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The Horn cone associated with symplectic eigenvalues

Paul-Emile Paradan*

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Abstract

In this note, we show that the Horn cone associated with symplectic eigenvalues admits the same inequalities as the classical Horn cone, except that the equality corresponding to $\text{Tr}(C) = \text{Tr}(A) + \text{Tr}(B)$ is replaced by the inequality corresponding to $\text{Tr}(C) \geq \text{Tr}(A) + \text{Tr}(B)$.

1 Introduction

We consider \mathbb{R}^{2n} equipped with its canonical symplectic structure $\Omega_n = \sum_{k=1}^n dx_k \wedge dx_{k+n}$. Recall that a family $(e_k)_{1 \leq k \leq 2n}$ is a symplectic basis of \mathbb{R}^{2n} , if $\Omega_n(e_k, e_\ell) = 0$ if $|k - \ell| \neq n$ and $\Omega_n(e_k, e_{k+n}) = 1, \forall k$.

Williamson's theorem [18] says that any positive definite quadratic form $q : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ can be written $q(v) = \sum_{k=1}^n \lambda_k (v_k^2 + v_{k+n}^2)$ where the (v_j) are the coordinates of the vector $v \in \mathbb{R}^{2n}$ relatively to a symplectic basis. The positive numbers λ_k , that one choose so that

$$\lambda(q) := (\lambda_1 \geq \dots \geq \lambda_n),$$

will be referred to as the *symplectic eigenvalues* of the quadratic form q . They correspond to the frequencies of the normal modes of oscillation for the linear Hamiltonian system generated by q .

The object of study of this note concerns the symplectic Horn cone, denoted $\text{Horn}_{\text{sp}}(n)$, that is defined as the set of triplets $(\lambda(q_1), \lambda(q_2), \lambda(q_1 + q_2))$ where q_1, q_2 are positive definite quadratic forms on \mathbb{R}^{2n} .

Example 1.1 *In dimension 2, the symplectic eigenvalue $\lambda(q)$ of a positive definite quadratic form $q(x_1, x_2) = ax_1^2 + bx_2^2 + cx_1x_2$ is equal to $\frac{1}{2}\sqrt{4ab - c^2}$. It is straightforward to show that $\text{Horn}_{\text{sp}}(1)$ is equal to the set of triplets (x, y, z) of positive numbers satisfying $x + y \leq z$.*

Our main Theorem states that $\text{Horn}_{\text{sp}}(n)$ is a convex polyhedral set. Before detailing it, let us recall some related results.

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In [17], A. Weinstein showed that for non-increasing n -tuples of positive real numbers a and b , the set $\Delta_{\text{sp}}(a, b) := \{\lambda(q_1 + q_2) \mid \lambda(q_1) = a, \lambda(q_2) = b\}$ is closed, convex and locally polyhedral.

Recently, several authors have realized that some inequalities obtained long ago in the context of eigenvalues of Hermitian matrices still apply to symplectic eigenvalues:

- T. Hiroshima proved in [7] an analogue of Ky Fan inequalities: $\sum_{j=1}^k \lambda_j(q_1 + q_2) \geq \sum_{j=1}^k \lambda_j(q_1) + \sum_{j=1}^k \lambda_j(q_2)$
- In [8], T. Jain and H. Mishra obtained an analogue of Lidskii inequalities: $\sum_{j=1}^k \lambda_{i_j}(q_1 + q_2) \geq \sum_{j=1}^k \lambda_{i_j}(q_1) + \sum_{j=1}^k \lambda_j(q_2)$ for any subset $\{i_1 < i_2 < \dots < i_k\}$.
- In [1], R. Bhatia and T. Jain obtained an analogue of the Weyl inequalities: $\lambda_{i+j-1}(q_1 + q_2) \geq \lambda_i(q_1) + \lambda_j(q_2)$.

As the previous results suggest, we now explain the strong relationship between $\text{Horn}_{\text{sp}}(n)$ with the classical Horn cone. If A is a Hermitian $n \times n$ matrix, we denote by $\text{s}(A) = (\text{s}_1(A) \geq \dots \geq \text{s}_n(A))$ its spectrum. The Horn cone $\text{Horn}(n)$ is defined as the set of triplets $(\text{s}(A), \text{s}(B), \text{s}(A+B))$ where A, B are Hermitian $n \times n$ matrices.

Denote the set of cardinality r subsets $I = \{i_1 < i_2 < \dots < i_r\}$ of $[n] := \{1, \dots, n\}$ by \mathcal{P}_r^n . To each $I \in \mathcal{P}_r^n$ we associate:

- a weakly decreasing sequence of non-negative integers $\lambda(I) = (\lambda_1 \geq \dots \geq \lambda_r)$ where $\lambda_a = n - r + a - i_a$ for $a \in [r]$.
- the irreducible representation $V_{\lambda(I)}$ of $GL_r(\mathbb{C})$ with highest weight $\lambda(I)$.

If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $I \subset [n]$, we define $|x|_I = \sum_{i \in I} x_i$ and $|x| = \sum_{i=1}^n x_i$. Let us denote by \mathbb{R}_+^n the set of non-increasing n -tuples of real numbers.

A. Klyachko [10] has shown that an element $(x, y, z) \in (\mathbb{R}_+^n)^3$ belongs to the cone $\text{Horn}(n)$ if and only if it satisfies $|x| + |y| = |z|$ and

$$((\star)_{I,J,K}) \quad |x|_I + |y|_J \leq |z|_K$$

for any $r < n$, for any $I, J, K \in \mathcal{P}_r^n$ such that the Littlewood-Richardson coefficient

$$c_{IJ}^K := \dim[V_{\lambda(I)} \otimes V_{\lambda(J)} \otimes V_{\lambda(K)}^*]^{GL_r(\mathbb{C})}$$

is non-zero. P. Belkale [2] showed that the inequalities $(\star)_{I,J,K}$ associated to the condition $c_{IJ}^K = 1$ are sufficient. Finally A. Knutson, T. Tao, and C. Woodward [11] have proved that this smaller list is actually minimal. We refer the reader to survey articles [5, 3] for details.

The main result of this note is the following Theorem. Let us denote by \mathbb{R}_{++}^n the set of non-increasing n -tuples of positive real numbers.

Theorem 1.2 *An element $(x, y, z) \in (\mathbb{R}_{++}^n)^3$ belongs to $\text{Horn}_{\text{sp}}(n)$ if and only if it satisfies*

1. $|x| + |y| \leq |z|$,
2. $(\star)_{I,J,K}$ for all (I, J, K) of cardinality $r < n$ such that $c_{IJ}^K = 1$.

Corollary 1.3 *Let $a, b \in \mathbb{R}_{++}^n$. An element $z \in \mathbb{R}_{++}^n$ belongs to $\Delta_{\text{sp}}(a, b)$ if and only if it satisfies $|a| + |b| \leq |z|$ and $|a|_I + |b|_J \leq |z|_K$ for all (I, J, K) of cardinality $r < n$ such that $c_{IJ}^K = 1$.*

2 The causal cone of the symplectic Lie algebra

The $2n \times 2n$ matrix $J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ defines a complex structure on \mathbb{R}^{2n} that is compatible with the symplectic structure Ω_n . The symplectic group $Sp(\mathbb{R}^{2n})$ is defined by the relation ${}^t g J_n g = J_n$. A matrix X belongs to the Lie algebra $\mathfrak{sp}(\mathbb{R}^{2n})$ of $Sp(\mathbb{R}^{2n})$ if and only the matrix $J_n X$ is symmetric. Moreover, $J_n X$ is positive if and only if $\Omega_n(Xv, v) \geq 0, \forall v \in \mathbb{R}^{2n}$.

We call an invariant convex cone C in $\mathfrak{sp}(\mathbb{R}^{2n})$ a causal cone if C is nontrivial, closed, and satisfies $C \cap -C = \{0\}$. A classical result [16, 13, 14] asserts that there are exactly two causal cones in $\mathfrak{sp}(\mathbb{R}^{2n})$: one, denoted $\mathbf{C}(n)$, containing $-J_n$ and its opposite $-\mathbf{C}(n)$. The causal cone $\mathbf{C}(n)$ is determined by the following equivalent conditions: for $X \in \mathfrak{sp}(\mathbb{R}^{2n})$, we have

$$X \in \mathbf{C}(n) \iff J_n X \text{ is positive} \iff \text{Tr}(XgJ_n g^{-1}) \geq 0, \forall g \in Sp(\mathbb{R}^{2n}).$$

Now we explain how is parameterized the interior $\mathbf{C}(n)^0$ of $\mathbf{C}(n)$. From the definition above, we see first that $X \in \mathbf{C}(n)^0$ if and only if $J_n X$ is positive definite.

The Lie algebra of the maximal compact subgroup $K = Sp(2n, \mathbb{R}) \cap O(2n)$ is

$$\mathfrak{k} := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, {}^t A = -A, {}^t B = B \right\}.$$

If $\mu := (\mu_1, \dots, \mu_n)$, we write $\Delta(\mu) = \text{Diag}(\mu_1, \dots, \mu_n)$ and $X(\mu) = \begin{pmatrix} 0 & \Delta(\mu) \\ -\Delta(\mu) & 0 \end{pmatrix}$. We work with the Cartan subalgebra $\mathfrak{t} := \{X(\mu), \mu \in \mathbb{R}^n\}$ of \mathfrak{k} and the corresponding maximal torus $T \subset K$. The set of roots \mathfrak{R} relatively to the action of T on $\mathfrak{sp}(\mathbb{R}^{2n}) \otimes \mathbb{C}$ are composed by the compact ones $\mathfrak{R}_c := \{\epsilon_i - \epsilon_j\}$ and the non compact ones $\mathfrak{R}_n = \{\pm(\epsilon_i + \epsilon_j)\}$. We work with the subsets of positive roots $\mathfrak{R}_c^+ := \{\epsilon_i - \epsilon_j, i < j\}$ and $\mathfrak{R}_n^+ := \{\epsilon_i + \epsilon_j\}$. The Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$ is defined by the relations $\langle \alpha, \mu \rangle \geq 0, \forall \alpha \in \mathfrak{R}_c^+$, namely $\mu_1 \geq \dots \geq \mu_n$. The subchamber $\mathcal{C}_n \subset \mathfrak{t}_+$ is defined by the conditions $\langle \beta, \mu \rangle > 0, \forall \beta \in \mathfrak{R}_n^+$. Thus $X(\mu) \in \mathcal{C}_n$ if and only if $\mu \in \mathbb{R}_{++}^n$.

If $M \in \mathfrak{sp}(\mathbb{R}^{2n})$, we denote by $\mathcal{O}_M := \{gMg^{-1}, g \in Sp(\mathbb{R}^{2n})\}$ the corresponding adjoint orbit.

Lemma 2.1 1. $M \in \mathbf{C}(n)^0$ if and only if there exists $X \in \mathcal{C}_n$ such that $M \in \mathcal{O}_X$.

2. Let $\mu \in \mathbb{R}_{++}^n$, and $M \in \mathcal{O}_{X(\mu)}$. The symplectic eigenvalues of the positive definite quadratic form $q(v) = {}^t v J_n M v = \Omega_n(Mv, v)$ are the $\mu_1 \geq \dots \geq \mu_n > 0$.

Proof: The first point is a classical fact [16, 14]. If $M = gX(\mu)g^{-1}$ with $g \in Sp(\mathbb{R}^{2n})$, we see that

$$\Omega_n(Mv, v) = \Omega_n(X(\mu)g^{-1}v, g^{-1}v) = \sum_{k=1}^n \mu_k (v_k^2 + v_{k+n}^2)$$

where each v_j is the j -th coordinate of the vector $g^{-1}v$. \square

Remark 2.2 In [15], we call the interior $\mathbf{C}(n)^0$ of $\mathbf{C}(n)$ the holomorphic cone, since any coadjoint orbit $\mathcal{O}_X \subset \mathbf{C}(n)^0$ admits a canonical structure of a Kähler manifold with a holomorphic action of K . These orbits are closely related to the holomorphic discrete series representations of the symplectic group $Sp(\mathbb{R}^{2n})$.

Thanks to the previous Lemma, we see that the symplectic Horn cone admits the alternative definition:

$$\text{Horn}_{\text{sp}}(n) = \{(x, y, z) \in (\mathbb{R}_{++}^n)^3 \mid \mathcal{O}_{X(z)} \subset \mathcal{O}_{X(x)} + \mathcal{O}_{X(y)}\}.$$

In the next section, we explain the result of [15] concerning the determination of $\text{Horn}_{\text{sp}}(n)$.

3 Convexity results

The trace on $\mathfrak{gl}(\mathbb{R}^{2n})$ provides an identification between $\mathfrak{sp}(\mathbb{R}^{2n})$ and its dual $\mathfrak{sp}(\mathbb{R}^{2n})^*$: to $X \in \mathfrak{sp}(\mathbb{R}^{2n})$ we associate $\xi_X \in \mathfrak{sp}(\mathbb{R}^{2n})^*$ defined by $\langle \xi_X, Y \rangle = -\text{Tr}(XY)$. Through this identification the causal cone $\mathbf{C}(n)$ becomes

$$\tilde{\mathbf{C}}(n) := \{\xi \in \mathfrak{sp}(\mathbb{R}^{2n})^*; \langle \xi, \text{Ad}(g)z \rangle \geq 0, \forall g \in Sp(\mathbb{R}^{2n})\}$$

where $z = \frac{-1}{2}J_n$. The identification $\mathfrak{sp}(\mathbb{R}^{2n}) \simeq \mathfrak{sp}(\mathbb{R}^{2n})^*$ induces several identifications $\mathfrak{k} \simeq \mathfrak{k}^*$, $\mathfrak{t} \simeq \mathfrak{t}^*$ and $\mathfrak{t}_+ \simeq \mathfrak{t}_+^*$. In the latter cases the identifications are done through an invariant scalar product $(-, -)$ on \mathfrak{k}^* . The subchamber $\tilde{\mathcal{C}}_n \subset \mathfrak{t}_+^*$ is defined by the conditions: $(\alpha, \xi) \geq 0, \forall \alpha \in \mathfrak{A}_c^+$, and $(\beta, \xi) > 0, \forall \beta \in \mathfrak{A}_n^+$.

Through $\mathfrak{sp}(\mathbb{R}^{2n}) \simeq \mathfrak{sp}(\mathbb{R}^{2n})^*$, the symplectic Horn cone becomes

$$\text{Horn}_{\text{hol}}(Sp(\mathbb{R}^{2n})) := \{(\xi_1, \xi_2, \xi_3) \in (\tilde{\mathcal{C}}_n)^3 \mid \mathcal{O}_{\xi_3} \subset \mathcal{O}_{\xi_1} + \mathcal{O}_{\xi_2}\}.$$

Here we have kept the notations of [15].

We have a Cartan decomposition $\mathfrak{sp}(\mathbb{R}^{2n}) = \mathfrak{k} \oplus \mathfrak{p}$ with

$$\mathfrak{p} := \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix}, {}^t A = A, {}^t B = B \right\}.$$

We denote by \mathfrak{p}^+ the vector space \mathfrak{p} equipped with the complex structure $\text{ad}(z)$ and the compatible symplectic structure $\Omega_{\mathfrak{p}^+}(Y, Y') := -\text{Tr}(J_n[Y, Y'])$: here $\Omega_{\mathfrak{p}^+}(Y, [z, Y]) > 0$ for any $Y \neq 0$.

The action of maximal compact subgroup $K \subset Sp(\mathbb{R}^{2n})$ on $(\mathfrak{p}^+, \Omega_{\mathfrak{p}^+})$ is Hamiltonian with moment map

$$\Phi_{\mathfrak{p}^+} : \mathfrak{p}^+ \rightarrow \mathfrak{k}^*$$

defined by $\langle \Phi_{\mathfrak{p}^+}(Y), X \rangle = \frac{1}{2}\Omega_{\mathfrak{p}^+}([X, Y], Y)$. If $Y = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$, we see that $\langle \Phi_{\mathfrak{p}^+}(Y), J_n \rangle = \text{Tr}(A^2 + B^2) = \frac{1}{2}\|Y\|^2$. Hence the moment map $\Phi_{\mathfrak{p}^+}$ is proper.

We consider the following action of the group K^3 on the manifold $K \times K$:

$$(k_1, k_2, k_3) \cdot (g, h) = (k_1 g k_3^{-1}, k_2 h k_3^{-1})$$

The action of K^3 on the cotangent bundle $N := T^*(K \times K)$ is Hamiltonian with moment map $\Phi_N : N \rightarrow \mathfrak{k}^* \times \mathfrak{k}^* \times \mathfrak{k}^*$ defined by the relations¹

$$\Phi_N(g_1, \eta_1; g_2, \eta_2) = (-g_1 \eta_1, -g_2 \eta_2, \eta_1 + \eta_2).$$

Finally we consider the Hamiltonian K^3 -manifold $N \times \mathfrak{p}^+$, where \mathfrak{p}^+ is equipped with the symplectic structure $\Omega_{\mathfrak{p}^+}$. The action is defined by the relations: $(k_1, k_2, k_3) \cdot (g, h, X) = (k_1 g k_3^{-1}, k_2 h k_3^{-1}, k_3 X)$. Let us denote by $\Phi : N \times \mathfrak{p}^+ \rightarrow \mathfrak{k}^* \times \mathfrak{k}^* \times \mathfrak{k}^*$ the moment map relative to the K^3 -action :

$$(1) \quad \Phi(g_1, \eta_1; g_2, \eta_2, Y) = (-g_1 \eta_1, -g_2 \eta_2, \eta_1 + \eta_2 + \Phi_{\mathfrak{p}^+}(Y)).$$

Since Φ is proper map, the Convexity Theorem [9, 12] tell us that

$$\Delta(N \times \mathfrak{p}^+) := \text{Image}(\Phi) \cap \mathfrak{t}_+^* \times \mathfrak{t}_+^* \times \mathfrak{t}_+^*$$

is a closed, convex, and locally polyhedral set.

The map $\mu \mapsto X(\mu)$ defines an isomorphism of \mathbb{R}^n with $\mathfrak{t} \simeq \mathfrak{t}^*$ that induces an identification of \mathbb{R}_{++}^n with $\mathcal{C}_n \simeq \tilde{\mathcal{C}}_n$. Recall that on $\mathfrak{t}^* \simeq \mathbb{R}^n$, we have a natural involution that sends $\mu = (\mu_1, \dots, \mu_n)$ to $\mu^* := (-\mu_n, \dots, -\mu_1)$. The following result is proved in [15] (see Theorem B).

Theorem 3.1 *An element $(x, y, z) \in (\mathbb{R}_{++}^n)^3$ belongs to $\text{Horn}_{\text{hol}}(Sp(\mathbb{R}^{2n}))$ if and only if*

$$(x, y, z^*) \in \Delta(N \times \mathfrak{p}^+).$$

Recall that a Hermitian matrix M majorizes another Hermitian matrix M' if $M - M'$ is positive semidefinite (its eigenvalues are all nonnegative). In this case, we write $M \geq M'$.

Proposition 3.2 *Let $(x, y, z) \in (\mathbb{R}_{++}^n)^3$. Then $(x, y, z^*) \in \Delta(N \times \mathfrak{p}^+)$ if and only if there exist Hermitian matrices A, B, C such that $s(A) = x$, $s(B) = y$, $s(C) = z$ and $C \geq A + B$.*

¹We use the identification $T^*K \simeq K \times \mathfrak{k}^*$ given by left translations.

Proof: The map $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto A - iB$ defines an isomorphism between K and the unitary group $U(n)$. Let us denote by $S^2(\mathbb{C}^n)$ the vector space of complex $n \times n$ symmetric matrices that is equipped with the following action of $U(n)$: $k \cdot M = kM^t k$. The map $\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mapsto A - iB$ defines an isomorphism between the K -module \mathfrak{p}^+ and the $U(n)$ -module $S^2(\mathbb{C}^n)$. Through this identifications the moment map $\Phi_{\mathfrak{p}^+} : \mathfrak{p}^+ \rightarrow \mathfrak{k}^*$ becomes the map $\Phi_{S^2} : S^2(\mathbb{C}^n) \rightarrow \mathfrak{u}(n)$ defined by the relations

$$\Phi_{S^2}(M) = -2iM\overline{M}.$$

So we know that the moment polytope Δ relative to the Hamiltonian action of $U(n)^3$ on $T^*U(n) \times T^*U(n) \times S^2(\mathbb{C}^n)$ is equal to $\Delta(N \times \mathfrak{p}^+)$. A small computation shows that $(x, y, z^*) \in \Delta$ if and only if there exists Hermitian matrices A, B, C and $M \in S^2(\mathbb{C}^n)$ such that

$$s(A) = x, \quad s(B) = y, \quad s(C) = z \quad \text{and} \quad A + B + 2M\overline{M} = C.$$

The existence of $M \in S^2(\mathbb{C}^n)$ satisfying the condition $A + B + 2M\overline{M} = C$ is equivalent to $C \geq A + B$. The proof is then completed. \square

S. Friedland [4] considered the following question: *which eigenvalues $(s(A), s(B), s(C))$ can occur if $C \geq A + B$* . His solution was in terms of linear inequalities, which includes Klyachko's inequalities, a trace inequality and some additional inequalities. Later, W. Fulton [6] proved the additional inequalities are unnecessary. Let us summarize their result in the following Theorem.

Theorem 3.3 ([4, 6]) *A triple $x, y, z \in \mathbb{R}_+^n$ occurs as the eigenvalues of n by n Hermitian matrices A, B, C with $C \geq A + B$ if and only if it satisfies $|x| + |y| \leq |z|$ and $(\star)_{I,J,K}$ for all (I, J, K) of cardinality $r < n$ such that $c_{I,J}^K = 1$.*

The combination of Theorems 3.1 and 3.3 with Proposition 3.2 completes the proof of Theorem 1.2.

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