The Horn cone associated with symplectic eigenvalues

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Abstract

In this note, we show that the Horn cone associated with symplectic eigenvalues admits the same inequalities as the classical Horn cone, except that the equality corresponding to $\mathrm{Tr}(C) = \mathrm{Tr}(A)+\mathrm{Tr}(B)$ is replaced by the inequality corresponding to $\mathrm{Tr}(C) \geq \mathrm{Tr}(A)+\mathrm{Tr}(B)$.

1 Introduction

We consider $\mathbb{R}^{2n}$ equipped with its canonical symplectic structure $\Omega_n = \sum_{k=1}^n dx_k \wedge dx_{k+n}$. Recall that a family $(e_k)_{1\leq k\leq 2n}$ is a symplectic basis of $\mathbb{R}^{2n}$, if $\Omega_n(e_k, e_\ell) = 0$ if $|k-\ell| \neq n$ and $\Omega_n(e_k, e_{k+n}) = 1$, $\forall k$.

Williamson's theorem [18] says that any positive definite quadratic form $q : \mathbb{R}^{2n} \to \mathbb{R}$ can be written $q(v) = \sum_{k=1} \lambda_k(v_k^2 + v_{k+n}^2)$ where the (v_j) are the coordinates of the vector $v \in \mathbb{R}^{2n}$ relatively to a symplectic basis. The positive numbers λ_k, that one choose so that

$$\lambda(q) := (\lambda_1 \geq \cdots \geq \lambda_n),$$

will be referred to as the *symplectic eigenvalues* of the quadratic form q. They correspond to the frequencies of the normal modes of oscillation for the linear Hamiltonian system generated by q.

The object of study of this note concerns the symplectic Horn cone, denoted $\mathrm{Horn}_{\mathrm{sp}}(n)$, that is defined as the set of triplets $(\lambda(q_1), \lambda(q_2), \lambda(q_1+q_2))$ where q_1, q_2 are positive definite quadratic forms on $\mathbb{R}^{2n}$.

Example 1.1 *In dimension* 2, *the symplectic eigenvalue* $\lambda(q)$ *of a positive definite quadratic form* $q(x_1, x_2) = ax_1^2 + bx_2^2 + cx_1x_2$ *is equal to* $\frac{1}{2}\sqrt{4ab-c^2}$. *It is straightforward to show that* $\mathrm{Horn}_{\mathrm{sp}}(1)$ *is equal to the set of triplets* (x, y, z) *of positive numbers satisfying* $x + y \leq z$.

Our main Theorem states that $\mathrm{Horn}_{\mathrm{sp}}(n)$ is a convex polyhedral set. Before detailing it, let us recall some related results.

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In [17], A. Weinstein showed that for non-increasing n-tuples of positive real numbers a and b, the set $\Delta_{\mathrm{sp}}(a,b) := \{\lambda(q_1+q_2) \,|\, \lambda(q_1)=a, \lambda(q_2)=b\}$ is closed, convex and locally polyhedral.

Recently, several authors have realized that some inequalities obtained long ago in the context of eigenvalues of Hermitian matrices still apply to symplectic eigenvalues:

- T. Hiroshima proved in [7] an analogue of Ky Fan inequalities: $\sum_{j=1}^k \lambda_j(q_1+q_2) \geq \sum_{j=1}^k \lambda_j(q_1) + \sum_{j=1}^k \lambda_j(q_2)$
- In [8], T. Jain and H. Mishra obtained an analogue of Lidskii inequalities: $\sum_{j=1}^k \lambda_{i_j}(q_1+q_2) \geq \sum_{j=1}^k \lambda_{i_j}(q_1) + \sum_{j=1}^k \lambda_j(q_2)$ for any subset $\{i_1 < i_2 < \cdots < i_k\}$.
- In [1], R. Bhatia and T. Jain obtained an analogue of the Weyl inequalities: $\lambda_{i+j-1}(q_1+q_2) \geq \lambda_i(q_1) + \lambda_j(q_2)$.

As the previous results suggest, we know explain the strong relationship between $\mathrm{Horn}_{\mathrm{sp}}(n)$ with the classical Horn cone. If A is a Hermitian $n\times n$ matrix, we denote by $\mathrm{s}(A) = (\mathrm{s}_1(A) \geq \cdots \geq \mathrm{s}_n(A))$ its spectrum. The Horn cone $\mathrm{Horn}(n)$ is defined as the set of triplets $(\mathrm{s}(A), \mathrm{s}(B), \mathrm{s}(A+B))$ where A, B are Hermitian $n\times n$ matrices.

Denote the set of cardinality r subsets $I = \{i_1 < i_2 < \cdots < i_r\}$ of $[n] := \{1,\ldots,n\}$ by $\mathcal{P}_r^n$. To each $I \in \mathcal{P}_r^n$ we associate:

- a weakly decreasing sequence of non-negative integers $\lambda(I) = (\lambda_1 \geq \cdots \geq \lambda_r)$ where $\lambda_a = n - r + a - i_a$ for $a \in [r]$.
- the irreducible representation $V_{\lambda(I)}$ of $GL_r(\mathbb{C})$ with highest weight $\lambda(I)$.

If $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ and $I \subset [n]$, we define $|\,x\,|_I = \sum_{i\in I} x_i$ and $|\,x\,| = \sum_{i=1}^n x_i$. Let us denote by $\mathbb{R}^n_+$ the set of non-increasing n-tuples of real numbers.

A. Klyachko [10] has shown that an element $(x,y,z) \in (\mathbb{R}^n_+)^3$ belongs to the cone $\mathrm{Horn}(n)$ if and only if it satisfies $|x| + |y| = |z|$ and

$$((\star)_{I,J,K}) \qquad |x|_I + |y|_J \leq |z|_K$$

for any $r < n$, for any $I, J, K \in \mathcal{P}_r^n$ such that the Littlewood-Richardson coefficient

$$c_{IJ}^K := \dim[V_{\lambda(I)} \otimes V_{\lambda(J)} \otimes V^*_{\lambda(K)}]^{GL_r(\mathbb{C})}$$

is non-zero. P. Belkale [2] showed that the inequalities $(\star)_{I,J,K}$ associated to the condition $c_{IJ}^K = 1$ are sufficient. Finally A. Knutson, T. Tao, and C. Woodward [11] have proved that this smaller list is actually minimal. We refer the reader to survey articles [5, 3] for details.

The main result of this note is the following Theorem. Let us denote by $\mathbb{R}^n_{++}$ the set of non-increasing n-tuples of positive real numbers.

Theorem 1.2 *An element $(x, y, z) \in (\mathbb{R}^n_{++})^3$ belongs to $\mathrm{Horn}_{\mathrm{sp}}(n)$ if and only if it satisfies*

1. $|x| + |y| \leq |z|$,
2. $(\star)_{I,J,K}$ *for all (I, J, K) of cardinality $r < n$ such that $c^K_{IJ} = 1$.*

Corollary 1.3 *Let $a, b \in \mathbb{R}^n_{++}$. An element $z \in \mathbb{R}^n_{++}$ belongs to $\Delta_{\mathrm{sp}}(a, b)$ if and only if it satisfies $|a| + |b| \leq |z|$ and $|a|_I + |b|_J \leq |z|_K$ for all (I, J, K) of cardinality $r < n$ such that $c^K_{IJ} = 1$.*

2 The causal cone of the symplectic Lie algebra

The $2n \times 2n$ matrix $J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ defines a complex structure on $\mathbb{R}^{2n}$ that is compatible with the symplectic structure Ω_n. The symplectic group $Sp(\mathbb{R}^{2n})$ is defined by the relation ${}^t g J_n g = J_n$. A matrix X belongs to the Lie algebra $\mathfrak{sp}(\mathbb{R}^{2n})$ of $Sp(\mathbb{R}^{2n})$ if and only the matrix $J_n X$ is symmetric. Moreover, $J_n X$ is positive if and only if $\Omega_n(Xv, v) \geq 0$, $\forall v \in \mathbb{R}^{2n}$.

We call an invariant convex cone C in $\mathfrak{sp}(\mathbb{R}^{2n})$ a causal cone if C is nontrivial, closed, and satisfies $C \cap -C = \{0\}$. A classical result [16, 13, 14] asserts that there are exactly two causal cones in $\mathfrak{sp}(\mathbb{R}^{2n})$: one, denoted $\mathbf{C}(n)$, containing $-J_n$ and its opposite $-\mathbf{C}(n)$. The causal cone $\mathbf{C}(n)$ is determined by the following equivalent conditions : for $X \in \mathfrak{sp}(\mathbb{R}^{2n})$, we have

$$X \in \mathbf{C}(n) \quad \Longleftrightarrow \quad J_n X \text{ is positive} \quad \Longleftrightarrow \quad \mathrm{Tr}(X g J_n g^{-1}) \geq 0,\ \forall g \in Sp(\mathbb{R}^{2n}).$$

Now we explain how is parameterized the interior $\mathbf{C}(n)^0$ of $\mathbf{C}(n)$. From the definition above, we see first that $X \in \mathbf{C}(n)^0$ if and only if $J_n X$ is positive definite.

The Lie algebra of the maximal compact subgroup $K = Sp(2n, \mathbb{R}) \cap O(2n)$ is

$$\mathfrak{k} := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix},\ {}^t A = -A,\ {}^t B = B \right\}.$$

If $\mu := (\mu_1, \cdots, \mu_n)$, we write $\Delta(\mu) = \mathrm{Diag}(\mu_1, \cdots, \mu_n)$ and $X(\mu) = \begin{pmatrix} 0 & \Delta(\mu) \\ -\Delta(\mu) & 0 \end{pmatrix}$. We work with the Cartan subalgebra $\mathfrak{t} := \{X(\mu), \mu \in \mathbb{R}^n\}$ of $\mathfrak{k}$ and the corresponding maximal torus $T \subset K$. The set of roots $\mathfrak{R}$ relatively to the action of T on $\mathfrak{sp}(\mathbb{R}^{2n}) \otimes \mathbb{C}$ are composed by the compact ones $\mathfrak{R}_c := \{\epsilon_i - \epsilon_j\}$ and the non compact ones $\mathfrak{R}_n = \{\pm(\epsilon_i + \epsilon_j)\}$. We work with the subsets of positive roots $\mathfrak{R}^+_c := \{\epsilon_i - \epsilon_j, i < j\}$ and $\mathfrak{R}^+_n := \{\epsilon_i + \epsilon_j\}$. The Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$ is defined by the relations $\langle \alpha, \mu \rangle \geq 0$, $\forall \alpha \in \mathfrak{R}^+_c$, namely $\mu_1 \geq \cdots \geq \mu_n$. The subchamber $\mathcal{C}_n \subset \mathfrak{t}_+$ is defined by the conditions $\langle \beta, \mu \rangle > 0$, $\forall \beta \in \mathfrak{R}^+_n$. Thus $X(\mu) \in \mathcal{C}_n$ if and only if $\mu \in \mathbb{R}^n_{++}$.

If $M \in \mathfrak{sp}(\mathbb{R}^{2n})$, we denote by $\mathcal{O}_M := \{gMg^{-1}, g \in Sp(\mathbb{R}^{2n})\}$ the corresponding adjoint orbit.

Lemma 2.1 *1. $M \in \mathbf{C}(n)^0$ if and only if there exists $X \in \mathcal{C}_n$ such that $M \in \mathcal{O}_X$.*

2. Let $\mu \in \mathbb{R}^n_{++}$, and $M \in \mathcal{O}_{X(\mu)}$. The symplectic eigenvalues of the positive definite quadratic form $q(v) = {}^t v J_n M v = \Omega_n(Mv, v)$ are the $\mu_1 \geq \cdots \geq \mu_n > 0$.

Proof: The first point is a classical fact [16, 14]. If $M = gX(\mu)g^{-1}$ with $g \in Sp(\mathbb{R}^{2n})$, we see that

$$\Omega_n(Mv, v) = \Omega_n(X(\mu)g^{-1}v, g^{-1}v) = \sum_{k=1}^{n} \mu_k (v_k^2 + v_{k+n}^2)$$

where each v_j is the j-th coordinate of the vector $g^{-1}v$. $\square$

Remark 2.2 *In [15], we call the interior $\mathbf{C}(n)^0$ of $\mathbf{C}(n)$ the holomorphic cone, since any coadjoint orbit $\mathcal{O}_X \subset \mathbf{C}(n)^0$ admits a canonical structure of a Kähler manifold with a holomorphic action of K. These orbits are closely related to the holomorphic discrete series representations of the symplectic group $Sp(\mathbb{R}^{2n})$.*

Thanks to the previous Lemma, we see that the symplectic Horn cone admits the alternative definition:

$$\mathrm{Horn}_{\mathrm{sp}}(n) = \left\{(x, y, z) \in (\mathbb{R}^n_{++})^3 \,|\, \mathcal{O}_{X(z)} \subset \mathcal{O}_{X(x)} + \mathcal{O}_{X(y)}\right\}.$$

In the next section, we explain the result of [15] concerning the determination of $\mathrm{Horn}_{\mathrm{sp}}(n)$.

3 Convexity results

The trace on $\mathfrak{gl}(\mathbb{R}^{2n})$ provides an identification between $\mathfrak{sp}(\mathbb{R}^{2n})$ and its dual $\mathfrak{sp}(\mathbb{R}^{2n})^*$: to $X \in \mathfrak{sp}(\mathbb{R}^{2n})$ we associate $\xi_X \in \mathfrak{sp}(\mathbb{R}^{2n})^*$ defined by $\langle \xi_X, Y \rangle = -\mathrm{Tr}(XY)$. Through this identification the causal cone $\mathbf{C}(n)$ becomes

$$\widetilde{\mathbf{C}}(n) := \left\{\xi \in \mathfrak{sp}(\mathbb{R}^{2n})^*; \langle \xi, \mathrm{Ad}(g)z \rangle \geq 0,\, \forall g \in Sp(\mathbb{R}^{2n})\right\}$$

where $z = \frac{-1}{2} J_n$. The identification $\mathfrak{sp}(\mathbb{R}^{2n}) \simeq \mathfrak{sp}(\mathbb{R}^{2n})^*$ induces several identifications $\mathfrak{k} \simeq \mathfrak{k}^*$, $\mathfrak{t} \simeq \mathfrak{t}^*$ and $\mathfrak{t}_+ \simeq \mathfrak{t}^*_+$. In the latter cases the identifications are done through an invariant scalar product $(-,-)$ on $\mathfrak{k}^*$.The subchamber $\widetilde{\mathcal{C}}_n \subset \mathfrak{t}^*_+$ is defined by the conditions: $(\alpha, \xi) \geq 0$, $\forall \alpha \in \mathfrak{R}^+_c$, and $(\beta, \xi) > 0$, $\forall \beta \in \mathfrak{R}^+_n$.

Through $\mathfrak{sp}(\mathbb{R}^{2n}) \simeq \mathfrak{sp}(\mathbb{R}^{2n})^*$, the symplectic Horn cone becomes

$$\mathrm{Horn}_{\mathrm{hol}}(Sp(\mathbb{R}^{2n})) := \left\{(\xi_1, \xi_2, \xi_3) \in (\widetilde{\mathcal{C}}_n)^3 \,|\, \mathcal{O}_{\xi_3} \subset \mathcal{O}_{\xi_1} + \mathcal{O}_{\xi_2}\right\}.$$

Here we have kept the notations of [15].

We have a Cartan decomposition $\mathfrak{sp}(\mathbb{R}^{2n}) = \mathfrak{k} \oplus \mathfrak{p}$ with

$$\mathfrak{p} := \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix},\ {}^tA = A,\ {}^tB = B \right\}.$$

We denote by $\mathfrak{p}^+$ the vector space $\mathfrak{p}$ equipped with the complex structure $\mathrm{ad}(z)$ and the compatible symplectic structure $\Omega_{\mathfrak{p}^+}(Y,Y') := -\mathrm{Tr}(J_n[Y,Y'])$: here $\Omega_{\mathfrak{p}^+}(Y,[z,Y]) > 0$ for any $Y \neq 0$.

The action of maximal compact subgroup $K \subset Sp(\mathbb{R}^{2n})$ on $(\mathfrak{p}^+, \Omega_{\mathfrak{p}^+})$ is Hamiltonian with moment map

$$\Phi_{\mathfrak{p}^+} : \mathfrak{p}^+ \to \mathfrak{k}^*$$

defined by $\langle \Phi_{\mathfrak{p}^+}(Y), X\rangle = \frac{1}{2}\Omega_{\mathfrak{p}^+}([X,Y],Y)$. If $Y = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$, we see that $\langle \Phi_{\mathfrak{p}^+}(Y), J_n\rangle = \mathrm{Tr}(A^2+B^2) = \frac{1}{2}\|Y\|^2$. Hence the moment map $\Phi_{\mathfrak{p}^+}$ is proper.

We consider the following action of the group K^3 on the manifold $K \times K$:

$$(k_1,k_2,k_3)\cdot(g,h) = (k_1 g k_3^{-1}, k_2 h k_3^{-1})$$

The action of K^3 on the cotangent bundle $N := T^*(K\times K)$ is Hamiltonian with moment map $\Phi_N : N \to \mathfrak{k}^* \times \mathfrak{k}^* \times \mathfrak{k}^*$ defined by the relations[1]

$$\Phi_N(g_1,\eta_1;g_2,\eta_2) = (-g_1\eta_1, -g_2\eta_2, \eta_1+\eta_2).$$

Finally we consider the Hamiltonian K^3-manifold $N \times \mathfrak{p}^+$, where $\mathfrak{p}^+$ is equipped with the symplectic structure $\Omega_{\mathfrak{p}^+}$. The action is defined by the relations: $(k_1,k_2,k_3)\cdot(g,h,X) = (k_1 g k_3^{-1}, k_2 h k_3^{-1}, k_3 X)$. Let us denote by $\Phi : N \times \mathfrak{p}^+ \to \mathfrak{k}^* \times \mathfrak{k}^* \times \mathfrak{k}^*$ the moment map relative to the K^3-action :

$$\Phi(g_1,\eta_1;g_2,\eta_2,Y) = (-g_1\eta_1, -g_2\eta_2, \eta_1+\eta_2+\Phi_{\mathfrak{p}^+}(Y)). \tag{1}$$

Since Φ is proper map, the Convexity Theorem [9, 12] tell us that

$$\Delta(N\times\mathfrak{p}^+) := \mathrm{Image}(\Phi) \bigcap \mathfrak{t}^*_+ \times \mathfrak{t}^*_+ \times \mathfrak{t}^*_+$$

is a closed, convex, and locally polyhedral set.

The map $\mu \mapsto X(\mu)$ defines an isomorphism of $\mathbb{R}^n$ with $\mathfrak{t} \simeq \mathfrak{t}^*$ that induces an identification of $\mathbb{R}^n_{++}$ with $\mathcal{C}_n \simeq \widetilde{\mathcal{C}}_n$. Recall that on $\mathfrak{t}^* \simeq \mathbb{R}^n$, we have a natural involution that sends $\mu = (\mu_1,\ldots,\mu_n)$ to $\mu^* := (-\mu_n,\ldots,-\mu_1)$. The following result is proved in [15] (see Theorem B).

Theorem 3.1 *An element $(x,y,z) \in (\mathbb{R}^n_{++})^3$ belongs to $\mathrm{Horn}_{\mathrm{hol}}(Sp(\mathbb{R}^{2n}))$ if and only if*

$$(x,y,z^*) \in \Delta(N\times\mathfrak{p}^+).$$

Recall that a Hermitian matrix M majorizes another Hermitian matrix M if $M - M'$ is positive semidefinite (its eigenvalues are all nonnegative). In this case, we write $M \geq M'$.

Proposition 3.2 *Let $(x,y,z) \in (\mathbb{R}^n_+)^3$. Then $(x,y,z^*) \in \Delta(N\times\mathfrak{p}^+)$ if and only if there exist Hermitian matrices A,B,C such that* $\mathrm{s}(A) = x$, $\mathrm{s}(B) = y$, $\mathrm{s}(C) = z$ *and* $C \geq A+B$.

[1] We use the identification $T^*K \simeq K \times \mathfrak{k}^*$ given by left translations.

Proof: The map $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto A - iB$ defines an isomorphism between K and the unitary group $U(n)$. Let us denoted by $S^2(\mathbb{C}^n)$ the vector space of complex $n \times n$ symmetric matrices that is equipped with the following action of $U(n)$: $k \cdot M = kM{}^t k$. The map $\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mapsto A - iB$ defines an isomorphism between the K-module $\mathfrak{p}^+$ and the $U(n)$-module $S^2(\mathbb{C}^n)$. Through this identifications the moment map $\Phi_{\mathfrak{p}^+} : \mathfrak{p}^+ \to \mathfrak{k}^*$ becomes the map $\Phi_{S^2} : S^2(\mathbb{C}^n) \to \mathfrak{u}(n)$ defined by the relations

$$\Phi_{S^2}(M) = -2iM\overline{M}.$$

So we know that the moment polytope Δ relative to the Hamiltonian action of $U(n)^3$ on $T^*U(n) \times T^*U(n) \times S^2(\mathbb{C}^n)$ is equal to $\Delta(N \times \mathfrak{p}^+)$. A small computation shows that $(x, y, z^*) \in \Delta$ if and only if there exists Hermitian matrices A, B, C and $M \in S^2(\mathbb{C}^n)$ such that

$$\mathrm{s}(A) = x, \quad \mathrm{s}(B) = y, \quad \mathrm{s}(C) = z \quad \text{and} \quad A + B + 2M\overline{M} = C.$$

The existence of $M \in S^2(\mathbb{C}^n)$ satisfying the condition $A + B + 2M\overline{M} = C$ is equivalent to $C \geq A + B$. The proof is then completed. $\square$

S. Friedland [4] considered the following question: *which eigenvalues* $(\mathrm{s}(A), \mathrm{s}(B), \mathrm{s}(C))$ *can occur if* $C \geq A + B$. His solution was in terms of linear inequalities, which includes Klyachko's inequalities, a trace inequality and some additional inequalities. Later, W. Fulton [6] proved the additional inequalities are unnecessary. Let us summarize their result in the following Theorem.

Theorem 3.3 ([4, 6]) *A triple $x, y, z \in \mathbb{R}^n_+$ occurs as the eigenvalues of n by n Hermitian matrices A, B, C with $C \geq A + B$ if and only it satisfies $|x| + |y| \leq |z|$ and $(\star)_{I,J,K}$ for all (I, J, K) of cardinality $r < n$ such that $c^K_{IJ} = 1$.*

The combination of Theorems 3.1 and 3.3 with Proposition 3.2 completes the proof of Theorem 1.2.

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