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## ON $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -ADDITIVE CONSTACYCLIC CODES

OM PRAKASH<sup>a</sup>, SHIKHA YADAV<sup>a</sup>, HABIBUL ISLAM<sup>a</sup>, AND PATRICK SOLÉ<sup>b1</sup>

<sup>a</sup> Department of Mathematics, Indian Institute of Technology Patna  
Patna- 801 106, India

<sup>b</sup>IM, (CNRS, Aix-Marseille University, Centrale Marseille) Marseille, France

E-mail: om@iitp.ac.in, 1821ma10@iitp.ac.in, habibul.pma17@iitp.ac.in, sole@enst.fr

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ABSTRACT. Let  $\mathbb{Z}_4$  be the ring of integers modulo 4. This paper studies mixed alphabets  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic and  $\lambda$ -constacyclic codes for units  $\lambda = 1 + 2u^2, 3 + 2u^2$ . First, we obtain the generator polynomials and minimal generating set of additive cyclic codes. Then we extend our study to determine the structure of additive constacyclic codes. Further, we define some Gray maps and obtain  $\mathbb{Z}_4$ -images of such codes. Finally, we present numerical examples that include six new and two best-known quaternary linear codes.

### 1. INTRODUCTION

Cyclic codes are one of the oldest and most studied classes of linear block codes due to strong algebraic properties which lead to efficient encoding and decoding algorithms. It also gained serious attention through several generalized classes like quasi-cyclic, constacyclic, skew cyclic, etc over the last few decades. Recall that a linear code is defined to be a subspace (resp. submodule) of a vector space (resp. module) over a finite field (resp. ring). Classically, a cyclic code is defined as a linear code invariant under the circular shift of coordinates. Therefore, it can be classified as an ideal of a quotient polynomial ring by identifying its codewords with polynomials. This leads to characterizing them algebraically over finite fields or rings. Recently, the linearity condition has been relaxed, and a new class of codes has been defined, namely, additive codes in [15]. In fact, an additive code has a group structure rather than a linear structure like a linear code. To be more specific, we can say that over a finite field  $\mathbb{F}_q$ , a linear code  $C$  is a subspace of  $\mathbb{F}_q^n$  whereas an additive code is a subgroup of  $\mathbb{F}_q^n$ . Therefore, every linear code is additive but not conversely. After the introduction of additive codes in [10], these codes were studied in both directions of single and mixed alphabets [11, 22, 23, 29]. One important application of such codes in steganography is shown in [24]. In 2010, Borges et al. [11] studied  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes to discuss their generator polynomials and dual codes while on the other hand, their rank and kernel are classified in [17]. Later, Abualrub et al. [1] studied  $\mathbb{Z}_2\mathbb{Z}_4$ -additive cyclic codes by identifying them as  $\mathbb{Z}_4[x]$ -submodules of  $\mathbb{Z}_2[x]/\langle x^r - 1 \rangle \times \mathbb{Z}_4[x]/\langle x^s - 1 \rangle$  for the length

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<sup>1</sup>Corresponding author

$(r, s)$ . Further, Borges et al. [12] revisited  $\mathbb{Z}_2\mathbb{Z}_4$ -additive cyclic codes and obtained their generator polynomials and dual codes. Later on, the study was extended to more general mixed alphabets like  $\mathbb{Z}_2\mathbb{Z}_{2^s}$  in [8],  $\mathbb{Z}_p\mathbb{Z}_{p^s}$  in [27, 30],  $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$  in [9, 31] subsequently. Also, a new alphabet, namely,  $\mathbb{Z}_2\mathbb{Z}_2[u]$  where  $u^2 = 0$  is considered in [4, 5, 13, 28]. Note that [5] is the first paper to study additive constacyclic codes in mixed alphabets setting and defined  $\mathbb{Z}_2\mathbb{Z}_2[u]$ -additive  $(1 + u)$ -cyclic codes for a unit  $1 + u$  as  $\mathbb{Z}_2[u][x]$ -submodules of  $\mathbb{Z}_2[x]/\langle x^r - 1 \rangle \times \mathbb{Z}_2[u][x]/\langle x^s - (1 + u) \rangle$ . Further, mixed alphabets additive constacyclic codes are extensively studied in [20, 21, 18]. Again, towards the generalization of these codes over mixed alphabets  $\mathbb{Z}_{2^r}\mathbb{Z}_{2^s}[u]$  where  $u^2 = 0$ , the reference [18] studied  $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic and constacyclic codes for  $s = r = 2$ . Therefore, it is still open and interesting to study these codes for  $s, r \geq 3$ . Besides these all, another set of alphabets  $\mathbb{Z}_2\mathbb{Z}_2[u^3]$  appeared in [6], where they determined the generator polynomials and dual codes of  $\mathbb{Z}_2\mathbb{Z}_2[u^3]$ -additive cyclic codes. As a computational result, they have also explored some binary optimal codes from the Gray images of such codes. Again, Aydogdu in [7] generalized  $\mathbb{Z}_2\mathbb{Z}_2[u]$ -linear codes over  $\mathbb{Z}_p[u]/\langle u^r \rangle \times \mathbb{Z}_p[u]/\langle u^s \rangle$  where  $p$  is a prime number. Along with their generator and parity check matrices, he presented binary Gray images for  $p = 2$ . Therefore, one may logically attempt to extend the alphabets of [6] to  $\mathbb{Z}_{2^r}\mathbb{Z}_{2^s}[u^3]$ , or more generally  $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}[u^3]$  where  $u^3 = 0$ . To address this problem, we particularly consider  $s = r = 2$  and study  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic and constacyclic codes. The cases for  $r, s \geq 3$  and prime  $p > 2$  are still open for investigation. Here, we first determine the generator polynomials and minimal spanning set for an additive cyclic code. Then we extend our study to additive  $\lambda$ -constacyclic codes for units  $\lambda = 1 + 2u^2$  and  $\lambda = 3 + 2u^2$ , respectively. In addition, we define some Gray maps and obtain quaternary cyclic, quasi-cyclic, and generalized quasi-cyclic (GQC) codes from their Gray images. Finally, we calculate their  $\mathbb{Z}_4$ -parameters, among them some are new and some are as good as the best-known codes. Note that along with the algebraic structure of these codes our defined Gray maps play a key role to obtain new and better  $\mathbb{Z}_4$ -parameters. There might exist some other Gray maps too, which would be capable to produce more new parameters. One can further investigate this direction, it would be interesting for future scope.

This paper is organized as follows: Section 2 contains some basic definitions and results for additive codes. Section 3 provides the generator polynomials and minimal generating sets of  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic codes. In Section 4, we obtain generator polynomials of  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive constacyclic codes. In Section 5, we define some Gray maps and obtain the Gray images of additive constacyclic codes. In Section 6, we provide several examples to validate our results. Section 7 concludes the paper.

## 2. PRELIMINARY

Let  $\mathbb{Z}_4$  be the ring of integers modulo 4. A *linear code*  $C$  of length  $n$  over  $\mathbb{Z}_4$  is defined as a  $\mathbb{Z}_4$ -submodule of the module  $\mathbb{Z}_4^n$ . For  $a = (a_0, \dots, a_{n-1})$  and  $b = (b_0, \dots, b_{n-1}) \in \mathbb{Z}_4^n$ , the *Euclidean inner product* is defined as  $a \cdot b = a_0b_0 + a_1b_1 + \dots + a_{n-1}b_{n-1} \pmod{4}$ . The *Euclidean dual* of a linear code  $C$  over  $\mathbb{Z}_4$  is denoted by  $C^\perp$  and defined as  $C^\perp = \{v \in \mathbb{Z}_4^n : v \cdot c = 0 \text{ for all } c \in C\}$ . We recall that the *Lee weight* of  $a \in \mathbb{Z}_4$  is  $w_L(a) = \min\{a, |4 - a|\}$ , i.e.,  $w_L(0) = 0, w_L(1) = 1, w_L(2) = 2, w_L(3) = 1$ .

Throughout this paper, we consider the ring  $\mathbb{Z}_4[u^3] = \{a + bu + cu^2 : a, b, c \in \mathbb{Z}_4\}$  where  $u^3 = 0$ ,  $\mathbb{Z}_4\mathbb{Z}_4[u^3] = \{(a, b) : a \in \mathbb{Z}_4, b \in \mathbb{Z}_4[u^3]\}$ . Let  $\gamma, \delta$  be odd positive integers. We define a projection map  $\sigma : \mathbb{Z}_4[u^3] \rightarrow \mathbb{Z}_4$  by

$$\sigma(a + bu + cu^2) = a$$

and a *scalar multiplication*

$$* : \mathbb{Z}_4[u^3] \times \mathbb{Z}_4\mathbb{Z}_4[u^3] \rightarrow \mathbb{Z}_4\mathbb{Z}_4[u^3]$$

by

$$r * (a, b) = (\sigma(r)a, rb), \text{ for } r \in \mathbb{Z}_4[u^3].$$

The above multiplication can be extended to  $\mathbb{Z}_4^\gamma \times \mathbb{Z}_4[u^3]^\delta$  as

$$r * (a_0, a_1, \dots, a_{\gamma-1}, b_0, b_1, \dots, b_{\delta-1}) = (\sigma(r)a_0, \dots, \sigma(r)a_{\gamma-1}, rb_0, \dots, rb_{\delta-1}).$$

Then  $\mathbb{Z}_4^\gamma \times \mathbb{Z}_4[u^3]^\delta$  forms a  $\mathbb{Z}_4[u^3]$ -module with respect to the usual addition and the scalar multiplication  $'*$  defined above.

**Definition 2.1.** A nonempty subset  $C$  of  $\mathbb{Z}_4^\gamma \times \mathbb{Z}_4[u^3]^\delta$  is said to be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive code of length  $(\gamma, \delta)$  if it is a  $\mathbb{Z}_4[u^3]$ -submodule of  $\mathbb{Z}_4^\gamma \times \mathbb{Z}_4[u^3]^\delta$ .

**Definition 2.2.** A  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive code  $C$  of length  $(\gamma, \delta)$  is said to be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic code if  $(a, b) = (a_0, a_1, \dots, a_{\gamma-1}, b_0, b_1, \dots, b_{\delta-1}) \in C$  implies that  $\tau_{\gamma, \delta}(a, b) = (a_{\gamma-1}, a_0, \dots, a_{\gamma-2}, b_{\delta-1}, b_0, \dots, b_{\delta-2}) \in C$ .

Now, we extend the projection map  $\sigma$  from  $\mathbb{Z}_4[u^3][x]$  to  $\mathbb{Z}_4[x]$  by

$$\sigma\left(\sum_i d_i x^i\right) = \sum_i \sigma(d_i) x^i.$$

Let  $R_\gamma = \mathbb{Z}_4[x]/\langle x^\gamma - 1 \rangle$  and  $S_\delta = \mathbb{Z}_4[u^3][x]/\langle x^\delta - 1 \rangle$ . Then  $R_\gamma \times S_\delta$  is a  $\mathbb{Z}_4[u^3][x]$ -module under the usual addition and scalar multiplication defined by

$$r(x) * (a(x), b(x)) = (\sigma(r(x))a(x), r(x)b(x)).$$

For a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive code  $C$  of length  $(\gamma, \delta)$ , we identify its codeword  $c = (a_0, a_1, \dots, a_{\gamma-1}, b_0, b_1, \dots, b_{\delta-1})$  to the pair of polynomials  $(a(x), b(x)) \in R_\gamma \times S_\delta$  where  $a(x) = \sum_{i=0}^{\gamma-1} a_i x^i \in R_\gamma$ ,  $b(x) = \sum_{i=0}^{\delta-1} b_i x^i \in S_\delta$ . Then  $C$  is a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic code if and only if  $C$  is a  $\mathbb{Z}_4[u^3][x]$ -submodule of  $R_\gamma \times S_\delta$ .

Now, by following [6], we consider an inner product for any two elements  $r = (a_0, \dots, a_{\gamma-1}, b_0, \dots, b_{\delta-1})$  and  $s = (c_0, \dots, c_{\gamma-1}, d_0, \dots, d_{\delta-1})$  in  $\mathbb{Z}_4^\gamma \times \mathbb{Z}_4[u^3]^\delta$  as

$$r \cdot s = u^2 \left( \sum_{i=0}^{\gamma-1} a_i c_i \right) + \left( \sum_{j=0}^{\delta-1} b_j d_j \right).$$

**Definition 2.3.** Let  $C$  be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive code. Then the dual  $C^\perp$  of  $C$  is defined as

$$C^\perp = \{s \in \mathbb{Z}_4^\gamma \times \mathbb{Z}_4[u^3]^\delta : s \cdot r = 0, \text{ for all } r \in C\}.$$

**Theorem 2.4.** Let  $C$  be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic code of length  $(\gamma, \delta)$ . Then  $C^\perp$  is also a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic code of length  $(\gamma, \delta)$ .

*Proof.* If  $r = (a_0, \dots, a_{\gamma-1}, b_0, \dots, b_{\delta-1}) \in C^\perp$ , then  $r \cdot s = 0$ , for all  $s = (c_0, \dots, c_{\gamma-1}, d_0, \dots, d_{\delta-1}) \in C$ . As  $C$  is an additive cyclic code, we have  $\tau_{\gamma, \delta}^{t-1}(s) = (c_1, c_2, \dots, c_{\gamma-1}, c_0, d_1, d_2, \dots, d_{\delta-1}, d_0) \in C$  where  $t = \text{lcm}(\gamma, \delta)$ . Now, in  $\mathbb{Z}_4^\gamma \times \mathbb{Z}_4[u^3]^\delta$ , we have

$$\begin{aligned} 0 &= r \cdot \tau_{\gamma, \delta}^{t-1}(s) = u^2(a_0c_1 + a_1c_2 + \dots + a_{\gamma-2}c_{\gamma-1} + a_{\gamma-1}c_0) \\ &\quad + (b_0d_1 + b_1d_2 + \dots + b_{\delta-2}d_{\delta-1} + b_{\delta-1}d_0) \\ &= \tau_{\gamma, \delta}(r) \cdot s. \end{aligned}$$

Therefore,  $\tau_{\gamma, \delta}(r) \in C^\perp$  and hence  $C^\perp$  is an additive cyclic code.  $\square$

**Definition 2.5.** Let  $C$  be a linear code of length  $n = lm$  over  $\mathbb{Z}_4$ . Then  $C$  is said to be a quasi-cyclic code of index  $l$ , if for each

$$s = (a_{0,0}, a_{0,1}, \dots, a_{0,m-1}, a_{1,0}, \dots, a_{1,m-1}, \dots, a_{l-1,0}, \dots, a_{l-1,m-1}) \in C,$$

its quasi-cyclic shift given by

$$\begin{aligned} \pi_l(s) &= (a_{0,m-1}, a_{0,0}, \dots, a_{0,m-2}, a_{1,m-1}, a_{1,0}, \dots, a_{1,m-2}, \dots, a_{l-1,m-1}, a_{l-1,0}, \\ &\quad \dots, a_{l-1,m-2}) \end{aligned}$$

is also in  $C$ . Moreover, if  $l = 1$ , then  $C$  is a cyclic code of length  $n$ .

### 3. ADDITIVE CYCLIC CODES

In this section, we are interested to determine the algebraic structure of  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic codes in terms of their generator polynomials and minimal spanning sets for these codes. The two main results of the present section are Theorem 3.2 and Theorem 3.4. In this direction, we first state a result from [19] which presents the structure of cyclic code and will be used to obtain the structure of  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic codes.

**Theorem 3.1.** [19, Theorem 11] *Let  $C_k$  be a cyclic code of odd length  $n$  over  $\mathbb{Z}_4 + u\mathbb{Z}_4 + \dots + u^{k-1}\mathbb{Z}_4$ , where  $u^k = 0$ . Then  $C_k = \langle g_1(x) + 2a_1(x) + up_1(x) + u^2p_2(x) + \dots + u^{k-1}p_{k-1}(x), u(g_2(x) + 2a_2(x)) + u^2q_1(x) + \dots + u^{k-1}q_{k-2}(x), \dots, u^{k-1}(g_k(x) + 2a_k(x)), a_i|g_i(x)|(x^n - 1) \pmod{4} \rangle$ .*

Let  $C$  be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic code of length  $(\gamma, \delta)$ . Define

$$\Gamma : R_\gamma \times S_\delta \rightarrow S_\delta$$

by  $\Gamma(p(x), q(x)) = q(x)$ . Then  $\ker(\Gamma) = (p(x), 0)$ . Since  $L = \{p(x) \in R_\gamma : p(x) \in \ker(\Gamma|_C)\}$  is an ideal of  $R_\gamma$ , it can be written as  $L = \langle g(x) + 2a(x) \rangle$ . Hence,  $\ker(\Gamma|_C) = \langle (g(x) + 2a(x), 0) \rangle$ , where  $a(x)|g(x)$ . Also,  $\Gamma(C)$  is an ideal in  $S_\delta$ . Therefore, by [19, Theorem 11] we get

$$\Gamma(C) = \langle g_1(x) + 2a_1(x) + up_1(x) + u^2p_2(x), u(g_2(x) + 2a_2(x)) + u^2q_1(x), u^2(g_3(x) + 2a_3(x)) \rangle$$

where  $a_i|g_i(x)|x^\delta - 1 \pmod{4}$ . Hence,

$$C = \langle (g(x) + 2a(x), 0), (f_1(x), g_1(x) + 2a_1(x) + up_1(x) + u^2p_2(x)), (f_2(x), u(g_2(x) + 2a_2(x)) + u^2q_1(x)), (f_3(x), u^2(g_3(x) + 2a_3(x))) \rangle$$

where  $p_1(x), p_2(x), q_1(x)$  and  $f_i(x) \in \mathbb{Z}_4[x]$  for  $i = 1, 2, 3$ . For simplicity, we write  $g$  for  $g(x)$ ,  $a$  for  $a(x)$ ,  $a_i$  for  $a_i(x)$  and so on. From the above discussion, we have the following theorem.

**Theorem 3.2.** A  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic code  $C$  of length  $(\gamma, \delta)$  for odd positive integers  $\gamma$  and  $\delta$  is given by  $C = \langle (g + 2a, 0), (f_1, g_1 + 2a_1 + up_1 + u^2p_2), (f_2, u(g_2 + 2a_2) + u^2q_1), (f_3, u^2(g_3 + 2a_3)) \rangle$ , where  $a|g|(x^\gamma - 1)$ ,  $a_i|g_i|(x^\delta - 1)$  and  $f_i(x) \in \mathbb{Z}_4[x]$  for  $i = 1, 2, 3$ .

To prepare the proof of the next theorem we require a technical lemma.

**Lemma 3.3.** Let  $C$  be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic code given in the above theorem. Let  $h_i = \frac{x^\delta - 1}{a_i}$ , for  $i = 1, 2$ ,  $l_1 = \gcd(h_1p_1 + uh_1p_2, x^\delta - 1)$ ,  $l_2 = \gcd(h_2q_1, x^\delta - 1)$ ,  $m_i = \frac{x^\delta - 1}{l_i}$  for  $i = 1, 2$ . Then  $(g + 2a) | \frac{x^\delta - 1}{a_3} f_3$  and  $(g + 2a) | m_j h_j f_j$ , for  $j = 1, 2$ .

*Proof.* Since  $\Gamma[\frac{x^\delta - 1}{a_3}(f_3, u^2(g_3 + 2a_3))] = \Gamma[\frac{x^\delta - 1}{a_3}f_3, 0] = 0$ , which implies that  $\frac{x^\delta - 1}{a_3}f_3 \in \ker(\Gamma|_C)$ , i.e.,  $(g + 2a) | \frac{x^\delta - 1}{a_3}f_3$ . Now,  $\Gamma[m_1h_1(f_1, g_1 + 2a_1 + up_1 + u^2p_2)] = \Gamma(m_1h_1f_1, m_1h_1(up_1 + u^2p_2)) = \Gamma[m_1h_1f_1, 0] = 0$ . Therefore,  $(m_1h_1f_1, 0) \in \ker(\Gamma|_C)$ , i.e.,  $(g + 2a) | m_1h_1f_1$ . Similarly,  $\Gamma[m_2h_2(f_2, u(g_2 + 2a_2))] = 0$ , which implies that  $(g + 2a) | m_2h_2f_2$ .  $\square$

The following theorem gives a minimal spanning set for an additive cyclic code  $C$ . By using the linear combination of elements in  $B$ , one can easily obtain all its codewords. In addition, the spanning set  $B$  helps to form its generator matrix which is useful to obtain its  $\mathbb{Z}_4$ -parameters under Gray image.

**Theorem 3.4.** Let  $C$  be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive code of length  $(\gamma, \delta)$  given in the above theorem. Take

$$\begin{aligned} B_1 &= \bigcup_{i=0}^{\gamma - \deg(a) - 1} \{x^i * (g + 2a, 0)\}, \\ B_2 &= \bigcup_{i=0}^{\delta - \deg(a_1) - 1} \{x^i * (f_1, g_1 + 2a_1 + up_1 + u^2p_2)\}, \\ B_3 &= \bigcup_{i=0}^{\delta - \deg(a_2) - 1} \{x^i * (f_2, u(g_2 + 2a_2) + u^2q_1)\}, \\ B_4 &= \bigcup_{i=0}^{\delta - \deg(a_3) - 1} \{x^i * (f_3, u^2(g_3 + 2a_3))\}, \\ B_5 &= \bigcup_{i=0}^{\delta - \deg(l_1) - 1} \{x^i * (h_1f_1, h_1(up_1 + u^2p_2))\}, \\ B_6 &= \bigcup_{i=0}^{\delta - \deg(l_2) - 1} \{x^i * (h_2f_2, u^2h_2q_1)\}. \end{aligned}$$

Then  $B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$  is a minimal spanning set for  $C$  as a  $\mathbb{Z}_4[u^3]$ -module. Moreover,  $|C| = 4^{\gamma + 9\delta - \deg(a) - 3\deg(a_1) - 2\deg(a_2) - \deg(a_3) - 2\deg(l_1) - \deg(l_2)}$ .

*Proof.* From Theorem 3.2, any  $c \in C$  can be written as

$$\begin{aligned} c &= c_1 * (g + 2a, 0) + c_2 * (f_1, g_1 + 2a_1 + up_1 + u^2p_2) + c_3 * (f_2, u(g_2 + 2a_2) + u^2q_1) \\ &\quad + c_4 * (f_3, u^2(g_3 + 2a_3)), \end{aligned}$$

where  $c_i \in \mathbb{Z}_4[u^3][x]$  for  $1 \leq i \leq 4$ . If  $\deg(c_1) \leq \gamma - \deg(a) - 1$ , then  $c_1 * (g + 2a, 0) \in \text{span}(B_1)$ . Otherwise, by division algorithm,  $c_1 = \frac{x^\gamma - 1}{a}r_0 + s_0$ , for some  $r_0, s_0 \in \mathbb{Z}_4[u^3][x]$  where  $\deg(s_0) \leq \gamma - \deg(a) - 1$ . Now,

$$\begin{aligned} c_1 * (g + 2a, 0) &= \left( \frac{x^\gamma - 1}{a}r_0 + s_0 \right) * (g + 2a, 0) \\ &= \left( \frac{x^\gamma - 1}{a}r_0 \right) * (g + 2a, 0) + s_0 * (g + 2a, 0) \\ &= s_0 * (g + 2a, 0) \in \text{span}(B_1). \end{aligned}$$

Also, if  $\deg(c_2) \leq \delta - \deg(a_1) - 1$ , then  $c_2 * (f_1, g_1 + 2a_1 + up_1 + u^2p_2) \in \text{span}(B_2)$ . Otherwise, by division algorithm,  $c_2 = \frac{x^\delta - 1}{a_1}r_1 + s_1$ , for some  $r_1, s_1 \in \mathbb{Z}_4[u^3][x]$  where  $\deg(s_1) \leq \delta - \deg(a_1) - 1$ . Again,

$$\begin{aligned} c_2 * (f_1, g_1 + 2a_1 + up_1 + u^2p_2) &= \left( \frac{x^\delta - 1}{a_1}r_1 + s_1 \right) * (f_1, g_1 + 2a_1 + up_1 + u^2p_2) \\ &= r_1 * (h_1f_1, h_1(up_1 + u^2p_2)) + s_1 * (f_1, g_1 + 2a_1 + up_1 + u^2p_2) \end{aligned}$$

where  $s_1 * (f_1, g_1 + 2a_1 + up_1 + u^2p_2) \in \text{span}(B_2)$ . If  $\deg(r_1) \leq \delta - \deg(l_1) - 1$ , then  $r_1 * (h_1f_1, h_1(up_1 + u^2p_2)) \in \text{span}(B_5)$ . Otherwise, by division algorithm,  $r_1 = \frac{x^\delta - 1}{l_1}r_2 + s_2$  for some  $r_2, s_2 \in \mathbb{Z}_4[u^3][x]$  where  $\deg(s_2) \leq \delta - \deg(l_1) - 1$ . Therefore,

$$\begin{aligned} r_1 * (h_1f_1, h_1(up_1 + u^2p_2)) &= \frac{x^\delta - 1}{l_1}r_2 * (h_1f_1, h_1(up_1 + u^2p_2)) + s_2 * (h_1f_1, h_1(up_1 + u^2p_2)) \\ &= r_2 * (m_1h_1f_1, m_1h_1(up_1 + u^2p_2)) + s_2 * (h_1f_1, h_1(up_1 + u^2p_2)). \end{aligned}$$

Since  $l_1 | (h_1p_1 + uh_1p_2)$ , we have  $(h_1p_1 + uh_1p_2) = l_1m_3$  for some  $m_3 \in \mathbb{Z}_4[u^3][x]$  and  $m_1(h_1p_1 + uh_1p_2) = m_1l_1m_3 = 0$ . Also,  $(g + 2a) | m_1h_1f_1$ . Therefore,

$$\begin{aligned} r_1 * (h_1f_1, h_1(up_1 + u^2p_2)) &= r_2 * (m_1h_1f_1, 0) + s_2 * (h_1f_1, h_1(up_1 + u^2p_2)) \\ &\in \text{span}(B_1 \cup B_5). \end{aligned}$$

Again, if  $\deg(c_3) \leq \delta - \deg(a_2) - 1$ , then  $c_3 * (f_2, u(g_2 + 2a_2) + u^2q_1) \in \text{span}(B_3)$ . Otherwise, by division algorithm,  $c_3 = \frac{x^\delta - 1}{a_2}r_3 + s_3$ , for some  $r_3, s_3 \in \mathbb{Z}_4[u^3][x]$  where  $\deg(s_3) \leq \delta - \deg(a_2) - 1$ . Therefore,

$$\begin{aligned} c_3 * (f_2, u(g_2 + 2a_2) + u^2q_1) &= \left( \frac{x^\delta - 1}{a_2}r_3 + s_3 \right) * (f_2, u(g_2 + 2a_2) + u^2q_1) \\ &= r_3 * (h_2f_2, u^2h_2q_1) + s_3 * (f_2, u(g_2 + 2a_2) + u^2q_1) \end{aligned}$$

where  $s_3 * (f_2, u(g_2 + 2a_2) + u^2q_1) \in \text{span}(B_3)$ . If  $\deg(r_3) \leq \delta - \deg(l_2) - 1$ , then  $r_3 * (h_2f_2, u^2h_2q_1) \in \text{span}(B_6)$ . Otherwise, by division algorithm,  $r_3 = \frac{x^\delta - 1}{l_2}r_4 + s_4$

for some  $r_4, s_4 \in \mathbb{Z}_4[u^3][x]$  where  $\deg(s_4) \leq \delta - \deg(l_2) - 1$ . Now,

$$\begin{aligned} r_3 * (h_2 f_2, u^2 h_2 q_1) &= \frac{x^\delta - 1}{l_2} r_4 * (h_2 f_2, u^2 h_2 q_1) + s_4 * (h_2 f_2, u^2 h_2 q_1) \\ &= r_4 * (m_2 h_2 f_2, u^2 m_2 h_2 q_1) + s_4 * (h_2 f_2, u^2 h_2 q_1). \end{aligned}$$

Since  $l_2 | h_2 q_1$ , we have  $h_2 q_1 = l_2 m_4$ , for some  $m_4 \in \mathbb{Z}_4[u^3][x]$  and  $m_2 h_2 q_1 = m_2 l_2 m_4 = 0$ . Also, by Lemma 3.3, we have  $(g + 2a) | m_2 h_2 f_2$ . Therefore,

$$\begin{aligned} r_3 * (h_2 f_2, u^2 h_2 q_1) &= r_4 * (m_2 h_2 f_2, 0) + s_4 * (h_2 f_2, u^2 h_2 q_1) \\ &\in \text{span}(B_1 \cup B_6). \end{aligned}$$

Again, if  $\deg(c_4) \leq \delta - \deg(a_3) - 1$ , then  $c_4 * (f_3, u^2(g_3 + 2a_3)) \in \text{span}(B_4)$ . Otherwise, division algorithm gives  $c_4 = \frac{x^\delta - 1}{a_3} r_5 + s_5$ , for some  $r_5, s_5 \in \mathbb{Z}_4[u^3][x]$  where  $\deg(s_5) \leq \delta - \deg(a_3) - 1$ . Now,

$$\begin{aligned} c_4 * (f_3, u^2(g_3 + 2a_3)) &= \left( \frac{x^\delta - 1}{a_3} r_5 + s_5 \right) * (f_3, u^2(g_3 + 2a_3)) \\ &= r_5 * \left( \frac{x^\delta - 1}{a_3} f_3, 0 \right) + s_5 * (f_3, u^2(g_3 + 2a_3)) \end{aligned}$$

where  $s_5 * (f_3, u^2(g_3 + 2a_3)) \in \text{span}(B_4)$ . Also, by Lemma 3.3 we have  $(g + 2a) | \left( \frac{x^\delta - 1}{a_3} \right) f_3$ . Therefore,  $c_4 * (f_3, u^2(g_3 + 2a_3)) \in \text{span}(B_1 \cup B_4)$  and we conclude that  $B$  spans  $C$ . Also, none of the elements of  $B$  are in the span of the remaining elements of  $B$ . Hence, we get the desired result.  $\square$

#### 4. ADDITIVE CONSTACYCLIC CODES

In this section, we extend our study to the algebraic structure of  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive  $\lambda$ -constacyclic codes. Here, we assume that  $\lambda$  be a unit in  $\mathbb{Z}_4[u^3]$ . We first recall the definition of such codes as below.

**Definition 4.1.** Let  $C$  be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive code of length  $(\gamma, \delta)$ . Then  $C$  is said to be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive constacyclic code if

$$(a, b) = (a_0, a_1, \dots, a_{\gamma-1}, b_0, b_1, \dots, b_{\delta-1}) \in C$$

implies

$$\rho_\lambda(a, b) = (a_{\gamma-1}, a_0, \dots, a_{\gamma-2}, \lambda b_{\delta-1}, b_0, \dots, b_{\delta-2}) \in C.$$

In particular, for  $\lambda = 1$ , it is an additive cyclic code.

We consider  $R_\gamma \times S_{\delta, \lambda} = \mathbb{Z}_4[x]/\langle x^\gamma - 1 \rangle \times \mathbb{Z}_4[u^3][x]/\langle x^\delta - \lambda \rangle$  and consider the scalar multiplication

$$* : \mathbb{Z}_4[u^3][x] \times (R_\gamma \times S_{\delta, \lambda}) \rightarrow R_\gamma \times S_{\delta, \lambda}$$

defined by

$$r(x) * (a(x), b(x)) = (\sigma(r(x))a(x), r(x)b(x)), \text{ for } r(x) \in \mathbb{Z}_4[u^3][x].$$

In this way  $R_\gamma \times S_{\delta, \lambda}$  forms a  $\mathbb{Z}_4[u^3][x]$ -module with respect to the scalar multiplication  $'*'$  and the usual addition. Now, identifying a vector

$$c = (a_0, a_1, \dots, a_{\gamma-1}, b_0, b_1, \dots, b_{\delta-1})$$



in  $\mathbb{Z}_4^\gamma \times \mathbb{Z}_4[u^3]^\delta$  to the polynomial  $c(x) = (a(x), b(x)) \in R_\gamma \times S_{\delta, \lambda}$  where  $a(x) = \sum_{i=0}^{\gamma-1} a_i x^i \in R_\gamma$ ,  $b(x) = \sum_{i=0}^{\delta-1} b_i x^i \in S_{\delta, \lambda}$ , we notice that  $\rho_\lambda(c)$  corresponds to the scalar multiplication  $x * (a(x), b(x))$ .

**Lemma 4.2.** *Let  $C$  be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive code of length  $(\gamma, \delta)$ . Then  $C$  is a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive constacyclic code if and only if  $C$  is a  $\mathbb{Z}_4[u^3][x]$ -submodule of  $R_\gamma \times S_{\delta, \lambda}$ .*

*Proof.* If  $C$  is a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive constacyclic code, then  $x^i * (p(x), q(x)) \in C$ , for  $c(x) = (p(x), q(x)) \in C$ . This implies that  $b(x) * (p(x), q(x)) \in C$ , for  $b(x) \in \mathbb{Z}_4[u^3][x]$ . Therefore,  $C$  is a  $\mathbb{Z}_4[u^3][x]$ -submodule of  $R_\gamma \times S_{\delta, \lambda}$ .

Conversely, let  $C$  be a  $\mathbb{Z}_4[u^3][x]$ -submodule of  $R_\gamma \times S_{\delta, \lambda}$  and  $c = (p(x), q(x)) \in C$ . Then  $x * (p(x), q(x)) = \rho_\lambda(c) \in C$ . Therefore,  $C$  is a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive constacyclic code.  $\square$

Let  $C$  be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive constacyclic code of length  $(\gamma, \delta)$ . Define a module homomorphism

$$\Gamma_1 : \mathbb{Z}_4[x]/\langle x^\gamma - 1 \rangle \times \mathbb{Z}_4[u^3][x]/\langle x^\delta - \lambda \rangle \rightarrow \mathbb{Z}_4[u^3][x]/\langle x^\delta - \lambda \rangle$$

by  $\Gamma_1(p(x), q(x)) = q(x)$ . Then, by the similar arguments as given for cyclic codes, we have  $\ker(\Gamma_1|_C) = \langle (g(x) + 2a(x), 0) \rangle$ , where  $a(x)|g(x)|(x^\gamma - 1)$ . Also,  $\Gamma_1(C)$  is an ideal in  $S_{\delta, \lambda}$ , therefore, by [19, Theorem 13], we get  $\Gamma_1(C) = \langle g_1(\hat{x}) + 2a_1(\hat{x}) + up_1(\hat{x}) + u^2p_2(\hat{x}), u(g_2(\hat{x}) + 2a_2(\hat{x})) + u^2q_1(\hat{x}), u^2(g_3(\hat{x}) + 2a_3(\hat{x})) \rangle$ , where  $a_i(x)|g_i(x)|(x^\delta - \lambda) \pmod{4}$  and  $\hat{x} = \lambda x$ . From the above discussion, we have the following theorem.

**Theorem 4.3.** *A  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive constacyclic code  $C$  of length  $(\gamma, \delta)$  for odd positive integers  $\gamma, \delta$  is given by  $C = \langle (g(x) + 2a(x), 0), (f_1(x), g_1(\hat{x}) + 2a_1(\hat{x}) + up_1(\hat{x}) + u^2p_2(\hat{x})), (f_2(x), u(g_2(\hat{x}) + 2a_2(\hat{x})) + u^2q_1(\hat{x})), (f_3(x), u^2(g_3(\hat{x}) + 2a_3(\hat{x}))) \rangle$ , where  $\hat{x} = \lambda x$ ,  $a(x)|g(x)|(x^\gamma - 1)$ ,  $a_i(x)|g_i(x)|(x^\delta - \lambda)$  and  $f_i(x), p_1(x), p_2(x), q_1(x) \in \mathbb{Z}_4[x]$ , for  $i = 1, 2, 3$ .*

## 5. GRAY MAPS AND $\mathbb{Z}_4$ -IMAGES OF CONSTACYCLIC CODES

In this section, we define some Gray maps and obtain the Gray images of  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive  $\lambda$ -constacyclic codes for two different values  $\lambda = 1 + 2u^2$  and  $3 + 2u^2$ . For this, we first define

$$\psi_1 : \mathbb{Z}_4[u^3] \rightarrow \mathbb{Z}_4^3$$

by

$$\psi_1(a + bu + cu^2) = (a, 2b, 2c),$$

which can be naturally extended as

$$\psi_1 : \mathbb{Z}_4[u^3]^\delta \rightarrow \mathbb{Z}_4^{3\delta}$$

defined by

$$(1) \quad \psi_1(s_0, s_1, \dots, s_{\delta-1}) = (a_0, a_1, \dots, a_{\delta-1}, 2b_0, 2b_1, \dots, 2b_{\delta-1}, 2c_0, \dots, 2c_{\delta-1}),$$

where  $s_i = a_i + ub_i + u^2c_i$  for  $0 \leq i \leq \delta - 1$ .

For the map  $\psi_1 : \mathbb{Z}_4[u^3]^\delta \rightarrow \mathbb{Z}_4^{3\delta}$  and  $v \in \mathbb{Z}_4[u^3]^\delta$ , the Gray weight is defined by  $w_G(v) = w_L(\psi_1(v))$  and the Gray distance between  $v$  and  $w$  is defined as  $d_G(v, w) =$

$d_L(\psi_1(v), \psi_1(w))$ , where  $w_L$  is the Lee weight and  $d_L$  is the Lee distance over  $\mathbb{Z}_4$ . Then it can be easily seen that  $\psi_1$  is a distance preserving linear bijective map from  $\mathbb{Z}_4[u^3]^\delta$  (Gray distance) to  $\mathbb{Z}_4^{3\delta}$  (Lee distance).

**Theorem 5.1.** *Let  $C$  be a  $(1+2u^2)$ -constacyclic code of length  $\delta$  over  $\mathbb{Z}_4[u^3]$ . Then  $\psi_1(C)$  is a quasi-cyclic code of index 3 and length  $3\delta$  over  $\mathbb{Z}_4$ .*

*Proof.* In order to demonstrate  $\psi_1(C)$  is a quasi-cyclic code it is sufficient to show that  $\psi_1\rho_{(1+2u^2)}(s) = \pi_3\psi_1(s)$  for all  $s \in C$ , where  $\pi_3, \rho_{(1+2u^2)}$  are the quasi-cyclic and constacyclic shifts, respectively. Let  $s = (s_0, s_1, \dots, s_{\delta-1}) \in C$ , where  $s_i = a_i + ub_i + u^2c_i$  for  $0 \leq i \leq \delta - 1$ . Now,

$$\begin{aligned} \psi_1\rho_{(1+2u^2)}(s) &= \psi_1((1+2u^2)s_{\delta-1}, s_0, s_1, \dots, s_{\delta-2}) \\ &= (a_{\delta-1}, a_0, a_1, \dots, a_{\delta-2}, 2b_{\delta-1}, 2b_0, 2b_1, \dots, 2b_{\delta-2}, 2c_{\delta-1}, \\ &\quad 2c_0, \dots, 2c_{\delta-2}), \end{aligned}$$

and

$$\begin{aligned} \pi_3\psi_1(s) &= \pi_3(a_0, a_1, \dots, a_{\delta-1}, 2b_0, 2b_1, \dots, 2b_{\delta-1}, 2c_0, \dots, 2c_{\delta-1}) \\ &= (a_{\delta-1}, a_0, a_1, \dots, a_{\delta-2}, 2b_{\delta-1}, 2b_0, 2b_1, \dots, 2b_{\delta-2}, 2c_{\delta-1}, 2c_0, \dots, 2c_{\delta-2}). \end{aligned}$$

Therefore,  $\psi_1\rho_{(1+2u^2)} = \pi_3\psi_1$  and we get the desired result.  $\square$

Now, we define

$$\psi_2 : \mathbb{Z}_4[u^3] \rightarrow \mathbb{Z}_4^3$$

by

$$\psi_2(a + bu + cu^2) = (2a, 2b, 2c)$$

which can be naturally extended as

$$\psi_2 : \mathbb{Z}_4[u^3]^\delta \rightarrow \mathbb{Z}_4^{3\delta}$$

by

$$\psi_2(s_0, s_1, \dots, s_{\delta-1}) = (2a_0, 2a_1, \dots, 2a_{\delta-1}, 2b_0, 2b_1, \dots, 2b_{\delta-1}, 2c_0, \dots, 2c_{\delta-1}),$$

where  $s_i = a_i + ub_i + u^2c_i$  for  $0 \leq i \leq \delta - 1$ . It is worth noting that we define two Gray maps  $\psi_1, \psi_2$  in such a way that we can classify  $\lambda$ -constacyclic codes ( $\lambda = 1 + 2u^2, 3 + 2u^2$ ) into a well-known class of quaternary quasi-cyclic codes. Also, we will see in Section 6 that these Gray maps eventually help to obtain new  $\mathbb{Z}_4$ -parameters. The way of defining Gray maps completely depends on the choice of  $\lambda$ , in other words, if we chose different  $\lambda$ , we need to define Gray maps suitably so that results corresponding to Theorem 5.1 and Theorem 5.2 are true.

**Remark 1.** From the definition, we must say that  $\psi_2$  is a linear map which carries quaternary linear (or additive) codes to a specific class of linear codes of type  $2C$  where  $C$  is a binary code. In other words, codes induced from Gray images under  $\psi_2$  are copies of binary codes.

**Theorem 5.2.** *Let  $C$  be a  $(3+2u^2)$ -constacyclic code of length  $\delta$  over  $\mathbb{Z}_4[u^3]$ . Then  $\psi_2(C)$  is a quasi-cyclic code of index 3 and length  $3\delta$  over  $\mathbb{Z}_4$ .*

*Proof.* In order to prove  $\psi_2(C)$  is a quasi-cyclic code, it is sufficient to show that  $\psi_2\rho_{(3+2u^2)}(s) = \pi_3\psi_2(s)$  for all  $s \in C$  where  $\pi_3, \rho_{(3+2u^2)}$  are the quasi-cyclic and

constacyclic shifts, respectively. Let  $s = (s_0, s_1, \dots, s_{\delta-1}) \in C$ , where  $s_i = a_i + ub_i + u^2c_i$  for  $0 \leq i \leq \delta - 1$ . Now,

$$\begin{aligned} \psi_2 \rho_{(3+2u^2)}(s) &= \psi_2((3+2u^2)s_{\delta-1}, s_0, s_1, \dots, s_{\delta-2}) \\ &= (2a_{\delta-1}, 2a_0, 2a_1, \dots, 2a_{\delta-2}, 2b_{\delta-1}, 2b_0, 2b_1, \dots, 2b_{\delta-2}, 2c_{\delta-1}, 2c_0, \dots, 2c_{\delta-2}), \end{aligned}$$

and

$$\begin{aligned} \pi_3 \psi_2(s) &= \pi_3(2a_0, 2a_1, \dots, 2a_{\delta-1}, 2b_0, 2b_1, \dots, 2b_{\delta-1}, 2c_0, \dots, 2c_{\delta-1}) \\ &= (2a_{\delta-1}, 2a_0, 2a_1, \dots, 2a_{\delta-2}, 2b_{\delta-1}, 2b_0, 2b_1, \dots, 2b_{\delta-2}, 2c_{\delta-1}, \\ &\quad 2c_0, \dots, 2c_{\delta-2}). \end{aligned}$$

Therefore,  $\psi_2 \rho_{(3+2u^2)} = \pi_3 \psi_2$  and hence we get the desired result.  $\square$

**Definition 5.3.** [16, Definition 1] Let  $n_1, n_2, \dots, n_r$  be positive integers and  $R_i = \mathbb{Z}_4[x]/\langle x^{n_i} - 1 \rangle$  for  $1 \leq i \leq r$ . Then a generalized quasi-cyclic (GQC) code of block length  $(n_1, n_2, \dots, n_r)$  is defined as a  $\mathbb{Z}_4[x]$ -submodule of the  $\mathbb{Z}_4[x]$ -module  $R = R_1 \times R_2 \cdots \times R_r$ . If  $n_1 = n_2 = \dots = n_r$ , then a GQC code is a quasi-cyclic (QC) code. Moreover, if  $r = 1$ , then it is a cyclic code of length  $n_1$ .

For  $j \in \{1, 2\}$ , we define  $\Psi_j$  in terms of  $\psi_j$  given as

$$\Psi_j : \mathbb{Z}_4 \times \mathbb{Z}_4[u^3] \rightarrow \mathbb{Z}_4^4$$

defined by

$$\Psi_j(d, a + bu + cu^2) = (d, \psi_j(a + bu + cu^2))$$

which can be naturally extended to

$$\Psi_j : \mathbb{Z}_4^\gamma \times \mathbb{Z}_4[u^3]^\delta \rightarrow \mathbb{Z}_4^{\gamma+3\delta}$$

by

$$\Psi_j(d_0, d_1, \dots, d_{\gamma-1}, s_0, s_1, \dots, s_{\delta-1}) = (d_0, d_1, \dots, d_{\gamma-1}, \psi_j(s_0, s_1, \dots, s_{\delta-1})),$$

where  $s_i = a_i + ub_i + u^2c_i$  for  $0 \leq i \leq \delta - 1$ .

**Theorem 5.4.** Let  $C$  be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive  $(1+2u^2)$ -constacyclic code of length  $(\gamma, \delta)$ . Then  $\Psi_1(C)$  is a GQC code of block length  $(\gamma, \delta, \delta, \delta)$  over  $\mathbb{Z}_4$ .

*Proof.* Let  $(d_0, d_1, \dots, d_{\gamma-1}, a_0, a_1, \dots, a_{\delta-1}, 2b_0, 2b_1, \dots, 2b_{\delta-1}, 2c_0, \dots, 2c_{\delta-1}) = \Psi_1(c) \in \Psi_1(C)$ , where  $c = (d_0, d_1, \dots, d_{\gamma-1}, s_0, s_1, \dots, s_{\delta-1}) \in C$ . As  $C$  is a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive constacyclic code, we have that  $\rho_{(1+2u^2), \delta}(c) = (d_{\gamma-1}, d_0, \dots, d_{\gamma-2}, (1+2u^2)s_{\delta-1}, s_0, \dots, s_{\delta-2}) \in C$ . Now,

$$\begin{aligned} \Psi_1 \rho_{(1+2u^2), \delta}(c) &= (d_{\gamma-1}, d_0, \dots, d_{\gamma-2}, a_{\delta-1}, a_0, a_1, \dots, a_{\delta-2}, 2b_{\delta-1}, 2b_0, 2b_1, \dots, 2b_{\delta-2}, \\ &\quad 2c_{\delta-1}, 2c_0, \dots, 2c_{\delta-2}) \\ &\in \Psi_1(C). \end{aligned}$$

This implies that  $\Psi_1(C)$  is a GQC code of block length  $(\gamma, \delta, \delta, \delta)$  over  $\mathbb{Z}_4$ .  $\square$

Using the above procedure, we get the following result.

**Theorem 5.5.** Let  $C$  be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive  $(3+2u^2)$ -constacyclic code of length  $(\gamma, \delta)$ . Then  $\Psi_2(C)$  is a GQC code of block length  $(\gamma, \delta, \delta, \delta)$  over  $\mathbb{Z}_4$ .

We define a new map

$$\hat{\psi}_1 : \mathbb{Z}_4[u^3] \rightarrow \mathbb{Z}_4^2$$

by

$$\hat{\psi}_1(a + bu + cu^2) = (3a + b + c, a + b + c)$$

whose natural extension is

$$\hat{\psi}_1 : \mathbb{Z}_4[u^3]^\delta \rightarrow \mathbb{Z}_4^{2\delta}$$

by

$$\begin{aligned} \hat{\psi}_1(s_0, s_1, \dots, s_{\delta-1}) &= (3a_0 + b_0 + c_0, 3a_1 + b_1 + c_1, \dots, 3a_{\delta-1} + b_{\delta-1} + c_{\delta-1}, \\ &\quad a_0 + b_0 + c_0, a_1 + b_1 + c_1, \dots, a_{\delta-1} + b_{\delta-1} + c_{\delta-1}), \end{aligned}$$

where  $s_i = a_i + ub_i + u^2c_i$  for  $0 \leq i \leq \delta - 1$ . Then  $\hat{\psi}_1$  is a distance preserving linear map and we have the following result.

**Theorem 5.6.** *Let  $C$  be a  $(1+2u^2)$ -constacyclic code of length  $\delta$  over  $\mathbb{Z}_4[u^3]$ . Then  $\hat{\psi}_1(C)$  is a cyclic code of length  $2\delta$  over  $\mathbb{Z}_4$ .*

*Proof.* Let  $C$  be a  $(1+2u^2)$ -constacyclic code of length  $\delta$  over  $\mathbb{Z}_4[u^3]$  and  $s = (s_0, s_1, \dots, s_{\delta-1}) \in C$ , where  $s_i = a_i + ub_i + u^2c_i$  for  $0 \leq i \leq \delta - 1$ . Then

$$\begin{aligned} \hat{\psi}_1\rho_{(1+2u^2)}(s) &= \hat{\psi}_1((1+2u^2)s_{\delta-1}, s_0, s_1, \dots, s_{\delta-2}) \\ &= (a_{\delta-1} + b_{\delta-1} + c_{\delta-1}, 3a_0 + b_0 + c_0, \dots, 3a_{\delta-2} + b_{\delta-2} + c_{\delta-2}, \\ &\quad 3a_{\delta-1} + b_{\delta-1} + c_{\delta-1}, a_0 + b_0 + c_0, \dots, a_{\delta-2} + b_{\delta-2} + c_{\delta-2}). \end{aligned}$$

Also,

$$\begin{aligned} \pi_1\hat{\psi}_1(s) &= \pi_1(3a_0 + b_0 + c_0, 3a_1 + b_1 + c_1, \dots, 3a_{\delta-1} + b_{\delta-1} + c_{\delta-1}, a_0 + b_0 + c_0, \\ &\quad a_1 + b_1 + c_1, \dots, a_{\delta-1} + b_{\delta-1} + c_{\delta-1}) \\ &= (a_{\delta-1} + b_{\delta-1} + c_{\delta-1}, 3a_0 + b_0 + c_0, \dots, 3a_{\delta-1} + b_{\delta-1} + c_{\delta-1}, a_0 + b_0 + \\ &\quad c_0, a_1 + b_1 + c_1, \dots, a_{\delta-2} + b_{\delta-2} + c_{\delta-2}) \end{aligned}$$

Therefore,  $\hat{\psi}_1\rho_{(1+2u^2)} = \pi_1\hat{\psi}_1$ , and hence  $\hat{\psi}_1(C)$  is a cyclic code of length  $2\delta$  over  $\mathbb{Z}_4$ .  $\square$

Now, we consider another new map

$$\hat{\psi}_2 : \mathbb{Z}_4[u^3] \rightarrow \mathbb{Z}_4^2$$

by

$$\hat{\psi}_2(a + bu + cu^2) = (2a + b + 2c, 2a + 3b + 2c)$$

whose extension is

$$\hat{\psi}_2 : \mathbb{Z}_4[u^3]^\delta \rightarrow \mathbb{Z}_4^{2\delta}$$

defined by

$$\begin{aligned} \hat{\psi}_2(s_0, s_1, \dots, s_{\delta-1}) &= (2a_0 + b_0 + 2c_0, 2a_1 + b_1 + 2c_1, \dots, 2a_{\delta-1} + b_{\delta-1} + 2c_{\delta-1}, \\ &\quad 2a_0 + 3b_0 + 2c_0, 2a_1 + 3b_1 + 2c_1, \dots, 2a_{\delta-1} + 3b_{\delta-1} + 2c_{\delta-1}), \end{aligned}$$

where  $s_i = a_i + ub_i + u^2c_i$  for  $0 \leq i \leq \delta - 1$ .

**Theorem 5.7.** *Let  $C$  be a  $(3+2u^2)$ -constacyclic code of length  $\delta$  over  $\mathbb{Z}_4[u^3]$ . Then  $\hat{\psi}_2(C)$  is a cyclic code of length  $2\delta$  over  $\mathbb{Z}_4$ .*

*Proof.* Let  $C$  be a  $(3 + 2u^2)$ -constacyclic code of length  $\delta$  over  $\mathbb{Z}_4[u^3]$  and  $s = (s_0, s_1, \dots, s_{\delta-1}) \in C$ , where  $s_i = a_i + ub_i + u^2c_i$  for  $0 \leq i \leq \delta - 1$ . Then

$$\begin{aligned} \hat{\psi}_{2\rho_{(3+2u^2)}}(s) &= \hat{\psi}_2((3 + 2u^2)s_{\delta-1}, s_0, s_1, \dots, s_{\delta-2}) \\ &= (2a_{\delta-1} + 3b_{\delta-1} + 2c_{\delta-1}, 2a_0 + b_0 + 2c_0, \dots, 2a_{\delta-1} + b_{\delta-1} + 2c_{\delta-1}, \\ &\quad 2a_0 + 3b_0 + 2c_0, 2a_1 + 3b_1 + 2c_1, \dots, 2a_{\delta-2} + 3b_{\delta-2} + 2c_{\delta-2}). \end{aligned}$$

Also,

$$\begin{aligned} \pi_1 \hat{\psi}_2(s) &= \pi_1(2a_0 + b_0 + 2c_0, 2a_1 + b_1 + 2c_1, \dots, 2a_{\delta-1} + b_{\delta-1} + 2c_{\delta-1}, \\ &\quad 2a_0 + 3b_0 + 2c_0, 2a_1 + 3b_1 + 2c_1, \dots, 2a_{\delta-1} + 3b_{\delta-1} + 2c_{\delta-1}) \\ &= (2a_{\delta-1} + 3b_{\delta-1} + 2c_{\delta-1}, 2a_0 + b_0 + 2c_0, \dots, 2a_{\delta-1} + b_{\delta-1} + 2c_{\delta-1}, \\ &\quad 2a_0 + 3b_0 + 2c_0, 2a_1 + 3b_1 + 2c_1, \dots, 2a_{\delta-1} + 3b_{\delta-1} + 2c_{\delta-1}). \end{aligned}$$

Therefore, we get  $\hat{\psi}_{2\rho_{(3+2u^2)}} = \pi_1 \hat{\psi}_2$  and hence  $\hat{\psi}_2(C)$  is a cyclic code of length  $2\delta$  over  $\mathbb{Z}_4$ .  $\square$

For  $j \in \{1, 2\}$ , we define  $\hat{\Psi}_j$  in terms of  $\hat{\psi}_j$  given by

$$\hat{\Psi}_j : \mathbb{Z}_4 \times \mathbb{Z}_4[u^3] \rightarrow \mathbb{Z}_4^3$$

defined as

$$\hat{\Psi}_j(d, a + bu + cu^2) = (d, \hat{\psi}_j(a + bu + cu^2))$$

whose natural extension is

$$\hat{\Psi}_j : \mathbb{Z}_4^\gamma \times \mathbb{Z}_4[u^3]^\delta \rightarrow \mathbb{Z}_4^{\gamma+2\delta}$$

defined by

$$\hat{\Psi}_j(d_0, d_1, \dots, d_{\gamma-1}, s_0, s_1, \dots, s_{\delta-1}) = (d_0, d_1, \dots, d_{\gamma-1}, \hat{\psi}_j(s_0, s_1, \dots, s_{\delta-1})),$$

where  $s_i = a_i + ub_i + u^2c_i$  for  $0 \leq i \leq \delta - 1$ .

**Theorem 5.8.** *Let  $C$  be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive  $(1 + 2u^2)$ -constacyclic code of length  $(\gamma, \delta)$ . Then  $\hat{\Psi}_1(C)$  is a GQC code of block length  $(\gamma, 2\delta)$  over  $\mathbb{Z}_4$ .*

*Proof.* Let  $(d_0, d_1, \dots, d_{\gamma-1}, 3a_0 + b_0 + c_0, \dots, 3a_{\delta-1} + b_{\delta-1} + c_{\delta-1}, a_0 + b_0 + c_0, \dots, a_{\delta-1} + b_{\delta-1} + c_{\delta-1}) = \hat{\Psi}_1(c) \in \hat{\Psi}_1(C)$ , where  $c = (d_0, d_1, \dots, d_{\gamma-1}, s_0, s_1, \dots, s_{\delta-1}) \in C$ . As  $C$  is a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive constacyclic code, we have  $\rho_{(1+2u^2), \delta}(c) = (d_{\gamma-1}, d_0, \dots, d_{\gamma-2}, (1 + 2u^2)s_{\delta-1}, s_0, \dots, s_{\delta-2}) \in C$ . Now,

$$\begin{aligned} \hat{\Psi}_1\rho_{(1+2u^2), \delta}(c) &= (d_{\gamma-1}, d_0, \dots, d_{\gamma-2}, a_{\delta-1} + b_{\delta-1} + c_{\delta-1}, 3a_0 + b_0 + c_0, 3a_1 + b_1 \\ &\quad + c_1, \dots, 3a_{\delta-2} + b_{\delta-2} + c_{\delta-2}, 3a_{\delta-1} + b_{\delta-1} + c_{\delta-1}, a_0 + b_0 + \\ &\quad c_0, a_1 + b_1 + c_1, \dots, a_{\delta-2} + b_{\delta-2} + c_{\delta-2}) \\ &\in \hat{\Psi}_1(C). \end{aligned}$$

Therefore, we can conclude that  $\hat{\Psi}_1(C)$  is a GQC code of block length  $(\gamma, 2\delta)$  over  $\mathbb{Z}_4$ .  $\square$

Using the same procedure, we get the following result.

**Theorem 5.9.** *Let  $C$  be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive  $(3 + 2u^2)$ -constacyclic code of length  $(\gamma, \delta)$ . Then  $\hat{\Psi}_2(C)$  is a GQC code of block length  $(\gamma, 2\delta)$  over  $\mathbb{Z}_4$ .*

6. EXAMPLES

In this section, we present three examples in support of our obtained results. By applying Gray maps, we obtain 6 new and 2 best-known quaternary codes. The computations involved here are carried out by the Magma computation system [14, 32].

**Example 6.1.** Let  $\delta = 7$  and  $C$  be a  $(1 + 2u^2)$ -constacyclic code of length 7 over  $\mathbb{Z}_4[u^3]$ . Then

$$C = \langle g_1(\hat{x}) + 2a_1(\hat{x}) + up_1(\hat{x}) + u^2p_2(\hat{x}), u(g_2(\hat{x}) + 2a_2(\hat{x})) + u^2q_1(\hat{x}), u^2(g_3(\hat{x}) + 2a_3(\hat{x})) \rangle,$$

where  $a_i(x)|g_i(x)|x^7 - 1 \pmod{4}$  and  $\hat{x} = (1 + 2u^2)x$ . Now,

$$x^7 - 1 = (x + 3)(x^3 + 2x^2 + x + 3)(x^3 + 3x^2 + 2x + 3) \in \mathbb{Z}_4[x].$$

Let

$$\begin{aligned} g_1(x) &= (x + 3)(x^3 + 2x^2 + x + 3) \\ &= x^4 + x^3 + 3x^2 + 2x + 1, \\ g_2(x) &= (x + 3)(x^3 + 3x^2 + 2x + 3) \\ &= x^4 + 2x^3 + 3x^2 + x + 1, \\ g_3(x) &= (x + 3)(x^3 + 2x^2 + x + 3) \\ &= x^4 + x^3 + 3x^2 + 2x + 1, \end{aligned}$$

and  $a_1(x) = a_2(x) = a_3(x) = x + 3, p_1(x) = p_2 = 1, q_1(x) = 0$ . Then  $C = \langle G_1(x), G_2(x), G_3(x) \rangle$  where

$$\begin{aligned} G_1(x) &= x^4 + (1 + 2u^2)x^3 + 3x^2 + 3 + u + u^2, \\ G_2(x) &= ux^4 + 2ux^3 + 3ux^2 + 3ux + 3u, \\ G_3(x) &= u^2x^4 + u^2x^3 + 3u^2x^2 + 3u^2. \end{aligned}$$

Now, by Theorem 5.1, we have  $\psi_1(C)$  is a quasi-cyclic code of index 3 with parameters  $(21, 4^3 2^{12}, 4)$  over  $\mathbb{Z}_4$ . Moreover, by Theorem 5.6,  $\hat{\psi}_1(C)$  is a cyclic code with parameters  $(14, 4^7 2^3, 4)$  over  $\mathbb{Z}_4$ . As per the database [2], both obtained codes are best-known.

**Example 6.2.** Let  $\delta = 9$  and  $C$  be a  $(3 + 2u^2)$ -constacyclic code of length 9. Then

$$C = \langle g_1(\hat{x}) + 2a_1(\hat{x}) + up_1(\hat{x}) + u^2p_2(\hat{x}), u(g_2(\hat{x}) + 2a_2(\hat{x})) + u^2q_1(\hat{x}), u^2(g_3(\hat{x}) + 2a_3(\hat{x})) \rangle,$$

where  $a_i(x)|g_i(x)|x^9 - 1 \pmod{4}$  and  $\hat{x} = (3 + 2u^2)x$ . Now,

$$x^9 - 1 = (x + 3)(x^2 + x + 1)(x^6 + x^3 + 1) \in \mathbb{Z}_4[x].$$

Let

$$\begin{aligned} g_1(x) &= (x^2 + x + 1)(x^6 + x^3 + 1) \\ &= x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1, \\ g_2(x) &= (x + 3)(x^2 + x + 1) = x^3 + 3, \\ g_3(x) &= x^2 + x + 1 \end{aligned}$$

and  $a_1(x) = x^2 + x + 1$ ,  $a_2(x) = x + 3$ ,  $a_3(x) = 1$ ,  $p_1(x) = p_2(x) = q_1(x) = 1$ .  
Therefore,  $C = \langle G_1(x), G_2(x), G_3(x) \rangle$  where

$$\begin{aligned} G_1(x) &= x^8 + (3 + 2u^2)x^7 + x^6 + (3 + 2u^2)x^5 + x^4 + (3 + 2u^2)x^3 + 3x^2 + (1 + 2u^2)x \\ &\quad + 3 + u + u^2, \\ G_2(x) &= 3ux^3 + 2ux + u + u^2, \\ G_3(x) &= u^2x^2 + 3u^2x + 3u^2. \end{aligned}$$

Hence, by Theorem 5.2,  $\psi_2(C)$  is a quasi-cyclic code with parameters  $(27, 4^{02^{12}}, 4)$ .  
Further, by Theorem 5.7,  $\hat{\psi}_2(C)$  is a cyclic code with parameters  $(18, 4^5 2^4, 3)$ . It is worth mentioning that both obtained codes are new linear codes over  $\mathbb{Z}_4$  as per the database [2].

**Example 6.3.** Let  $(\gamma, \delta) = (7, 9)$  and  $C$  be a  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic code of length  $(7, 9)$ . Then by Theorem 3.2, we have

$$C = \langle (g + 2a, 0), (f_1, g_1 + 2a_1 + up_1 + u^2p_2), (f_2, u(g_2 + 2a_2) + u^2q_1), (f_3, u^2(g_3 + 2a_3)) \rangle,$$

where  $a|g|(x^7 - 1)$  and  $a_i|g_i|(x^9 - 1)$  for  $i = 1, 2, 3$ . Now,

$$\begin{aligned} x^7 - 1 &= (x + 3)(x^3 + 2x^2 + x + 3)(x^3 + 3x^2 + 2x + 3) \in \mathbb{Z}_4[x] \\ x^9 - 1 &= (x + 3)(x^2 + x + 1)(x^6 + x^3 + 1) \in \mathbb{Z}_4[x]. \end{aligned}$$

Let

$$\begin{aligned} g &= x^4 + x^3 + 3x^2 + 2x + 1, a = x + 3, \\ g_1 &= (x^2 + x + 1)(x^6 + x^3 + 1) \\ &= x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1, a_1 = x^2 + x + 1, \\ g_2 &= (x + 3)(x^6 + x^3 + 1) \\ &= x^7 + 3x^6 + x^4 + 3x^3 + x + 3, a_1 = x + 3, \\ g_3 &= x^2 + x + 1, a_3 = 1, \\ p_1 = p_2 = q_1 &= 1, f_1 = x^3 + x^2 + x + 1, f_2 = x^2 + 2x + 3, f_3 = x + 1. \end{aligned}$$

Then  $h_1 = \frac{x^9 - 1}{x^2 + x + 1} = g_2 = l_1$ ,  $h_2 = \frac{x^9 - 1}{x + 3} = g_1 = l_2$ . Therefore, by Theorem 3.4,  $B = B_1 \cup B_2 \cup \dots \cup B_6$  is a minimal spanning set of  $C$  where

$$\begin{aligned} B_1 &= \bigcup_{i=0}^5 \{x^i * (x^4 + x^3 + 3x^2 + 3, 0)\}, \\ B_2 &= \bigcup_{i=0}^6 \{x^i * (x^3 + x^2 + x + 1, x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + 3x^2 + 3x + 3 + u + u^2)\}, \end{aligned}$$

$$B_3 = \bigcup_{i=0}^7 \{x^i * (x^2 + 2x + 3, ux^7 + 3ux^6 + ux^4 + 3ux^3 + 3ux + u + u^2)\},$$

$$B_4 = \bigcup_{i=0}^8 \{x^i * (x + 1, u^2x^2 + u^2x + 3u^2)\},$$

$$B_5 = \bigcup_{i=0}^1 \{x^i * (3x^6 + x^4, (u + u^2)(x^7 + 3x^6 + x^4 + 3x^3 + x + 3))\},$$

$$B_6 = \{(2x^6 + 2x^5 + 2x^4 + 3x^3 + x^2 + 3x + 1, u^2x^8 + u^2x^7 + u^2x^6 + u^2x^5 + u^2x^4 + u^2x^3 + u^2x^2 + u^2x + u^2)\}.$$

Hence,  $\Psi_1(C)$  has parameters  $(34, 4^{11}2^{15}, 5)$  and  $\Psi_2(C)$  has parameters  $(34, 4^82^{18}, 3)$ , respectively. Again,  $\hat{\Psi}_1(C)$  is  $(25, 4^{14}2^2, 4)$  and  $\hat{\Psi}_2(C)$  is a  $(25, 4^{10}2^6, 3)$  linear codes over  $\mathbb{Z}_4$ , respectively. It is worth mentioning that all these four codes are new  $\mathbb{Z}_4$  codes as per the database [2].

## 7. CONCLUSION AND OPEN PROBLEMS

In this paper, we have studied  $\mathbb{Z}_4\mathbb{Z}_4[u^3]$ -additive cyclic and constacyclic codes and determined their algebraic structure. By applying Gray maps, we have obtained some good  $\mathbb{Z}_4$ -codes, some of them improve the best-known codes as per the table [2]. As far as we know, the mixed alphabets  $(\mathbb{Z}_4, \mathbb{Z}_4[u^3])$  is new in the literature. Therefore, it would be a worthy study to discuss other aspects, like dual codes of such mixed alphabet codes. Another important and interesting direction could be to obtain distance bounds on the Gray images of these codes. In addition, as mentioned in the introduction, the discussion of  $\mathbb{Z}_{2^r}\mathbb{Z}_{2^s}[u^3]$ -additive codes is a meaningful open direction.

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