$

The direct image of the virtual class by the above morphism produces an element $f_\beta \in (I \otimes \mathbb{Q}_\ell)^3$. By Poincaré duality, this element can be paired with an element $x \in I \otimes \mathbb{Q}_\ell$ in two different manners, $x.f_\beta$ and $f_\beta.x$, in order to get an endomorphism of $I \otimes \mathbb{Q}_\ell$.

The following proposition is the analogue of the splitting/associativity axiom in GW theory (see e.g. [BM96, Def. 7.1-(3)], taking into account our vector field ν .

Proposition 3.1.3. *Assume, as before, that X is convex and that the zeros of ν are isolated and simple. Then, for each $\beta \in N$ we have*

$$\sum_{\beta_1 + \beta_2 = \beta} d_{\beta_1}.d_{\beta_2} + e_{\beta_1}.f_{\beta_2} + f_{\beta_1}.e_{\beta_2} = d_\beta^\infty.$$

Proof. This follows easily from the natural identification of \mathcal{M}_β^∞ with

$$\coprod_{\beta_1 + \beta_2 = \beta} \mathcal{M}_{\beta_1} \times_X \mathcal{M}_{\beta_2} \coprod \mathbb{R}\bar{\mathcal{M}}_{0,1}(X, \beta_1) \times_X \mathbb{R}\bar{\mathcal{M}}_{0,3}(X, \beta_2)$$

$$\coprod \mathbb{R}\bar{\mathcal{M}}_{0,3}(X, \beta_1) \times_X \mathbb{R}\bar{\mathcal{M}}_{0,1}(X, \beta_2),$$

and the fact that the identification respects the induced vector fields. \square

An easy example. We finish this section by the most simple example, namely $X = \mathbb{P}^1$ and $\beta = 1$. In this case, $\mathcal{M}_{0,2}(X, \beta) \simeq \mathbb{P}^1 \times \mathbb{P}^1$, the isomorphism being given by the evaluation map. Given $(a, b) \in \mathbb{P}^1 \times \mathbb{P}^1$ outside of the diagonal, the corresponding stable map $C \rightarrow \mathbb{P}^1$ is given

by any linear endomorphism of \mathbb{P}^1 sending 0 to a and ∞ to b . When $a = b$, the corresponding stable map is such that C has two components, one with the two marked points entirely contracted sent to a , and a second one identified with \mathbb{P}^1 with the identity map. Let $\nu = z\partial_z$ be the standard vector field on \mathbb{P}^1 with two simple zeros at 0 and ∞ . The induced vector field $\nu_{0,2}$ is the direct sum $\nu \oplus \nu$ as a vector field on $\mathbb{P}^1 \times \mathbb{P}^1$. This vector field has 4 simple zeros, namely $(0, 0)$, $(0, \infty)$, $(\infty, 0)$ and (∞, ∞) . As a result, $I \otimes \mathbb{Q}_\ell = \mathbb{Q}_\ell \cdot 0 \oplus \mathbb{Q}_\ell \cdot \infty$, and the endomorphism d_1 defined above is given by

$$d_1(0) = d_1(\infty) = 0 + \infty.$$

Therefore, we see that the equation satisfied by d_1 is here $d_1^2 = 2 \cdot d_1$.

3.2. An algebraic dynamical Zeta function. We now turn to the case of genus 1. Let ν be again a vector field on a smooth and proper scheme X . For each curve class $\beta \in N$ we have the total Euler series of the vector field $\nu_{1,0}$ on $\mathbb{R}\bar{\mathcal{M}}_{1,0}(X, \beta)$ of stable maps of genus 1 with no marked points. The zeros of $\nu_{1,0}$ corresponds to "periodic orbits" of ν in X , that is elliptic curves $f : E \rightarrow X$ whose image is globally invariant by ν . Let us fix a polarization ω on X , so that we have a degree map $\beta \mapsto |\beta|$

$$|-| := \deg(\omega \cdot -) : N \rightarrow \mathbb{N}.$$

We suggest to form the following generating series

$$Z(X, \nu; (t, z)) := \exp \left(- \sum_{\beta \in N} \frac{N_\beta}{|\beta|} z^\beta \right),$$

where $N_\beta \in \mathbb{Q}_\ell(t)$ is the total Euler series of $\nu_{1,0}$ on $\mathbb{R}\bar{\mathcal{M}}_{1,0}(X, \beta)$. This generating series is an element of $\mathbb{Q}_\ell(t)[[N]]$, the completed group ring of N over the field of rational function $\mathbb{Q}_\ell(t)$.

We like to think of $Z(X, \nu; (t, z))$ as the algebraic analogue of Ruelle's dynamical zeta function (see e.g. [Rue92, §1.4-(B)]) that counts periodic orbit of the flow associated to a global vector field. It is tempting to ask if $Z(X, \nu; (t, z))$ is rational, but we currently have not enough intuition whether this is a reasonable statement to expect or not. We also note that similar generating series, without any vector field considerations, already appear in [Man98].

3.3. Higher genus considerations. So far we have started with a global vector field ν on X , so that (irreducible) invariant curves can only be of genus 0 or 1. In particular, the zero locus of the induced vector field $\nu_{g,n}$ on $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X, \beta)$ is not a very pertinent object to consider

when $g > 1$, as this corresponds to stable maps that factor through the zero locus of ν in X . In order to get something interesting for all genres, we have to allow ourselves to replace vector fields by more general *foliations*.

It is a general fact from derived foliation theory, see [TVc, TVb], that any foliation \mathcal{F} on X induces a derived foliation $\mathcal{F}_{g,n}$ on $\mathbb{R}\mathcal{M}_{g,n}(X, \beta)$, but again, when $g > 1$ the singularities of this induced foliation is a rather poor geometric object (it consists essentially of pointwise invariant curves, i.e. stable maps to the singular locus of the original foliation on X). It is however possible to replace the zero locus of $\nu_{g,n}$ on $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X, \beta)$ with a more relevant derived stack $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X, \beta)^{\mathcal{F}}$ of \mathcal{F} -equivariant stable maps defined as follows. For simplicity of exposition, we suppose that \mathcal{F} is a one dimensional foliation on X , defined by a line bundle \mathcal{L} together with a morphism of \mathcal{O}_X -modules $s : \mathcal{L} \rightarrow T_X$. This defines a one dimensional foliation \mathcal{F} on X , possibly with singularities at those points in X where s vanishes. By definition, an $S = \mathbf{Spec} A$ -point in the derived stack $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X, \beta)^{\mathcal{F}}$ consists of a pair (f, u) , where $f : C \rightarrow X$ is stable map (where $C \rightarrow S$ is a relative curve of genus g with n marked points), and u is a morphism of \mathcal{O}_C -modules $u : f^*(\mathcal{L}) \rightarrow \mathbb{T}_C(-\Sigma)$, making commutative the diagram

$$\begin{array}{ccc} f^*(\mathcal{L}) & \xrightarrow{s} & f^*(\mathbb{T}_X) \\ u \downarrow & & \uparrow Tf \\ \mathbb{T}_{C/S}(-\Sigma) & \longrightarrow & T_{C/S}, \end{array}$$

where $\Sigma = \coprod_n S \hookrightarrow C$ is the relative divisor of marked points in C . This will define a derived Deligne-Mumford stack $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X, \beta)^{\mathcal{F}}$, endowed with a forgetful map to $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X, \beta)$, from which we can indeed extract interesting numerical invariants. As in the case studied in this paper, these derived stacks are not quasi-smooth, and thus only Euler series as in Section 2.2 will make sense in general, unless specific conditions on \mathcal{F} ensures that we have quasi-smoothness. When \mathcal{L} is the trivial line bundle, so s is global vector field on X , the derived stack $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X, \beta)^{\mathcal{F}}$ coincides with the derived locus of $\nu_{g,n}$. Essentially all of our results and constructions in the previous sections extend from vector fields to more general one dimensional foliations, simply by using $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X, \beta)^{\mathcal{F}}$. The general study of the derived stack of \mathcal{F} -equivariant stable maps will appear elsewhere.

4. APPENDIX: K-THEORY AND G-THEORY OF QUASI-SMOOTH DERIVED STACKS

We collect in this Appendix a few facts about the K -theory and G -theory of derived stacks that we have used in the main text. The interested reader will find more details e.g. in [Kha22].

Recall that for a k -cdga A (non-positively graded and Noetherian, for simplicity), an A -dg-module M is said to be *coherent* if it is cohomologically bounded, and each $H^i(M)$ is finitely generated over $H^0(A)$. If X is a Noetherian derived Artin stack, we denote by $\mathbf{QCoh}(X)$ the derived ∞ -category of quasi-coherent complexes on X (see [Toë14], 3.1). $\mathbf{QCoh}(X)$ is a symmetric monoidal ∞ -category (via the derived tensor product over \mathcal{O}_X), and we denote by $\mathbf{Perf}(X)$ its full subcategory consisting of objects which are dualizable with respect to this tensor product. We write $\mathbf{Coh}(X)$ (respectively $\mathbf{Coh}(X)_{\geq 0}$, respectively $\mathbf{Coh}(X)_{\leq 0}$) for the full subcategory of $\mathbf{QCoh}(X)$ consisting of complexes E such that, for any smooth map $u : S = \mathrm{Spec} A \rightarrow X$, u^*E is a coherent (respectively, and connective, respectively, and coconnective) A -dg module. The pair $(\mathbf{Coh}(X)_{\geq 0}, \mathbf{Coh}(X)_{\leq 0})$ defines a t-structure on $\mathbf{Coh}(X)$ (i.e. a usual t-structure on the homotopy category $\mathbf{hCoh}(X)$, [Lur17, 1.2]), whose heart will be denoted by $\mathbf{Coh}_{\heartsuit}(X)$. Note that $\mathbf{Coh}_{\heartsuit}(X)$ is equivalent to the abelian category of coherent complexes on the truncation $\tau_0(X)$ of X .

Both $\mathbf{Perf}(X)$ and $\mathbf{Coh}(X)$ are stable ∞ -categories ([Lur17, 1.1]), and therefore ([Bar16, BGT13]) we can consider their (connective) K -theory spectra, $\mathbf{K}(X) := \mathbf{K}(\mathbf{Perf}(X))$, $\mathbf{G}(X) := \mathbf{K}(\mathbf{Coh}(X))$.

Recall that a morphism $f : X \rightarrow Y$ of derived Artin stacks is (derived) *lci* if it is locally of finite and the relative cotangent complex \mathbb{L}_f is of Tor-amplitude ≤ 1 . Then, X is *quasi-smooth* (or, synonymously, *lci*) if $X \rightarrow \mathrm{Spec} \mathbb{C}$ is lci. Note that an arbitrary (a priori non-noetherian) derived Artin stack that is lci over an arbitrary regular noetherian base derived Artin stack is automatically noetherian and its structure sheaf is coherent.

It is easy to see that an lci morphism $X \rightarrow Y$ is of finite Tor-amplitude (e.g. [Kha22, Lemma 1.15]). The pullback $f^* : \mathbf{K}(Y) \rightarrow \mathbf{K}(X)$ exists for any morphism $f : X \rightarrow Y$ of derived Artin stacks (because the inverse image preserves perfect complexes), and the pushforward $f_* : \mathbf{K}(X) \rightarrow \mathbf{K}(Y)$ exists if f is proper and lci. On G -theory, a morphism $f : X \rightarrow Y$ induces a pullback $f^* : \mathbf{G}(Y) \rightarrow \mathbf{G}(X)$ if f has finite Tor-dimension (e.g. it is lci) because the inverse image preserves

coherence, and f induces a pushforward $f_* : \mathbf{G}(X) \rightarrow \mathbf{G}(Y)$ if f is proper and of finite cohomological dimension.

The pushforward along the inclusion $\tau_0(X) \rightarrow X$ induces an equivalence $\mathbf{G}(\tau_0(X)) \simeq \mathbf{G}(X)$ ([Bar16, Thm. 6.1]).

Note that, if $\mathcal{E} \in \mathbf{Perf}(X)$ and $\mathcal{F} \in \mathbf{Coh}(X)$, then $\mathcal{E} \otimes \mathcal{F} \in \mathbf{Coh}(X)$ (derived tensor product), so that we get a well defined map of spectra

$$\cap : \mathbf{K}(X) \otimes \mathbf{G}(X) \rightarrow \mathbf{G}(X)$$

inducing a $\mathbf{K}(X)$ -module structure on $\mathbf{G}(X)$.

We have the following *projection formula* ([Kha22, Proposition 3.7]). If $f : X \rightarrow Y$ is proper and of finite cohomological dimension, we have the following equality in $\mathbf{G}(Y)$

$$y \cap f_*(x) = f_*(f^*(y) \cap x)$$

for any $x \in \mathbf{G}(X)$, and $y \in \mathbf{K}(Y)$.

Given a (homotopy) pullback diagram of derived Artin stacks

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

with g proper and of finite cohomological dimension, and g is of finite Tor-amplitude, we have a canonical homotopy $f^*g_* \simeq g'_*f'^*$ of morphisms $\mathbf{G}(Y') \rightarrow \mathbf{G}(X)$ (see [Kha22, Proposition 3.8]).

For a quasi-smooth derived Artin stack X of finite type (over k), there is a Grothendieck-Riemann-Roch transformation

$$\tau_X : \mathbf{G}_0(X) \longrightarrow \mathbf{H}_{2*}^{BM}(X, \mathbb{Q}_\ell(-*)) := \bigoplus_n \mathbf{H}_{2n}^{BM}(X, \mathbb{Q}_\ell(-n))$$

where $\mathbf{H}_{2n}^{BM}(X, \mathbb{Q}_\ell(-n))$ denotes the $2n$ -th Borel-Moore homology of the truncation $\tau_0 X$ ([Kha22, 6.3], [Kha, Def. 2.1]).

Note that an arbitrary (a priori non-noetherian) derived Artin stack that is lci over an arbitrary regular noetherian base derived Artin stack is automatically noetherian and its structure sheaf is coherent. In particular, a quasi-smooth derived Artin stack X over k has a fundamental class $[X] = [\mathcal{O}_X] \in \mathbf{G}_0(X)$, which under the isomorphism $\mathbf{G}_0(X) \simeq \mathbf{G}_0(\tau_0 X)$ corresponds to a class $[X]_{vir} \in \mathbf{G}_0(\tau_0 X)$, called the

virtual fundamental class of X . We have

$$[X]^{vir} = \sum_i (-1)^i [H^i(\mathcal{O}_X)]$$

(the sum being finite since \mathcal{O}_X is bounded). Analogously ([Kha, Construction 3.6]), there is a fundamental class $[X]_{\text{BM}} \in H_{2d}^{BM}(X, \mathbb{Q}_\ell(d))$ in Borel-Moore homology, where d is the virtual dimension of X (i.e. the Euler characteristic of its cotangent complex). The classes $\tau_X([X])$ and $[X]_{\text{BM}}$ differ by multiplication by the inverse of the Todd class of the cotangent complex of X (Kontsevich Formula, see [Kha22, Thm. 6.12]).

There is a Chern character ring morphism

$$Ch : K_0(X) \rightarrow H^{2*}(X, \mathbb{Q}_\ell(*)) := \bigoplus_n H^{2n}(X, \mathbb{Q}_\ell(n))$$

to ℓ -adic cohomology of X , that enters in a Grothendieck-Riemann-Roch (GRR) formula for lci morphisms; we do not need such a GRR formula in the main text. ¹

REFERENCES

- [AG15] D. Arinkin and D. Gaitsgory. Singular support of coherent sheaves and the geometric Langlands conjecture. *Selecta Math. (N.S.)*, 21(1):1–199, 2015.
- [Bar16] Clark Barwick. On the algebraic K-theory of higher categories. *Journal of Topology*, 9(1):245–347, 2016.
- [BGT13] Andrew J Blumberg, David Gepner, and Gonalo Tabuada. A universal characterization of higher algebraic K-theory. *Geometry & Topology*, 17(2):733 – 838, 2013.
- [BM] Kai Behrend and Yuri I. Manin. Stacks of stable maps and gromov-witten invariants. *Duke Math. J.*
- [BM96] K. Behrend and Yu. Manin. Stacks of stable maps and Gromov-Witten invariants. *Duke Math. J.*, 85(1):1–60, 1996.
- [Bot88] Raoul Bott. Morse theory indomitable. *Publications Mathématiques de l’IHÉS*, 68:99–114, 1988.
- [BRTV18] Anthony Blanc, Marco Robalo, Bertrand Toën, and Gabriele Vezzosi. Motivic realizations of singularity categories and vanishing cycles. *Journal de l’École polytechnique — Mathématiques*, 5:651–747, 2018.
- [CPT⁺17] Damien Calaque, Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted Poisson structures and deformation quantization. *J. Topol.*, 10(2):483–584, 2017.
- [Dav17] Ben Davison. The critical CoHA of a quiver with potential. *Q. J. Math.*, 68(2):635–703, 2017.
- [GP99] T. Graber and R. Pandharipande. Localization of virtual classes. *Invent. Math.*, 135(2):487–518, 1999.

¹The reader may look at [Kha22, Thm. 6.22] where a version of the GRR formula is proved (with values in a different cohomology theory).

- [Kha] Adeel A. Khan. Virtual fundamental classes of derived stacks I. Preprint arXiv:1909.01332.
- [Kha22] Adeel A. Khan. K-theory and G-theory of derived algebraic stacks. *Japanese Journal of Mathematics*, 2022.
- [KL13] Y-H. Kiem and J Li. Localizing virtual cycles by cosections. *Journal of the American Mathematical Society*, 26(4):1025–1050, 2013.
- [LT18] Jacob Lurie and Hiro Lee Tanaka. Associative algebras and broken lines, 2018.
- [Lur17] Jacob Lurie. *Higher Algebra*. 2017.
- [Man98] Yuri I. Manin. Stable maps of genus zero to flag spaces. *Topol. Methods Nonlinear Anal.*, 11(2):207–217, 1998.
- [Mon21] Ludovic Monier. A note on linear stacks, 2021.
- [MR18] E. Mann and M. Robalo. Brane actions, categorifications of Gromov–Witten theory and quantum K–theory. *Geometry & Topology*, (22):1759–1836, 2018.
- [Rue92] David Ruelle. Dynamical zeta functions: where do they come from and what are they good for? In *Mathematical physics, X (Leipzig, 1991)*, pages 43–51. Springer, Berlin, 1992.
- [STV15] Timo Schürg, Bertrand Toën, and Gabriele Vezzosi. Derived algebraic geometry, determinants of perfect complexes, and applications to obstruction theories for maps and complexes. *J. Reine Angew. Math.*, 702:1–40, 2015.
- [Toë99] Bertrand Toën. Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford. *K-Theory*, 18(1):33–76, 1999.
- [Toë12] Bertrand Toën. Derived Azumaya algebras and generators for twisted derived categories. *Invent. Math.*, 189(3):581–652, 2012.
- [Toë14] Bertrand Toën. Derived algebraic geometry. *EMS Surv. Math. Sci.*, 1(2):153–240, 2014.
- [TVa] Bertrand Toën and Gabriele Vezzosi. Algebraic foliations and derived geometry. in preparation.
- [TVb] Bertrand Toën and Gabriele Vezzosi. Algebraic foliations and derived geometry: GRR theorems. Preprint arXiv 2007.09251.
- [TVc] Bertrand Toën and Gabriele Vezzosi. Algebraic foliations and derived geometry: the Riemann-Hilbert correspondence. to appear in *Selecta Math.*, available as arXiv:2001.05450, October 2021.
- [TV08] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008.