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BROWNIAN CONTINUUM RANDOM TREE CONDITIONED TO BE LARGE

ROMAIN ABRAHAM, JEAN-FRANÇOIS DELMAS, AND HUI HE

ABSTRACT. We consider a Feller diffusion $(Z_s, s \geq 0)$ (with diffusion coefficient $\sqrt{2\beta}$ and drift $\theta \in \mathbb{R}$) that we condition on $\{Z_t = a_t\}$, where a_t is a deterministic function, and we study the limit in distribution of the conditioned process and of its genealogical tree as $t \rightarrow +\infty$. When a_t does not increase too rapidly, we recover the standard size-biased process (and the associated genealogical tree given by the Kesten's tree). When a_t behaves as $\alpha\beta^2 t^2$ when $\theta = 0$ or as $\alpha e^{2\beta|\theta|t}$ when $\theta \neq 0$, we obtain a new process whose distribution is described by a Girsanov transformation and equivalently by a SDE with a Poissonian immigration. Its associated genealogical tree is described by an infinite discrete skeleton (which does not satisfy the branching property) decorated with Brownian continuum random trees given by a Poisson point measure.

As a by-product of this study, we introduce several sets of trees endowed with a Gromov-type distance which are of independent interest and which allow here to define in a formal and measurable way the decoration of a backbone with a family of continuum random trees.

1. INTRODUCTION

In [1], for the geometric reproduction law, and in [5], for general super-critical reproduction laws with finite mean and some special sub-critical reproduction laws, the authors consider the limit of a Galton-Watson (GW) process $(Z_n, n \in \mathbb{N})$ started at $Z_0 = 1$ conditionally on $Z_n = a_n$ as n goes to infinity, provided the event $\{Z_n = a_n\}$ has positive probability. They also consider more generally the local limit of the GW tree, which in particular allows to study condensation phenomenon (on this latter subject, see [26, 25, 4]). According to the different growth rate of a_n as n goes to infinity, they observe different regimes for the limiting random tree: if $a_n = 0$ for n large, the limiting tree corresponds to the GW tree conditioned on the extinction event; if a_n is strictly positive but grows slowly (including the case a_n bounded), then the limit is the so called Kesten tree, which consists in an infinite spine decorated with independent GW trees with the initial reproduction law; if a_n grows at a moderate speed (given in the super-critical case of finite variance by $a_n \sim \alpha m^n$ with $\alpha > 0$ and m the mean of the reproduction law), then the limit is a skeleton given by an immigration process decorated again with independent GW trees with the initial reproduction law; if a_n grows faster than m^n (that is $\lim_{n \rightarrow \infty} m^{-n} a_n = \infty$) then results are known only for the geometric reproduction law (the limit exhibits a condensation at the root, that is, the root has an infinite number of children, and then those children generate independent trees) and for bounded reproduction laws (the limit is the regular b -ary tree, with b the possible maximum number of children).

This work is a first step to extend those results to random real trees called Lévy trees introduced by Duquesne and Le Gall in [18, 19] which are scaling limits of (sub)critical GW trees and can be seen as genealogical trees for (sub)critical continuous state branching processes (CSBP); see also [3, 20] for the extension of this latter representation to the super-critical case. We shall only

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consider Feller diffusions, which correspond to CSBPs with quadratic branching mechanism and whose genealogy can be described using the Brownian continuum random tree introduced by Aldous [9]. Our results belong also to the family of works dedicated to the description of limits of conditioned random real trees, in this direction, see [29, 28, 17, 2].

We now present informally our result. We denote by \mathcal{T} the rooted Brownian continuum tree which describes the genealogy of a population started at time 0, and by Z_t the “size” of the population at time $t \geq 0$. The process $Z = (Z_t, t \geq 0)$ is a quadratic CSBP solution to the following stochastic differential equation (SDE):

$$dZ_t = \sqrt{2\beta Z_t} dB_t - 2\beta\theta Z_t dt, \quad \text{for } t \geq 0,$$

where $(B_t, t \geq 0)$ is some standard Brownian motion, $\beta > 0$ and $\theta \in \mathbb{R}$. The CSBP is associated with the branching mechanism

$$\psi_\theta(\lambda) = \beta\lambda^2 + 2\beta\theta\lambda.$$

The CSBP is sub-critical (resp. super-critical) if $\theta > 0$ (resp. $\theta < 0$). We denote by \mathbb{N}^θ the σ -finite excursion measure associate with the Brownian tree \mathcal{T} and the process Z , stressing the dependence in the parameter θ , whereas the time scaling parameter β is fixed. Intuitively, under \mathbb{N}^θ , the population starts from an infinitesimal individual which is the root of \mathcal{T} , and the non-zero process Z starts from an infinitesimal mass. In the introduction, we simply denote by \mathbf{t}_t the real tree \mathbf{t} truncated at level t . We denote by \mathcal{G}_t the σ -field generated by \mathcal{T}_t for $t \geq 0$; in particular the process Z is adapted to the filtration $(\mathcal{G}_t, t \geq 0)$. Let $a = (a_t, t \geq 0)$ be a non-negative function. We shall consider for $s > 0$, the possible limiting distribution of \mathcal{T}_s , conditionally on $\{Z_t = a_t\}$ as t goes to infinity. Let F be any bounded continuous function defined on the set of trees (see Section 5 for the topology on the set of rooted locally compact tree).

- **Extinction case:** $a_t = 0$ for t large. We have:

$$\lim_{t \rightarrow \infty} \mathbb{N}^\theta [F(\mathcal{T}_s) \mid Z_t = a_t] = \mathbb{N}^{|\theta|} [F(\mathcal{T}_s)].$$

We recall that the measure of the non extinction event is given by $\mathbb{N}^\theta [Z_t > 0 \text{ for all } t > 0] = 2 \max(0, -\theta)$. So the conditioning on the extinction event does not change anything in the (sub)critical case. In the super-critical case, this result is a direct consequence of the Girsanov transformation used in [3] to define the super-critical Lévy tree, see also (78). This result holds in general for any compact Lévy trees (but for the above value of the measure of the non extinction event).

- **Low regime:** a is positive and $a_t = o(t^2)$ if $\theta = 0$ or $a_t = o(e^{2\beta|\theta|t})$ if $\theta \neq 0$. We have (see Theorem 6.13):

$$\lim_{t \rightarrow \infty} \mathbb{N}^\theta [F(\mathcal{T}_s) \mid Z_t = a_t] = \mathbb{E} \left[F \left(\mathcal{T}_s^{0,|\theta|} \right) \right],$$

where for $q \geq 0$, $\mathcal{T}^{0,q}$ is a Kesten tree with parameter (β, q) , which is informally obtained by grafting the trees $(\mathcal{T}_i, i \in I)$ respectively at levels $(h_i, i \in I)$ on an infinite spine, where the point measure $\sum_{i \in I} \delta_{h_i, \mathcal{T}_i}$ is a Poisson point measure with intensity rate $2\beta \mathbf{1}_{\{h > 0\}} dh \mathbb{N}^q [d\mathcal{T}]$. See Section 6.2 for a more formal definition of the Kesten tree. When conditioning instead by $\{Z_t > 0\}$, this result appears already in [2] for general compact Lévy trees.

- **Moderate regime:** $a_t \sim \alpha\beta^2 t^2$ if $\theta = 0$ or $a_t \sim \alpha(2\theta)^{-2} e^{2\beta|\theta|t}$ if $\theta \neq 0$, where $\alpha \in (0, +\infty)$. We have (see Theorem 6.12):

$$\lim_{t \rightarrow \infty} \mathbb{N}^\theta [F(\mathcal{T}_s) \mid Z_t = a_t] = \mathbb{E} \left[F \left(\mathcal{T}_s^{\alpha,|\theta|} \right) \right],$$

where for $q \geq 0$, $\mathcal{T}^{\alpha,q}$ is a random tree informally obtained by grafting the tree \mathcal{T}_i at vertex x_i of a backbone tree $\mathfrak{T}^{\alpha,q}$, with the point measure $\sum_{i \in I} \delta_{x_i, \tau_i}$ being, conditionally given $\mathfrak{T}^{\alpha,q}$, a Poisson point measure with intensity rate $2\beta \mathcal{L}(dx) \mathbb{N}^q[d\mathcal{T}]$, and $\mathcal{L}(dx)$ being the length measure on $\mathfrak{T}^{\alpha,q}$. The backbone tree $\mathfrak{T}^{\alpha,q}$ represents in some sense the genealogy associated to an inhomogeneous Poissonian immigration with finite rate, see Section 6.4 for a more precise description. Let us stress that the backbone tree does not enjoy the branching property, as already observed by [1, 5] in a discrete setting.

- **High regime:** $\lim_{t \rightarrow \infty} t^{-2} a_t = +\infty$ if $\theta = 0$ or $\lim_{t \rightarrow \infty} e^{-2\beta|\theta|t} a_t = +\infty$ if $\theta \neq 0$. The description of the possible limit in this regime is still an open question. As in the discrete setting studied in [1], one expects to have a condensation phenomenon at the root. However, such limiting tree would not be locally compact (at least at the root), and the study of such trees would require a non trivial extension of the current topology developed for locally compact trees.

The present result on the convergence in distribution of the conditioned Brownian tree in the low and moderate regime is given in Section 6.4. It relies on:

- (i) The extremal time-space harmonic functions for the process Z given by Overbeck [31]. Those harmonic functions appear in the limit when conditioning the process Z by the events $\{Z_t = a_t\}$ in the moderate regime, see Section 3.
- (ii) A nice representation of the Doob h -transform using those extremal harmonic functions, based on Feller diffusion with an increasing immigration rate given by an inhomogeneous Poisson process. Roughly speaking, for $\theta \geq 0$ and $\alpha > 0$, the local time process $Z^\alpha = (Z_t^\alpha, t \geq 0)$ of the tree $\mathcal{T}^{\alpha,\theta}$ is a Markov process which satisfies the following SDE:

$$dZ_t^\alpha = \sqrt{2\beta Z_t^\alpha} dB_t - 2\beta\theta Z_t^\alpha dt + 2\beta (S_t^{\alpha,\theta} + 1) dt \quad \text{for } t \geq 0, \quad \text{and } Z_0^\alpha = 0,$$

where $B = (B_t, t \geq 0)$ is a standard Brownian motion and $S^{\alpha,\theta} = (S_t^{\alpha,\theta}, t \geq 0)$ is a Poisson process with intensity $\alpha\beta e^{2\beta\theta t} dt$ independent of B . See the main result of Section 4 in Proposition 4.1 for more details. Its proof, given in Section 4.2, uses a result from Rogers and Pitman [33] for a transformation of a Markov process (which in our case amount to not be able to distinguish the immigration $S^{\alpha,\theta}$) to still be a Markov process.

- (iii) An extension of a result from Duquesne and Le Gall [19, Theorem 4.5] on the description of the Brownian tree \mathcal{T} as a decorated sub-tree spanned by n vertices chosen randomly at level t , see Theorem 6.8 in Section 6.3. We believe that this theorem is of independent interest and complete the description of [20] where one chooses these vertices at random without condition on their level.
- (iv) A transcription of the genealogy for immigration process from point (ii) above into a backbone tree, see Section 6.1 and the combinatorial Lemma 6.1 on the distribution of a tree built sequentially by grafting uniformly branches at random levels.

Let us mention here that there have been several works on skeletal/backbone decompositions for (spatial) branching processes and their corresponding genealogical trees, for example see [2, 10, 12, 20, 23, 24, 27] and the references therein. In particular, in [23], coupled systems of SDEs were established to represent the skeletal decompositions for continuous-state branching processes (conditioned on survival), where the skeletons are determined by continuous-time Galton-Watson processes. And we refer to [20], for reconstruction of a Lévy tree from a backbone tree, which could be formed by leaves taking at random in a Poissonian way from the Lévy tree according to the so-called mass measure; see Remark 5.4 there and [18]. For representations of branching processes (with immigrations) via SDEs, we also refer to [15] and references therein.

To complete the outline of the paper, let us mention that Section 2 is devoted to some notations and elementary facts for quadratic CSBP and Section 6.2 to known results on the Brownian tree. Eventually the large Section 5 is devoted to various topological results on the spaces of trees. The main objective of this section is to define the grafting, the splitting, as well as the decorating of trees in a measurable way on the set of equivalent classes of locally compact rooted real trees. An index of all the (numerous) relevant notations of this section is provided at the end of the document.

2. GENERAL QUADRATIC CSBP

2.1. Notations. We set $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_+^* = (0, +\infty)$, $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, \dots\}$.

For $x \in \mathbb{R}$, we set $x_+ = \max(0, x)$ and $x_- = \max(0, -x)$, so that $x = x_+ - x_-$. We write δ_x for the Dirac mass at x .

Let (E, d) be a metric space. We denote by $\mathcal{M}_+(E)$ the space of non-negative measures on E endowed with the vague topology. For $\mu \in \mathcal{M}_+(E)$ and A a Borel subset of E , we denote by $\mu|_A(dx)$ the measure $\mathbf{1}_A(x)\mu(dx)$. We write $\mu(f) = \langle f, \mu \rangle = \int f d\mu = \langle f, \mu \rangle$ for the integral of the measurable real-valued function f with respect to the measure μ , whenever it is meaningful.

We say that a function from a measurable space to a measurable space is bi-measurable if it is measurable and the image of any measurable set is a measurable set (when the function is one-to-one this is equivalent to the function and its inverse being measurable).

2.2. Quadratic CSBP. Let $\beta > 0$ be fixed. Let $\theta \in \mathbb{R}$. We consider the quadratic branching mechanism ψ_θ given for $\lambda \in \mathbb{R}$ by:

$$(1) \quad \psi_\theta(\lambda) = \beta\lambda^2 + 2\beta\theta\lambda.$$

The corresponding CSBP $Z = (Z_t, t \geq 0)$ is the unique strong solution to the following stochastic differential equation (SDE):

$$(2) \quad dZ_t = \sqrt{2\beta Z_t} dB_t - 2\beta\theta Z_t dt \quad \text{for } t \geq 0,$$

where $B = (B_t, t \geq 0)$ is a standard Brownian motion and $Z_0 = x \geq 0$. For $t \geq 0$, let \mathcal{F}_t be the σ -field generated by $(Z_s, s \in [0, t])$. We write \mathbb{P}_x^θ to stress the value of the parameter θ , and the initial value of the process Z , $Z_0 = x$. We denote by \mathbb{N}^θ the canonical measure of the process Z , normalized in such a way that for $\lambda \geq 0$:

$$\mathbb{N}^\theta \left[1 - e^{-\lambda\sigma} \right] = \psi_\theta^{-1}(\lambda),$$

where $\sigma = \int_0^\infty Z_t dt$ is the total size of the population under the canonical measure \mathbb{N}^θ and $\psi_\theta^{-1}(\lambda)$ is the only root t to $\psi_\theta(t) = \lambda$ such that $t \geq 2\theta_-$. In particular, the process $(Z_t, t \geq 0)$ under \mathbb{P}_x^θ is distributed as the process $\left(\sum_{i \in I} Z_t^{(i)}, t \geq 0 \right)$ where $\sum_{i \in I} \delta_{Z^{(i)}}$ is a Poisson point measure with intensity $x\mathbb{N}^\theta(dZ)$. We refer to [18] for $\theta \geq 0$ (critical and sub-critical case) and to [16, 3, 20] for $\theta < 0$ (super-critical case) for a detailed presentation of the CSBP process Z and the corresponding continuum Brownian random tree \mathcal{T} .

In order to recall the Laplace transform of Z_t , we introduce the following positive functions c^θ and \tilde{c}^θ defined for $t \in (0, +\infty)$ by:

$$(3) \quad c_t^\theta = \frac{2\theta}{e^{2\beta\theta t} - 1} \quad \text{and} \quad \tilde{c}_t^\theta = \frac{2\theta}{1 - e^{-2\beta\theta t}},$$

with the convention $c_t^0 = \tilde{c}_t^0 = 1/\beta t$. The functions c^θ and \tilde{c}^θ are decreasing with $\lim_{t \rightarrow 0^+} c_t^\theta = \lim_{t \rightarrow 0^+} \tilde{c}_t^\theta = +\infty$, $\lim_{t \rightarrow +\infty} c_t^\theta = 2\theta_-$ and $\lim_{t \rightarrow +\infty} \tilde{c}_t^\theta = 2\theta_+$. We also have for $t > 0$:

$$(4) \quad \tilde{c}_t^\theta = c_t^{-\theta} = c_t^\theta + 2\theta.$$

Remark 2.1 (Scaling property of Z). In this remark, we write $Z^{[\beta, \theta]}$ for Z under \mathbb{N}^θ or \mathbb{P}_x^θ in order to stress the dependence in $\beta > 0$ and $\theta \in \mathbb{R}$. Let $Y = (Y_s, s \geq 0)$ be a Feller diffusion, *i.e.*, a CSBP with parameters $\beta = 1$ and $\theta = 0$; given as the unique strong solution to the SDE:

$$(5) \quad dY_s = \sqrt{2Y_s} dB_s, \quad \text{for } s \geq 0.$$

Let $\beta > 0$ and $\theta \in \mathbb{R}$, and define the process $Y' = (Y'_s, s \in [0, 1/(2\theta_-)])$ by:

$$(6) \quad Y'_s = e^{2\beta\theta t} Z_t^{[\beta, \theta]} \quad \text{with } s = \frac{1}{c_t^\theta}.$$

Then, we get that Y' is distributed as $(Y_s, s \in [0, 1/(2\theta_-)])$ under \mathbb{N}^θ or \mathbb{P}_x^θ provided $Y'_0 = Y_0 = x \geq 0$.

We define for $t > 0$ and $\lambda > -\tilde{c}_t^\theta$:

$$(7) \quad u^\theta(\lambda, t) = \frac{\lambda c_t^\theta}{\tilde{c}_t^\theta + \lambda} = c_t^\theta - \frac{c_t^\theta \tilde{c}_t^\theta}{\tilde{c}_t^\theta + \lambda},$$

and set $u^\theta(\lambda, 0) = \lambda$ for $t = 0$. This gives that for $t > 0$ and $\lambda > -\tilde{c}_t^\theta$:

$$u^\theta(\lambda, t) = \begin{cases} \frac{2\theta\lambda}{(2\theta + \lambda)e^{2\beta\theta t} - \lambda}, & \text{if } \theta \neq 0, \\ \lambda/(1 + \lambda\beta t), & \text{if } \theta = 0. \end{cases}$$

For $r > 0$ and $t \geq 0$, we have that:

$$u^\theta(c_r^\theta, t) = c_{t+r}^\theta.$$

We recall from the above mentioned references ([18, 16, 3, 20]) for $\lambda \geq 0$ and by analytic prolongation for $\lambda < 0$, that for $t > 0$ and $x \geq 0$:

$$(8) \quad \mathbb{N}^\theta \left[1 - e^{-\lambda Z_t} \right] = u^\theta(\lambda, t) \quad \text{and} \quad \mathbb{E}_x^\theta \left[e^{-\lambda Z_t} \right] = e^{-xu^\theta(\lambda, t)} \quad \text{for all } \lambda > -\tilde{c}_t^\theta.$$

We denote by $\zeta = \inf\{t > 0; Z_t = 0\}$ the lifetime of the process Z . We recall that for all $t > 0$:

$$\mathbb{N}^\theta[\zeta > t] = \lim_{\lambda \rightarrow \infty} u^\theta(\lambda, t) = c^\theta(t).$$

By considering the series in λ in (7) and (8), we deduce that for all $t > 0$ and $n \in \mathbb{N}^*$:

$$(9) \quad \mathbb{N}^\theta \left[\left(\tilde{c}_t^\theta Z_t \right)^n \right] = n! c_t^\theta.$$

We now give a martingale related to the CSBP Z . Recall that \tilde{c}_t^θ is decreasing in t , and thus $-\tilde{c}_{t+r}^\theta > -\tilde{c}_r^\theta$. The next lemma is an easy consequence of (8) and the following elementary equality:

$$u(-\tilde{c}_{t+r}^\theta, t) = -\tilde{c}_r^\theta \quad \text{for all } t \geq 0 \text{ and } r > 0.$$

Lemma 2.2. *Let $\theta \in \mathbb{R}$, $x \in \mathbb{R}_+$, $r > 0$ and the quadratic CSBP $(Z_t, t \geq 0)$ solution of (2). The process $\left(e^{\tilde{c}_{t+r}^\theta Z_t}, t \in I \right)$ is a martingale under \mathbb{N}^θ with $I = (0, +\infty)$ and under \mathbb{P}_x^θ with $I = \mathbb{R}_+$ with respect to the filtration $(\mathcal{F}_t, t \geq 0)$.*

We end this section with the computation of the densities of the entrance law $q_t^\theta(dx)$ and the transition kernel $q_t^\theta(x, dy)$ of the CSBP Z under its excursion measure, where for $t, s > 0$, $x > 0$ and $y \geq 0$:

$$q_t^\theta(dx) = d\mathbb{N}^\theta[Z_t = x, \zeta > t] \quad \text{and} \quad q_{t,s}^\theta(x, dy) = d\mathbb{N}^\theta[Z_{t+s} = y | Z_s = x].$$

Lemma 2.3 (Entrance law and transition densities of Z). *Let $\theta \in \mathbb{R}$. Let $t, s > 0$, $x > 0$ and $y \geq 0$. We have:*

$$q_t^\theta(dx) = q_t^\theta(x) dx \quad \text{and} \quad q_{t,s}^\theta(x, dy) = e^{-xc_t^\theta} \delta_0(dy) + q_t^\theta(x, y) dy,$$

where:

$$(10) \quad q_t^\theta(x) = c_t^\theta \tilde{c}_t^\theta e^{-\tilde{c}_t^\theta x},$$

$$(11) \quad q_t^\theta(x, y) = x c_t^\theta \tilde{c}_t^\theta e^{-(x+y)c_t^\theta - 2\theta y} \sum_{k \in \mathbb{N}} \frac{(xy c_t^\theta \tilde{c}_t^\theta)^k}{k!(k+1)!}.$$

Proof. We omit the parameter θ in the proofs. On one hand, from the definition of $q_t(dx)$, we get that for $\lambda \geq 0$:

$$\int_0^{+\infty} e^{-\lambda x} q_t(dx) = \mathbb{N} \left[e^{-\lambda Z_t} \mathbf{1}_{\{\zeta > t\}} \right] = -\mathbb{N} \left[1 - e^{-\lambda Z_t} \right] + \mathbb{N}[\zeta > t] = c(t) - u(\lambda, t).$$

On the other hand, using (7), we get:

$$\int_0^\infty c_t \tilde{c}_t e^{-(\tilde{c}_t + \lambda)x} dx = c(t) - u(\lambda, t).$$

Then use that finite positive measures on \mathbb{R}_+ are characterized by their Laplace transform to obtain that $q_t(dx) = q_t(x) dx$ with q_t given by (10).

From the definition of $q_t(x, dy)$, we get that for $\lambda \geq 0$:

$$\int_0^{+\infty} e^{-\lambda y} q_t(x, dy) = \mathbb{N} \left[e^{-\lambda Z_{t+s}} \mid Z_s = x \right] = e^{-xu(\lambda, t)} = e^{-\frac{a}{b} + \frac{a}{b+\lambda}},$$

where, thanks to (7), $a = x c_t \tilde{c}_t$ and $b = \tilde{c}_t$. Notice that:

$$e^{\frac{a}{b+\lambda}} = 1 + \sum_{k \in \mathbb{N}} \frac{1}{(k+1)!} \left(\frac{a}{b+\lambda} \right)^{k+1} = 1 + a \sum_{k \in \mathbb{N}} \int_0^{+\infty} \frac{(ay)^k}{k!(k+1)!} e^{-by - \lambda y} dy.$$

Using (3), we deduce that $q_t(x, dy) = e^{-xc_t} \delta_0(dy) + q_t(x, y) dy$, with $q_t(x, y)$ given by (11). \square

3. LOCAL LIMITS FOR THE PROCESS Z

3.1. Some martingales. We present in this section two martingales which will naturally appear in the local limits for the Brownian continuum random tree (CRT). Recall $\theta \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$. We define the non-negative process $M^{\alpha, \theta} = (M_t^{\alpha, \theta}, t > 0)$ by:

$$(12) \quad M_t^{\alpha, \theta} = Z_t e^{-\alpha/c_t^\theta} \sum_{i \in \mathbb{N}} \frac{(\alpha Z_t)^i}{i!(i+1)!} e^{(i+1)2\beta\theta t}.$$

Remark 3.1. Using the Bessel function $I_1(x) = \sum_{i \in \mathbb{N}} (x/2)^{2i+1} / i!(i+1)!$, we can rewrite $M_t^{\alpha, \theta}$ as:

$$M_t^{\alpha, \theta} = \sqrt{\frac{e^{2\beta\theta t} Z_t}{\alpha}} e^{-\alpha/c_t^\theta} I_1 \left(2\sqrt{\alpha e^{2\beta\theta t} Z_t} \right).$$

Using the process Y' defined in (6), which is a time change of the process Z (and which is distributed as a CSBP with parameter $\beta = 1$ and $\theta = 0$), we have for $t > 0$:

$$(13) \quad M_t^{\alpha, \theta} = \sqrt{\frac{Y'_s}{\alpha}} e^{-\alpha s} I_1 \left(2\sqrt{\alpha Y'_s} \right) = Y'_s e^{-\alpha s} \sum_{i \in \mathbb{N}} \frac{(\alpha Y'_s)^i}{i!(i+1)!}.$$

Using Theorem 3 of [31] and Remark 3.1, we immediately get the following result.

Proposition 3.2. *Let $\alpha, \theta \in \mathbb{R}$, $x \in \mathbb{R}_+$. The process $(M_t^{\alpha, \theta}, t \in I)$ is a martingale under \mathbb{N}^θ with $I = (0, +\infty)$ and under \mathbb{P}_x^θ with $I = \mathbb{R}_+$.*

Moreover, for all $t > 0$, we have, using (9) and $\tilde{c}_t^\alpha / c_t^\theta = e^{2\beta\theta t}$, that:

$$\mathbb{N}^\theta \left[M_t^{\alpha, \theta} \right] = 1.$$

Since $u^\theta(-2\theta, t) = -2\theta$, see (4) and (7), and $-2\theta > -\tilde{c}_t^\theta$, we deduce that $(e^{2\theta Z_t}, t > 0)$ is a martingale under \mathbb{N}^θ . According to [3, Section 4], we have that for $\theta \in \mathbb{R}$, $t > 0$:

$$(14) \quad \mathbb{N}^{-\theta} [dZ]_{|\mathcal{F}_t} = e^{2\theta Z_t} \mathbb{N}^\theta [dZ]_{|\mathcal{F}_t}.$$

For $\alpha, \theta \in \mathbb{R}$, we set $\tilde{M}^{\alpha, \theta} = (\tilde{M}_t^{\alpha, \theta}, t > 0)$ with:

$$(15) \quad \tilde{M}_t^{\alpha, \theta} = e^{2\theta Z_t} M_t^{\alpha, -\theta}.$$

Using (12), we get:

$$(16) \quad \tilde{M}_t^{\alpha, \theta} = Z_t e^{2\theta Z_t} e^{-\alpha / \tilde{c}_t^\theta} \sum_{i \in \mathbb{N}} \frac{(\alpha Z_t)^i}{i!(i+1)!} e^{-(i+1)2\beta\theta t}.$$

We then deduce from Proposition 3.2 the following corollary.

Corollary 3.3. *Let $\theta, \alpha \in \mathbb{R}$. The process $\tilde{M}^{\alpha, \theta}$ is a martingale under \mathbb{N}^θ , and for $t > 0$ and any non-negative \mathcal{F}_t -measurable random variable H_t , we have:*

$$(17) \quad \mathbb{N}^\theta [H_t \tilde{M}_t^{\alpha, \theta}] = \mathbb{N}^{-\theta} \left[H_t M_t^{\alpha, -\theta} \right].$$

Remark 3.4 (The case $\theta = 0$ and $\alpha = 0$). Let $t > 0$. For $\theta = 0$, we have:

$$\tilde{M}_t^{\alpha, 0} = M_t^{\alpha, 0} = Z_t e^{-\alpha\beta t} \sum_{i \in \mathbb{N}} \frac{(\alpha Z_t)^i}{i!(i+1)!}.$$

For $\alpha = 0$, we have:

$$(18) \quad M_t^{0, \theta} = Z_t e^{2\beta\theta t} \quad \text{and} \quad \tilde{M}_t^{0, \theta} = Z_t e^{2\theta(Z_t - \beta t)}.$$

Then for $\alpha = \theta = 0$, we have:

$$\tilde{M}_t^{0, 0} = M_t^{0, 0} = Z_t.$$

3.2. Local limit. We first consider the Poisson regime, whose name is inherited from the representation given in Proposition 4.1 based on a Poisson immigration. Let $a = (a_t, t > 0)$ be a positive function.

Proposition 3.5 (Poisson regime). *Let $\theta \in \mathbb{R}$, $s > 0$ and H_s be a bounded non-negative \mathcal{F}_s -measurable random variable. Let $\alpha \in (0, +\infty)$. Assume the function a is such that as $t \rightarrow \infty$ large:*

$$a_t \sim \begin{cases} \alpha \beta^2 t^2, & \text{if } \theta = 0; \\ \alpha (2\theta)^{-2} e^{2\beta|\theta|t}, & \text{if } \theta \neq 0. \end{cases}$$

Then we have:

$$(19) \quad \lim_{t \rightarrow \infty} \mathbb{N}^\theta[H_s | Z_t = a_t] = \mathbb{N}^{|\theta|} [H_s M_s^{\alpha, |\theta|}] = \begin{cases} \mathbb{N}^\theta[H_s M_s^{\alpha, \theta}], & \text{if } \theta \geq 0; \\ \mathbb{N}^\theta[H_s \tilde{M}_s^{\alpha, \theta}], & \text{if } \theta \leq 0. \end{cases}$$

Remark 3.6. Contrary to Proposition 3.2, it is not possible to deduce the result for any θ from the result for $\theta = 0$ in Proposition 3.5, since a_t is not continuous at $\theta = 0$.

Proof. Let $s > 0$ and H_s be fixed. For $t > 0$, we set:

$$A_t = \mathbb{N}^\theta[H_s | Z_{t+s} = a_{t+s}].$$

We omit most of the time the superscript θ in this proof. Thanks to (10) and (11), we have:

$$\begin{aligned} A_t &= \frac{\mathbb{N}[H_s q_t(Z_s, a_{t+s})]}{q_{t+s}(a_{t+s})} \\ &= \frac{c_t \tilde{c}_t}{c_{t+s} \tilde{c}_{t+s}} e^{-a_{t+s}(c_t + 2\theta - \tilde{c}_{t+s})} \mathbb{N} \left[H_s Z_s e^{-Z_s c_t} \sum_{i \in \mathbb{N}} \frac{(Z_s a_{t+s} c_t \tilde{c}_t)^i}{i!(i+1)!} \right]. \end{aligned}$$

As t goes to infinity, we have $\tilde{c}_{t+r} c_{t+r} \sim (2\theta)^2 e^{-2\beta|\theta|t} e^{-2\beta|\theta|r}$ for $\theta \neq 0$ and $\tilde{c}_{t+r} c_{t+r} \sim (\beta t)^{-2}$ for $\theta = 0$. Thanks to the condition on a , it is tedious but elementary to check that:

$$\begin{aligned} \lim_{t \rightarrow \infty} c_t \tilde{c}_t / c_{t+s} \tilde{c}_{t+s} &= e^{2\beta|\theta|s}, \\ \lim_{t \rightarrow \infty} a_{t+s}(c_t + 2\theta - \tilde{c}_{t+s}) &= \lim_{t \rightarrow \infty} a_{t+s}(\tilde{c}_t - \tilde{c}_{t+s}) = \begin{cases} \alpha/c_s & \text{if } \theta \geq 0, \\ \alpha/\tilde{c}_s & \text{if } \theta \leq 0, \end{cases} \\ \lim_{t \rightarrow \infty} a_{t+s} c_t \tilde{c}_t &= \alpha e^{2\beta|\theta|s}. \end{aligned}$$

In particular, we get $\lim_{t \rightarrow +\infty} D_t = D_\infty$, where:

$$\begin{aligned} D_t &= \frac{c_t \tilde{c}_t}{c_{t+s} \tilde{c}_{t+s}} Z_s e^{-Z_s c_t} \sum_{i \in \mathbb{N}} \frac{(Z_s a_{t+s} c_t \tilde{c}_t)^i}{i!(i+1)!}, \\ D_\infty &= Z_s \sum_{i \in \mathbb{N}} \frac{(\alpha Z_s)^i}{i!(i+1)!} \times \begin{cases} e^{(i+1)2\beta\theta s}, & \text{if } \theta \geq 0; \\ e^{2\theta Z_s - (i+1)2\beta\theta s}, & \text{if } \theta \leq 0. \end{cases} \end{aligned}$$

Using the definitions of $M^{\alpha, \theta}$ and $\tilde{M}^{\alpha, \theta}$, see (12) and (16), we get that:

$$D_\infty = \begin{cases} e^{\alpha/c_s} M_s^{\alpha, \theta}, & \text{if } \theta \geq 0; \\ e^{\alpha/\tilde{c}_s} \tilde{M}_s^{\alpha, \theta}, & \text{if } \theta \leq 0. \end{cases}$$

There exists a finite constant C such that, for t large enough:

$$a_{t+s} c_t \tilde{c}_t \leq C e^{2\beta|\theta|s}.$$

Recall $-c_t \leq 2\theta$ (as $c_t + 2\theta = \tilde{c}_t \geq 0$). We deduce that there exists a finite constant C_0 such that, for t large enough:

$$D_t \leq C_0 \left(M_s^{C,\theta} + \tilde{M}_s^{C,\theta} \right).$$

Since $M_s^{C,\theta}$ and $\tilde{M}_s^{C,\theta}$ are integrable, we then deduce that the first equality in (19) holds using the dominated convergence theorem. Then second equality in (19) is a consequence of (17). \square

We now consider the Kesten regime, whose proof is left to the reader as it is very similar to the proof in the Poisson regime when one takes $\alpha = 0$ and uses (18).

Proposition 3.7 (Kesten regime). *Let $\theta \in \mathbb{R}$, $s > 0$ and H_s be a bounded non-negative \mathcal{F}_s -measurable random variable. Assume the function a is positive ($a_t > 0$) and such that as $t \rightarrow \infty$:*

$$a_t = \begin{cases} o(t^2), & \text{if } \theta = 0; \\ o(e^{2\beta|\theta|t}), & \text{if } \theta \neq 0. \end{cases}$$

Then we have:

$$(20) \quad \lim_{t \rightarrow \infty} \mathbb{N}^\theta[H_s | Z_t = a_t] = \mathbb{N}^{|\theta|} \left[H_s Z_s e^{2\beta|\theta|s} \right] = \begin{cases} \mathbb{N}^\theta[H_s M_s^{0,\theta}], & \text{if } \theta \geq 0; \\ \mathbb{N}^\theta[H_s \tilde{M}_s^{0,\theta}], & \text{if } \theta \leq 0. \end{cases}$$

For completeness, we add the well known extinction case, that is the function $a_t = 0$ for large t , which is a direct consequence of (14).

Proposition 3.8 (Extinction regime). *Let $\theta \in \mathbb{R}$, $s > 0$ and H_s be a bounded non-negative \mathcal{F}_s -measurable random variable. Then we have:*

$$(21) \quad \lim_{t \rightarrow \infty} \mathbb{N}^\theta[H_s | Z_t = 0] = \mathbb{N}^{|\theta|}[H_s] = \begin{cases} \mathbb{N}^\theta[H_s], & \text{if } \theta \geq 0; \\ \mathbb{N}^{-\theta}[H_s], & \text{if } \theta \leq 0. \end{cases}$$

4. h -TRANSFORM

We give a representation of the distribution of the process Z under the h -transform given by the martingale $M^{\alpha,\theta}$. The proof will be done for $\beta = 1$ and $\theta = 0$, and then use a time-change, see (6), to get $\theta \in \mathbb{R}$.

4.1. SDE representation. Let $\beta > 0$ and $\theta \in \mathbb{R}$. Let $B = (B_t, t \geq 0)$ be a standard Brownian motion. Let $\alpha > 0$ and $S^{\alpha,\theta}(dt)$ be a Poisson point measure on \mathbb{R}_+ with intensity $\alpha\beta e^{2\beta\theta t} dt$ independent of the Brownian motion B . We set $S_t^{\alpha,\theta} = S^{\alpha,\theta}([0, t])$ for $t \in \mathbb{R}_+$. We define the process $Z^\alpha = (Z_t^\alpha, t \geq 0)$ under \mathbb{P}^θ as the unique strong solution (conditionally on S) of the following SDE:

$$dZ_t^\alpha = \sqrt{2\beta Z_t^\alpha} dB_t - 2\beta\theta Z_t^\alpha dt + 2\beta(S_t^{\alpha,\theta} + 1) dt \quad \text{for } t \geq 0, \quad \text{and } Z_0^\alpha = 0.$$

Proposition 4.1. *Let $\alpha > 0$, $\theta \in \mathbb{R}$ and $t_0 > 0$. The process $(Z_t, t \in [0, t_0])$ under $\mathbb{N}^\theta \left[\bullet M_{t_0}^{\alpha,\theta} \right]$ (resp. under $\mathbb{N}^\theta \left[\bullet \tilde{M}_{t_0}^{\alpha,\theta} \right]$) is distributed as the process $(Z_t^\alpha, t \in [0, t_0])$ under \mathbb{P}^θ (resp. $\mathbb{P}^{-\theta}$).*

The proof of this proposition is detailed in the next subsection.

4.2. Proof of Proposition 4.1. Following Remark 2.1, we first use a scaling argument to remove the parameters β and θ .

Let $\alpha > 0$. Let $S^\alpha = (S_t^\alpha, t \geq 0)$ be a Poisson process with parameter α independent of the Brownian motion B . Let $Y^\alpha = (Y_t^\alpha, t \geq 0)$ be the unique strong solution (conditionally on S) of the following SDE:

$$(22) \quad dY_t^\alpha = \sqrt{2Y_t^\alpha} dB_t + 2(S_t^\alpha + 1) dt \quad \text{for } t \geq 0, \quad \text{and } Y_0^\alpha = 0.$$

Let $\beta, \alpha > 0$ and $\theta \in \mathbb{R}$, and write $Z^{[\beta, \theta, \alpha]}$ for the process Z^α under \mathbb{N}^θ or \mathbb{P}^θ to stress the dependence in β and θ . Define the process $(Y'^\alpha, S'^\alpha) = ((Y'_s{}^\alpha, S'_s{}^\alpha), s \in [0, 1/(2\theta_-)])$ by:

$$(23) \quad Y'_s{}^\alpha = e^{2\beta\theta t} Z_t^{[\beta, \theta, \alpha]} \quad \text{and} \quad S'_s{}^\alpha = S_t^{\alpha, \theta}, \quad \text{with} \quad s = \frac{1}{c_t^\theta}.$$

Then, it is elementary that this deterministic time change yields the following result.

Lemma 4.2. *Let $\beta, \alpha > 0$ and $\theta \in \mathbb{R}$. The process (Y'^α, S'^α) under \mathbb{P}^θ (whose law depends on (β, θ) and α) is distributed as $((Y_s^\alpha, S_s^\alpha), s \in [0, 1/(2\theta_-)])$.*

Let $(P_t, t \geq 0)$ be the transition semi-group on $\mathbb{R}_+ \times \mathbb{N}$ of the Markov process (Y^α, S^α) .

Lemma 4.3. *The semi-group $(P_t, t \geq 0)$ is Feller, that is for all $t \geq 0$ and all bounded continuous function f defined on $\mathbb{R}_+ \times \mathbb{N}$, the function $P_t(f)$ is also bounded and continuous.*

Proof. Let $((Y_t^{\alpha, (x, s)}, S_t^{\alpha, (x, s)}), t \geq 0)$ denote the solution of the SDE (22) starting from (x, s) . Let $(X_t^x, t \geq 0)$ be a Feller process starting from x (it is distributed as a solution to the SDE (5)) independent of the $(Y_t^{\alpha, (x, s)}, S_t^{\alpha, (x, s)})_{t \geq 0}$. We denote by Q_t the semi-group of the process X^x and recall that Q_t is a Feller semi-group. By the branching property, we have the equality in distribution for the processes:

$$\left((Y_t^{\alpha, (x, s)}, S_t^{\alpha, (x, s)}), t \geq 0 \right) \stackrel{(d)}{=} \left((Y_t^{\alpha, (0, s)} + X_t^x, S_t^{\alpha, (0, s)}), t \geq 0 \right).$$

Then for every $t \geq 0$, $x, y \in \mathbb{R}_+$, $s \in \mathbb{N}$ and every bounded continuous function f defined on $\mathbb{R}_+ \times \mathbb{N}$, we have:

$$\begin{aligned} P_t f(x, s) - P_t f(y, s) &= \mathbb{E} \left[f \left(Y_t^{\alpha, (x, s)}, S_t^{\alpha, (x, s)} \right) - f \left(Y_t^{\alpha, (y, s)}, S_t^{\alpha, (y, s)} \right) \right] \\ &= \mathbb{E} \left[f \left(Y_t^{\alpha, (0, s)} + X_t^x, S_t^{\alpha, (0, s)} \right) - f \left(Y_t^{\alpha, (0, s)} + X_t^y, S_t^{\alpha, (0, s)} \right) \right] \\ &= \mathbb{E} \left[Q_t f_{(Y_t^{\alpha, (0, s)}, S_t^{\alpha, (0, s)})}(x) - Q_t f_{(Y_t^{\alpha, (0, s)}, S_t^{\alpha, (0, s)})}(y) \right] \end{aligned}$$

where $f_{(y, s)}$ is the continuous map $x \mapsto f(y + x, s)$. By the Feller property of the semi-group Q_t and the dominated convergence theorem, we deduce that $\lim_{y \rightarrow x} P_t f(x, s) - P_t f(y, s) = 0$. This gives the Feller property of the kernel P_t . \square

We now give the density of (Y_t^α, S_t^α) . Recall that $Y_0^\alpha = S_0^\alpha = 0$. Let \mathbb{N} be the counting measure on \mathbb{N} .

Lemma 4.4. *Let $t > 0$. The random variable (Y_t^α, S_t^α) has a density f on $\mathbb{R}_+ \times \mathbb{N}$ with respect to $dy \otimes \mathbb{N}(dk)$ given by:*

$$(24) \quad f(y, k) = \frac{1}{t^2} \frac{\alpha^k y^{k+1}}{k!(k+1)!} e^{-(\alpha t + t^{-1}y)}, \quad y \geq 0, k \in \mathbb{N}.$$

Proof. Conditionally on S , by Definition (22), we can see Y^α as a quadratic CSBP process (with $\beta = 1$) with immigration whose rate is $2(S_t^\alpha + 1)dt$. This implies that, conditionally on S^α , the process Y^α is distributed as $\left(\sum_{i \in I} \mathbf{1}_{\{h_i \leq t\}} Z_{t-h_i}^{(i)}, t \geq 0\right)$, where $\sum_{i \in I} \delta_{(h_i, Z^{(i)})}(dt, dZ)$ is a Poisson point measure on $\mathbb{R}_+ \times \mathcal{C}[0, \mathbb{R}_+)$ with intensity $2(S_t^\alpha + 1)dt \mathbb{N}[dZ]$ and \mathbb{N} is the excursion measure of a CSBP with branching mechanism $\psi(\lambda) = \lambda^2$.

We deduce that for $\lambda, \mu \geq 0$:

$$\mathbb{E} \left[e^{-\lambda Y_t^\alpha - \mu S_t^\alpha} \right] = \mathbb{E} \left[e^{-\mu S_t^\alpha - \int_0^t 2(S_r^\alpha + 1) \mathbb{N}[1 - e^{-\lambda Z_{t-r}}] dr} \right] = \mathbb{E} \left[e^{-\mu S_t^\alpha - 2 \int_0^t (S_r^\alpha + 1) \frac{\lambda}{1 + (t-r)\lambda} dr} \right],$$

where we used (8) for the last equality (with $\beta = 1$ and $\theta = 0$). Denote by $(\xi_i, i \in \mathbb{N}^*)$ the increasing sequence of the jumping times of the Poisson process S^α , and set $\xi_0 = 0$. Then, we have on $\{S_t^\alpha = k\}$:

$$\begin{aligned} \int_0^t (S_r^\alpha + 1) \frac{\lambda}{1 + (t-r)\lambda} dr &= \sum_{i=0}^k (i+1) \int_{\xi_i}^{\xi_{i+1} \wedge t} \frac{\lambda}{1 + (t-r)\lambda} dr \\ &= - \sum_{i=0}^k (i+1) \log(1 + (t-r)\lambda) \Big|_{\xi_i}^{\xi_{i+1} \wedge t} \\ &= \sum_{i=0}^k \log(1 + (t - \xi_i)\lambda). \end{aligned}$$

Conditionally on $\{S_t^\alpha = k\}$, the random set $\{\xi_1, \dots, \xi_k\}$ is distributed as $\{tU_1, \dots, tU_k\}$ (notice the order is unimportant and is not preserved), where U_1, \dots, U_k are independent random variables uniformly distributed on $[0, 1]$. We deduce that:

$$\begin{aligned} \mathbb{E} \left[e^{-\lambda Y_t^\alpha - \mu S_t^\alpha} \right] &= \sum_{k \in \mathbb{N}} \frac{(\alpha t)^k e^{-\alpha t - \mu k}}{k!} \mathbb{E} \left[\prod_{i=1}^k (1 + t(1 - U_i)\lambda)^{-2} \right] (1 + t\lambda)^{-2} \\ &= \sum_{k \in \mathbb{N}} \frac{(\alpha t)^k e^{-\alpha t - \mu k}}{k!} (1 + t\lambda)^{-k-2} \\ &= \sum_{k \in \mathbb{N}} \int_{\mathbb{R}_+} dy f(y, k) e^{-\lambda y - \mu k}, \end{aligned}$$

where for the last equality, we used the definition of f given in (24). This finishes the proof. \square

Let q'_t be the distribution of Y_t^α for $t \in \mathbb{R}_+$. We have $q'_0 = \delta_0$ the Dirac mass at 0, and for $t > 0$, we deduce from Lemma 4.4 that $q'_t(dy)$ has a density, also denoted by q'_t , on \mathbb{R}_+ with respect to the Lebesgue measure given by:

$$q'_t(y) = t^{-2} e^{-(\alpha t + t^{-1}y)} \sum_{k \in \mathbb{N}} \frac{\alpha^k y^{k+1}}{k!(k+1)!}, \quad t > 0, y \geq 0.$$

We now give some properties of the conditional law of S_t given Y_t .

Lemma 4.5. *Let $y \in \mathbb{R}_+$. The law of S_t^α conditionally on $\{Y_t^\alpha = y\}$ does not depend on t . More precisely, we get for all $t \geq 0$, $k \in \mathbb{N}$ and $y \geq 0$:*

$$(25) \quad \mathbb{P}(S_t^\alpha = k | Y_t^\alpha = y) = C^{-1} \frac{(\alpha y)^k}{k!(k+1)!} \quad \text{with} \quad C = \sum_{j \in \mathbb{N}} \frac{(\alpha y)^j}{j!(j+1)!} \in [1, +\infty).$$

Proof. Using Lemma 4.4, we directly have (25) for $t > 0$. Notice that for $y = 0$, we have $C = 1$ and

$$C^{-1} \frac{(\alpha y)^k}{k!(k+1)!} = \mathbf{1}_{\{k=0\}}.$$

As $(Y_0^\alpha, S_0^\alpha) = (0, 0)$, we deduce that (25) also holds for $t = 0$. \square

We can now prove the Markov property of the process $Y = (Y_t, t \geq 0)$.

Lemma 4.6. *The process Y^α is Markov, and its transition semi-group $(Q_t, t \in \mathbb{R}_+)$ is the unique Feller semi-group such that $q'_t = q'_0 Q_t$ for $t \in \mathbb{R}_+$, with q'_t the distribution of Y_t^α .*

Proof. We say a probability kernel K is continuous if for all continuous and bounded function f , Kf is also continuous (and bounded). We shall check hypothesis from [33, Lemma 1]. With the notation therein ($X = (Y^\alpha, S^\alpha)$ and $\phi(y, s) = y$), the semi-group $(P_t, t \geq 0)$ is Feller, see Lemma 4.3. The probability kernel $\Lambda(y; dz, dk) = \mathbb{P}(S_t^\alpha = k | Y_t^\alpha = y) \delta_y(dz) \mathbb{N}(dk)$ is clearly continuous and does not depend on t . The probability kernel $\Phi(y, k; dz) = \delta_y(dz)$ is also clearly continuous. Lemma 4.5 gives exactly condition (i) in [33, Lemma 1]. We now check condition (ii) in [33, Lemma 1], that is the one-dimensional marginal distributions of Y^α , $(q'_t, t \in \mathbb{R}_+)$, are determining, that is if h and g are bounded continuous functions defined on \mathbb{R}_+ , then $\mathbb{E}[h(Y_t^\alpha)] = \mathbb{E}[g(Y_t^\alpha)]$ for all $t \in \mathbb{R}_+$ implies $h = g$. To prove this, notice that:

$$t^2 e^{\alpha t} \mathbb{E}[h(Y_t^\alpha)] = \int_{\mathbb{R}_+} e^{-t^{-1}y} H(y) dy,$$

where $H(y) = h(y) \sum_{k \in \mathbb{N}} \frac{\alpha^k y^{k+1}}{k!(k+1)!}$. As the Laplace transform characterizes the bounded continuous function, we deduce that if $\mathbb{E}[h(Y_t^\alpha)] = \mathbb{E}[g(Y_t^\alpha)]$ for all $t \in \mathbb{R}_+$, then $H = G$ (with $G(y) = g(y) \sum_{k \in \mathbb{N}} \frac{\alpha^k y^{k+1}}{k!(k+1)!}$) and thus $h = g$ on $(0, +\infty)$ and by continuity on \mathbb{R}_+ .

As the assumption of [33, Lemma 1] are satisfied, we deduce that Y^α is a Markov process, and that its transition semi-group $(Q_t, t \in \mathbb{R}_+)$ is the unique Feller semi-group such that $q'_t = q'_0 Q_t$ for $t \in \mathbb{R}_+$, with q'_t the distribution of Y_t^α . \square

We now compare the distribution of Y^α and the distribution of the Feller diffusion Y defined in Remark 2.1, which is a CSBP with parameter $\beta = 1$ and $\theta = 0$. Following (13), we set for $t > 0$:

$$M_t^\alpha = \sqrt{\frac{Y_t}{\alpha}} e^{-\alpha t} I_1 \left(2\sqrt{\alpha Y_t} \right).$$

Let \mathbb{N} denote the canonical measure of Y .

Lemma 4.7. *Let $\alpha > 0$. Let $t_0 > 0$. The process $(Y_t^\alpha, t \in [0, t_0])$ has the same distribution as the process $(Y_t, t \in [0, t_0])$ under $\mathbb{N}[\bullet M_{t_0}^\alpha]$.*

Proof. We first check the two processes have the same one-dimensional marginals. Clearly $Y_0^\alpha = Y_0 = 0$. Let $t > 0$. According to Lemma 2.3, the entrance law of Y_t under \mathbb{N} has density $y \mapsto t^{-2} e^{-y/t}$. We deduce that for $\lambda \geq 0$:

$$\begin{aligned} \mathbb{N} \left[e^{-\lambda Y_t} M_t^\alpha \right] &= \int_{\mathbb{R}_+} e^{-\lambda y} y e^{-\alpha t} \sum_{i \in \mathbb{N}} \frac{(\alpha y)^i}{i!(i+1)!} t^{-2} e^{-y/t} dy \\ &= \int_{\mathbb{R}_+} e^{-\lambda y} t^{-2} e^{-(\alpha t + t^{-1}y)} \sum_{i \in \mathbb{N}} \frac{\alpha^i y^{i+1}}{i!(i+1)!} dy \\ &= \mathbb{E} \left[e^{-\lambda Y_t^\alpha} \right]. \end{aligned}$$

Since the Laplace transform characterizes the probability distribution on \mathbb{R}_+ , we obtain that Y_t^α has the same distribution as Y_t under $\mathbb{N}[\bullet M_t^\alpha]$.

Using Doob's h -transform, we get that the process $(Y_t, t \in [0, t_0])$ under $\mathbb{N}[\bullet M_{t_0}^\alpha]$ is Markov. Using that M^α is a martingale under \mathbb{N} (see Proposition 3.2 and use that Y is distributed as Z when $\beta = 1, \theta = 0$), that M_t^α is a function of Y_t , and that Y is Feller under \mathbb{N} , we get that $(Y_t, t \in [0, t_0])$ under $\mathbb{N}[\bullet M_{t_0}^\alpha]$ is also Feller. We deduce from the uniqueness property of Lemma 4.6 and the identification of the one-dimensional marginals from the first step of the proof, that $(Y_t^\alpha, t \in [0, t_0])$ has the same distribution as $(Y_t, t \in [0, t_0])$ under $\mathbb{N}[\bullet M_{t_0}^\alpha]$. \square

We can now give the proof of Proposition 4.1. Let $\beta, \alpha > 0, \theta \in \mathbb{R}$ and $t_0 > 0$. Using the time changes given by (6), (13) and (23), we deduce that the process $(Z_t^\alpha, t \in [0, t_0])$ under \mathbb{P}^θ is distributed as the process $(Z_t, t \in [0, t_0])$ under $\mathbb{N}^\theta[\bullet M_{t_0}^{\alpha, \theta}]$. Then, using Corollary 3.3, we also deduce that the process $(Z_t^\alpha, t \in [0, t_0])$ under $\mathbb{P}^{-\theta}$ is distributed as the process $(Z_t, t \in [0, t_0])$ under $\mathbb{N}^\theta[\bullet \tilde{M}_{t_0}^{\alpha, \theta}]$. This finishes the proof of Proposition 4.1.

5. TOPOLOGY ON SET OF TREES

In Section 5.1, we recall the usual basic definitions and notations for rooted real trees. In Section 5.2 (resp. Section 5.3), we consider the Polish space of equivalent classes of compact (resp. locally compact) rooted trees with distinguished vertices endowed with the Gromov-Hausdorff distance. We define various grafting measurable operations (denoted by \otimes^*) of a tree on an another tree in Section 5.4. Motivated by the fact that some random trees are obtained as decorated backbone trees, we introduce in Section 5.5 the space of marked trees, that is of trees with a distinguished sub-tree (or backbone tree). We also establish in this section the measurability of various truncation maps. The short Section 5.6 is devoted to special case of the backbone tree being reduced to an infinite spine (this is the case for the Kesten tree). In Section 5.7, we consider specifically discrete trees which are spanned by n distinguished vertices, and describe them as a set of branches indexed by all the possible subsets of the n distinguished vertices. This description is then used in Section 5.8 to split (with a function Split_n) a locally tree with n distinguished vertices as sub-trees supported by the different branches of the discrete tree spanned by the distinguished vertices. Then, we provide in a sense the inverse construction in Section 5.9 where (with a function Graft_n) we decorate the branches of a discrete trees with subtrees. In Section 5.10, we describe a measurable procedure to decorate a branch with a family of sub-trees given by the atoms of a point measure on the set of trees (the function Tree) and a measurable procedure to describe the decoration of a distinguished branch of a tree (the function \mathcal{M}) through a point measure on the set of trees.

In a nutshell, the main objective of this section is to define the grafting and splitting functions, as well as the decorating and de-decorating functions in a measurable way on the set of locally compact rooted real trees. An index of all the (numerous) relevant notations of this section is provided at the end of the document.

5.1. Notations and definitions for trees. We use as usual the framework of real trees to encode the genealogy of a continuous state branching process. We refer to [21] for a detailed introduction to real trees.

A real tree (or simply a tree in the rest of the text) is a metric space (T, d) that satisfies the two following properties for every $u, v \in T$:

(i) There is a unique isometric map $f_{u,v}$ from $[0, d(u, u)]$ into T such that

$$f_{u,v}(0) = u \quad \text{and} \quad f_{u,v}(d(u, v)) = v.$$

(ii) If φ is a continuous injective map from $[0, 1]$ into T such that $\varphi(0) = u$ and $\varphi(1) = v$, then the range of φ is also the range of $f_{u,v}$.

The range of the map $f_{u,v}$ is denoted by $\llbracket u, v \rrbracket$. It is the unique continuous path that links u to v in the tree. We will write $\llbracket u, v \llbracket$ (resp. $\llbracket u, v \rrbracket \llbracket$) for $\llbracket u, v \rrbracket \setminus \{v\}$ (resp. $\llbracket u, v \rrbracket \setminus \{u\}$, $\llbracket u, v \rrbracket \setminus \{u, v\}$).

A rooted tree is a tree (T, d) with a distinguished vertex denoted by ϱ and called the root. We always consider rooted trees in this work. For an element x of a rooted tree (T, d, ϱ) , we denote by $H(x) = d(\varrho, x)$ its height, and we set $H(T) = \sup_{x \in T} H(x)$ the height of the tree T .

The set of leaves $\text{Lf}(T)$ of T is $\{\varrho\}$ if $T = \{\varrho\}$ is reduced to its root and the set of $x \in T \setminus \{\varrho\}$ such that $T \setminus \{x\}$ has only one connected component. The skeleton of the tree is the set $\text{Sk}(T) = T \setminus \text{Lf}(T)$. The set of branching points (or vertices) $\text{Br}(T)$ is the set of $x \in T$ such that $T \setminus \{x\}$ has at least 3 connected components if $x \neq \varrho$ or at least 2 components if $x = \varrho$.

For a vertex $x \in T$, we define the subtree T_x ‘‘above’’ x as:

$$T_x = \{y \in T : x \in \llbracket \varrho, y \rrbracket\}.$$

The real tree T_x is endowed with the distance induced by T and will be rooted at x .

We define a partial order \prec (called the genealogical order) on a rooted tree (T, d, ϱ) by:

$$u \prec v \iff u \in \llbracket \varrho, v \rrbracket$$

and we say in this case that u is an ancestor of v . If $u, v \in T$, we denote by $u \wedge v$ the most recent common ancestor of u and v , *i.e.* the unique vertex of T such that:

$$\llbracket \varrho, u \rrbracket \cap \llbracket \varrho, v \rrbracket = \llbracket \varrho, u \wedge v \rrbracket.$$

The trace of the Borel σ -field of T on $\text{Sk}(T)$ is generated by the sets $\llbracket s, s' \rrbracket$, $s, s' \in \text{Sk}(T)$ (see [22]). Hence, there exists a σ -finite Borel measure \mathcal{L}^T on T , such that

$$\mathcal{L}^T(\text{Lf}(T)) = 0 \quad \text{and} \quad \mathcal{L}^T(\llbracket s, s' \rrbracket) = d(s, s').$$

This measure \mathcal{L}^T is called the length measure on T . When there is no ambiguity, we simply write \mathcal{L} for \mathcal{L}^T .

5.2. Set of (equivalence classes of) n -pointed compact trees. Let $n \in \mathbb{N}$. A rooted n -pointed tree (T, d, \mathbf{v}) is a tree (T, d) with a root ϱ and n -distinguished (possibly equal) vertices $v_1, \dots, v_n \in T$, with the notation $\mathbf{v} = (v_0 = \varrho, v_1, \dots, v_n)$.

A correspondence \mathcal{R} between two rooted n -pointed trees (T, d, \mathbf{v}) and (T', d', \mathbf{v}') is a subset of $T \times T'$ such that for all $x \in T$ (resp. $x' \in T'$), there exists $x' \in T'$ (resp. $x \in T$) such that $(x, x') \in \mathcal{R}$, and for all $0 \leq k \leq n$, we have $(v_k, v'_k) \in \mathcal{R}$, where $\mathbf{v} = (v_0 = \varrho, \dots, v_n)$ and $\mathbf{v}' = (v'_0 = \varrho', \dots, v'_n)$. The distortion of \mathcal{R} is defined as:

$$\text{dist}(\mathcal{R}) = \sup \left\{ |d(x, y) - d'(x', y')| : (x, x'), (y, y') \in \mathcal{R} \right\}.$$

For two compact rooted n -pointed trees $T = (T, d, \mathbf{v})$ and $T' = (T', d', \mathbf{v}')$, we set:

$$d_{\text{GH}}^{(n)}(T, T') = \inf \frac{1}{2} \text{dist}(\mathcal{R}),$$

where the infimum is taken over all the correspondences \mathcal{R} between (T, d, \mathbf{v}) and (T', d', \mathbf{v}') . The function $d_{\text{GH}}^{(n)}$ is the so-called Gromov-Hausdorff pseudo-distance, see [30]. Furthermore, we have that $d_{\text{GH}}^{(n)}(T, T') = 0$ if and only if there exists an isometric one-to-one map φ from (T, d) to (T', d')

which preserves the root and the distinguished vertices (that is $\varphi(v_k) = v'_k$ for all $0 \leq k \leq n$). The relation $d_{\text{GH}}^{(n)}(T, T') = 0$ defines an equivalence relation between compact rooted n -pointed trees. The set $\mathbb{T}_{\text{K}}^{(n)}$ of equivalence classes of compact rooted n -pointed trees endowed with $d_{\text{GH}}^{(n)}$ is then a metric Polish space, see [30, Proposition 9]. We simply write (T, \mathbf{v}) for (T, d, \mathbf{v}) , and unless specified otherwise, we shall denote also by (T, \mathbf{v}) its equivalence class. For $n = 0$, we simply write \mathbb{T}_{K} and d_{GH} for $\mathbb{T}_{\text{K}}^{(n)}$ and $d_{\text{GH}}^{(n)}$ and T for (T, d, ϱ) .

For a rooted n -pointed tree (T, d, \mathbf{v}) , with $\mathbf{v} = (v_0 = \varrho, \dots, v_n)$, we denote the corresponding spanned tree $\text{Span}^\circ(T, \mathbf{v})$ as:

$$(26) \quad \text{Span}^\circ(T, \mathbf{v}) = \bigcup_{k=1}^n \llbracket \varrho, v_k \rrbracket.$$

The tree $(\text{Span}^\circ(T, \mathbf{v}), d, \varrho)$ will be simply denoted by $\text{Span}^\circ(T, \mathbf{v})$, whereas we will denote by $\text{Span}(T, \mathbf{v})$ the rooted n -pointed tree $(\text{Span}^\circ(T, \mathbf{v}), d, \mathbf{v})$. For $y \in T$, we also define $p_{\mathbf{v}}(y)$, the projection of y on $\text{Span}^\circ(T, \mathbf{v})$, as the only point of $\text{Span}^\circ(T, \mathbf{v})$ such that:

$$(27) \quad \llbracket \varrho, y \rrbracket \cap \text{Span}^\circ(T, \mathbf{v}) = \llbracket \varrho, p_{\mathbf{v}}(y) \rrbracket.$$

Let us state a technical result which will be used several times in what follows.

Lemma 5.1. *Let $n \in \mathbb{N}$. Let (T, d, \mathbf{v}) and (T', d', \mathbf{v}') be two compact rooted n -pointed trees and let \mathcal{R} be a correspondence between them. For every $(x, x') \in \mathcal{R}$ with $x' \in \text{Span}^\circ(T', \mathbf{v}')$, we have:*

$$d(x, p_{\mathbf{v}}(x)) \leq \frac{3}{2} \text{dist}(\mathcal{R}).$$

Proof. Let $(x, x') \in \mathcal{R}$ with $x' \in \text{Span}^\circ(T', \mathbf{v}')$. First remark that there exist $k, \ell \in \{0, \dots, n\}$ such that $p_{\mathbf{v}}(x) \in \llbracket v_k, v_\ell \rrbracket$ and $x' \in \llbracket v'_k, v'_\ell \rrbracket$. Indeed, let us set:

$$A = \{v_k : p_{\mathbf{v}}(x) \in \llbracket \varrho, v_k \rrbracket\} \quad \text{and} \quad A' = \{v'_k : x' \in \llbracket \varrho', v'_k \rrbracket\}.$$

Notice that $A \neq \emptyset$ and $A' \neq \emptyset$. If there exists $k \geq 1$ such that $v_k \in A$ and $v'_k \in A'$, then one can take $\ell = 0$ so that $v_\ell = \varrho$ and $v'_\ell = \varrho'$. Otherwise, take k and ℓ with $k \neq \ell$ such that $v_k \in A$ and $v'_\ell \in A'$. In this case, we get $v_\ell \notin A$. Clearly we have $p_{\mathbf{v}}(x) \in \llbracket v_k, v_\ell \rrbracket$ and by a similar argument, $x' \in \llbracket v'_k, v'_\ell \rrbracket$. Therefore, we have:

$$2d(x, p_{\mathbf{v}}(x)) = d(x, v_k) + d(x, v_\ell) - d(v_k, v_\ell) \leq d'(x', v'_k) + d'(x', v'_\ell) - d'(v'_k, v'_\ell) + 3 \text{dist}(\mathcal{R}).$$

Then, use that $d'(x', v'_k) + d'(x', v'_\ell) - d'(v'_k, v'_\ell) = 0$, as $x' \in \llbracket v'_k, v'_\ell \rrbracket$, to conclude. \square

If (T, \mathbf{v}) and (T', \mathbf{v}') belong to the same equivalence class in $\mathbb{T}_{\text{K}}^{(n)}$, then so do $\text{Span}(T, \mathbf{v})$ and $\text{Span}(T', \mathbf{v}')$ in $\mathbb{T}_{\text{K}}^{(n)}$. Therefore, the function $(T, \mathbf{v}) \mapsto \text{Span}(T, \mathbf{v})$ is well defined from $\mathbb{T}_{\text{K}}^{(n)}$ to $\mathbb{T}_{\text{K}}^{(n)}$. A first consequence of Lemma 5.1 is that this function is Lipschitz continuous; this result will be completed in Lemma 5.6.

Lemma 5.2 (Continuity of the map Span). *Let $n \in \mathbb{N}$. The map $(T, \mathbf{v}) \mapsto \text{Span}(T, \mathbf{v})$ is 4-Lipschitz continuous from $\mathbb{T}_{\text{K}}^{(n)}$ to $\mathbb{T}_{\text{K}}^{(n)}$.*

Proof. Let $(T, \mathbf{v}), (T', \mathbf{v}')$ be two compact rooted n -pointed trees and let \mathcal{R} be a correspondence between them. Let us set with obvious notations:

$$(28) \quad \tilde{\mathcal{R}} = \left\{ (x, p'_{\mathbf{v}'}(x')) : (x, x') \in \mathcal{R}, x \in \text{Span}^\circ(T, \mathbf{v}) \right\} \\ \cup \left\{ (p_{\mathbf{v}}(x), x') : (x, x') \in \mathcal{R}, x' \in \text{Span}^\circ(T', \mathbf{v}') \right\}.$$

Clearly, $\tilde{\mathcal{R}}$ is a correspondence between $\text{Span}(T, \mathbf{v})$ and $\text{Span}(T', \mathbf{v}')$. We now compute its distortion. We consider the case $x \in \text{Span}(T, \mathbf{v})$, $y' \in \text{Span}(T', \mathbf{v}')$ and $(x, x'), (y, y') \in \mathcal{R}$, so that $(x, p'_{\mathbf{v}'}(x'))$ and $(p_{\mathbf{v}}(y), y')$ belong to $\tilde{\mathcal{R}}$. We have:

$$\begin{aligned} \left| d(x, p_{\mathbf{v}}(y)) - d'(p'_{\mathbf{v}'}(x'), y') \right| &= \left| d(x, y) - d(y, p_{\mathbf{v}}(y)) - d'(x', y') + d'(x', p'_{\mathbf{v}'}(x')) \right| \\ &\leq \left| d(x, y) - d'(x', y') \right| + d(y, p_{\mathbf{v}}(y)) + d'(x', p'_{\mathbf{v}'}(x')) \\ &\leq 4 \text{dist}(\mathcal{R}), \end{aligned}$$

where we used Lemma 5.1 for the last inequality. The other cases can be treated similarly. This implies that $\text{dist}(\tilde{\mathcal{R}}) \leq 4 \text{dist}(\mathcal{R})$ and thus, by definition of $d_{\text{GH}}^{(n)}$:

$$d_{\text{GH}}^{(n)}(\text{Span}(T, \mathbf{v}), \text{Span}(T', \mathbf{v}')) \leq 4 d_{\text{GH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}')).$$

□

5.3. Set of (equivalence classes of) rooted n -pointed locally compact trees. Recall the definition of the height $H(x) = d(\varrho, x)$ of a vertex x in a rooted tree (T, d, ρ) . For a rooted n -pointed tree (T, d, \mathbf{v}) and $t \geq 0$, we define the rooted n -pointed tree T truncated at level t as $(r_t(T, \mathbf{v}), d, \mathbf{v})$ with:

$$(29) \quad r_t(T, \mathbf{v}) = \{x \in T : H(x) \leq t\} \cup \{\text{Span}^\circ(T, \mathbf{v})\},$$

and the distance on $r_t(T, \mathbf{v})$ is given by the restriction of the distance d . We shall simply write $r_t(T, \mathbf{v})$ for $(r_t(T, \mathbf{v}), d, \mathbf{v})$. If (T, \mathbf{v}) and (T', \mathbf{v}') are in the same equivalence class of $\mathbb{T}_{\mathbb{K}}^{(n)}$, so are $r_t(T, \mathbf{v})$ and $r_t(T', \mathbf{v}')$. Thus the function r_t can be seen as a map from $\mathbb{T}_{\mathbb{K}}^{(n)}$ to itself. When $n = 0$, we shall simply write $r_t(T)$ for $r_t(T, \varrho)$. The next lemma is about the continuity of r_t .

Lemma 5.3 (Continuity of r_t). *Let $n \in \mathbb{N}$. For $s, t \geq 0$ and $(T, \mathbf{v}), (T', \mathbf{v}') \in \mathbb{T}_{\mathbb{K}}^{(n)}$, we have:*

$$(30) \quad d_{\text{GH}}^{(n)}(r_t(T, \mathbf{v}), r_{t+s}(T', \mathbf{v}')) \leq 4 d_{\text{GH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}')) + s.$$

The map $(t, (T, \mathbf{v})) \mapsto r_t(T, \mathbf{v})$ is continuous from $\mathbb{R}_+ \times \mathbb{T}_{\mathbb{K}}^{(n)}$ to $\mathbb{T}_{\mathbb{K}}^{(n)}$.

Proof. Let $(T, d, \mathbf{v}), (T', d', \mathbf{v}')$ be two compact rooted n -pointed trees. Firstly, notice that $d_{\text{GH}}^{(n)}(r_{t+s}(T, \mathbf{v}), r_t(T, \mathbf{v})) \leq s$. Secondly, recall Definition (27) of the projection $p_{\mathbf{v}}$ on $\text{Span}^\circ(T, \mathbf{v})$. For $y \in T$, we also define the projection $p_t(y)$ on $r_t(T, \mathbf{v})$ as the only point of $r_t(T, \mathbf{v})$ such that:

$$\llbracket \varrho, y \rrbracket \cap r_t(T, \mathbf{v}) = \llbracket \varrho, p_t(y) \rrbracket.$$

We first prove the analogue of Lemma 5.1. Let \mathcal{R} be a correspondence between (T, \mathbf{v}) and (T', \mathbf{v}') . Let $(x, x') \in \mathcal{R}$ with $x' \in r_t(T', \mathbf{v}')$. By construction, we have $p_t(x) \in \llbracket p_{\mathbf{v}}(x), x \rrbracket$. If $x' \in \text{Span}(T', \mathbf{v}')$, then we deduce from Lemma 5.1 that $d(x, p_t(x)) \leq d(x, p_{\mathbf{v}}(x)) \leq \frac{3}{2} \text{dist}(\mathcal{R})$. If $x' \in r_t(T', \mathbf{v}') \setminus \text{Span}(T', \mathbf{v}')$, then we have $H(x') \leq t$ and thus $H(x) = d(\varrho, x) \leq d'(\varrho', x') + \text{dist}(\mathcal{R}) \leq t + \text{dist}(\mathcal{R})$, which implies that $d(x, p_t(x)) \leq \text{dist}(\mathcal{R})$. In conclusion, we get $d(x, p_t(x)) \leq \frac{3}{2} \text{dist}(\mathcal{R})$. Now, arguing as in the proof of Lemma 5.2, we deduce that $d_{\text{GH}}^{(n)}(r_t(T, \mathbf{v}), r_t(T', \mathbf{v}')) \leq 4 d_{\text{GH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}'))$. This gives the result. □

A rooted n -pointed tree (T, d, \mathbf{v}) is locally compact if $r_t(T, \mathbf{v})$ is a compact rooted tree for all $t \geq 0$. Following [6], we set for two locally compact rooted n -pointed trees (T, \mathbf{v}) and (T', \mathbf{v}') :

$$d_{\text{LGH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}')) = \int_0^\infty e^{-t} dt \left(1 \wedge d_{\text{GH}}^{(n)}(r_t(T, \mathbf{v}), r_t(T', \mathbf{v}')) \right).$$

Furthermore, we have that $d_{\text{LGH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}')) = 0$ if and only if there exists an isometric one-to-one map φ from (T, d) to (T', d') which preserves the distinguished vertices. The relation $d_{\text{LGH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}')) = 0$ defines an equivalence relation. The set $\mathbb{T}_{\text{loc-K}}^{(n)}$ of equivalence classes of locally compact rooted trees endowed with $d_{\text{LGH}}^{(n)}$ is then a metric Polish space. Furthermore, $\mathbb{T}_{\text{K}}^{(n)}$ is an open dense subset of $\mathbb{T}_{\text{loc-K}}^{(n)}$. For $n = 0$, we simply write $\mathbb{T}_{\text{loc-K}}$ and d_{LGH} for $\mathbb{T}_{\text{loc-K}}^{(n)}$ and $d_{\text{LGH}}^{(n)}$. We provide a short proof for the following inequalities.

Lemma 5.4 (Inequalities for $d_{\text{GH}}^{(n)}$ and $d_{\text{LGH}}^{(n)}$). *Let $n \in \mathbb{N}$. For $(T, \mathbf{v}), (T', \mathbf{v}') \in \mathbb{T}_{\text{K}}^{(n)}$, we have:*

$$(31) \quad d_{\text{LGH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}')) \leq 1 \wedge 4 d_{\text{GH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v})).$$

For $(T, \mathbf{v}), (T', \mathbf{v}') \in \mathbb{T}_{\text{loc-K}}^{(n)}$ and $s, t \geq 0$, we have:

$$(32) \quad d_{\text{LGH}}^{(n)}(r_t(T, \mathbf{v}), r_{t+s}(T', \mathbf{v}')) \leq 4 d_{\text{LGH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}')) + s,$$

$$(33) \quad d_{\text{GH}}^{(n)}(r_t(T, \mathbf{v}), r_t(T', \mathbf{v}')) \leq 4 e^t d_{\text{LGH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}')).$$

The map $(t, (T, \mathbf{v})) \mapsto r_t(T, \mathbf{v})$ is continuous from $\mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{(n)}$ (and to $\mathbb{T}_{\text{K}}^{(n)}$).

Proof. Equation (31) is a direct consequence of (30) with $s = 0$ and the definition of $d_{\text{LGH}}^{(n)}$. Equation (32) follows from similar arguments, using also that $r_{t'} \circ r_u = r_u \circ r_{t'} = r_{t' \wedge u}$. For $t \leq s$, we have $4^{-1} d_{\text{GH}}^{(n)}(r_t(T, \mathbf{v}), r_t(T', \mathbf{v}')) \leq d_{\text{GH}}^{(n)}(r_s(T, \mathbf{v}), r_s(T', \mathbf{v}'))$. Integrating with respect to $e^{-s} ds$ gives (33). The continuity of the map $(t, (T, \mathbf{v})) \mapsto r_t(T, \mathbf{v})$ is a direct consequence of (32). \square

We deduce from (31) and (33) that all the measurable sets of $(\mathbb{T}_{\text{K}}^{(n)}, d_{\text{GH}}^{(n)})$ are measurable sets of $(\mathbb{T}_{\text{loc-K}}^{(n)}, d_{\text{LGH}}^{(n)})$, and that a converging sequence in $(\mathbb{T}_{\text{K}}^{(n)}, d_{\text{GH}}^{(n)})$ is also converging in $(\mathbb{T}_{\text{loc-K}}^{(n)}, d_{\text{LGH}}^{(n)})$. We also we deduce from (31) that the restriction to $\mathbb{T}_{\text{K}}^{(n)}$ of a continuous function defined on $(\mathbb{T}_{\text{loc-K}}^{(n)}, d_{\text{LGH}}^{(n)})$ is also continuous on $(\mathbb{T}_{\text{K}}^{(n)}, d_{\text{GH}}^{(n)})$.

Removing from \mathbf{v} some of the pointed vertices (but the root) is continuous, see the next lemma. For $(T, \mathbf{v} = (v_0 = \varrho, \dots, v_n)) \in \mathbb{T}_{\text{loc-K}}^{(n)}$ and $0 \in A \subset \{0, \dots, n\}$, we set:

$$(34) \quad \Pi_n^{\circ, A}(T, \mathbf{v}) = (T, \mathbf{v}_A) \quad \text{with} \quad \mathbf{v}_A = (v_i, i \in A).$$

For simplicity, we shall write Π_n° for $\Pi_n^{\circ, A}$ when A is reduced to $\{0\}$, so that Π_n° corresponds to removing all the pointed vertices but the root.

Lemma 5.5 (Removing some pointed vertices is continuous). *Let $n \in \mathbb{N}$ and $0 \in A \subset \{0, \dots, n\}$. The map $\Pi_n^{\circ, A}$ from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{(k)}$, with k the cardinal of A , is 1-Lipschitz continuous.*

Proof. First, notice that the equivalence class of (T, \mathbf{v}_A) in $\mathbb{T}_{\text{loc-K}}^{(k)}$ does not depend of the choice of (T, \mathbf{v}) in its equivalence class in $\mathbb{T}_{\text{loc-K}}^{(n)}$. Thus the map $\Pi_n^{\circ, A}$ is well defined from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{(k)}$. It is clearly 1-Lipschitz continuous since a correspondence between the trees (T, \mathbf{v}) and (T', \mathbf{v}') is also a correspondence between (T, \mathbf{v}_A) and (T', \mathbf{v}'_A) . \square

We give an immediate consequence on the continuity of the maps Span and Span° .

Lemma 5.6 (Continuity of the maps Span and Span°). *Let $n \in \mathbb{N}$. The map $(T, \mathbf{v}) \mapsto \text{Span}(T, \mathbf{v})$ and $(T, \mathbf{v}) \mapsto \text{Span}^\circ(T, \mathbf{v})$ are 4-Lipschitz continuous from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{(n)}$ and to $\mathbb{T}_{\text{loc-K}}$ respectively.*

Proof. Notice that $d_{\text{LGH}}^{(n)}(\text{Span}(T, \mathbf{v}), \text{Span}(T', \mathbf{v})) = d_{\text{GH}}^{(n)}(\text{Span}(T, \mathbf{v}), \text{Span}(T', \mathbf{v}))$, and thus the map Span from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{(n)}$ is 4-Lipschitz continuous, thanks to Lemma 5.2. Then use Lemma 5.5 on the continuity of Π_n° and the fact that $\text{Span}^\circ = \Pi_n^\circ \circ \text{Span}$ to conclude. \square

Next, we check that rerooting or reordering the pointed vertices is a continuous operation. For a vector $\mathbf{v} = (v_0, \dots, v_n)$ and a permutation π of $\{0, \dots, n\}$, we set $\mathbf{v}^\pi = (v_{\pi(0)}, \dots, v_{\pi(n)})$.

Remark 5.7. One can see that the map $(T, \mathbf{v}) \mapsto (T, \mathbf{v}^\pi)$ is an isometry on $\mathbb{T}_{\text{K}}^{(n)}$. The next lemma is an extension to locally compact case.

Lemma 5.8 (Permuting the pointed vertices is continuous). *Let $n \in \mathbb{N}$ and let π be a permutation on $\{0, \dots, n\}$. The map $(T, \mathbf{v}) \mapsto (T, \mathbf{v}^\pi)$ defined on $\mathbb{T}_{\text{loc-K}}^{(n)}$ is continuous.*

Proof. First notice that if (T, \mathbf{v}) and (T', \mathbf{v}') are rooted n -pointed trees belonging to the same equivalence class of $\mathbb{T}_{\text{loc-K}}^{(n)}$, so do (T, \mathbf{v}^π) and (T', \mathbf{v}'^π) . Thus, the map $(T, \mathbf{v}) \mapsto (T, \mathbf{v}^\pi)$ is indeed well-defined on $\mathbb{T}_{\text{loc-K}}^{(n)}$. We shall use the following notation: we denote by r_t° the truncation r_t when one forgets about the pointed vertices (but the root): $r_t^\circ = \Pi_n^\circ \circ r_t$. (Take care that $\Pi_n^\circ \circ r_t \neq r_t \circ \Pi_n^\circ$.) To prove the continuity of the map, we consider two cases.

1st case: No rerooting, $\pi(0) = 0$. In that case, for every $t \geq 0$ and every $(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(n)}$, we have that $r_t^\circ(T, \mathbf{v}) = r_t^\circ(T, \mathbf{v}^\pi)$ and thus we get that:

$$d_{\text{LGH}}^{(n)}((T, \mathbf{v}^\pi), (T', \mathbf{v}'^\pi)) = d_{\text{LGH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}')).$$

This trivially implies the continuity of the map.

2nd case: With rerooting, $\pi(k_0) = 0$ for some $k_0 \neq 0$. Let $(T, \mathbf{v}), (T', \mathbf{v}') \in \mathbb{T}_{\text{loc-K}}^{(n)}$, with $\mathbf{v} = (v_0 = \varrho, \dots, v_n)$ and $\mathbf{v}' = (v'_0 = \varrho', \dots, v'_n)$, such that $d_{\text{LGH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}')) < 1/2$. As v_{k_0} and v'_{k_0} are always in correspondence as well as ϱ and ϱ' , we have, for every $t \geq 0$ that:

$$|H(v_{k_0}) - H(v'_{k_0})| \leq 2d_{\text{GH}}^{(n)}(r_t(T, \mathbf{v}), r_t(T', \mathbf{v}')).$$

Multiplying by e^{-t} and integrating yields:

$$1 \wedge |H(v_{k_0}) - H(v'_{k_0})| \leq 2d_{\text{LGH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}')) < 1,$$

and hence:

$$H(v'_{k_0}) \leq H(v_{k_0}) + 1.$$

We set $h_0 = H(v_{k_0}) + 1$. Then, for every $t \geq 0$, we have:

$$r_t^\circ(T, \mathbf{v}^\pi) \subset r_{t+h_0}^\circ(T, \mathbf{v}) \quad \text{and thus} \quad r_t(T, \mathbf{v}^\pi) = r_t(r_{t+h_0}^\circ(T, \mathbf{v}), \mathbf{v}^\pi),$$

and the same holds for T' . Consequently, applying Lemma 5.3, we have:

$$\begin{aligned} d_{\text{LGH}}^{(n)}((T, \mathbf{v}^\pi), (T', \mathbf{v}'^\pi)) &\leq 4 \int_0^{+\infty} dt e^{-t} \left(1 \wedge d_{\text{GH}}^{(n)}\left((r_{t+h_0}^\circ(T, \mathbf{v}), \mathbf{v}^\pi), (r_{t+h_0}^\circ(T', \mathbf{v}'), \mathbf{v}'^\pi)\right) \right) \\ &= 4 \int_0^{+\infty} dt e^{-t} \left(1 \wedge d_{\text{GH}}^{(n)}(r_{t+h_0}(T, \mathbf{v}), r_{t+h_0}(T', \mathbf{v}')) \right) \\ &\leq 4e^{h_0} d_{\text{LGH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}')), \end{aligned}$$

where we used for the second inequality that $d_{\text{GH}}^{(n)}((\tilde{T}, \mathbf{v}^\pi), (\tilde{T}', \mathbf{v}'^\pi)) = d_{\text{GH}}^{(n)}((\tilde{T}, \mathbf{v}), (\tilde{T}', \mathbf{v}'))$ for $(\tilde{T}, \mathbf{v}), (\tilde{T}', \mathbf{v}') \in \mathbb{T}_{\text{K}}^{(n)}$. The continuity of the map follows. \square

We shall also consider the set of trees whose root is not a branching vertex:

$$(35) \quad \mathbb{T}_{\text{loc-K}}^{(n),0} = \{(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(n)} : \varrho \notin \text{Br}(T)\}.$$

We shall simply write $\mathbb{T}_{\text{loc-K}}^0$ for $\mathbb{T}_{\text{loc-K}}^{(n),0}$ when $n = 0$.

Lemma 5.9. *The set $\mathbb{T}_{\text{loc-K}}^{(n),0}$ is a Borel subset of $\mathbb{T}_{\text{loc-K}}^{(n)}$.*

Proof. For a rooted tree T , we define its diameter by $\text{diam}(T) = \sup\{d(x, y) : x, y \in T\}$. Notice that $H(T) \leq \text{diam}(T) \leq 2H(T)$. Clearly the function diam is constant on all equivalent classes of $\mathbb{T}_{\text{K}}^{(n)}$ and thus of $\mathbb{T}_{\text{loc-K}}^{(n)}$. If $\text{diam}(T) = 2H(T) < +\infty$, then we deduce that the root is a branching vertex. Recall Π_n° for (34). More generally, we get that:

$$\mathbb{T}_{\text{loc-K}}^{(n),0} = \bigcup_{n \in \mathbb{N}^*} D_{1/n} \quad \text{with} \quad D_t = \left\{ T \in \mathbb{T}_{\text{loc-K}}^{(n)} : \text{diam}(r_t \circ \Pi_n^\circ(T)) = 2t \right\}.$$

Since the functions diam , r_t and Π_n° are continuous, we deduce that D_t is closed, and hence $\mathbb{T}_{\text{loc-K}}^{(n),0}$ is a Borel subset of $\mathbb{T}_{\text{loc-K}}^{(n)}$. \square

We now define the set of discrete trees. We say that a rooted n -pointed tree (T, d, \mathbf{v}) is a discrete tree if T is equal to the tree spanned by its distinguished vertices: $T = \text{Span}^\circ(T, \mathbf{v})$. We define the set of (equivalence classes of) discrete trees with at most n leaves as:

$$(36) \quad \mathbb{T}_{\text{dis}}^{(n)} = \{(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(n)} : (T, \mathbf{v}) = \text{Span}(T, \mathbf{v})\}.$$

As a direct consequence of the continuity of the map Span we get the following result.

Lemma 5.10. *Let $n \in \mathbb{N}$. The set of discrete trees $\mathbb{T}_{\text{dis}}^{(n)}$ is a closed subset of $\mathbb{T}_{\text{K}}^{(n)}$ and of $\mathbb{T}_{\text{loc-K}}^{(n)}$.*

We end this section with partial measurability result on the number of vertices at a given height of a tree.

Remark 5.11. It is immediate to check that the map $(T, \mathbf{v}) \mapsto (d(v_i, v_j), i, j \in \{0, \dots, n\})$ is injective $1/2$ -Lipschitz continuous from $(\mathbb{T}_{\text{loc-K}}^{(n)}, d_{\text{LGH}}^{(n)})$ to $\mathbb{R}^{(n+1) \times (n+1)}$ endowed with the supremum norm (*i.e.* the maximum of the distances between coordinates). It is also bi-measurable thanks to Lusin's theorem from [32] or [11, Exercise 6.10.54 p.60].

Let $\mathbb{T}_{\text{loc-K}}^{\text{no leaf}}$ be the set of trees with no leaves:

$$\mathbb{T}_{\text{loc-K}}^{\text{no leaf}} = \{T \in \mathbb{T}_{\text{loc-K}} : \text{Lf}(T) = \emptyset\}.$$

For $T \in \mathbb{T}_{\text{loc-K}}^{\text{no leaf}}$ and $t \geq 0$, let $\tilde{N}_t(T)$ denotes the finite number of vertices at height t of T :

$$(37) \quad \tilde{N}_t(T) = \text{Card} \left(\{x \in T : H(x) = t\} \right).$$

We have the following result.

Lemma 5.12 (Measurability of \tilde{N}_t). *The set $\mathbb{T}_{\text{loc-K}}^{\text{no leaf}}$ is a Borel subset of $\mathbb{T}_{\text{loc-K}}$ and the map $(t, T) \mapsto \tilde{N}_t(T)$ is measurable from $\mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^{\text{no leaf}}$ to \mathbb{N} .*

Proof. Let $t \geq 0$ and let $\Theta_n(t)$ be the set of discrete trees such that all the pointed vertices (but the root) are leaves at height t :

$$\Theta_n(t) = \{T \in \mathbb{T}_{\text{dis}}^{(n)} : d(\varrho, v_i) = t \text{ and } d(v_i, v_j) > 0 \text{ for all } i, j \in \{1, \dots, n\}\}.$$

Thanks to Remark 5.11, $\Theta_n(t)$ is a Borel set of $\mathbb{T}_{\text{dis}}^{(n)} \subset \mathbb{T}_{\text{K}}^{(n)} \subset \mathbb{T}_{\text{loc-K}}^{(n)}$. For $T \in \mathbb{T}_{\text{loc-K}}^{(n)}$, we get that $\{T' \in \mathbb{T}_{\text{dis}}^{(n)} : \Pi_n^\circ(T') = T\}$ is finite. We deduce from Lusin's theorem, see [32], that

Π_n° restricted to $\mathbb{T}_{\text{dis}}^{(n)}$ is bi-measurable. This implies that the set $\Pi_n^\circ(\Theta_n(t))$ is a Borel subset of $\mathbb{T}_{\text{loc-K}}$. We deduce that the set of trees with no leaves, $\mathbb{T}_{\text{loc-K}}^{\text{no leaf}}$, which is formally defined by:

$$\mathbb{T}_{\text{loc-K}}^{\text{no leaf}} = \bigcap_{k \in \mathbb{N}^*} \bigcup_{n \in \mathbb{N}} r_k^{-1} \left(\Pi_n^\circ(\Theta_n(k)) \right),$$

is a Borel subset of $\mathbb{T}_{\text{loc-K}}$. We also get that $\{T \in \mathbb{T}_{\text{loc-K}}^{\text{no leaf}} : \tilde{N}_t(T) = n\} = r_t^{-1} \left(\Pi_n^\circ(\Theta_n(t)) \right)$; this implies that the map \tilde{N}_t is measurable. Since $t \mapsto \tilde{N}_t(T)$ is non-decreasing and left-continuous, we deduce that the map $(t, T) \mapsto \tilde{N}_t(T)$ is measurable from $\mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^{\text{no leaf}}$ to \mathbb{N} . \square

5.4. Grafting a tree on another one. We start by recalling the grafting operation of [2] which is slightly different from the function Graft which will be introduced in Section 5.9.

Let $n, k \in \mathbb{N}$ and $i \in \{0, \dots, n\}$ be given. Let (T, d, \mathbf{v}) be a locally compact rooted n -pointed tree with $\mathbf{v} = (v_0 = \varrho, \dots, v_n)$ and (T', d', \mathbf{v}') be a locally compact rooted k -pointed tree with $\mathbf{v}' = (v'_0 = \varrho', \dots, v'_k)$. We define the tree $T \otimes_i T'$ as the tree obtained by grafting T' on the i -th distinguished vertex of the tree T . We set $\mathbf{v} \otimes \mathbf{v}'$ the concatenation of the vector \mathbf{v} and the vector (v'_1, \dots, v'_k) which is the vector \mathbf{v}' where the coordinate $v'_0 = \varrho'$ is removed, and:

$$T \otimes_i T' = T \sqcup (T' \setminus \{\varrho'\}),$$

$$\forall x, x' \in T \otimes_i T', \quad d^\otimes(x, x') = \begin{cases} d(x, x') & \text{if } x, x' \in T, \\ d'(x, x') & \text{if } x, x' \in T', \\ d(x, v_i) + d'(\varrho', x') & \text{if } x \in T, x' \in T', \end{cases}$$

where \sqcup denotes the disjoint union of two sets. By construction $(T \otimes_i T', d^\otimes, \mathbf{v} \otimes \mathbf{v}')$ is a locally compact rooted $n+k$ pointed tree. It is easy to see that the equivalence class of $T \otimes_i T'$ in $\mathbb{T}_{\text{loc-K}}^{(n+k)}$ does not depend of the choice of the representatives in the equivalence classes of T and T' and hence the map $((T, d, \mathbf{v}), (T', d', \mathbf{v}')) \mapsto (T \otimes_i T', d^\otimes, \mathbf{v} \otimes \mathbf{v}')$ is well defined from $\mathbb{T}_{\text{loc-K}}^{(n)} \times \mathbb{T}_{\text{loc-K}}^{(k)}$ into $\mathbb{T}_{\text{loc-K}}^{(n+k)}$. We shall simply write $(T \otimes_i T', \mathbf{v} \otimes \mathbf{v}')$, or even $T \otimes_i T'$, for $(T \otimes_i T', d^\otimes, \mathbf{v} \otimes \mathbf{v}')$. The next lemma asserts that this grafting procedure is continuous.

Lemma 5.13 (Continuity of the grafting map). *Let $n, k \in \mathbb{N}$ and $i \in \{0, \dots, n\}$. The map $((T, \mathbf{v}), (T', \mathbf{v}')) \mapsto (T \otimes_i T', \mathbf{v} \otimes \mathbf{v}')$, is continuous from $\mathbb{T}_{\text{loc-K}}^{(n)} \times \mathbb{T}_{\text{loc-K}}^{(k)}$ to $\mathbb{T}_{\text{loc-K}}^{(n+k)}$.*

Proof. Let $(T_1, \mathbf{v}_1), (T'_1, \mathbf{v}'_1) \in \mathbb{T}_{\text{loc-K}}^{(n)}$ and $(T_2, \mathbf{v}_2), (T'_2, \mathbf{v}'_2) \in \mathbb{T}_{\text{loc-K}}^{(k)}$. Set $T = T_1 \otimes_i T_2$, $T' = T'_1 \otimes_i T'_2$, $\mathbf{v} = \mathbf{v}_1 \otimes \mathbf{v}_2$, and $\mathbf{v}' = \mathbf{v}'_1 \otimes \mathbf{v}'_2$.

First suppose that the trees are compact, that is $(T_1, \mathbf{v}_1), (T'_1, \mathbf{v}'_1) \in \mathbb{T}_{\text{K}}^{(n)}$ and $(T_2, \mathbf{v}_2), (T'_2, \mathbf{v}'_2) \in \mathbb{T}_{\text{K}}^{(k)}$. Let \mathcal{R}_1 be a correspondence between (elements of the classes) (T_1, \mathbf{v}_1) and (T'_1, \mathbf{v}'_1) and let \mathcal{R}_2 be a correspondence between (elements of the classes) (T_2, \mathbf{v}_2) and (T'_2, \mathbf{v}'_2) . We set $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ with ϱ_2 and ϱ'_2 replaced respectively by v_i and v'_i . It defines a correspondence between (T, \mathbf{v}) and (T', \mathbf{v}') . For every $(x, x'), (y, y') \in \mathcal{R}$, we have:

$$|d^\otimes(x, y) - d^\otimes(x', y')| = \begin{cases} |d_1(x, y) - d'_1(x', y')| \leq \text{dist}(\mathcal{R}_1) & \text{if } (x, x'), (y, y') \in \mathcal{R}_1, \\ |d_2(x, y) - d'_2(x', y')| \leq \text{dist}(\mathcal{R}_2) & \text{if } (x, x'), (y, y') \in \mathcal{R}_2, \end{cases}$$

and if $(x, x') \in \mathcal{R}_1$ and $(y, y') \in \mathcal{R}_2$, we have:

$$\begin{aligned} |d^{\otimes}(x, y) - d^{\otimes}(x', y')| &= |d_1(x, v_i) + d_2(\rho_2, y) - d'_2(\rho'_2, y') - d'_1(x', v'_i)| \\ &\leq |d_1(x, v_i) - d_1(x', v'_i)| + |d_2(\rho_2, y) - d'_2(\rho'_2, y')| \\ &\leq \text{dist}(\mathcal{R}_1) + \text{dist}(\mathcal{R}_2). \end{aligned}$$

This gives:

$$(38) \quad d_{\text{GH}}^{(n+k)}((T, \mathbf{v}), (T', \mathbf{v}')) \leq d_{\text{GH}}^{(n)}((T_1, \mathbf{v}_1), (T'_1, \mathbf{v}'_1)) + d_{\text{GH}}^{(k)}((T_2, \mathbf{v}_2), (T'_2, \mathbf{v}'_2)).$$

Now consider $(T_1, \mathbf{v}_1), (T'_1, \mathbf{v}'_1) \in \mathbb{T}_{\text{loc-K}}^{(n)}$ and $(T_2, \mathbf{v}_2), (T'_2, \mathbf{v}'_2) \in \mathbb{T}_{\text{loc-K}}^{(k)}$. Without loss of generality we assume that $H(v'_i) \geq H(v_i)$. Remark that, for every $t \geq 0$, we have, with $a_+ = \max(a, 0)$:

$$r_t(T, \mathbf{v}) = r_t(T_1, \mathbf{v}_1) \otimes_i r_{(t-H(v_i))_+}(T_2, \mathbf{v}_2).$$

Therefore, we have:

$$\begin{aligned} &d_{\text{LGH}}^{(n+k)}((T, \mathbf{v}), (T', \mathbf{v}')) \\ &= \int_0^{+\infty} dt e^{-t} \left(1 \wedge d_{\text{GH}}^{(n+k)}(r_t(T, \mathbf{v}), r_t(T', \mathbf{v}')) \right) \\ &= \int_0^{+\infty} dt e^{-t} \left(1 \wedge d_{\text{GH}}^{(n+k)} \left(r_t(T_1, \mathbf{v}_1) \otimes_i r_{(t-H(v_i))_+}(T_2, \mathbf{v}_2), r_t(T'_1, \mathbf{v}'_1) \otimes_i r_{(t-H(v'_i))_+}(T'_2, \mathbf{v}'_2) \right) \right) \\ &\leq \int_0^{+\infty} dt e^{-t} \left(1 \wedge d_{\text{GH}}^{(n)}(r_t(T_1, \mathbf{v}_1), r_t(T'_1, \mathbf{v}'_1)) \right) \\ &\quad + \int_0^{+\infty} dt e^{-t} \left(1 \wedge d_{\text{GH}}^{(k)} \left(r_{(t-H(v_i))_+}(T_2, \mathbf{v}_2), r_{(t-H(v'_i))_+}(T'_2, \mathbf{v}'_2) \right) \right) \\ &\leq d_{\text{LGH}}^{(n)}((T_1, \mathbf{v}_1), (T'_1, \mathbf{v}'_1)) + 4e^{-H(v'_i)} d_{\text{LGH}}^{(k)}((T_2, \mathbf{v}_2), (T'_2, \mathbf{v}'_2)) + H(v'_i) - H(v_i) \\ &\leq 3d_{\text{LGH}}^{(n)}((T_1, \mathbf{v}_1), (T'_1, \mathbf{v}'_1)) + 4d_{\text{LGH}}^{(k)}((T_2, \mathbf{v}_2), (T'_2, \mathbf{v}'_2)), \end{aligned}$$

where we used Equation (38) for the first inequality and Lemma 5.3 for the second one. We are done. \square

Remark 5.14. We shall use a version of the grafting procedure where, instead of grafting on v_i , we shall graft on the branch $[\varrho, v_i]$ at height h provided that $H(v_i) \geq h$.

Let $n \in \mathbb{N}$ and $i \in \{0, \dots, n\}$ be given. For $h \in \mathbb{R}_+$ and $(T, \mathbf{v}) \in \mathbb{T}_{\text{K}}^{(n)}$, we denote by $x_{i,h}$ the unique vertex of T that satisfies $x_{i,h} \in [\varrho, v_i]$ and $H(x_{i,h}) = H(v_i) \wedge h$. Then, the map $(h, (T, \mathbf{v})) \mapsto (T, (\mathbf{v}, x_{i,h}))$ is clearly continuous from $\mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{(n+1)}$. We then define the grafting map $\otimes_{i,h}$ by:

$$(39) \quad (h, (T, \mathbf{v}), (T', \mathbf{v}')) \mapsto T \otimes_{i,h} T' = (T \otimes_{i,h} T', \mathbf{v} \otimes \mathbf{v}'),$$

as the composition of

$$\text{[adding the vertex } x_{i,h}\text{]: } (h, (T, \mathbf{v})) \mapsto (T, \tilde{\mathbf{v}}) \text{ with } \mathbf{v} = (v_0 = \varrho, \dots, v_n) \text{ and } \tilde{\mathbf{v}} = (\mathbf{v}, x_{i,h}) = (\tilde{v}_0 = \varrho, \dots, \tilde{v}_n = v_n, \tilde{v}_{n+1} = x_{i,h}),$$

$$\text{[grafting]: } ((T, \tilde{\mathbf{v}}), (T', \mathbf{v}')) \mapsto (T \otimes_{n+1} T', \tilde{\mathbf{v}} \otimes \mathbf{v}') \text{ and}$$

$$\text{[removing the } (n+1)\text{-th pointed vertex]: } (T'' = T \otimes_{n+1} T', \tilde{\mathbf{v}} \otimes \mathbf{v}') \mapsto (T'', \mathbf{v} \otimes \mathbf{v}').$$

Since all those maps are continuous, we get that the map $\otimes_{i,h}$ is continuous (and hence measurable) from $\mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^{(n)} \times \mathbb{T}_{\text{loc-K}}^{(k)}$ to $\mathbb{T}_{\text{loc-K}}^{(n+k)}$.

We shall also be interested in a grafting on the left or on the right of $i \in \{1, \dots, n\}$, which is the same as the grafting (39), but for the order of the coordinates of the vector $\mathbf{v} \otimes \mathbf{v}'$. For $\epsilon \in \{\text{g}, \text{d}\}$, we define the grafting map $\otimes_{i,h}^\epsilon$ by:

$$(40) \quad (h, (T, \mathbf{v}), (T', \mathbf{v}')) \mapsto T \otimes_{i,h}^\epsilon T' = (T \otimes_{i,h} T', \mathbf{v}^{\epsilon,i}),$$

where, with $\mathbf{v} = (v_0 = \varrho, \dots, v_n)$ and $\mathbf{v}' = (v'_0 = \varrho, \dots, v'_k)$, we set:

$$\begin{aligned} \mathbf{v}^{\text{g},i} &= (v_0 = \varrho, \dots, v_{i-1}, v'_1, \dots, v'_k, v_i, \dots, v_n) \\ \mathbf{v}^{\text{d},i} &= (v_0 = \varrho, \dots, v_i, v'_1, \dots, v'_k, v_{i+1}, \dots, v_n). \end{aligned}$$

Thanks to Lemma 5.8, we deduce from the continuity of the map $\otimes_{i,h}$, that the maps $\otimes_{i,h}^\epsilon$ are continuous.

We summarize the results from Remark 5.14 in the following lemma.

Lemma 5.15 (Continuity of the grafting maps). *Let $n, k \in \mathbb{N}$, $i \in \{0, \dots, n\}$ and $\epsilon \in \{\text{g}, \text{d}\}$. The maps $(h, (T, \mathbf{v}), (T', \mathbf{v}')) \mapsto T \otimes_{i,h} T'$ and $(h, (T, \mathbf{v}), (T', \mathbf{v}')) \mapsto T \otimes_{i,h}^\epsilon T'$, are continuous from $\mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^{(n)} \times \mathbb{T}_{\text{loc-K}}^{(k)}$ to $\mathbb{T}_{\text{loc-K}}^{(n+k)}$.*

5.5. Set of (equivalence classes of) marked trees. We shall consider trees with a marked infinite branch; for this reason we introduce the notion of marked trees. In this part, we do not record an order on the marked vertices as in the n -pointed trees.

We say that (T, S, d, ϱ) is a marked rooted tree if (T, d, ϱ) is a rooted tree and the set of marks S is a sub-tree of T with the same root (that is $\varrho \in S$) endowed with the restriction of the distance d . A correspondence between two compact marked rooted trees (T, S, d, ϱ) and (T', S', d', ϱ') is a set $\mathcal{R} \subset T \times T'$ such that \mathcal{R} is a correspondence between (T, d, ϱ) and (T', d', ϱ') and $\mathcal{R} \cap (S \times S')$ is also a correspondence between (S, d, ϱ) and (S', d', ϱ') . Then, we set:

$$d_{\text{GH}}^{[2]}((T, S), (T', S')) = \inf \frac{1}{2} \text{dist}(\mathcal{R}),$$

where the infimum is taken over all the correspondences \mathcal{R} between (T, S, d, ϱ) and (T', d', S', ϱ') . An easy extension of [6] gives that $d_{\text{GH}}^{[2]}$ is a pseudo-distance, and that $d_{\text{GH}}^{[2]}(T, T') = 0$ if and only if there exists an isometric one-to-one map φ from (T, d) to (T', d') which preserves the root and which is also one-to-one from S to S' . The relation $d_{\text{GH}}^{[2]}((T, S), (T', S')) = 0$ defines an equivalence relation. The set $\mathbb{T}_{\text{K}}^{[2]}$ of equivalence classes of compact marked rooted trees (T, S, d, ϱ) endowed with $d_{\text{GH}}^{[2]}$ is then a metric Polish space. We simply write (T, S) for (T, S, d, ϱ) , and unless specified otherwise, we shall denote also by (T, S) its equivalence class. Since $d_{\text{GH}}^{[2]}((T, S), (T', S')) \geq d_{\text{GH}}(T, T') \vee d_{\text{GH}}(S, S')$, we deduce that the map $(T, S) \mapsto (T, S)$ from $\mathbb{T}_{\text{K}}^{[2]}$ to $(\mathbb{T}_{\text{K}})^2$ (endowed with the maximum distance on the coordinates) is continuous. For $t \geq 0$, we define the truncation function $r_t^{[2]}$ of a marked rooted tree (T, S, d, ϱ) as the marked rooted tree $r_t^{[2]}(T, S) = (r_t(T), r_t(S), d, \varrho)$, where we recall that $r_t(T) = \{x \in T : H(x) \leq t\}$. If (T, S) and (T', S') are in the same equivalence class of $\mathbb{T}_{\text{K}}^{[2]}$, so are $r_t^{[2]}(T, S)$ and $r_t^{[2]}(T', S')$; thus the function $r_t^{[2]}$ can be seen as a map from $\mathbb{T}_{\text{K}}^{[2]}$ to itself. Similarly to (30), we have for $t, s \geq 0$ and $(T, S), (T', S') \in \mathbb{T}_{\text{K}}^{[2]}$:

$$(41) \quad d_{\text{GH}}^{[2]} \left(r_t^{[2]}(T, S), r_{t+s}^{[2]}(T', S') \right) \leq 4 d_{\text{GH}}^{[2]}((T, S), (T', S')) + s.$$

This implies that the map $(t, (T, S)) \mapsto r_t^{[2]}(T, S)$ is continuous from $\mathbb{R}_+ \times \mathbb{T}_K^{[2]}$ to $\mathbb{T}_K^{[2]}$.

A marked rooted tree (T, S, d, ϱ) is locally compact if $r_t^{[2]}(T, S)$ is a compact marked rooted tree for all $t \geq 0$. Following [6], we consider for two locally compact marked rooted trees (T, S) and (T', S') :

$$(42) \quad d_{\text{LGH}}^{[2]}((T, S), (T', S')) = \int_0^\infty e^{-t} dt \left(1 \wedge d_{\text{GH}}^{[2]} \left(r_t^{[2]}(T, S), r_t^{[2]}(T', S') \right) \right).$$

Furthermore, we have that $d_{\text{LGH}}^{[2]}((T, S), (T', S')) = 0$ if and only if there exists an isometric one-to-one map φ from (T, d) to (T', d') which is one-to-one from S to S' and preserves the roots. Thus the relation $d_{\text{LGH}}^{[2]}((T, S), (T', S')) = 0$ defines an equivalence relation, see [7, Proposition 5.3]. The set $\mathbb{T}_{\text{loc-K}}^{[2]}$ of equivalence classes of locally compact marked rooted trees (T, S, d, ϱ) endowed with $d_{\text{LGH}}^{[2]}$ is then a metric Polish space. Furthermore, $\mathbb{T}_K^{[2]}$ is an open dense subset of $\mathbb{T}_{\text{loc-K}}^{[2]}$. Similar equations to (31), (32) and (33) holds with $d_{\text{LGH}}^{(n)}$ and $d_{\text{GH}}^{(n)}$ replaced by $d_{\text{LGH}}^{[2]}$ and $d_{\text{GH}}^{[2]}$. For future use, let us give the equations corresponding to (32) and (33). For $(T, S), (T', S') \in \mathbb{T}_{\text{loc-K}}^{[2]}$ and $s, t \geq 0$, we have:

$$(43) \quad d_{\text{LGH}}^{[2]} \left(r_t^{[2]}(T, S), r_{t+s}^{[2]}(T', S') \right) \leq 4 d_{\text{LGH}}^{[2]}((T, S), (T', S')) + s,$$

$$(44) \quad d_{\text{GH}}^{[2]} \left(r_t^{[2]}(T, S), r_t^{[2]}(T', S') \right) \leq 4 e^t d_{\text{LGH}}^{[2]}((T, S), (T', S')).$$

We also we have the following result consequences of (41) and (43).

Lemma 5.16 (Continuity of the truncation map). *Let $n \in \mathbb{N}$. The map $(t, (T, S)) \mapsto r_t^{[2]}(T, S)$ is continuous from $\mathbb{R}_+ \times \mathbb{T}_K^{[2]}$ to $\mathbb{T}_K^{[2]}$ and from $\mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^{[2]}$ to $\mathbb{T}_{\text{loc-K}}^{[2]}$ (and to $\mathbb{T}_K^{[2]}$).*

We give in the next lemma an example of a $\mathbb{T}_K^{[2]}$ and $\mathbb{T}_{\text{loc-K}}^{[2]}$ valued function.

Lemma 5.17 (Continuity of Span°). *Let $n \in \mathbb{N}$. The map $(T, d, \mathbf{v}) \mapsto (\Pi_n^\circ(T), \text{Span}^\circ(T, \mathbf{v}), d, \varrho)$ from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{[2]}$ (resp. from $\mathbb{T}_K^{(n)}$ to $\mathbb{T}_K^{[2]}$) is injective, bi-measurable and 16-Lipschitz (resp. 4-Lipschitz) continuous.*

Proof. We first consider the compact case. Let (T, \mathbf{v}) and (T', \mathbf{v}') be rooted n -pointed compact trees and let \mathcal{R} be a correspondence between them. Recall the definition of $p_{\mathbf{v}}$ in (27) as the projection on $\text{Span}^\circ(T, \mathbf{v})$ and the correspondence $\tilde{\mathcal{R}}$ from (28). We set $\mathcal{R}^{[2]} = \mathcal{R} \cup \tilde{\mathcal{R}}$. By construction $\mathcal{R}^{[2]}$ is a correspondence between $(T, \text{Span}^\circ(T, \mathbf{v}))$ and $(T', \text{Span}^\circ(T', \mathbf{v}'))$. From the proof of Lemma 5.2, we get that $\text{dist}(\mathcal{R}^{[2]}) \leq 4 \text{dist}(\mathcal{R})$. This directly implies that:

$$(45) \quad d_{\text{GH}}^{[2]} \left((T, \text{Span}^\circ(T, \mathbf{v})), (T', \text{Span}^\circ(T', \mathbf{v}')) \right) \leq 4 d_{\text{GH}}^{(n)}((T, \mathbf{v}), (T', \mathbf{v}')).$$

This gives that the map $(T, d, \mathbf{v}) \mapsto (T, \text{Span}^\circ(T, \mathbf{v}), d, \varrho)$ from $\mathbb{T}_K^{(n)}$ to $\mathbb{T}_K^{[2]}$ is 4-Lipschitz continuous.

We now consider the locally compact case. Let (T, \mathbf{v}) and (T', \mathbf{v}') belong to $\mathbb{T}_{\text{loc-K}}^{[2]}$. We have:

$$\begin{aligned}
& d_{\text{LGH}}^{[2]} \left((T, \text{Span}^\circ(T, \mathbf{v})), (T', \text{Span}^\circ(T', \mathbf{v}')) \right) \\
&= \int_0^\infty e^{-t} dt \left(1 \wedge d_{\text{GH}}^{[2]} \left(r_t^{[2]} \left((T, \text{Span}^\circ(T, \mathbf{v})), r_t^{[2]} (T', \text{Span}^\circ(T', \mathbf{v}')) \right) \right) \right) \\
&\leq 4 \int_0^\infty e^{-t} dt \left(1 \wedge d_{\text{GH}}^{[2]} \left((r_t(T, \mathbf{v}), \text{Span}^\circ(T, \mathbf{v})), (r_t(T', \mathbf{v}'), \text{Span}^\circ(T', \mathbf{v}')) \right) \right) \\
&\leq 16 \int_0^\infty e^{-t} dt \left(1 \wedge d_{\text{GH}}^{(n)} (r_t(T, \mathbf{v}), r_t(T', \mathbf{v}')) \right) \\
&= 16 d_{\text{LGH}}^{(n)} ((T, \mathbf{v}), (T', \mathbf{v}')),
\end{aligned}$$

where we used (41) (with T and S replaced respectively by $r_t(T, \mathbf{v})$ and $\text{Span}^\circ(T, \mathbf{v})$ and similarly for T' and S') for the first inequality, and (45) (with (T, \mathbf{v}) replaced by $r_t(T, \mathbf{v})$) as well as the relation $\text{Span}^\circ(r_t(T, \mathbf{v})) = \text{Span}^\circ(T, \mathbf{v})$ for the second. This gives that the map $(T, d, \mathbf{v}) \mapsto (T, \text{Span}^\circ(T, \mathbf{v}), d, \varrho)$ from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{[2]}$ is 16-Lipschitz continuous.

Clearly those maps are injective and thus bi-measurable thanks to Lusin's theorem [32]. \square

Remark 5.18. Let us stress that for (T, \mathbf{v}) a rooted n -pointed compact tree, the rooted tree $r_t^{[2]}(T, \text{Span}^\circ(T, \mathbf{v})) = (r_t(T), r_t(\text{Span}^\circ(T, \mathbf{v})))$ and the rooted tree $(r_t(T), \text{Span}^\circ(r_t(T, \mathbf{v}))) = (r_t(T), \text{Span}^\circ(T, \mathbf{v}))$ differ if and only if t is smaller than the height of $\text{Span}^\circ(T, \mathbf{v})$.

Let (T, S, d, ϱ) be a marked locally compact rooted tree. To simplify, we shall only write (T, S) for (T, S, d, ϱ) . We define the projection of $z \in T$ on S , $p_S(z) \in S$, as the element of S uniquely defined by:

$$\llbracket \varrho, p_S(z) \rrbracket = \llbracket \varrho, z \rrbracket \cap S.$$

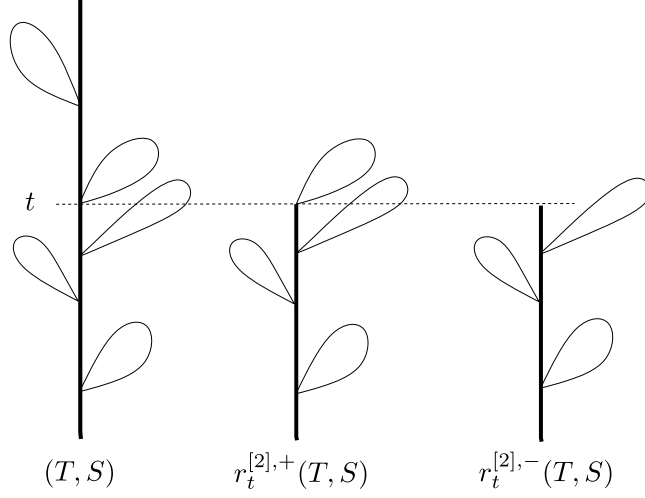
Now, we consider the truncation of a marked tree at a given height, say t , of the marked sub-tree. For $t \geq 0$ and $\varepsilon \in \{-, +\}$, we set:

$$r_t^{[2], \varepsilon}(T, S) = \left(r_{t,1}^{[2], \varepsilon}(T, S), r_t(S) \right)$$

with:

$$\begin{aligned}
r_{t,1}^{[2], +}(T, S) &= \left\{ x \in T : H(p_S(x)) \leq t \right\}, \\
r_{t,1}^{[2], -}(T, S) &= \left\{ x \in T : H(p_S(x)) < t \right\} \cup \left\{ x \in S : H(x) = t \right\}.
\end{aligned}$$

See Figure 1 for an instance of $r_t^{[2], \varepsilon}(T, S)$, where S is an infinite branch. For $\varepsilon \in \{+, -\}$, we also denote by $r_t^{[2], \varepsilon}(T, S)$ the marked rooted tree $(r_{t,1}^{[2], \varepsilon}(T, S), d, \varrho)$ endowed with the restriction of the distance d and the root ϱ . Furthermore, if (T, S) and (T', S') belong to the same equivalence class of $\mathbb{T}_{\text{loc-K}}^{[2]}$ or $\mathbb{T}_{\text{K}}^{[2]}$, then so do $r_t^{[2], \varepsilon}(T, S)$ and $r_t^{[2], \varepsilon}(T', S')$. Thus the map $(t, (T, S)) \mapsto r_t^{[2], \varepsilon}(T, S)$ is a well defined map from $\mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^{[2]}$ to $\mathbb{T}_{\text{loc-K}}^{[2]}$ for $\varepsilon \in \{+, -\}$.

FIGURE 1. Example of restrictions of a tree T with a marked spine S (in bold).

Remark 5.19 (Examples). We give elementary examples. For $\varepsilon \in \{+, -\}$ and $t > 0$, we have that $r_t^{[2],\varepsilon}(T, \{\varrho\}) = (T, \{\varrho\})$ and $r_0^{[2],-}(T, \{\varrho\}) = (\{\varrho\}, \{\varrho\})$ as well as $r_0^{[2],+}(T, \{\varrho\}) = (T, \{\varrho\})$. We also have for $t \in \mathbb{R}_+$ that $r_t^{[2],\varepsilon}(T, T) = (r_t(T), r_t(T))$.

Remark 5.20 (The map $r_t^{[2],\varepsilon}$ is not continuous). Let $\varepsilon \in \{+, -\}$ and $t > 0$. The function $r_t^{[2],\varepsilon}$ is not continuous from $\mathbb{T}_{\text{loc-K}}^{[2]}$ to itself. Indeed take $t = 1$ without loss of generality and consider $T = [0, 2]$ and $S_\delta = [0, \delta]$, with $\delta \in [0, 2]$, $\varrho = 0$ and the Euclidean distance. Notice that $([0, 1], [0, 1]) = (S_1, S_1) \neq (T, S_1)$. Then we have that $\lim_{\delta \rightarrow 1} d_{\text{GH}}^{[2]}((T, S_\delta), (T, S_1)) = 0$, $r_1^{[2],\varepsilon}(T, S_\delta) = (T, S_\delta)$ for $\delta < 1$, $r_1^{[2],\varepsilon}(T, S_\delta) = (S_1, S_1)$ for $\delta > 1$, $r_1^{[2],-}(T, S_1) = (S_1, S_1)$ and $r_1^{[2],+}(T, S_1) = (T, S_1)$.

We have the following measurability result.

Lemma 5.21 (Measurability of some truncation maps). *Let $\varepsilon \in \{+, -\}$. The map $(t, (T, S)) \mapsto r_t^{[2],\varepsilon}(T, S)$ is measurable from $\mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^{[2]}$ to $\mathbb{T}_{\text{loc-K}}^{[2]}$.*

Proof. Let $a > 0$. For a marked tree $(T, S) = (T, S, d, \varrho)$, we define its partial dilatation $R_a(T, S) = (T, S, d_a, \varrho)$ as the marked tree with $d_a(x, y) = ad(x, p_S(x)) + d(p_S(x), p_S(y)) + ad(y, p_S(y))$. Intuitively the distances on T are multiplied by a outside S . The equivalence class of $R_a(T, S)$ in $\mathbb{T}_{\text{loc-K}}^{[2]}$ does not depend of the choice of (T, S) in its equivalence class in $\mathbb{T}_{\text{loc-K}}^{[2]}$; so the map R_a is well defined on $\mathbb{T}_{\text{loc-K}}^{[2]}$ to itself. Notice that the map R_a is continuous and one-to-one with inverse $R_{1/a}$. It is immediate to check that, for $t \geq 0$:

$$r_t^{[2],-} = \lim_{a \rightarrow 0^+} R_{1/a} \circ r_t^{[2]} \circ R_a.$$

This and Lemma 5.16 imply the measurability of the map $(t, (T, S)) \mapsto r_t^{[2],-}(T, S)$. Then, notice that $\lim_{s \downarrow t} r_s^{[2],-} = r_t^{[2],+}$ to get the measurability of the map $(t, (T, S)) \mapsto r_t^{[2],+}(T, S)$. \square

We end this section by proving (in a very similar way) that the map $r_*^{[2]}$ below, which consists in cleaning the root, that is, in erasing the bushes at the root of a marked tree is measurable.

For $(T, S) = (T, S, d, \varrho)$ a marked locally compact rooted tree, we set:

$$(46) \quad r_*^{[2]}(T, S) = \left(r_{*,1}^{[2]}(T, S), S \right) \quad \text{with} \quad r_{*,1}^{[2]}(T, S) = \{x \in T : p_S(x) \neq \varrho\} \cup \{\varrho\}.$$

We also denote by $r_*^{[2]}(T, S)$ the marked rooted tree $(r_*^{[2]}(T, S), d, \varrho)$ endowed with the restriction of the distance d and the root ϱ . Furthermore, if (T, S) and (T', S') belong to the same equivalence class of $\mathbb{T}_{\text{loc-K}}^{[2]}$, then so do $r_*^{[2]}(T, S)$ and $r_*^{[2]}(T', S')$. Thus the map $r_*^{[2]}$ is well-defined from $\mathbb{T}_{\text{loc-K}}^{[2]}$ to $\mathbb{T}_{\text{loc-K}}^{[2]}$.

Lemma 5.22 (Measurability of the root cleaning map). *The map $r_*^{[2]}$ is measurable from $\mathbb{T}_{\text{loc-K}}^{[2]}$ to $\mathbb{T}_{\text{loc-K}}^{[2]}$.*

Proof. Let $a > 0$. For a marked tree $(T, S) = (T, S, d, \varrho)$, we define its partial dilatation $R'_a(T, S) = (T, S, d'_a, \varrho)$ as the marked tree with $d'_a(x, y) = F_a(t)d(x, y)$ if $p_S(x) = p_S(y)$ with $t = H(p_S(x))$, and otherwise $d'_a(x, y) = F_a(t)d(x, p_S(x)) + ad(p_S(x), p_S(y)) + F_a(s)d(y, p_S(y))$ with $t = H(p_S(x))$, $s = H(p_S(y))$, and the function F_a defined for $t \geq 0$ by $F_a(t) = t \wedge a + a^{-2}(a - t)_+$ if $a \leq 1$, and $F_a(t) = 1/F_{1/a}(at)$ if $a > 1$. The equivalence class of $R'_a(T, S)$ in $\mathbb{T}_{\text{loc-K}}^{[2]}$ does not depend of the choice of (T, S) in its equivalence class in $\mathbb{T}_{\text{loc-K}}^{[2]}$; so the map R'_a is well defined on $\mathbb{T}_{\text{loc-K}}^{[2]}$ to itself. Notice that the map R'_a is continuous and one-to-one with inverse $R'_{1/a}$. It is immediate to check that for $t > 0$:

$$r_*^{[2]} = \lim_{a \rightarrow 0^+} R_{1/a} \circ r_t^{[2]} \circ R_a.$$

This and Lemma 5.16 imply the measurability of the map $r_*^{[2]}$. \square

5.6. Set of (equivalence classes of) trees with one infinite marked branch. Let us denote by $T_0 = (\varrho, \{\varrho\})$ the rooted tree reduced to its root. Notice that $r_0^{[2],+}(T, S) = \{(T_0, T_0)\}$ if and only if $\llbracket \varrho, x \rrbracket \cap S = \{\varrho\}$ implies $x = \varrho$. Let $T_1 = ([0, \infty), d, 0)$ be the tree consisting of only one infinite branch. We consider the set (of equivalence classes) of locally compact rooted trees with one infinite marked branch and its subset of trees whose root is not a branching vertex:

$$(47) \quad \mathbb{T}_{\text{loc-K}}^{\text{spine}} = \left\{ (T, S) \in \mathbb{T}_{\text{loc-K}}^{[2]} : S = T_1 \text{ in } \mathbb{T}_{\text{loc-K}} \right\},$$

$$(48) \quad \mathbb{T}_{\text{loc-K}}^{\text{spine},0} = \left\{ (T, S) \in \mathbb{T}_{\text{loc-K}}^{\text{spine}} : \varrho \notin \text{Br}(T) \right\}.$$

Lemma 5.23. *The sets $\mathbb{T}_{\text{loc-K}}^{\text{spine}}$ and $\mathbb{T}_{\text{loc-K}}^{\text{spine},0}$ are Borel subsets of $\mathbb{T}_{\text{loc-K}}^{[2]}$.*

Proof. Consider the projection $\tilde{\Pi} : (T, S) \mapsto S$ from $\mathbb{T}_{\text{loc-K}}^{[2]}$ to $\mathbb{T}_{\text{loc-K}}$, which is by construction 1-Lipschitz and thus continuous. As $\mathbb{T}_{\text{loc-K}}^{\text{spine}} = \tilde{\Pi}^{-1}(\{T_1\})$, we get that $\mathbb{T}_{\text{loc-K}}^{\text{spine}}$ is Borel.

Notice that for $(T, S) \in \mathbb{T}_{\text{loc-K}}^{\text{spine}}$, then, by definition of $r_t^{[2],+}$, we get that the root is not a branching vertex of (T, S) if and only if $r_0^{[2],+}(T, S) = (T_0, T_0)$. Then, the set $\mathbb{T}_{\text{loc-K}}^{\text{spine},0} = \mathbb{T}_{\text{loc-K}}^{\text{spine}} \cap (r_0^{[2],+})^{-1}(\{(T_0, T_0)\})$ is Borel as the map $r_0^{[2],+}$ is measurable according to Lemma 5.21. \square

We shall be mainly consider elements of $\mathbb{T}_{\text{loc-K}}^{\text{spine},0}$ in what follows. For simplicity, we shall write $T^* = (T, S)$ for an element of $\mathbb{T}_{\text{loc-K}}^{\text{spine},0}$. For $t \geq 0$ and $T^* = (T, S)$ in $\mathbb{T}_{\text{loc-K}}^{\text{spine},0}$, we have $r_t^{[2],+}(T^*) = (r_{t,1}^{[2],+}(T), r_t(S))$ where the rooted tree $r_t(S)$ is given by $(\llbracket \varrho, x \rrbracket, \varrho)$ with $x \in S$

uniquely characterized by $d(\varrho, x) = t$. We shall consider a slight modification of $r_t^{[2],+}$ on $\mathbb{T}_{\text{loc-K}}^{\text{spine},0}$, say $\tilde{r}_t^{[2],+}$, where one keeps track only of (ϱ, x) instead of $r_t(S)$:

$$(49) \quad \tilde{r}_t^{[2],+}(T^*) = (r_{t,1}^{[2],+}(T), (\varrho, x)).$$

It is left to the reader to check that $\tilde{r}_t^{[2],\varepsilon}$ is defined on $\mathbb{T}_{\text{loc-K}}^{\text{spine},0}$ and $\mathbb{T}_{\text{loc-K}}^{(1)}$ -valued. Similarly to Lemma 5.21, we get the following result.

Lemma 5.24. *The function $(t, T^*) \mapsto \tilde{r}_t^{[2],+}(T^*)$ from $\mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^{\text{spine},0}$ to $\mathbb{T}_{\text{loc-K}}^{(1)}$ is measurable.*

5.7. Another representation for discrete trees. Let $n \in \mathbb{N}$ be fixed. Let (T, \mathbf{v}) , with $\mathbf{v} = (v_0 = \varrho, \dots, v_n)$, be a locally compact rooted n -pointed tree. We will decompose the tree $\text{Span}(T, \mathbf{v})$ as a sequence of edges. To do so, we introduce some notations. Let $A \subset \{0, \dots, n\}$ be non-empty. We set $\mathbf{v}_A = (v_i, i \in A)$. We denote by v_A the most recent common ancestor of \mathbf{v}_A , which is the only element of T such that:

$$(50) \quad \llbracket \varrho, v_A \rrbracket = \bigcap_{k \in A} \llbracket \varrho, v_k \rrbracket.$$

Notice that $v_{\{i\}} = v_i$. Recall that for $x \in T$, T_x is the sub-tree of T above x and rooted at x . Let \mathcal{P}_n^+ be the set of all subsets $A \subset \{1, \dots, n\}$ such that $A \neq \emptyset$. For $A \in \mathcal{P}_n^+$, if $T_{v_A} \cap \text{Span}^\circ(T, \mathbf{v}_{A^c}) \neq \emptyset$ with $A^c = \{0, 1, 2, \dots, n\} \setminus A$, we set $w_A = v_A$, otherwise we define $w_A \in \llbracket \varrho, v_A \rrbracket$ as the only element of T such that:

$$(51) \quad \llbracket \varrho, w_A \rrbracket = \text{Span}^\circ(T, \mathbf{v}_{A^c}) \cap \text{Span}^\circ(T, (\varrho, \mathbf{v}_A)).$$

Equivalently w_A is the only element in $\llbracket \varrho, v_A \rrbracket$ such that $w_A = v_{A \cup \{k_0\}}$ for some $k_0 \in A^c$ and for all $k \in A^c$, we have $v_{A \cup \{k\}} \in \llbracket \varrho, w_A \rrbracket$. Notice that $w_{\{1, \dots, n\}} = \varrho$. We also record the lengths of all the branches $\llbracket w_A, v_A \rrbracket$:

$$(52) \quad \mathbf{L}_n(T, \mathbf{v}) = (\ell_A(T, \mathbf{v}), A \in \mathcal{P}_n^+) \quad \text{with} \quad \ell_A(T, \mathbf{v}) = d(w_A, v_A).$$

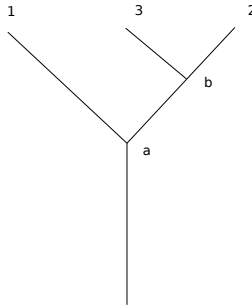


FIGURE 2. A discrete trees spanned by the leaves $\{1, 2, 3\}$.

TABLE 1. Quantities of interest for the discrete tree from Figure 2.

$A \subset \mathcal{P}_3^+$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
v_A	1	2	3	a	a	b	a
w_A	a	b	b	a	a	a	ϱ
ℓ_A	$d(a, 1)$	$d(b, 2)$	$d(b, 3)$	0	0	$d(a, b)$	$d(\varrho, a)$

For instance, we record the quantity of interest in Table 1 for the discrete tree spanned by the leaves $\{1, 2, 3\}$ from Figure 2. We can see that each branch of the discrete tree appears (through their length) once and only once in $\mathbf{L}_3(T, \mathbf{v})$.

Set $\hat{\mathbf{v}} = (\hat{\mathbf{v}}_0 = \varrho, (v_A, A \in \mathcal{P}_n^+)) \in T^{2^n}$, so that $(T, \hat{\mathbf{v}})$ is a locally compact rooted $(2^n - 1)$ -pointed tree with the same root ϱ as T . Notice that all the vertices in \mathbf{v} appear in $\hat{\mathbf{v}}$ (possibly more than once), and that w_A also appears in $\hat{\mathbf{v}}$ for all $A \in \mathcal{P}_n^+$. Recall the set of discrete trees defined at the end of Section 5.3. The next lemma states that \mathbf{L}_n codes continuously for discrete trees. Set $\text{Im}(\mathbf{L}_n) \subset \mathbb{R}_+^{\mathcal{P}_n^+}$ (with $\mathbb{R}_+^{\mathcal{P}_n^+} = \mathbb{R}_+^{2^n-1}$) for the image of \mathbf{L}_n .

Lemma 5.25 (Regularity of the branch lengths as a function of the tree). *Let $n \in \mathbb{N}^*$. The map $(T, \mathbf{v}) \mapsto (T, \hat{\mathbf{v}})$ is well defined from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{(2^n-1)}$, and it is continuous. The function \mathbf{L}_n is well defined from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\text{Im}(\mathbf{L}_n) \subset \mathbb{R}_+^{\mathcal{P}_n^+}$ and is continuous. Furthermore, $\text{Im}(\mathbf{L}_n)$ is closed and \mathbf{L}_n is a one-to-one bi-measurable map from $\mathbb{T}_{\text{dis}}^{(n)}$ to $\text{Im}(\mathbf{L}_n)$.*

Proof. If (T, \mathbf{v}) and (T', \mathbf{v}') belong to the same equivalence class in $\mathbb{T}_{\text{loc-K}}^{(n)}$, then we deduce from (50) and (51) that $(T, \hat{\mathbf{v}})$ and $(T', \hat{\mathbf{v}}')$ belong also to the same equivalence class. This implies that the function $(T, \mathbf{v}) \mapsto (T, \hat{\mathbf{v}})$ is well defined from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{(2^n-1)}$. We deduce from (50) and (51) that this function is in fact continuous on $\mathbb{T}_{\text{loc-K}}^{(n)}$. We also get that the function \mathbf{L}_n is well defined from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{R}_+^{\mathcal{P}_n^+}$.

We shall now precise the image of the function \mathbf{L}_n and prove its continuity. Recall $x_+ = \max(x, 0)$ denotes the positive part of $x \in \mathbb{R}$. We define the function L from $\mathbb{R}_+^{(n+1) \times (n+1)}$ to $\mathbb{R}_+^{\mathcal{P}_n^+}$ by, for $\mathbf{d} = (d_{ij}, 0 \leq i, j \leq n)$ and $A \in \mathcal{P}_n^+$:

$$L_A(\mathbf{d}) = \frac{1}{4} \inf \left\{ (d_{ii'} + d_{ij'} + d_{j'i'} + d_{jj'} - 2d_{ij} - 2d_{i'j'})_+ : i, j \in A \text{ and } i', j' \in A^c \right\},$$

where $A^c = \{0, \dots, n\} \setminus A$. We also define the function D from $\mathbb{R}_+^{(n+1) \times (n+1)}$ to $\mathbb{R}_+^{\mathcal{P}_n^+}$ by, for $\ell = (\ell_A, A \in \mathcal{P}_n^+)$ and $i, j \in \{0, \dots, n\}$:

$$(53) \quad D_{ij}(\ell) = \sum_{A \in \mathcal{P}_n^+} \ell_A (\mathbf{1}_{\{i \in A, j \notin A\}} + \mathbf{1}_{\{i \notin A, j \in A\}}).$$

The functions L and D are continuous. Consider the closed subset $\mathcal{Q}^{(n)}$ of $\mathbb{R}_+^{(n+1) \times (n+1)}$ satisfying the so-called four-point condition, that is the set of all $(d_{ij}, 0 \leq i, j \leq n) \in \mathbb{R}_+^{(n+1) \times (n+1)}$ such that:

$$d_{ij} + d_{i'j'} \leq \max(d_{ii'} + d_{jj'}, d_{ij'} + d_{j'i'}) \quad \text{for all } i, j, i', j' \in \{0, \dots, n\}.$$

Notice that the four-point condition is also used to characterize metric spaces which are real trees, see Evans[?]. Then, one can check that the function L is one-to-one from $\mathcal{Q}^{(n)}$ to $L(\mathcal{Q}^{(n)})$ with inverse D . We also get that $L(\mathcal{Q}^{(n)})$ is closed (indeed if $(\ell^k = L(\mathbf{d}^k), k \in \mathbb{N})$ is a sequence of elements of $L(\mathcal{Q}^{(n)})$ converging to a limit, say ℓ , then it is bounded and thus the sequence $(\mathbf{d}^k, k \in \mathbb{N})$ is also bounded. Hence there is a converging sub-sequence, and denote by \mathbf{d} its limit which belongs to $\mathcal{Q}^{(n)}$ as this set is closed. Since L is continuous, we get that $L(\mathbf{d}) = \ell$ and thus ℓ belongs to $L(\mathcal{Q}^{(n)})$, which gives that $L(\mathcal{Q}^{(n)})$ is closed). Since for $(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(n)}$, we have that $\mathbf{L}_n(T, \mathbf{v}) = L(d(v_i, v_j), 0 \leq i, j \leq n)$, we deduce that the function \mathbf{L}_n is continuous from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $L(\mathcal{Q}^{(n)})$.

We now prove that $\text{Im}(\mathbf{L}_n) = L(\mathcal{Q}^{(n)})$ and that \mathbf{L}_n is one-to-one from $\mathbb{T}_{\text{dis}}^{(n)}$ to $L(\mathcal{Q}^{(n)})$. Let $\ell = (\ell_A, A \in \mathcal{P}_n^+) \in L(\mathcal{Q}^{(n)})$. Thus, there exists a sequence $d = (d_{ij}, 0 \leq i, j \leq n) \in \mathcal{Q}^{(n)}$ which satisfies the four-point condition and such that $L(d) = \ell$. Since d satisfies the four-point condition, we get that there exists a discrete tree $(T, d, \mathbf{v}) \in \mathbb{T}_{\text{dis}}^{(n)}$ such that $d(v_i, v_j) = d_{ij}$ for all $i, j \in \{0, \dots, n\}$. This proves that $\text{Im}(\mathbf{L}_n) = L(\mathcal{Q}^{(n)})$. Then use that L is one-to-one from $\mathcal{Q}^{(n)}$ to $L(\mathcal{Q}^{(n)})$ with inverse D and that two discrete trees (T, d, \mathbf{v}) and (T', d', \mathbf{v}') are equal in $\mathbb{T}_{\text{dis}}^{(n)}$ if and only if $d(v_i, v_j) = d'(v'_i, v'_j)$ for all $i, j \in \{0, \dots, n\}$ to deduce that \mathbf{L}_n is one-to-one from $\mathbb{T}_{\text{dis}}^{(n)}$ to $L(\mathcal{Q}^{(n)})$ and thus bi-measurable thanks to Lusin's theorem from [32]. \square

5.8. The splitting operator for a pointed tree. We want now to decompose the pointed tree (T, \mathbf{v}) along the branches of $\text{Span}^\circ(T, \mathbf{v})$. We keep notations from Section 5.7.

Let (T, \mathbf{v}) , with $\mathbf{v} = (v_0 = \varrho, \dots, v_n)$, be a locally compact rooted n -pointed tree. Recall Definition (27) of the projection $p_{\mathbf{v}}$ on $\text{Span}(T, \mathbf{v})$. For $A \in \mathcal{P}_n^+$, consider the rooted 1-pointed tree:

$$(54) \quad \hat{T}_A(T, \mathbf{v}) = (T_A(T, \mathbf{v}), (\varrho_A, v_A)) \in \mathbb{T}_{\text{loc-K}}^{(1)}$$

with root $\rho_A = w_A$ and

$$T_A(T, \mathbf{v}) = \{x \in T : p_{\mathbf{v}}(x) \in \llbracket w_A, v_A \rrbracket\} \cup \{w_A\}.$$

By construction, we have that $\ell_A(T, \mathbf{v}) = d(\varrho_A, v_A)$.

Notice that $\ell_A(T, \mathbf{v}) = 0$ if and only if $\hat{T}_A(T, \mathbf{v})$ is reduced to its root, that is, $(\{\varrho_A\}, (\varrho_A, \varrho_A))$. Notice also that $\ell_A(T, \mathbf{v}) > 0$ implies that \hat{T}_A belongs to $\mathbb{T}_{\text{loc-K}}^{(1),0}$, the set of trees in $\mathbb{T}_{\text{loc-K}}^{(1)}$ such that the root is not a branching point (see Definition (35)). We also define the rooted 1-pointed tree $\hat{T}_{\{0\}}(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(1)} = (T_{\{0\}}(T, \mathbf{v}), (\varrho, \varrho))$ by:

$$T_{\{0\}}(T, \mathbf{v}) = \{x \in T : \llbracket \varrho, x \rrbracket \cap \text{Span}^\circ(T, \mathbf{v}) = \emptyset\},$$

with root ϱ and distinguished vertex also ϱ . If (T, \mathbf{v}) and (T', \mathbf{v}') belong to the same equivalence class in $\mathbb{T}_{\text{loc-K}}^{(n)}$, then we get that $\hat{T}_A(T, \mathbf{v})$ and $\hat{T}_A(T', \mathbf{v}')$ belong also to the same equivalent class in $\mathbb{T}_{\text{loc-K}}^{(1)}$ for $A \in \mathcal{P}_n = \mathcal{P}_n^+ \cup \{\{0\}\}$. Thus, the map Split_n defined on $\mathbb{T}_{\text{loc-K}}^{(n)}$ by:

$$(55) \quad \text{Split}_n(T, \mathbf{v}) = \left(\hat{T}_A(T, \mathbf{v}), A \in \mathcal{P}_n \right)$$

takes values in $\left(\mathbb{T}_{\text{loc-K}}^{(1)} \right)^{2^n}$. We give an instance of the function Split_n in Figure 3.

Lemma 5.26 (Measurability of the splitting map). *Let $n \in \mathbb{N}^*$. The map Split_n from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\left(\mathbb{T}_{\text{loc-K}}^{(1)} \right)^{2^n}$ is measurable.*

Proof. The proof is divided into three steps.

Step 1: The map $\hat{T}_{\{0\}}$ is measurable. Let $(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(n)}$. By construction, we have that $r_0^{[2],+}(T, \text{Span}^\circ(T, \mathbf{v})) = (T_{\{0\}}(T, \mathbf{v}), T_0)$. We deduce from Lemma 5.21 on the measurability of $r_t^{[2],\varepsilon}$, that the map $(T, \mathbf{v}) \mapsto \hat{T}_{\{0\}} = (T_{\{0\}}(T, \mathbf{v}), (\varrho, \varrho))$ is measurable.

Step 2: A measurable truncation function. Let $n \geq 1$. Let (T, \mathbf{v}) be a rooted n -pointed tree. Recall the definition of $\hat{T}_A(T, \mathbf{v})$ from (54). We set $q(T, \mathbf{v}) = \hat{T}_{\{1,2,\dots,n\}}(T, \mathbf{v})$ so that q is a map from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{(1)}$. Recall the measurable truncation functions $r_t^{[2],+}$ and $r_*^{[2]}$ from (49) and (46), respectively.

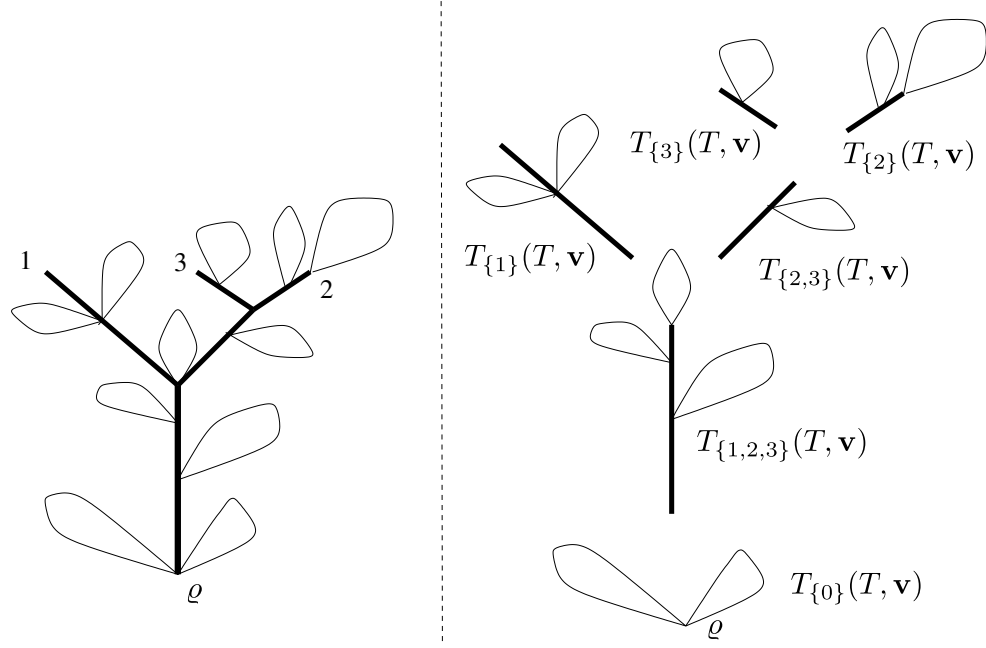


FIGURE 3. The splitting of the left hand tree with respect to $\mathbf{v} = \{\varrho, 1, 2, 3\}$. In this instance, $T_{\{1,2\}}$ and $T_{\{1,3\}}$ are reduced to their own root.

We set:

$$q'(T, \mathbf{v}) = r_*^{[2]} \circ r_{d(\varrho, w_{\{1, \dots, b\}})}^{[2],+}(T, \text{Span}^\circ(T, \mathbf{v})).$$

Thanks to Lemma 5.17, the map $(T, \mathbf{v}) \mapsto (T, \text{Span}^\circ(T, \mathbf{v}))$ is continuous from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{[2]}$. Thanks to Lemma 5.25 and Remark 5.11, we get that the map $(T, \mathbf{v}) \mapsto d(\varrho, w_{\{1, \dots, b\}})$ is continuous from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to \mathbb{R}_+ . Then, use Lemmas 5.21 and 5.22 on the measurability of $r_t^{[2],\varepsilon}$ and $r_*^{[2]}$ to conclude that the map q' from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{[2]}$ is measurable and it has the same image as the map $(T, (\varrho, v)) \mapsto (T, \llbracket \varrho, v \rrbracket)$ from $\mathbb{T}_{\text{loc-K}}^{(1)}$ to $\mathbb{T}_{\text{loc-K}}^{[2]}$. According to Lemma 5.17 (with $n = 1$), this latter map is injective and measurable. Hence the map q , which is the composition of q' and this latter map, is measurable.

Step 3: Conclusion. Let $A \subset \{1, \dots, n\}$ be non-empty. Notice that \hat{T}_A is the image of (T, \mathbf{v}) by: the expansion procedure $(T, \mathbf{v}) \mapsto (T, \hat{\mathbf{v}})$ from the first part of Lemma 5.25, the rerooting at w_A from Lemma 5.8, the reducing procedure from Lemma 5.5 where one forgets about all $w_{A'}$ and $v_{A'}$ for $A' \subset A^c$, and then the function q from Step 2. This implies that the function $(T, \mathbf{v}) \mapsto \hat{T}_A(T, \mathbf{v})$ is measurable from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{(1)}$. \square

5.9. The grafting procedure. Let $n \in \mathbb{N}^*$. Let $\ell = (\ell_A, A \in \mathcal{P}_n^+) \in \text{Im}(\mathbf{L}_n)$. According to Lemma 5.25, there exists a unique (up to the equivalence in $\mathbb{T}_K^{(n)}$) rooted n -pointed discrete tree (S, \mathbf{v}) (that is $S = \text{Span}^\circ(S, \mathbf{v})$) such that $\mathbf{L}_n(S, \mathbf{v}) = \ell$. Recall v_A and w_A defined in Section 5.7 for $A \in \mathcal{P}_n^+$ so that:

$$(56) \quad S = \bigcup_{A \in \mathcal{P}_n^+} \llbracket w_A, v_A \rrbracket,$$

where the sets $(\llbracket w_A, v_A \rrbracket, A \in \mathcal{P}_n^+)$ are pairwise disjoint.

Recall that $\mathbb{T}_{\text{loc-K}}^{\text{spine},0}$ denotes the set (of equivalence classes) of locally compact rooted trees with one infinite marked branch such that the root is not a branching vertex. Let $T^* = (T_A^*, A \in \mathcal{P}_n^+)$ be a family of elements of equivalence classes in $\mathbb{T}_{\text{loc-K}}^{\text{spine},0}$. Then, we define the tree $(T, \mathbf{v}) = \text{Graft}_n(\ell, T^*)$, where T is the tree S with that the branches $\llbracket w_A, v_A \rrbracket$ are replaced by the trees given by the first component of $r_{\ell_A}^{[2],+}(T_A^*)$ (where the second component has been identified to $\llbracket w_A, v_A \rrbracket$).

We now provide a more formal construction of $\text{Graft}_n(\ell, T^*)$. Let $\ell \in \text{Im}(\mathbf{L}_n)$, and consider the rooted n -pointed discrete tree $(S, \mathbf{v}) = \mathbf{L}_n^{-1}(\ell) \in \mathbb{T}_{\text{dis}}^{(n)}$ and $\mathbf{v} = (v_0 = \varrho, \dots, v_n)$. Set $\hat{\mathbf{v}} = (\hat{v}_0 = \varrho, (v_A, A \in \mathcal{P}_n^+)) \in T^{2^n}$, with v_A the most recent common ancestor of $(v_i, i \in A)$ defined in (50). Thus, we get that $(S, \hat{\mathbf{v}}) \in \mathbb{T}_{\text{dis}}^{(2^n-1)}$ is a rooted $(2^n - 1)$ -pointed discrete tree with the same root ϱ as S .

In a first step, we build by a backward induction an ‘‘increasing’’ sequence of discrete trees $((S_k, \mathbf{v}_k), k \in \{0, \dots, 2^n - 1\})$ such that $(S_k, \mathbf{v}_k) \in \mathbb{T}_{\text{dis}}^{(k)}$ with root ϱ . We set $(S_{2^n-1}, \mathbf{v}_{2^n-1}) = (S, \hat{\mathbf{v}})$. Recall that x is a leaf of a tree T with root ϱ if $x \in \llbracket \varrho, y \rrbracket \subset T$ implies $y = x$. Assume that $(S_{k+1}, \mathbf{v}_{k+1})$ is defined for some $k \geq 0$. We endow the sets \mathcal{P}_n^+ and $\mathcal{P}_n = \mathcal{P}_n^+ \cup \{\{0\}\}$ with the lexicographical order, so that the maximum of the subset of \mathcal{P}_n^+ is well-defined. We set:

$$A_{k+1} = \max\{A \in \mathcal{P}_n^+, v_A \in \mathbf{v}_{k+1} \text{ and } v_A \text{ is a leaf of } (S_{k+1}, \mathbf{v}_{k+1})\}.$$

Then, we define \mathbf{v}_k as the sequence \mathbf{v}_{k+1} where $v_{A_{k+1}}$ has been removed (notice that the first element of \mathbf{v}_k is still the root ϱ), and we set $(S_k, \mathbf{v}_k) = \text{Span}(S, \mathbf{v}_k) \in \mathbb{T}_{\text{dis}}^{(k)}$. We also set $B_k = \max\{B \in \mathcal{P}_n : v_B = w_{A_{k+1}}\}$. By construction, $v_{B_k} = w_{A_{k+1}}$ belongs to the sequence \mathbf{v}_k and is therefore an element of \mathbf{v} for some index, and, with a slight abuse of notation, we simply denote this index by B_k . We have, using the grafting operation from Section 5.4 that:

$$(57) \quad (S_{k+1}, \mathbf{v}_{k+1}) = (S_k, \mathbf{v}_k) \otimes_{B_k} [0, \ell_{A_{k+1}}],$$

where the equality holds in $\mathbb{T}_{\text{loc-K}}^{(k+1)}$ (and in $\mathbb{T}_{\text{dis}}^{(k+1)}$) and by convention $[0, t]$ denotes the discrete 1-pointed tree $([0, t], (0, t))$ with root 0. Notice that $\ell_{A_{k+1}} = 0$ if and only if $\text{Span}^\circ(S, \mathbf{v}_k) = \text{Span}^\circ(S, \mathbf{v}_{k+1})$. Eventually, notice that $(S_0, \mathbf{v}_0) = (\{\varrho\}, \varrho)$ is the rooted tree reduced to its root $\varrho = v_{\{0\}}$ and $B_0 = \{0\}$. Let us stress, that in Section 5.4, the vector \mathbf{v}_{k+1} is obtained by adding the distinguished vertex $\ell_{A_{k+1}}$ of $[0, \ell_{A_{k+1}}]$ to \mathbf{v}_k . However here we identify $[0, \ell_{A_{k+1}}]$ with $\llbracket v_{B_k} = w_{A_{k+1}}, v_{A_{k+1}} \rrbracket$ and add the distinguished vertex $v_{A_{k+1}}$ to \mathbf{v}_k in order to obtain \mathbf{v}_{k+1} .

For instance, we give in Table 2 the sequences $(A_k, 1 \leq k \leq 2^n - 1)$ and $(B_k, 0 \leq k \leq 2^n - 2)$ for the tree of Figure 2.

TABLE 2. The sequences $(A_{k+1}, 0 \leq k \leq 6)$, $(B_k, 0 \leq k \leq 6)$ and $(\ell_{A_{k+1}}, 0 \leq k \leq 6)$ for the tree of Figure 2.

k	0	1	2	3	4	5	6
A_{k+1}	{1, 2}	{1, 2, 3}	{1, 3}	{1}	{2, 3}	{2}	{3}
B_k	{0}	{1, 3}	{1, 2}	{1, 2, 3}	{1, 2, 3}	{2, 3}	{2, 3}
$\ell_{A_{k+1}}$	$d(\varrho, a)$	0	0	$d(1, a)$	$d(a, b)$	$d(2, b)$	$d(3, b)$

Remark 5.27. The family $\{A_k, k \in \{1, 2^n - 1\}\}$ is exactly equal to \mathcal{P}_n^+ . Furthermore the sequence $\ell \in \text{Im}(\mathbf{L}_n) \subset \mathbb{R}_+^{2^n-1}$ provides implicitly two unique ordered sequences $\mathcal{A}(\ell) = (A_k, k \in \{1, 2^n - 1\})$ (of all elements of \mathcal{P}_n^+) and $\mathcal{B}(\ell) = (B_k, k \in \{0, 2^n - 2\})$ (of elements of $\mathcal{P}_n = \mathcal{P}_n^+ \cup \{\{0\}\}$),

and an “increasing” way to build $\mathbf{L}_n^{-1}(\ell)$ recursively by adding at step $k \in \{0, 2^n - 2\}$ a branch of length $\ell_{A_{k+1}}$ (and graft it on v_{B_k} chosen among \mathbf{v}_k). It is obvious from the construction that if ℓ and ℓ' are two sequences in $\text{Im}(\mathbf{L}_n)$ with the same zeros (that is, $\ell_A = 0$ if and only if $\ell'_A = 0$), then we have $\mathcal{A}(\ell) = \mathcal{A}(\ell')$ and $\mathcal{B}(\ell) = \mathcal{B}(\ell')$. Thus, the sets $\mathcal{A}(\ell)$ and $\mathcal{B}(\ell)$ are implicitly coded by the zeros of ℓ .

In a second step, given $\mathcal{A}(\ell)$ and $\mathcal{B}(\ell)$ from Remark 5.27 and a sequence $T^* = (T_A^*, A \in \mathcal{P}_+^n)$ in $\mathbb{T}_{\text{loc-K}}^{\text{spine},0}$, we build by a forward induction an “increasing” sequence of marked locally compact trees $((T_k, \mathbf{v}_k), k \in \{0, \dots, 2^n - 1\})$ such that (T_k, \mathbf{v}_k) belongs to $\mathbb{T}_{\text{loc-K}}^{(k)}$, has root ϱ , and the components of the vector \mathbf{v}_k can be ranked as the root $\varrho = v_{\{0\}}$ and $(v_{A_i}, 1 \leq i \leq k)$. Recall also the truncation function $\tilde{r}_t^{[2],+}$ given in (49). We set $(T_0, \mathbf{v}_0) = (\{\varrho\}, \varrho)$ and for $k \in \{0, 2^n - 2\}$:

$$(58) \quad (T_{k+1}, \mathbf{v}_{k+1}) = (T_k, \mathbf{v}_k) \otimes_{B_k} \tilde{r}_{\ell_{A_{k+1}}}^{[2],+}(T_{A_{k+1}}^*),$$

where the distinguished vertex of $\tilde{r}_{\ell_{A_{k+1}}}^{[2],+}(T_{A_{k+1}}^*)$ is identified with $v_{A_{k+1}}$ (and its root with v_{B_k}). Then, we set:

$$(59) \quad \text{Graft}_n(\ell, T^*) = (T_{2^n-1}, \mathbf{v}) \quad \text{with} \quad \mathbf{v} = (v_{\{k\}}, 0 \leq k \leq n).$$

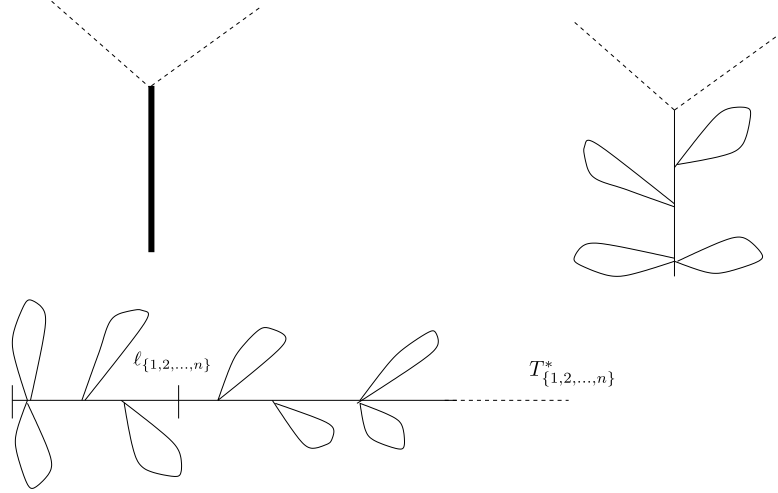


FIGURE 4. Example of a replacement of the branch $\llbracket w_{\{1,\dots,n\}}, v_{\{1,\dots,n\}} \rrbracket$.
 Upper left: The tree S with the branch $\llbracket w_{\{1,\dots,n\}}, v_{\{1,\dots,n\}} \rrbracket$ in bold.
 Upper right: The branch $\llbracket w_{\{1,\dots,n\}}, v_{\{1,\dots,n\}} \rrbracket$ replaced by the first component of the marked tree $r_{\ell_{\{1,\dots,n\}}}^{[2],+}(T_{\{1,\dots,n\}}^*)$.
 Lower: The tree $T_{\{1,\dots,n\}}^*$ with its marked infinite branch.

It is easy to check that the equivalence class of (T_{2^n-1}, \mathbf{v}) in $\mathbb{T}_{\text{loc-K}}^{(n)}$ does not depend on the choice of $T^* = (T_A^*, A \in \mathcal{P}_+^n)$ in their own equivalence class. Thus, the map Graft_n defined by:

$$(\ell, T^*) \mapsto \text{Graft}_n(\ell, T^*)$$

is well defined from $\text{Im}(\mathbf{L}_n) \times \left(\mathbb{T}_{\text{loc-K}}^{\text{spine},0}\right)^{\mathcal{P}_+^n}$ to $\mathbb{T}_{\text{loc-K}}^{(n)}$. The main result of this section is the measurability of the map Graft_n .

Lemma 5.28 (Measurability of the grafting map). *Let $n \in \mathbb{N}^*$. The map Graft_n from $\text{Im}(\mathbf{L}_n) \times \left(\mathbb{T}_{\text{loc-K}}^{\text{spine},0}\right)^{\mathcal{P}_+^n}$ to $\mathbb{T}_{\text{loc-K}}^{(n)}$ is measurable.*

Proof. For $J \subset \mathcal{P}_+^n$, we write $I_J = \{\ell \in \text{Im}(\mathbf{L}_n) : \ell_A = 0 \text{ if and only if } A \in J\}$. Thus, the closed set $\text{Im}(\mathbf{L}_n)$ of $\mathbb{R}_+^{\mathcal{P}_+^n}$ can be written as the union of I_J over all the subsets J of \mathcal{P}_+^n . Furthermore, the sets $(I_J, J \subset \mathcal{P}_+^n)$ are Borel sets (as $\text{Im}(\mathbf{L}_n)$ is a Borel set), and they are pairwise disjoint. Thanks to Remark 5.27, the maps $\ell \mapsto \mathcal{A}(\ell)$ and $\ell \mapsto \mathcal{B}(\ell)$ are constant over I_J . We deduce from Equation (59) and recursion (58), Lemma 5.13 on the continuity of the grafting procedure and Lemma 5.24 on the measurability of $(t, T) \mapsto \tilde{r}_t^{[2],+}(T)$ that the function Graft_n from $I_J \times \left(\mathbb{T}_{\text{loc-K}}^{\text{spine},0}\right)^{\mathcal{P}_+^n}$ to $\mathbb{T}_{\text{loc-K}}^{(n)}$ is measurable (as long as I_J is not empty). Since there is a finite number of such sets I_J , we deduce that the function Graft_n from $\text{Im}(\mathbf{L}_n) \times \left(\mathbb{T}_{\text{loc-K}}^{\text{spine},0}\right)^{\mathcal{P}_+^n}$ to $\mathbb{T}_{\text{loc-K}}^{(n)}$ is measurable. \square

Remark 5.29. Since the map \mathbf{L}_n is continuous one-to-one from $\mathbb{T}_{\text{dis}}^{(n)}$ to $\text{Im}(\mathbf{L}_n)$, we deduce that the map:

$$(T, T^*) \mapsto \text{Graft}_n(\mathbf{L}_n(T), T^*)$$

from $\mathbb{T}_{\text{dis}}^{(n)} \times \left(\mathbb{T}_{\text{loc-K}}^{\text{spine},0}\right)^{\mathcal{P}_+^n}$ to $\mathbb{T}_{\text{loc-K}}^{(n)}$ is measurable. Without ambiguity, we shall simply write $\text{Graft}_n(T, T^*)$ for $\text{Graft}_n(\mathbf{L}_n(T), T^*)$.

Remark 5.30. Intuitively, the maps Graft_n and Split_n should be the inverse one of the other. More precisely, we have the following result. For every $(T, (\varrho, v)) \in \mathbb{T}_{\text{loc-K}}^{(1)}$, we define the tree $\text{Sp}(T) = (T', S') \in \mathbb{T}_{\text{loc-K}}^{\text{spine},0}$ by $T' = \Pi_1^\circ(T \otimes_1 [0, \infty))$ with the marked spine $S = \Pi_1^\circ([[\varrho, v]] \otimes_1 [0, \infty))$. Then then we have, for every $(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(n),0}$ (that is, the root of T is not a branching vertex, see Definition (35)), that the following equality hold in $\mathbb{T}_{\text{loc-K}}^{(n)}$:

$$(60) \quad \text{Graft}_n\left(\text{Span}_n(T, \mathbf{v}), \text{Sp}(\text{Split}_n(T, \mathbf{v}))\right) = (T, \mathbf{v}),$$

where $\text{Sp}(T_A, A \in \mathcal{P}_n) = (\text{Sp}(T_A), A \in \mathcal{P}_n^+)$.

5.10. A measure associated with trees in $\mathbb{T}_{\text{loc-K}}^{\text{spine},0}$ or $\mathbb{T}_{\text{loc-K}}^{(1)}$. Recall $T_0 = (\{\varrho\}, \varrho) \in \mathbb{T}_{\text{loc-K}}$ is the tree reduced to its root. We define

$$(61) \quad \mathbb{T}_{\text{loc-K}}^* = \mathbb{T}_{\text{loc-K}} \setminus \{T_0\}$$

endowed with the distance:

$$d_{\text{LGH}}^*(T, T') = d_{\text{LGH}}(T, T') + |H(T)^{-1} - H(T')^{-1}|.$$

Clearly $(\mathbb{T}_{\text{loc-K}}^*, d_{\text{LGH}}^*)$ is Polish with the topology induced by the topology on $\mathbb{T}_{\text{loc-K}}$ (as H is continuous on $\mathbb{T}_{\text{loc-K}}$), and for all $\varepsilon > 0$, the sets $B_{\mathbb{T}_{\text{loc-K}}^*}(\varepsilon) = \{T \in \mathbb{T}_{\text{loc-K}}^* : H(T) \geq \varepsilon\}$ are closed and bounded. Furthermore, every bounded set is a subset of $B_{\mathbb{T}_{\text{loc-K}}^*}(\varepsilon)$ for $\varepsilon > 0$ small enough. Set $E = \mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^*$ endowed with the distance $d_E((u, T), (u', T')) = |u - u'| + d_{\text{LGH}}^*(T, T')$, so that (E, d_E) is a Polish space. Every bounded set of E is a subset of $B_E(\varepsilon) = [0, \varepsilon^{-1}] \times B_{\mathbb{T}_{\text{loc-K}}^*}(\varepsilon)$ for $\varepsilon > 0$ small enough. We define $\mathbb{M}(E)$, the set of point measures on E which are bounded on bounded sets, that is finite on $B_E(\varepsilon)$ for all $\varepsilon > 0$. We say that a sequence $(\mathcal{M}_n, n \in \mathbb{N})$ of elements of $\mathbb{M}(E)$ converges to a limit \mathcal{M} , if $\lim_{n \rightarrow \infty} \mathcal{M}_n(f) = \mathcal{M}(f)$

for all continuous functions on E with bounded support. According to [14, Proposition 9.1.IV] the space $\mathbb{M}(E)$ is Polish and the Borel σ -field is the smallest σ -field such that the application $\mathcal{M} \mapsto \mathcal{M}(A)$ is measurable for every Borel set A of E .

We build a tree from a point measure $\mathcal{M} = \sum_{i \in I} \delta_{(h_i, T_i)} \in \mathbb{M}(E)$ by grafting T_i at height h_i on an infinite spine. Recall the infinite spine $T_1 = ([0, \infty), 0)$ endowed with the Euclidean distance is an element of $\mathbb{T}_{\text{loc-K}}^{\text{spine}, 0} \subset \mathbb{T}_{\text{loc-K}}$. For $T \in \mathbb{T}_{\text{loc-K}}$, let (\tilde{T}, d, ϱ) denote a rooted locally compact tree in the equivalent class T . With obvious notation, we define the tree T' as follow:

$$T' = \tilde{T}_1 \sqcup_{i \in I} (\tilde{T}_i \setminus \{\varrho_i\}),$$

$$\forall x, x' \in T', \quad d(x, x') = \begin{cases} d_i(x, x') & \text{if } x, x' \in \tilde{T}_i, i \in I \\ |x - x'| & \text{if } x, x' \in \tilde{T}_1, \\ d_i(x, \varrho_i) + |h_i - x| & \text{if } x \in \tilde{T}_i, x' \in \tilde{T}_1, i \in I, \\ d_i(x, \varrho_i) + d_j(x', \varrho_j) + |h_i - h_j| & \text{if } x \in \tilde{T}_i, x' \in \tilde{T}_j \text{ with } i \neq j, i, j \in I, \end{cases}$$

where \sqcup denotes the disjoint union. By construction T' is a tree rooted at $\varrho = \varrho_1$, the root of \tilde{T}_1 . Because \mathcal{M} is finite on bounded sets of E , it is not difficult to check that T' is locally compact. It is easy to see that the equivalence class of $\text{Tree}(\mathcal{M}) = (T', \tilde{T}_1)$ in $\mathbb{T}_{\text{loc-K}}^{[2]}$ does not depend of the choice of the representatives in the equivalence classes of T_1 and T_i for $i \in I$. Hence, identifying $\text{Tree}(\mathcal{M})$ with its equivalence class, we get that the map Tree is well defined from $\mathbb{M}(E)$ into $\mathbb{T}_{\text{loc-K}}^{[2]}$.

Lemma 5.31 (Regularity of the map Tree). *The map Tree from $\mathbb{M}(E)$ to $\mathbb{T}_{\text{loc-K}}^{[2]}$ (or $\mathbb{T}_{\text{loc-K}}^{\text{spine}}$) is continuous.*

Proof. We only give the principal arguments of the proof. Let $(\mathcal{M}_n, n \in \mathbb{N})$ a sequence of point measures, elements of $\mathbb{M}(E)$, which converges to \mathcal{M} . Let $\varepsilon > 0$ be fixed such that $\mathcal{M}(\partial B_E(\varepsilon)) = 0$. For n large enough, we have $\mathcal{M}_n(B_E(\varepsilon)) = \mathcal{M}(B_E(\varepsilon))$ and the atoms of \mathcal{M}_n in $B_E(\varepsilon)$ converge to the atoms of \mathcal{M} in $B_E(\varepsilon)$. Using correspondence between the representations of the atoms, and similar arguments as in the proof of Lemma 5.13, we deduce that the distance between $\text{Tree}(\mathcal{M}_n)$ and $\text{Tree}(\mathcal{M})$ (in $\mathbb{T}_{\text{loc-K}}^{[2]}$) is small if $\varepsilon > 0$ is small (to prove this statement in detail, one can use the distance on $\mathbb{M}(E)$ given in [13, Equation (A2.6.1)]). This means that $\lim_{n \rightarrow \infty} d_{\text{LGH}}^{[2]}(\text{Tree}(\mathcal{M}_n), \text{Tree}(\mathcal{M})) = 0$, and thus the map Tree is continuous on $\mathbb{T}_{\text{loc-K}}^{[2]}$. \square

We shall now prove that the restriction of the map Tree to a subset of $\mathbb{M}(E)$ is injective and bi-measurable. For this reason, we consider the subset of $\mathbb{T}_{\text{loc-K}}$ of (equivalence classes of) trees not reduced to its root and such that the root is not a branching vertex (recall Definitions (61) and (35) with $n = 0$):

$$(62) \quad \mathbb{T}_{\text{loc-K}}^{0,*} = \mathbb{T}_{\text{loc-K}}^* \cap \mathbb{T}_{\text{loc-K}}^0.$$

As a direct consequence of Lemma 5.9, $\mathbb{T}_{\text{loc-K}}^{0,*}$ is a Borel subset of $\mathbb{T}_{\text{loc-K}}$ and thus of $\mathbb{T}_{\text{loc-K}}^*$. In particular, the following subset of $\mathbb{M}(E)$ is a Borel set:

$$\tilde{\mathbb{M}}(E) = \left\{ \mathcal{M} \in \mathbb{M}(E) : \mathcal{M}(\mathbb{R}_+ \times (\mathbb{T}_{\text{loc-K}}^{0,*})^c) = 0 \right\}.$$

We now introduce a map \mathcal{M} from $\mathbb{T}_{\text{loc-K}}^{\text{spine}}$ to $\mathbb{M}(E)$ as follow. Let $T^* = (T, T_1)$ be a rooted locally compact tree with an infinite marked spine. In particular, we have $T_1 \subset T$ and T_1 is equivalent to $([0, \infty), d, 0)$. Let $(T_i^\circ, i \in I)$ be the family of the connected components of $T \setminus T_1$. For every $i \in I$, let us denote by x_i the MRCA of T_i° , that is, the unique point of T_1 such that

for every $x \in T_i^\circ$, $[[\varrho, x] \cap T_1 = [[\varrho, x_i]$. We then set $T_i = T_i^\circ \cup \{x_i\}$ viewed as a locally compact tree rooted at x_i . Then, we define the point measure $\mathcal{M}(T^*)$ on $\mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^* \subset \mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}$ by:

$$\mathcal{M}(T^*) = \sum_{i \in I} \delta_{(H(x_i), T_i)}.$$

As $\mathcal{M}(T^*)$ does not depends on the representatives chosen in the equivalence class of T^* in $\mathbb{T}_{\text{loc-K}}^{\text{spine}}$, we deduce that $\mathcal{M} : T^* \mapsto \mathcal{M}(T^*)$ is a map from $\mathbb{T}_{\text{loc-K}}^{\text{spine}}$ to $\mathbb{M}(E)$. We now give the main result of this section.

Proposition 5.32 (Regularity of the maps Tree and \mathcal{M}). *The map \mathcal{M} is bi-measurable from $\mathbb{T}_{\text{loc-K}}^{\text{spine}}$ to $\tilde{\mathbb{M}}(E)$ with $\tilde{\mathbb{M}}(E) = \text{Im}(\mathcal{M})$. The map Tree is bi-measurable from $\tilde{\mathbb{M}}(E)$ to $\mathbb{T}_{\text{loc-K}}^{\text{spine}}$. Furthermore, the map $\text{Tree} \circ \mathcal{M}$ is the identity map on $\mathbb{T}_{\text{loc-K}}^{\text{spine}}$ and $\mathcal{M} \circ \text{Tree}$ is the identity map on $\tilde{\mathbb{M}}(E)$.*

Proof. By construction, the roots of all the trees T_i in the point measure $\mathcal{M}(T^*)$ are not branching vertices, so that $\mathcal{M}(T^*)$ belongs to $\tilde{\mathbb{M}}(E) \subset \mathbb{M}(E)$. We also get by construction that $\text{Tree}(\mathcal{M}(T^*)) = T^*$. This implies that \mathcal{M} is injective and thus bi-measurable thanks to Lusin's theorem [32].

We also have by construction that $\mathcal{M} \circ \text{Tree}(\mathcal{M}) = \mathcal{M}$ for $\mathcal{M} \in \tilde{\mathbb{M}}(E)$. This implies that $\text{Im}(\mathcal{M}) = \tilde{\mathbb{M}}(E)$ and also that Tree restricted to $\tilde{\mathbb{M}}(E)$ is injective and thus bi-measurable thanks to Lusin's theorem. \square

We extend the map $T^* \mapsto \mathcal{M}(T^*)$ to $\mathbb{T}_{\text{loc-K}}^{(1)}$ in the following way. For $(T, \mathbf{v} = (\varrho, v_1)) \in \mathbb{T}_{\text{loc-K}}^{(1)}$, we graft the infinite spine T_1 on v_1 and consider the rooted locally compact tree with an infinite marked spine $\text{Sp}(T) \in \mathbb{T}_{\text{loc-K}}^{\text{spine}}$ defined in Remark 5.30. Then, we define $\mathcal{M}(T, \mathbf{v})$ as $\mathcal{M}(\text{Sp}(T))$. From the continuity of the grafting procedure, see Lemma 5.13 and the continuity of Π_1° , see Lemma 5.5, and the measurability of the map \mathcal{M} , we deduce that the map $(T, \mathbf{v}) \mapsto \mathcal{M}(T, \mathbf{v})$, which we still denote by \mathcal{M} is measurable. In fact, we have the stronger following result. Consider the set of (equivalence class of) n -pointed rooted locally compact tree such that the root is not a branching vertex and the pointed vertices are not equal to the root:

$$(63) \quad \mathbb{T}_{\text{loc-K}}^{(n),0,*} = \{(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(n),0} : d(\varrho, v_i) > 0 \text{ for all } i \in \{1, \dots, n\}\},$$

where $\mathbf{v} = (\varrho, v_1, \dots, v_n)$. According to Lemma 5.9 and Remark 5.11, the set $\mathbb{T}_{\text{loc-K}}^{(n),0,*}$ is a Borel subset of $\mathbb{T}_{\text{loc-K}}^{(n)}$. Recall from (62) that the Borel set $\mathbb{T}_{\text{loc-K}}^{0,*}$ is the set of (equivalence class of) 1-pointed rooted locally compact trees such that the root is not a branching vertex and the pointed vertex is not equal to the root.

Corollary 5.33 (Recovering (T, \mathbf{v}) from $\mathcal{M}(T, \mathbf{v})$). *The following map from $\mathbb{T}_{\text{loc-K}}^{(1)}$ to $\mathbb{R}_+ \times \mathbb{M}(E)$ defined by:*

$$(T, \mathbf{v}) \mapsto (d(\varrho, v), \mathcal{M}(T, \mathbf{v}))$$

is measurable and its restriction to $\mathbb{T}_{\text{loc-K}}^{(1),0,}$ is injective and bi-measurable.*

Proof. Set $\mathbb{M}^*(E) = \{\mathcal{M} \in \mathbb{M}(E) : \mathcal{M}(\{0\} \times \mathbb{T}_{\text{loc-K}}^*) = 0\}$. For $\mathcal{M} \in \mathbb{M}^*(E)$, we get that $\text{Tree}(\mathcal{M})$ belongs to $\mathbb{T}_{\text{loc-K}}^{\text{spine},0}$. So, we can define the map g on $\mathbb{R}_+ \times \mathbb{M}^*(E)$ by $g(a, \mathcal{M}) = \text{Graft}_1([0, a], \text{Tree}(\mathcal{M}))$, where the tree $[0, a]$ has root 0 and pointed vertex a . Thanks to the continuity of the grafting procedure, see Lemma 5.28 and of the function Tree, see Lemma 5.31, we deduce that g is continuous.

Let $(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(1),0,*}$. As the root of T is not a branching vertex, we get that $\mathcal{M}(T, \mathbf{v})$ belongs to $\mathbb{M}^*(E)$, and thus $g(d(\varrho, v), \mathcal{M}(T, \mathbf{v}))$, where $\mathbf{v} = (\varrho, v)$, is well defined and in fact equal to (T, \mathbf{v}) thanks to (60) with $n = 1$. This implies that the map $(T, \mathbf{v}) \mapsto (d(\varrho, v), \mathcal{M}(T, \mathbf{v}))$ defined on $\mathbb{T}_{\text{loc-K}}^{(1),0,*}$ is injective, and thus bi-measurable by Lusin's theorem [32]. \square

We extend this result to n -pointed trees. Recall from (55) that, for $(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(n)}$, we have $\text{Split}_n(T, \mathbf{v}) = (\hat{T}_A(T, \mathbf{v}), A \in \mathcal{P}_n)$ and set $\mathcal{M}_A[T, \mathbf{v}] = \mathcal{M}(\hat{T}_A(T, \mathbf{v}))$ for $A \in \mathcal{P}_n^+$.

Corollary 5.34 (Recovering (T, \mathbf{v}) from the $\mathcal{M}_A[T, \mathbf{v}]$). *The following map from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{dis}}^{(n)} \times \mathbb{M}(E)^{\mathcal{P}_n^+}$ defined by:*

$$(T, \mathbf{v}) \mapsto \left(\text{Span}_n(T, \mathbf{v}), (\mathcal{M}_A[T, \mathbf{v}], A \in \mathcal{P}_n^+) \right)$$

is measurable and its restriction to $\mathbb{T}_{\text{loc-K}}^{(n),0,*}$ is injective and bi-measurable.

Proof. Using the measurability of the functions Span from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{(n)}$ (see Lemma 5.6), \mathbf{L}_n from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{R}_+^{\mathcal{P}_n^+}$ (see Lemma 5.25), Split_n from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\left(\mathbb{T}_{\text{loc-K}}^{(1)}\right)^{2^n}$ (see Lemma 5.26) and the map $(T, \mathbf{v}) \mapsto \mathcal{M}(T, \mathbf{v})$ from $\mathbb{T}_{\text{loc-K}}^{(1)}$ to $\mathbb{M}(E)$ (see Corollary 5.33), we deduce that the following map, say g_1 , from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{(n)} \times (\mathbb{R}_+ \times \mathbb{M}(E))^{\mathcal{P}_n^+}$ is measurable:

$$g_1 : (T, \mathbf{v}) \mapsto \left(\text{Span}(T, \mathbf{v}), ((\ell_A(T, \mathbf{v}), \mathcal{M}_A[T, \mathbf{v}]), A \in \mathcal{P}_n^+) \right).$$

Notice that $(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(n),0,*}$ implies that $\hat{T}_{\{0\}}$ is reduced to its root. Using the measurable functions Graft_n and the map defined in Corollary 5.33, we easily deduce that g_1 restricted to $\mathbb{T}_{\text{loc-K}}^{(n),0,*}$ is injective and thus bi-measurable by Lusin's theorem [32]. Since $\mathbf{L}_n(T, \mathbf{v})$ is also equal to $\mathbf{L}_n(\text{Span}(T, \mathbf{v}))$, we deduce that the following map g_2 , from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to $\mathbb{T}_{\text{loc-K}}^{(n)} \times \mathbb{M}(E)^{\mathcal{P}_n^+}$ is measurable:

$$g_2 : (T, \mathbf{v}) \mapsto \left(\text{Span}(T, \mathbf{v}), (\mathcal{M}_A[T, \mathbf{v}], A \in \mathcal{P}_n^+) \right).$$

Furthermore, its restriction to $\mathbb{T}_{\text{loc-K}}^{(n),0,*}$ is also injective and thus bi-measurable. \square

6. BACKBONE DECOMPOSITION

6.1. A discrete random tree constructed by successive grafts.

6.1.1. *A random tree.* In this section, for $a \geq 0$, we denote by $([0, a], (\varrho = 0, a)) \in \mathbb{T}_{\text{dis}}^{(1)}$ the (equivalent class of the) tree $[0, a]$ endowed with the usual distance on \mathbb{R} , rooted at $\varrho = 0$ and pointed at a ; and when there is no possible confusion we simply denote it by $[0, a]$.

Let $t > 0$ and let ν be a probability measure on $[0, t]$. Let $\xi = (\xi_k, k \in \mathbb{N}^*)$ be a sequence of independent random variables with distribution ν and let $((K_k, \varepsilon_k), k \in \mathbb{N}^*)$ be a sequence of independent random variables independent of the sequence ξ , with K_k uniformly distributed on $\{1, \dots, k\}$ independent of ε_k uniformly distributed on $\{g, d\}$. For every integer $n \geq 2$, we set $(\xi_1^{(n)}, \dots, \xi_{n-1}^{(n)})$ the increasing order statistic of $(\xi_1, \dots, \xi_{n-1})$. Then we define the family of pointed trees $((\mathbf{T}_1^{(n)}, \mathbf{v}_1^{(n)}), \dots, (\mathbf{T}_n^{(n)}, \mathbf{v}_n^{(n)}))$, with $(\mathbf{T}_k^{(n)}, \mathbf{v}_k^{(n)}) \in \mathbb{T}_{\text{dis}}^{(k)}$, recursively by:

- $\mathbf{T}_1^{(n)} = [0, t]$, that is, $(\mathbf{T}_1^{(n)}, \mathbf{v}_1^{(n)}) = ([0, t], (0, t)) \in \mathbb{T}_{\text{dis}}^{(k)} \subset \mathbb{T}_{\text{loc-K}}^{(1)}$.

- For every $k \in \{1, \dots, n-1\}$, conditionally given the random variable $(\mathbf{T}_k^{(n)}, \mathbf{v}_k^{(n)})$ in $\mathbb{T}_{\text{dis}}^{(k)}$, we define the $\mathbb{T}_{\text{loc-K}}^{(k+1)}$ -valued random variable $(\mathbf{T}_{k+1}^{(n)}, \mathbf{v}_{k+1}^{(n)})$ by grafting a branch of length $t - \xi_{k+1}^{(n)}$ uniformly on the left or on the right of a uniformly chosen vertex among the k vertices of $\mathbf{T}_k^{(n)}$ at level $\xi_{k+1}^{(n)}$, and the new leaf (which is, as all the other leaves, at level t) is added to the vector recording the pointed vertices. Formally, using the grafting procedure (39) from Remark 5.14, we set:

$$(64) \quad (\mathbf{T}_{k+1}^{(n)}, \mathbf{v}_{k+1}^{(n)}) = (\mathbf{T}_k^{(n)}, \mathbf{v}_k^{(n)}) \otimes_{K_k, \xi_{k+1}^{(n)}}^{\varepsilon_{k+1}} [0, t - \xi_{k+1}^{(n)}].$$

According to Lemma 5.15 the grafting procedure is continuous and thus measurable, we deduce that $(\mathbf{T}_k^{(n)}, \mathbf{v}_k^{(n)})$ is a $\mathbb{T}_K^{(k)}$ -valued random variable for every $1 \leq k \leq n$.

By construction, we also get that $(\mathbf{T}_k^{(n)}, \mathbf{v}_k^{(n)})$ belongs to $\mathbb{T}_{\text{dis}}^{(k)}$, and is thus a $\mathbb{T}_{\text{dis}}^{(k)}$ -valued random variable, for all $k \in \{1, \dots, n\}$. To simplify the notations, we set $\mathbf{T}_n = (\mathbf{T}_n^{(n)}, \mathbf{v}_n^{(n)})$ for the $\mathbb{T}_K^{(n)}$ -valued (and $\mathbb{T}_{\text{dis}}^{(n)}$ -valued) random variable. By construction, we also get that \mathbf{T}_n is a planar tree with the leaves v_1, \dots, v_n ranked from left to right, where $\mathbf{v}_n^{(n)} = (v_0 = \varrho, \dots, v_n)$.

Recall that for a rooted tree T , \mathcal{L}^T denotes its length measure; and we simply write \mathcal{L} when there is no ambiguity. Informally, for $x \in T$ chosen according to $\mathcal{L}(dx)$, we denote by $T \otimes_x [0, a]$ the tree T on which a segment of length a is grafted at the vertex x . It is possible to make this construction in a measurable way, in this direction see for example [8, Equation (1.6)] and Section 3 therein. As we shall consider the discrete tree \mathbf{T}_n for T , we refer to Remark 6.2 below for a direct proof. The next lemma relates the distributions of \mathbf{T}_n and of \mathbf{T}_{n+1} ; its proof is given in the next section.

Lemma 6.1. *Let $t \geq 0$. Assume that the probability distribution ν has a positive density f_{dens} with respect to the Lebesgue measure on $[0, t]$. For $n \in \mathbb{N}^*$, G a measurable non-negative function defined on $\mathbb{T}_{\text{dis}}^{(n+1)}$, and ε a $\{\text{g}, \text{d}\}$ -valued uniform random variable independent of \mathbf{T}_n , we have:*

$$(65) \quad \mathbb{E} \left[\int_{\mathbf{T}_n} \mathcal{L}(dx) f_{\text{dens}}(H(x)) G(\mathbf{T}_n \otimes_x^\varepsilon [0, t - H(x)]) \right] = \frac{n+1}{2} \mathbb{E} [G(\mathbf{T}_{n+1})].$$

Remark 6.2. We comment on the left-hand side of (65), and more precisely we check that the integral $\mathcal{I} = \int_{\mathbf{T}_n} \mathcal{L}(dx) f_{\text{dens}}(H(x)) G(\mathbf{T}_n \otimes_x^\varepsilon [0, t - H(x)])$ is a non-negative random variable. Recall that $\mathbf{T}_n = (\mathbf{T}_n^{(n)}, \mathbf{v}_n^{(n)})$. Then, we can write \mathcal{I} as follows:

$$\mathcal{I} = \sum_{k=1}^n \int_{\xi_{k-1}^{(n)}}^t dh f_{\text{dens}}(h) G(\mathbf{T}_n \otimes_{k,h}^\varepsilon [0, t - h]),$$

with the convention that $\xi_0^{(n)} = 0$. Therefore, using the continuity of the grafting function, see Lemma 5.15, we obtain that \mathcal{I} is a non-negative real-valued random variable, and thus its expectation is well defined.

6.1.2. *Proof of Lemma 6.1.* The proof is based on two technical lemmas. We first consider the case $t = 1$ and ν the uniform distribution on $[0, 1]$. Let us denote by $\mathbf{T}_n^{\text{unif}}$ for \mathbf{T}_n when ν is the uniform distribution on $[0, 1]$.

Lemma 6.3. For $n \in \mathbb{N}^*$, G a measurable non-negative functional defined on $\mathbb{T}_{\text{dis}}^{(n+1)}$, and ε a $\{\mathfrak{g}, \mathfrak{d}\}$ -valued uniform random variable independent of $\mathbf{T}_n^{\text{unif}}$, we have:

$$(66) \quad \mathbb{E} \left[\int_{\mathbf{T}_n^{\text{unif}}} \mathcal{L}(dx) G \left(\mathbf{T}_n^{\text{unif}} \otimes_x^\varepsilon [0, t - H(x)] \right) \right] = \frac{n+1}{2} \mathbb{E} \left[G(\mathbf{T}_{n+1}^{\text{unif}}) \right].$$

Remark 6.4. From (66), we see $\frac{n+1}{2}$ is just the mean length of $\mathbf{T}_n^{\text{unif}}$.

Proof. To simplify notation, we write \mathbf{T}_n for $\mathbf{T}_n^{\text{unif}}$. We give a proof by induction. For $n = 1$, this is a direct consequence of the construction of $\mathbf{T}_2 = \mathbf{T}_2^{(2)}$ from $\mathbf{T}_1^{(2)} = \mathbf{T}_1 = [0, 1]$ given by (64).

Let $n \in \mathbb{N}^*$ and assume that (66) holds for n replaced by any $k \in \{1, \dots, n-1\}$. We will use for the proof a special representation of planar binary trees. Let T be a compact planar binary tree rooted at ϱ , with all leaves at height 1. Let us denote by x its lowest branching vertex, set $h = H(x)$ and set $\tilde{T}^{\mathfrak{g}}$ (resp. $\tilde{T}^{\mathfrak{d}}$) the left (resp. right) sub-tree above x . In our settings, we have:

$$h = H(x) \quad \text{and} \quad T = ([\varrho, x] \otimes_x \tilde{T}^{\mathfrak{g}}) \otimes_x^{\mathfrak{d}} \tilde{T}^{\mathfrak{d}} = ([\varrho, x] \otimes_x \tilde{T}^{\mathfrak{d}}) \otimes_x^{\mathfrak{g}} \tilde{T}^{\mathfrak{g}},$$

where one remove the vertex x from the pointed vertices after the graftings. For convenience, we consider the scaled left and right trees $T^{\mathfrak{g}} = (1-h)^{-1} \tilde{T}^{\mathfrak{g}}$ and $T^{\mathfrak{d}} = (1-h)^{-1} \tilde{T}^{\mathfrak{d}}$, so that $T^{\mathfrak{g}}$ and $T^{\mathfrak{d}}$ are rooted bounded binary planar trees with all their leaves at height 1. We call $(h, T^{\mathfrak{g}}, T^{\mathfrak{d}})$ the decomposition of T according to its lowest branching vertex.

Let $(\xi_1^{(n+1)}, \mathbf{T}_{n+1}^{\mathfrak{g}}, \mathbf{T}_{n+1}^{\mathfrak{d}})$ be the decomposition of \mathbf{T}_{n+1} according to its lowest branching vertex (which is indeed at height $\xi_1^{(n+1)}$ by construction). Denote by I_{n+1} the number of leaves of $\mathbf{T}_{n+1}^{\mathfrak{g}}$. Using a Pólya urn starting with two balls of color \mathfrak{g} and \mathfrak{d} , we get that, by construction, I_{n+1} is the number of balls of color \mathfrak{g} in the urn after n draws. Thus I_{n+1} is uniform on $\{1, \dots, n\}$ and independent of $\xi_1^{(n+1)}$. Notice that if U is a uniform random variable on $[0, 1]$, for every $h \in (0, 1)$, conditionally given $\{U \geq h\}$, the random variable $(1-h)^{-1}(U-h)$ is still uniformly distributed on $[0, 1]$. This gives that, conditionally on $\{\xi_1^{(n+1)} = h\}$ and $\{I_{n+1} = i\}$, the two trees $\mathbf{T}_{n+1}^{\mathfrak{g}}$ and $\mathbf{T}_{n+1}^{\mathfrak{d}}$ are independent and distributed respectively as \mathbf{T}_i and \mathbf{T}_{n+1-i} .

We consider a measurable non-negative functional G defined on the space of rooted compact binary planar trees with a finite number of leaves, all of them at height 1 of the form:

$$(67) \quad G(T) = g_1(h) g_2(T^{\mathfrak{g}}) g_3(T^{\mathfrak{d}}),$$

where the g_i 's are measurable non-negative functionals and $(h, T^{\mathfrak{g}}, T^{\mathfrak{d}})$ is the decomposition of T according to its lowest branching vertex. Setting $f_j(i) = \mathbb{E} [g_j(\mathbf{T}_i)]$ for $j \in \{2, 3\}$, we have since $\xi_1^{(n+1)}$ is distributed according to a $\beta(1, n)$ distribution:

$$(68) \quad \begin{aligned} \mathbb{E} [G(\mathbf{T}_{n+1})] &= \left(\int_0^1 g_1(h) n(1-h)^{n-1} dh \right) \frac{1}{n} \sum_{i=1}^n f_2(i) f_3(n+1-i) \\ &= \left(\int_0^1 g_1(h) (1-h)^{n-1} dh \right) \sum_{i=1}^n f_2(i) f_3(n+1-i). \end{aligned}$$

On the other hand, let $(\xi_1^{(n)}, \mathbf{T}_n^{\mathfrak{g}}, \mathbf{T}_n^{\mathfrak{d}})$ be the decomposition of \mathbf{T}_n according to its lowest branching vertex. Let $x \in \mathbf{T}_n$ and set $h = H(x)$.

- If $h < \xi_1^{(n)}$, the decomposition of $\mathbf{T}_n \otimes_x^g [0, 1-h]$ according to its lowest branching vertex is given by $(h, [0, 1], (1-h)^{-1}\mathbf{T}'_n)$ where \mathbf{T}'_n is as the tree \mathbf{T}_n but for its lowest branch whose length is $\xi_1^{(n)} - h$ instead of $\xi_1^{(n)}$. Notice that the shapes of the tree \mathbf{T}'_n and \mathbf{T}_n are the same. Then using again the property of conditioned uniform random variables, we deduce that conditionally on $\{\xi_1^{(n)} \geq h\}$, the tree $(1-h)^{-1}\mathbf{T}'_n$ is distributed as \mathbf{T}_n . Thus, we get:

$$(69) \quad \mathbb{E} \left[\int_{\mathbf{T}_n} \mathbf{1}_{\{H(x) < \xi_1^{(n)}\}} \mathcal{L}(dx) G(\mathbf{T}_n \otimes_x^g [0, 1-h]) \right] = \mathbb{E} \left[\int_0^{\xi_1^{(n)}} g_1(h) dh \right] f_2(1) f_3(n).$$

By symmetry, we have:

$$(70) \quad \mathbb{E} \left[\int_{\mathbf{T}_n} \mathbf{1}_{\{H(x) < \xi_1^{(n)}\}} \mathcal{L}(dx) G(\mathbf{T}_n \otimes_x^d [0, 1-h]) \right] = \mathbb{E} \left[\int_0^{\xi_1^{(n)}} g_1(h) dh \right] f_2(n) f_3(1).$$

- For $x \in \tilde{\mathbf{T}}_n^g$, the decomposition of $\mathbf{T}_n \otimes_x^\varepsilon [0, 1-h]$ according to its lowest branching vertex is given by $(\xi_1^{(n)}, (1-h)^{-1}\mathcal{T}', \mathbf{T}_n^d)$, where $\mathcal{T}' = \tilde{\mathbf{T}}_n^g \otimes_x^\varepsilon [0, 1-h]$. Notice that the length measure on the tree \mathbf{T}_n^g is obtained by scaling by $(1-\xi_1^{(n)})^{-1}$ the length measure on \mathbf{T}_n restricted to $\tilde{\mathbf{T}}_n^g$. We deduce that:

$$\begin{aligned} & \mathbb{E} \left[\int_{\mathbf{T}_n} \mathbf{1}_{\{x \in \tilde{\mathbf{T}}_n^g\}} \mathcal{L}^{\mathbf{T}_n}(dx) G(\mathbf{T}_n \otimes_x^\varepsilon [0, 1-h]) \right] \\ &= \mathbb{E} \left[g_1(\xi_1^{(n)}) g_3(\mathbf{T}_n^d) \int_{\mathbf{T}_n} \mathbf{1}_{\{x \in \tilde{\mathbf{T}}_n^g\}} \mathcal{L}^{\mathbf{T}_n}(dx) g_2 \left((1-\xi_1^{(n)})^{-1} (\tilde{\mathbf{T}}_n^g \otimes_x^\varepsilon [0, 1-H(x)]) \right) \right] \\ &= \mathbb{E} \left[g_1(\xi_1^{(n)}) g_3(\mathbf{T}_n^d) \int_{\mathbf{T}_n^g} (1-\xi_1^{(n)}) \mathcal{L}^{\mathbf{T}_n^g}(dy) g_2 \left(\mathbf{T}_n^g \otimes_y^\varepsilon [0, 1-H(y)] \right) \right] \\ &= \mathbb{E} \left[(1-\xi_1^{(n)}) g_1(\xi_1^{(n)}) \right] \frac{1}{n-1} \sum_{i=1}^{n-1} f_3(n-i) \mathbb{E} \left[\int_{\mathbf{T}_i} \mathcal{L}^{\mathbf{T}_i}(dy) g_2 \left(\mathbf{T}_i \otimes_y^\varepsilon [0, 1-H(y)] \right) \right]. \end{aligned}$$

where we used the distribution of $(\mathbf{T}_n^g, \mathbf{T}_n^d)$ conditionally on $\xi_1^{(n)}$ and I_n for the last equality. Using that, by induction, (66) holds for $n = i$, we get:

$$(71) \quad \begin{aligned} & \mathbb{E} \left[\int_{\mathbf{T}_n} \mathbf{1}_{\{x \in \tilde{\mathbf{T}}_n^g\}} \mathcal{L}(dx) G(\mathbf{T}_n \otimes_x^\varepsilon [0, 1-h]) \right] \\ &= \mathbb{E} \left[(1-\xi_1^{(n)}) g_1(\xi_1^{(n)}) \right] \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{i+1}{2} f_2(i+1) f_3(n-i) \\ &= \mathbb{E} \left[(1-\xi_1^{(n)}) g_1(\xi_1^{(n)}) \right] \frac{1}{n-1} \sum_{i=2}^n \frac{i}{2} f_2(i) f_3(n-i+1). \end{aligned}$$

- By symmetry, for $x \in \tilde{\mathbf{T}}_n^d$, we get:

$$\begin{aligned}
(72) \quad \mathbb{E} \left[\int_{\mathbf{T}_n} \mathbf{1}_{\{x \in \tilde{\mathbf{T}}_n^d\}} \mathcal{L}(dx) G(\mathbf{T}_n \otimes_x^\varepsilon [0, 1 - h]) \right] \\
= \mathbb{E} \left[(1 - \xi_1^{(n)}) g_1(\xi_1^{(n)}) \right] \frac{1}{n-1} \sum_{i=2}^n \frac{i}{2} f_3(i) f_2(n-i+1) \\
= \mathbb{E} \left[(1 - \xi_1^{(n)}) g_1(\xi_1^{(n)}) \right] \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{n-i+1}{2} f_2(i) f_3(n-i+1).
\end{aligned}$$

Summing (69) times $\mathbb{P}(\varepsilon = g) = 1/2$, (70) times $\mathbb{P}(\varepsilon = d) = 1/2$, (71) and (72), and using that $\xi_1^{(n)}$ has distribution $\beta(1, n-1)$ so that:

$$\mathbb{E} \left[\int_0^{\xi_1^{(n)}} g_1(h) dh \right] = \frac{1}{n-1} \mathbb{E} \left[(1 - \xi_1^{(n)}) g_1(\xi_1^{(n)}) \right] = \int_0^1 g_1(h) (1-h)^{n-1} dh,$$

we deduce that:

$$\begin{aligned}
\mathbb{E} \left[\int_{\mathbf{T}_n} \mathcal{L}(dx) G(\mathbf{T}_n \otimes_x^\varepsilon [0, 1 - H(x)]) \right] \\
= \left(\int_0^1 g_1(h) (1-h)^{n-1} dh \right) \sum_{i=1}^n \frac{n+1}{2} f_2(i) f_3(n+1-i).
\end{aligned}$$

Thanks to (68), we deduce that (66) holds for G given by (67). Then use a monotone class argument to conclude that (66) holds for any measurable non-negative G . This concludes the proof by induction. \square

We now consider $t \geq 0$ and assume that the probability distribution ν has a positive density f_{dens} with respect to the Lebesgue measure on $[0, t]$. Let F denote the cumulative distribution function of ν . By the assumptions on f_{dens} , F is one to one from $[0, t]$ onto $[0, 1]$ and its inverse F^{-1} is continuous. For a compact rooted real tree (T, d, ϱ) , we define:

$$\begin{aligned}
\forall x \in T, H^{f_{\text{dens}}}(x) &= F^{-1}(H(x)), \\
\forall x, y \in T, d^{f_{\text{dens}}}(x, y) &= H^{f_{\text{dens}}}(x) + H^{f_{\text{dens}}}(y) - 2H^{f_{\text{dens}}}(x \wedge y).
\end{aligned}$$

The scaling map $R^{f_{\text{dens}}} : (T, d, \varrho) \mapsto (T, d^{f_{\text{dens}}}, \varrho)$ is then well-defined from $\{T \in \mathbb{T}_K : H(T) \leq 1\}$ to \mathbb{T}_K . We shall now prove it is continuous.

Lemma 6.5. *The map $R^{f_{\text{dens}}}$ from $\{T \in \mathbb{T}_K, H(T) \leq 1\}$ to \mathbb{T}_K is continuous.*

Proof. Let $\varepsilon > 0$. As F^{-1} is uniformly continuous with our assumptions, there exists $\delta > 0$ such that, for every $x, y \in [0, 1]$:

$$|x - y| < \delta \implies |F^{-1}(x) - F^{-1}(y)| \leq \frac{\varepsilon}{2}.$$

Let $T, T' \in \mathbb{T}_K$ with $H(T) \leq 1$ and $H(T') \leq 1$ such that $d_{\text{GH}}(T, T') < \delta/8$. Then, there exists a correspondence \mathcal{R} between (elements in the equivalence classes) T and T' such that $\text{dist}(\mathcal{R}) \leq 2d_{\text{GH}}(T, T') + \delta/4 < \delta/2$.

For every $(x, x'), (y, y') \in \mathcal{R}$, we have:

$$\begin{aligned} |d^{f_{\text{dens}}}(x, y) - d'^{f_{\text{dens}}}(x', y')| &= |H^{f_{\text{dens}}}(x) + H^{f_{\text{dens}}}(y) - 2H^{f_{\text{dens}}}(x \wedge y) - H^{f_{\text{dens}}}(x') \\ &\quad - H^{f_{\text{dens}}}(y') + 2H^{f_{\text{dens}}}(x' \wedge y')| \\ &\leq \left| F^{-1}(H(x)) - F^{-1}(H(x')) \right| + \left| F^{-1}(H(y)) - F^{-1}(H(y')) \right| \\ &\quad + 2 \left| F^{-1}(H(x \wedge y)) - F^{-1}(H(x' \wedge y')) \right|. \end{aligned}$$

As $(x, x') \in \mathcal{R}$, we have $|H(x) - H(x')| \leq \text{dist}(\mathcal{R}) < \delta$ and consequently, $\left| F^{-1}(H(x)) - F^{-1}(H(x')) \right| < \varepsilon/2$. Similarly, we have $\left| F^{-1}(H(y)) - F^{-1}(H(y')) \right| < \varepsilon/2$. We also have:

$$\begin{aligned} |H(x \wedge y) - H(x' \wedge y')| &= \frac{1}{2} |H(x) + H(y) - d(x, y) - H(x') - H(y') + d'(x', y')| \\ &\leq \frac{1}{2} |H(x) - H(x')| + \frac{1}{2} |H(y) - H(y')| + \frac{1}{2} |d(x, y) - d'(x', y')| \\ &\leq \frac{3}{2} \text{dist}(\mathcal{R}) \\ &< \delta. \end{aligned}$$

This gives $\left| F^{-1}(H(x \wedge y)) - F^{-1}(H(x' \wedge y')) \right| < \varepsilon/2$.

To conclude, we have $\text{dist}^{f_{\text{dens}}}(\mathcal{R}) < 2\varepsilon$ which implies that $d_{\text{GH}}^{f_{\text{dens}}}(T, T') < \varepsilon$. This gives the continuity of the map $R^{f_{\text{dens}}}$. \square

We now prove Lemma 6.1. Recall that \mathbf{T}_n denotes the trees constructed with the probability measure $\nu(dx) = f_{\text{dens}}(x) dx$ and $\mathbf{T}_n^{\text{unif}}$ the trees constructed with the uniform distribution on $[0, 1]$ as studied in the first step. By construction, for all $n \in \mathbb{N}^*$, the random variables $R^{f_{\text{dens}}}(\mathbf{T}_n^{\text{unif}})$ and \mathbf{T}_n have the same distribution. Notice also that, for every $T \in \mathbb{T}_{\mathbb{K}}$ and every non-negative measurable function g on $\mathbb{R}_+ \times \mathbb{T}_{\text{loc}-\mathbb{K}}$, we have:

$$\int_T \mathcal{L}^T(dy) g(H(y), T) = \int_{R^{f_{\text{dens}}}(T)} \mathcal{L}^{R^{f_{\text{dens}}}(T)}(dx) f_{\text{dens}}(H^{f_{\text{dens}}}(x)) g(H^{f_{\text{dens}}}(x), R^{f_{\text{dens}}}(T)).$$

Let G be a measurable non-negative functional defined on the space of rooted compact binary planar trees with a finite number of leaves, all of them at height t . We first have:

$$\begin{aligned} &\mathbb{E} \left[\int_{\mathbf{T}_n} \mathcal{L}^{\mathbf{T}_n}(dx) f_{\text{dens}}(H(x)) G\left(\mathbf{T}_n \otimes_x^\varepsilon [0, t - H(x)]\right) \right] \\ &= \mathbb{E} \left[\int_{R^{f_{\text{dens}}}(\mathbf{T}_n^{\text{unif}})} \mathcal{L}^{R^{f_{\text{dens}}}(\mathbf{T}_n^{\text{unif}})}(dx) f_{\text{dens}}(H^{f_{\text{dens}}}(x)) G\left(R^{f_{\text{dens}}}(\mathbf{T}_n^{\text{unif}}) \otimes_x^\varepsilon [0, t - H^{f_{\text{dens}}}(x)]\right) \right] \\ &= \mathbb{E} \left[\int_{\mathbf{T}_n^{\text{unif}}} \mathcal{L}^{\mathbf{T}_n^{\text{unif}}}(dy) G \circ R^{f_{\text{dens}}}\left(\mathbf{T}_n^{\text{unif}} \otimes_y^\varepsilon [0, 1 - H(y)]\right) \right]. \end{aligned}$$

Applying Lemma 6.1, and then that $R^{f_{\text{dens}}}(\mathbf{T}_{n+1}^{\text{unif}})$ and \mathbf{T}_{n+1} have the same distribution, we get the result.

6.1.3. *An infinite tree with no leaves.* Let f_{int} be a positive locally integrable function on $[0, +\infty)$. Let S be a Poisson point measure on \mathbb{R}_+ with intensity $f_{\text{int}}(t) dt$. We denote by $(\xi_i, i \geq 1)$ the increasing sequence of the atoms of S and by N the process $(N_t = S([0, t]), t \geq 0)$.

Let $(\varepsilon_n, n \geq 1)$ be independent random variables uniformly distributed on $\{g, d\}$ and let $(K_n, n \geq 1)$ be independent random variables uniformly distributed on $\{1, 2, \dots, n\}$ respectively, all these variables being independent and independent of S .

We define a tree-valued process $(\mathfrak{T}_t, t \geq 0)$ where, for every $t \geq 0$, the random tree \mathfrak{T}_t has height t and $N_t + 1$ leaves, all of them at this height t . Before going into this construction, we first define a function Growth_n on rooted n -pointed trees. We define $\text{Growth}_n((T, \mathbf{v}), h)$ as the tree obtained by grafting on all the pointed vertices of T a branch of length h and pointing the new leaves with the order naturally induced by \mathbf{v} . Set $A_n = \{0, n + 1, \dots, 2n\}$. Formally, we have:

$$\text{Growth}_n((T, \mathbf{v}), h) = \Pi_{2n}^{\circ, A_n} \left(\left((T, \mathbf{v}) \otimes_1 [0, h] \right) \otimes_2 [0, h] \right) \cdots \otimes_n [0, h],$$

where Π_{2n}°, A_n} , defined in (34), removes the first n pointed vertices: for $(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(2n)}$, we have $\Pi_{2n}^{\circ, A_n}(T, \mathbf{v}) = (T, \mathbf{v}')$, with $\mathbf{v} = (v_0 = \varrho, v_1, \dots, v_{2n})$ and $\mathbf{v}' = (v'_0 = \varrho, v'_1 = v_{n+1}, \dots, v'_n = v_{2n})$. Thanks to the continuity of the grafting procedure (see Lemma 5.15) and the continuity of Π_{2n}°, A_n} (see Lemma 5.5), we get that Growth_n is a continuous map from $\mathbb{T}_{\text{loc-K}}^{(n)}$ to itself.

We can now construct the process $(\mathfrak{T}_t, t \geq 0)$ inductively. For $0 \leq t \leq \xi_1$, we set $\mathfrak{T}_t = ([0, t], (0, t))$ and $N_t = 0$.

Let $n \in \mathbb{N}^*$ and assume that $(\mathfrak{T}_{\xi_n}, \mathbf{v}_n)$ is a tree of height ξ_n with n leaves, all of them at height ξ_n and pointed (*i.e.* the vector \mathbf{v}_n is composed of the root of \mathfrak{T}_{ξ_n} and all its leaves). Then, we define the process on $(\xi_n, \xi_{n+1}]$ by setting, for every $t \in (\xi_n, \xi_{n+1}]$,

$$\mathfrak{T}_t = \text{Growth}_n(\mathfrak{T}_{\xi_n}, t - \xi_n) \otimes_{K_n, \xi_n}^{\varepsilon_n} [0, t - \xi_n] \quad \text{and} \quad N_t = n.$$

Standard properties of Poisson processes give the following result.

Lemma 6.6. *For every $n \geq 1$ and every $t > 0$, conditionally given $N_t = n - 1$, the tree \mathfrak{T}_t is distributed as the tree \mathbf{T}_n of Section 6.1.1 associated with the density f_{dens} on $[0, t]$ given by:*

$$(73) \quad f_{\text{dens}}(u) = \frac{f_{\text{int}}(u)}{F(t)} \mathbf{1}_{[0, t]}(u) \quad \text{with} \quad F(t) = \int_0^t f_{\text{dens}}(u) du.$$

Recall the definition of the function Π_n° which removes the pointed vertices. It is easy to see that the process $(\Pi_{N_t+1}^\circ(\mathfrak{T}_t), t \geq 0)$ satisfies the Cauchy property in $\mathbb{T}_{\text{loc-K}}$ as $r_s(\Pi_{N_t+1}^\circ(\mathfrak{T}_t)) = r_s(\Pi_{N_{t'}+1}^\circ(\mathfrak{T}_{t'}))$ for every $s \leq t \leq t'$. Thus this sequence converges a.s. in $\mathbb{T}_{\text{loc-K}}$, and we write:

$$(74) \quad \mathfrak{T}^{\text{ske}} = \lim_{t \rightarrow +\infty} \Pi_{N_t+1}^\circ(\mathfrak{T}_t).$$

The tree $\mathfrak{T}^{\text{ske}}$ is a $\mathbb{T}_{\text{loc-K}}$ -valued random variable which has no leaves and thus belongs to $\mathbb{T}_{\text{loc-K}}^{\text{no leaf}}$, and the process $(\mathfrak{T}_t^{\text{ske}} = r_t(\mathfrak{T}^{\text{ske}}), t \geq 0)$ is distributed as the process $(\Pi_{N_t+1}^\circ(\mathfrak{T}_t), t \geq 0)$. The tree $\mathfrak{T}^{\text{ske}}$ will serve as a backbone for the description of the genealogical tree of the conditioned CSBP.

We present now an ancillary result which is a consequence of Lemma 6.1 on two tree-valued processes that have the same one-dimensional marginal.

We first consider the process $(\mathfrak{T}_t, t \geq 0)$ associated with the intensity $f_{\text{int}} \equiv 1$, that is, $f_{\text{int}}(t) = 1$ for all $t \geq 0$. Then we construct a sequence $\mathbf{t} = (\mathbf{t}_n, n \geq 1)$ of increasing real trees, with $\mathbf{t}_n \in \mathbb{T}_K^{(n)}$ for every $n \geq 1$, all of them of height 1. Let $(\varepsilon_k, k \geq 1)$ be independent random

variables uniformly distributed on $\{g, d\}$. We define the sequence \mathbf{t} by induction by setting first $\mathbf{t}_1 = ([0, 1], (0, 1))$. Let $n \geq 1$ and assume that $(\mathbf{t}_n, \mathbf{v}_n)$ is a tree of $\mathbb{T}_K^{(n)}$ with height 1 and with n leaves all of them at height 1. Conditionally given \mathbf{t}_n , let V_{n+1} be a random element on \mathbf{t}_n uniformly chosen according to the length measure; that is V_{n+1} is distributed according to the measure $c_n \mathcal{L}$, with \mathcal{L} the length measure on \mathbf{t}_n and the normalization $c_n = 1/\mathcal{L}(\mathbf{t}_n)$. Notice that V_{n+1} is a.s. not a leaf nor the root of \mathbf{t}_n . Then we set:

$$\mathbf{t}_{n+1} = \mathbf{t}_n \otimes_{V_{n+1}}^{\varepsilon_{n+1}} [0, 1 - H(V_{n+1})].$$

In particular, for every measurable nonnegative function G , we have:

$$(75) \quad \mathbb{E}[G(\mathbf{t}_{n+1}) | \mathbf{t}_1, \dots, \mathbf{t}_n, \varepsilon_{n+1}] = \int_{\mathbf{t}_n} \frac{\mathcal{L}(dx)}{\mathcal{L}(\mathbf{t}_n)} G(\mathbf{t}_n \otimes_x^{\varepsilon_{n+1}} [0, 1 - H(x)]).$$

Recall the measurable function \tilde{N}_t from Definition (37) which records the number of vertices at level t of a tree without leaves. We have the following result.

Proposition 6.7. *Let $n \geq 1$ and $f_{\text{int}} \equiv 1$. For all measurable non-negative functional G defined on $\mathbb{T}_{\text{dis}}^{(n)}$, we have, with \mathcal{L} the length measure on \mathbf{t}_n :*

$$(76) \quad \mathbb{E} \left[G(\mathfrak{T}_1) \mid N_1 = n - 1 \right] = \frac{2^{n-1}}{n!} \mathbb{E} \left[G(\mathbf{t}_n) \prod_{k=1}^{n-1} \mathcal{L}(\mathbf{t}_k) \right],$$

and for all measurable non-negative functional G defined on \mathbb{T}_K (or on $\mathbb{T}_{\text{loc-K}}$):

$$(77) \quad \mathbb{E} \left[G(\mathfrak{T}_1^{\text{ske}}) \mid \tilde{N}_1(\mathfrak{T}^{\text{ske}}) = n \right] = \mathbb{E} \left[G \circ \Pi_n^\circ(\mathfrak{T}_1) \mid N_1 = n - 1 \right].$$

Proof. By construction, we have that the process $((\mathfrak{T}_t^{\text{ske}}, \tilde{N}_t(\mathfrak{T}^{\text{ske}})), t \geq 0)$ is distributed as the process $((\Pi_{N_t+1}^\circ(\mathfrak{T}_t), N_t + 1), t \geq 0)$. This gives (77).

We now prove (76) by induction. Thanks to Lemma 6.6, conditionally given $N_1 = n - 1$, the tree \mathfrak{T}_1 is distributed as $\mathbf{T}_n^{\text{unif}}$. For $n = 1$, we have $\mathbf{T}_n^{\text{unif}} = \mathbf{t}_1 = ([0, 1], (0, 1))$ hence Equation (76) holds. Let us suppose that (76) holds for some $n \geq 1$. Applying Lemma 6.1, one gets:

$$\mathbb{E}[G(\mathbf{T}_{n+1}^{\text{unif}})] = \frac{2}{n+1} \mathbb{E} \left[\mathcal{L}(\mathbf{T}_n^{\text{unif}}) \int_{\mathbf{T}_n^{\text{unif}}} \frac{\mathcal{L}^{\mathbf{T}_n^{\text{unif}}}(dx)}{\mathcal{L}^{\mathbf{T}_n^{\text{unif}}}(\mathbf{T}_n^{\text{unif}})} G(\mathbf{T}_n^{\text{unif}} \otimes_x^\varepsilon [0, 1 - H(x)]) \right].$$

Now we apply the induction assumption for the right-hand side of the previous equation to get:

$$\begin{aligned} \mathbb{E}[G(\mathbf{T}_{n+1}^{\text{unif}})] &= \frac{2}{n+1} \frac{2^{n-1}}{n!} \left[\mathcal{L}^{\mathbf{t}_n}(\mathbf{t}_n) \int_{\mathbf{t}_n} \frac{\mathcal{L}^{\mathbf{t}_n}(dx)}{\mathcal{L}^{\mathbf{t}_n}(\mathbf{t}_n)} G(\mathbf{t}_n \otimes_x^{\varepsilon_{n+1}} [0, 1 - H(x)]) \prod_{k=1}^{n-1} \mathcal{L}^{\mathbf{t}_n}(\mathbf{t}_k) \right] \\ &= \frac{2^n}{(n+1)!} \mathbb{E} \left[G(\mathbf{t}_{n+1}) \prod_{k=1}^n \mathcal{L}^{\mathbf{t}_{n+1}}(\mathbf{t}_k) \right] \end{aligned}$$

by definition of the tree \mathbf{t}_{n+1} and by (75). This gives that (76) holds with n replaced by $n + 1$. This concludes the proof by induction. \square

6.2. Brownian CRTs and Kesten trees. Brownian CRTs are random trees in $\mathbb{T}_{\text{loc-K}}$ that code for the genealogy of continuous-state branching processes.

Before recalling the definition of such trees, we give some additional notations. For a locally compact rooted tree \mathbf{t} , we define the population at level a as the sub-set:

$$\mathcal{Z}_{\mathbf{t}}(a) = \{u \in \mathbf{t}, H(u) = a\}.$$

We denote by $(\mathbf{t}^{(i),*}, i \in I)$ the connected components of the open set $\mathbf{t} \setminus r_a(\mathbf{t})$. For every $i \in I$, let ϱ_i be the MRCA of $\mathbf{t}^{(i),*}$, which is equivalently characterized by $[[\varrho, \varrho_i]] = \cap_{u \in \mathbf{t}^{(i),*}} [[\varrho, u]]$; notice that $\varrho_i \in \mathcal{Z}_{\mathbf{t}}(a)$. We then set $\mathbf{t}^{(i)} = \mathbf{t}^{(i),*} \cup \{\varrho_i\}$ so that $\mathbf{t}^{(i)}$ is a locally compact rooted tree with root ϱ_i , and we consider the point measure on $\mathcal{Z}_{\mathbf{t}}(a) \times \mathbb{T}_{\text{loc-K}}$:

$$\mathcal{N}_a^{\mathbf{t}} = \sum_{i \in I} \delta_{(\varrho_i, \mathbf{t}^{(i)})}.$$

We then recall the definition of the excursion measure \mathbb{N}^θ for $\beta > 0$ $\theta \geq 0$ associated with a Brownian CRT from [19]. There exists a measure \mathbb{N}^θ on \mathbb{T}_{K} (and hence on $\mathbb{T}_{\text{loc-K}}$) such that:

- (i) **Existence of a local time.** For every $a \geq 0$ and for $\mathbb{N}^\theta[\text{d}\mathcal{T}]$ -a.e. \mathcal{T} , there exists a finite measure Λ_a on \mathcal{T} such that
 - (a) $\Lambda_0 = 0$ and, for every $a > 0$, Λ_a is supported on $\mathcal{Z}_{\mathcal{T}}(a)$.
 - (b) For every $a > 0$, $\mathbb{N}^\theta[\text{d}\mathcal{T}]$ -a.e., we have $\{\Lambda_a \neq 0\} = \{H(\mathcal{T}) > a\}$.
 - (c) For every $a > 0$, $\mathbb{N}^\theta[\text{d}\mathcal{T}]$ -a.e., we have for every continuous function φ on \mathcal{T} :

$$\begin{aligned} \langle \Lambda_a, \varphi \rangle &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{c_\varepsilon^\theta} \int \mathcal{N}_a^{\mathcal{T}}(du, \text{d}\mathcal{T}') \varphi(u) \mathbf{1}_{\{H(\mathcal{T}') \geq \varepsilon\}} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{c_\varepsilon^\theta} \int \mathcal{N}_{a-\varepsilon}^{\mathcal{T}}(du, \text{d}\mathcal{T}') \varphi(u) \mathbf{1}_{\{H(\mathcal{T}') \geq \varepsilon\}}. \end{aligned}$$

- (ii) **Branching property.** For every $a > 0$, the conditional distribution of the point measure $\mathcal{N}_a^{\mathcal{T}}(du, \text{d}\mathcal{T}')$, under the probability measure $\mathbb{N}^\theta[\text{d}\mathcal{T} \mid H(\mathcal{T}) > a]$ and given $r_a(\mathcal{T})$, is that of a Poisson point measure on $\mathcal{Z}_{\mathcal{T}}(a) \times \mathbb{T}_{\text{loc-K}}$ with intensity $\Lambda_a(du) \mathbb{N}^\theta[\text{d}\mathcal{T}']$.
- (iii) **Regularity of the local time process.** We can choose a modification of the process $(\Lambda_a, a \geq 0)$ in such a way that the mapping $a \mapsto \Lambda_a$ is $\mathbb{N}^\theta[\text{d}\mathcal{T}]$ -a.e. continuous for the weak topology of finite measures on \mathcal{T} .
- (iv) **Link with CSBP.** Under $\mathbb{N}^\theta[\text{d}\mathcal{T}]$, the process $(\langle \Lambda_a^{\mathcal{T}}, 1 \rangle, a \geq 0)$ is distributed as a CSBP under its canonical measure with branching mechanism:

$$\psi(\lambda) = \beta \lambda^2 + 2\beta \theta \lambda, \quad \lambda \geq 0.$$

We now extend the definition of the measure \mathbb{N}^θ (only on $\mathbb{T}_{\text{loc-K}}$) for $\theta < 0$ by a Girsanov transformation, following [7]. For $t \geq 0$, set $\mathcal{G}_t = \sigma(r_t(\mathcal{T}))$ and $Z_t = \Lambda_t(\mathcal{T})$, the latter notation is consistent with Section 2.2. The CSBP process $Z = (Z_t, t \geq 0)$ is Markov with respect to the filtration $(\mathcal{G}_t, t \geq 0)$. For $\theta < 0$ and $t > 0$, we set:

$$(78) \quad \mathbb{N}^{-\theta}[\text{d}\mathcal{T}]|_{\mathcal{G}_t} = e^{2\theta Z_t} \mathbb{N}^\theta[\text{d}\mathcal{T}]|_{\mathcal{G}_t}.$$

Then properties (i) to (iv) still hold for every $\theta \in \mathbb{R}$. This Girsanov transformation is consistent with the Girsanov transformation of CSBP given by (14). Let us stress that the measure \mathbb{N}^θ on $\mathbb{T}_{\text{loc-K}}$ depends also on the parameter $\beta > 0$.

The so-called Kesten tree with parameters $(\beta, \theta) \in \mathbb{R}_+^* \times \mathbb{R}_+$ can be defined as the genealogical tree associated with the continuous-state branching process with the same parameters, conditioned on non-extinction (see for instance [28]). This latter process can also be defined by

adding to the initial process a particular immigration. We use this second approach to extend the definition of the Kesten tree for $\theta < 0$.

Using our framework, the Kesten tree with parameters $(\beta, \theta) \in \mathbb{R}_+^* \times \mathbb{R}$ is the random tree $\mathcal{T}^* = \text{Tree}(\mathcal{M})$ (see Section 5.10 for the definition of the map Tree), with $\mathcal{M}(dh, dT)$ a Poisson point measure on $E = \mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^*$ with intensity $2\beta \mathbf{1}_{\{h>0\}} dh \mathbb{N}^\theta[dT]$. The Kesten tree belongs a.s. to the space $\mathbb{T}_{\text{loc-K}}^{\text{spine},0}$.

6.3. Backbone decomposition. The decomposition of a (sub)critical Brownian CRT \mathcal{T} according to a spine $[[\emptyset, x]]$, where $x \in \mathcal{T}$ is a leaf picked at random at level $t > 0$, that is according to the local time $\Lambda_t(dx)$, is given in Theorem 4.5 in [19]. In our setting, it can be rephrased in the next theorem. Recall that, for $t > 0$, the (discrete) 1-pointed tree $[0, t] \in \mathbb{T}_{\text{loc-K}}^{(1)}$ denotes the segment $[0, t]$ endowed with the Euclidean distance, with the root 0 and the distinguished vertex t . Recall that $\mathbb{T}_{\text{loc-K}}^{\text{spine},0}$ defined in Section 5.6 is the set of locally compact rooted trees with one infinite marked branch such that the root is not a branching vertex. From Remark 5.29, we recall that informally the 1-pointed tree $(T, (\varrho, v_1)) = \text{Graft}_1([0, t], T^*)$, with $T^* \in \mathbb{T}_{\text{loc-K}}^{\text{spine},0}$ is the tree where the marked branch has been cut at height t (the part above t being removed), and the cut marked branch is identified with $[0, t]$, so that the root ϱ of T is identified with 0 and the distinguished vertex v_1 of T is also identified with t .

Theorem 6.8 ([19]). *Let $\beta > 0$, $\theta \geq 0$ and $t > 0$. Let \mathcal{T}^* be under \mathbb{E} a Kesten tree with parameter (β, θ) . For every non-negative measurable functional F on $\mathbb{T}_{\text{loc-K}}^{(1)}$ (or $\mathbb{T}_{\text{K}}^{(1)}$), we have, with ϱ the root of \mathcal{T} :*

$$(79) \quad \mathbb{N}^\theta \left[\int_{\mathcal{T}} \Lambda_t(dv) F(\mathcal{T}, (\varrho, v)) \right] = e^{-2\beta\theta t} \mathbb{E} [F(\text{Graft}_1([0, t], \mathcal{T}^*))].$$

We extend this result to the super-critical case $\theta < 0$.

Corollary 6.9. *Let $\beta > 0$, $\theta \in \mathbb{R}$ and $t > 0$. Let \mathcal{T}^* be under \mathbb{E} a Kesten tree with parameter (β, θ) . For every non-negative measurable functional F on $\mathbb{T}_{\text{loc-K}}^{(1)}$, Equation (79) holds.*

Proof. We first to prove (79) for functionals F of the form:

$$(80) \quad F(T, \mathbf{v}) = e^{-\langle \Phi, \mathcal{M}(T, \mathbf{v}) \rangle},$$

where $(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(1)}$ and Φ is a continuous non-negative function with bounded support defined on $\mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^*$ (with $\mathbb{T}_{\text{loc-K}}^* = \mathbb{T}_{\text{loc-K}} \setminus \{T_0\}$ where $T_0 \in \mathbb{T}_{\text{loc-K}}$ is the tree reduced to its root, see Section 5.10). And the measure $\mathcal{M}(T, \mathbf{v})$ on $\mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^*$ is defined at the end of Section 5.10.

For simplicity, we simply write (\mathcal{T}, v) for the 1-pointed tree $(\mathcal{T}, (\varrho, v))$. Let $\theta > 0$. Using (78), we have for every $s > t$ that:

$$\mathbb{N}^{-\theta} \left[\int_{\mathcal{T}} \Lambda_t(dv) e^{\langle \Phi, \mathcal{M}_{r_s}(\mathcal{T}, v) \rangle} \right] = \mathbb{N}^\theta \left[\int_{\mathcal{T}} \Lambda_t(dv) e^{2\theta Z_s - \langle \Phi, \mathcal{M}(r_s(\mathcal{T}, v)) \rangle} \right].$$

We apply then (79) to get:

$$\begin{aligned}
\mathbb{N}^{-\theta} \left[\int_{\mathcal{T}} \Lambda_t(dv) e^{-\langle \Phi, \mathcal{M}(r_s(\mathcal{T}, v)) \rangle} \right] &= e^{-2\beta\theta t} \mathbb{E} \left[e^{2\theta Z_s} F(r_s(\text{Graft}_1([0, t], \mathcal{T}^*)) \right] \\
&= \exp \left\{ -2\beta\theta t - 2\beta \int_0^t da \mathbb{N}^\theta \left[1 - e^{-\Phi(a, r_{s-a}(\mathcal{T})) + 2\theta Z_{s-a}} \right] \right\} \\
&= \exp \left\{ -2\beta\theta t - 2\beta \int_0^t da \left(\mathbb{N}^{-\theta} \left[1 - e^{-\Phi(a, r_{s-a}(\mathcal{T}))} \right] + \mathbb{N}^\theta \left[1 - e^{2\theta Z_{s-a}} \right] \right) \right\} \\
&= \exp \left\{ 2\beta\theta t - 2\beta \int_0^t da \mathbb{N}^{-\theta} \left[1 - e^{-\Phi(a, r_{s-a}(\mathcal{T}))} \right] \right\},
\end{aligned}$$

where we used the definition of the Kesten tree for the second equality, (78) again for the third one, and that $\mathbb{N}^\theta [1 - e^{2\theta Z_a}] = u(-2\theta, a) = -2\theta$, see (4) and (7), for the last one. As Φ has bounded support, we get taking s large enough:

$$\mathbb{N}^{-\theta} \left[\int_{\mathcal{T}} \Lambda_t(dv) e^{-\langle \Phi, \mathcal{M}(\mathcal{T}, v) \rangle} \right] = \exp \left\{ 2\beta\theta t - 2\beta \int_0^t da \mathbb{N}^{-\theta} \left[1 - e^{-\Phi(a, \mathcal{T})} \right] \right\}.$$

Then the result follows from the definition of the Kesten tree, that is (79) holds for F given by (80).

Recall $\mathbb{T}_{\text{loc-K}}^{(n),0,*}$ defined in (63) is the Borel subset of $\mathbb{T}_{\text{loc-K}}^{(1)}$ of the trees such that the root is not a branching vertex and the pointed vertex is distinct from the root. The map:

$$(T, \mathbf{v}) \mapsto (d(\varrho, v), \mathcal{M}(T, \mathbf{v})),$$

with $\mathbf{v} = (\varrho, v)$, defined on $\mathbb{T}_{\text{loc-K}}^{(n),0,*}$ is injective and bi-measurable, see Corollary 5.33. Furthermore the set $\mathbb{T}_{\text{loc-K}}^{(n),0,*}$ is of full measure with respect to the distribution of $(\mathcal{T}, \mathbf{v})$ under $\mathbb{N}^\theta[d\mathcal{T}] \Lambda_t(dv)$, with $\mathbf{v} = (\varrho, v)$, as \mathbb{N}^θ -a.e. the root of \mathcal{T} is not a branching vertex and $d(\varrho, v) = t > 0$. Thus, as $t > 0$ is fixed, we get that (\mathcal{T}, v) is a measurable function of $\mathcal{M}(\mathcal{T}, v)$. We then conclude by the monotone class theorem that Equation (79) holds for any non-negative measurable function F defined on $\mathbb{T}_{\text{loc-K}}^{(n)}$. \square

Let $\beta > 0$, $\theta \in \mathbb{R}$ and $t > 0$. Recall $\tilde{c}_t^\theta = (2\theta)/(1 - e^{-2\beta\theta t})$ defined in (3). We consider the probability measure on $[0, t]$:

$$(81) \quad \nu(ds) = \frac{2\beta\theta e^{2\beta\theta s}}{e^{2\beta\theta t} - 1} \mathbf{1}_{[0,t]}(s) ds = \beta \tilde{c}_t^\theta e^{-2\beta\theta(t-s)} \mathbf{1}_{[0,t]}(s) ds.$$

Let $(\mathbf{T}_n, \mathbf{v}_n)$ be, under $\mathbb{P}^{\theta,t}$, the tree of $\mathbb{T}_{\text{dis}}^{(n)}$ defined in Section 6.1.1 associated with the measure ν and $t > 0$ (recall that all the distinguished vertices from \mathbf{v}_n but the root are at distance t from the root). The following theorem is a generalization of Theorem 6.8 when picking n leaves uniformly at random at level t .

Theorem 6.10. *Let $\beta > 0$, $\theta \in \mathbb{R}$, $t > 0$ and $n \in \mathbb{N}^*$. For every non-negative measurable function F defined on $\mathbb{T}_{\text{loc-K}}^{(n)}$, we have:*

$$(82) \quad \mathbb{N}^\theta \left[\int_{\mathcal{T}_n} \Lambda_t^{\otimes n}(d\mathbf{v}^*) F(\mathcal{T}, \mathbf{v}) \right] = n! \left(\tilde{c}_t^\theta \right)^{1-n} e^{-2\beta\theta t} \mathbb{E}^{\theta,t} \left[F \left(\text{Graft}_n((\mathbf{T}_n, \mathbf{v}_n), \mathcal{T}^*) \right) \right],$$

where $\mathbf{v} = (\varrho, \mathbf{v}^*) \in \mathcal{T}^{n+1}$, with ϱ the root of \mathcal{T} , and $\mathcal{T}^* = (\mathcal{T}_A^*, A \in \mathcal{P}_n^+)$ is under $\mathbb{P}^{\theta, t}$ a family of independent Kesten trees with parameter (β, θ) , independent of \mathbf{T}_n .

Proof. We prove Formula (82) by induction. For $n = 1$, as $\mathbf{T}_1 = [0, t]$ (with root $\varrho = 0$ and distinguished vertex $v_1 = t$), this is Corollary 6.9.

Let $k \in \mathbb{N}^*$. Recall the maps \mathbf{L}_k , from (52) in Section 5.7, and Split_k from (55) in Section 5.8. For $(T, \mathbf{v}) \in \mathbb{T}_{\text{loc-K}}^{(k)}$ and $A \in \mathcal{P}_k^+$, we write $\mathcal{M}_A[T, \mathbf{v}](dh, dt)$ for the measure $\mathcal{M}(\hat{T}_A(T, \mathbf{v}))$ on $E = \mathbb{R}_+ \times \mathbb{T}_{\text{loc-K}}^*$, where $(\hat{T}_A(T, \mathbf{v}), A \in \mathcal{P}_k) = \text{Split}_k(T, \mathbf{v})$ and the measure $\mathcal{M}(T, \mathbf{v})$ is defined at the end of Section 5.10. We also recall the notation $(\ell_A(T, \mathbf{v}), A \in \mathcal{P}_k^+) = \mathbf{L}_k(T, \mathbf{v})$, and notice that $\ell_A(T, \mathbf{v}) = 0$ implies that $\mathcal{M}_A[T, \mathbf{v}] = 0$. Let $n \in \mathbb{N}^*$ and $(\Phi_A, A \in \mathcal{P}_n^+)$ be a family of non-negative measurable functions defined on E . Let f be a bounded non-negative measurable function defined on $\mathbb{T}_{\text{loc-K}}^{(n)}$ (or more simply on $\mathbb{T}_{\text{dis}}^{(n)}$). We shall first prove (82) for a non-negative function F defined on $\mathbb{T}_{\text{loc-K}}^{(n)}$ of the form:

$$F(T, \mathbf{v}) = f(\text{Span}(T, \mathbf{v})) \exp\left\{-\sum_{A \in \mathcal{P}_n^+} \langle \Phi_A, \mathcal{M}_A[T, \mathbf{v}] \rangle\right\}.$$

Let $n \geq 2$ and suppose that (82) holds for $n - 1$. For $k \in \{1, \dots, n\}$, we denote by $\mathcal{T}^{[k]} = \text{Span}(\mathcal{T}, \mathbf{v}_k) \in \mathbb{T}_{\text{loc-K}}^{(k)}$, where $\mathbf{v}_k = (v_0 = \varrho, \mathbf{v}_k^*)$ and $\mathbf{v}_k^* = (v_1, \dots, v_k)$; and we simply write $\mathcal{M}_A^{[k]}$ for $\mathcal{M}_A[\mathcal{T}, \mathbf{v}_k]$ and $\ell_A^{[k]}$ for $\ell_A(\mathcal{T}, \mathbf{v}_k)$, so that under $\mathbb{N}^\theta[d\mathcal{T}] \ell_t^{\otimes n}(d\mathbf{v}^*)$:

$$F(\mathcal{T}, \mathbf{v}_n) = f(\mathcal{T}^{[n]}) \exp\left\{-\sum_{A \in \mathcal{P}_n^+} \langle \Phi_A, \mathcal{M}_A^{[n]} \rangle\right\}.$$

We also write $v_A^{[k]}$ and $w_A^{[k]}$ for v_A and w_A from (50) and (51) with (T, \mathbf{v}) replaced by $(\mathcal{T}^{[k]}, \mathbf{v}_k)$; and thus we have $\ell_A^{[k]} = d(w_A^{[k]}, v_A^{[k]})$.

Similarly, under $\mathbb{E}^{\theta, t}$, for $k \geq 2$, we write also $\hat{\mathcal{M}}_A^{[k]}$ for the measure $\mathcal{M}(T_A^*)$ restricted to $[0, \ell_A(\mathbf{T}_k)] \times \mathbb{T}_{\text{loc-K}}^*$, $\hat{v}_A^{[k]}$ and $\hat{w}_A^{[k]}$ for v_A and w_A from (50) and (51) with (T, \mathbf{v}) replaced by $(\mathbf{T}_k, \mathbf{v}_k)$, and $\hat{\ell}_A^{[k]} = d(\hat{w}_A^{[k]}, \hat{v}_A^{[k]}) = \ell_A(\mathbf{T}_k)$. For $n \geq 2$, simply writing \mathbf{T}_n for $(\mathbf{T}_n, \mathbf{v}_n)$, we have:

$$F(\text{Graft}_n(\mathbf{T}_n, \mathcal{T}^*)) = f(\mathbf{T}_n) \exp\left\{-\sum_{A \in \mathcal{P}_n^+} \langle \Phi_A, \hat{\mathcal{M}}_A^{[n]} \rangle\right\}.$$

Using the definition of Kesten tree via Poisson point measures and the definition of the function Graft_n , we obtain in particular that:

$$(83) \quad \mathbb{E}^{\theta, t} \left[F(\text{Graft}_n(\mathbf{T}_n, \mathcal{T}^*)) \right] = \mathbb{E}^{\theta, t} \left[F'(\mathbf{T}_n) \right],$$

where

$$(84) \quad F'(\mathbf{T}_n) = f(\mathbf{T}_n) \exp\left\{-2\beta \sum_{A \in \mathcal{P}_n^+} \int_0^{\hat{\ell}_A^{[n]}} da \mathbb{N}^\theta \left[1 - e^{-\Phi_A(a, \mathcal{T})} \right]\right\}.$$

Recall (27). Set $p_n = p_{\mathbf{v}_{n-1}}(v_n)$ for the projection of v_n on $\mathcal{T}^{[n-1]}$. Since \mathbb{N}^θ -a.e. $p_n \neq \varrho$, we deduce that there exists \mathbb{N}^θ -a.e. a unique (random) $B \in \mathcal{P}_{n-1}^+$ such that $p_n \in \llbracket w_B^{[n-1]}, v_B^{[n-1]} \rrbracket \subset \mathcal{T}^{[n-1]}$, and write $h_n = d(p_n, w_B^{[n-1]})$. Recall the function Tree , defined in Section 5.10 just before Lemma 5.31, from $\mathbb{M}(E)$ into $\mathbb{T}_{\text{loc-K}}^{[2]}$ and the projection $\tilde{\Pi}$ from $\mathbb{T}_{\text{loc-K}}^{[2]}$ to $\mathbb{T}_{\text{loc-K}}$, defined just

before Lemma 5.23, which forgets about the marked sub-tree defined in Section 5.6. We simply write $\text{Tree}' = \tilde{\Pi} \circ \text{Tree}$. On the one hand, we have:

$$(85) \quad \begin{aligned} \mathcal{T}^{[n]} &= \mathcal{T}^{[n-1]} \otimes_{\min B, H(p_n)} [0, t - H(p_n)], \\ \ell_B^{[n-1]} &= \ell_B^{[n]} + \ell_{B \cup \{n\}}^{[n]}, \\ \mathcal{M}_B^{[n-1]} &= \mathcal{M}_{B \cup \{n\}}^{[n]} + \mathcal{M}_B^{[n]}(\cdot + h_n, \cdot) + \delta_{(h_n, \text{Tree}'(\mathcal{M}_{\{n\}}^{[n]}))}; \end{aligned}$$

and, to fix notation, we shall write:

$$\mathcal{M}_B^{[n-1]} = \mathcal{M}_B[\mathcal{T}, \mathbf{v}_{n-1}] = \sum_{i \in I_{n-1}^B} \delta_{h_i^{[n-1], B}, \mathcal{T}_i^{[n-1], B}}.$$

On the other hand, for $A \in \mathcal{P}_{n-1}^+$ and $A \neq B$, we have:

$$(86) \quad B \subset A \implies \mathcal{M}_A^{[n-1]} = \mathcal{M}_{A \cup \{n\}}^{[n]}, \quad \mathcal{M}_A^{[n]} = 0, \quad \ell_A^{[n-1]} = \ell_{A \cup \{n\}}^{[n]} \quad \text{and} \quad \ell_A^{[n]} = 0,$$

$$(87) \quad A \cap B \in \{\emptyset, A\} \implies \mathcal{M}_A^{[n-1]} = \mathcal{M}_A^{[n]}, \quad \mathcal{M}_{A \cup \{n\}}^{[n]} = 0, \quad \ell_A^{[n-1]} = \ell_A^{[n]} \quad \text{and} \quad \ell_{A \cup \{n\}}^{[n]} = 0,$$

$$(88) \quad A \cap B \notin \{\emptyset, B, A\} \implies \mathcal{M}_A^{[n-1]} = \mathcal{M}_A^{[n]} = \mathcal{M}_{A \cup \{n\}}^{[n]} = 0 \quad \text{and} \quad \ell_A^{[n-1]} = \ell_A^{[n]} = \ell_{A \cup \{n\}}^{[n]} = 0.$$

It is also easy to rebuild $(\mathcal{M}_A^{[n]}, A \in \mathcal{P}_n^+)$ from $(\mathcal{M}_A^{[n-1]}, A \in \mathcal{P}_{n-1}^+)$ and v_n .

Set

$$F_n = \mathbb{N}^\theta \left[\int_{\mathcal{T}^n} \Lambda_t^{\otimes n} (d\mathbf{v}_n^*) F(\mathcal{T}, \mathbf{v}_n) \right].$$

Considering that $\mathcal{T}_i^{[n-1], B}$ is a subset of \mathcal{T} , we have:

$$\begin{aligned} F_n &= \mathbb{N}^\theta \left[\int_{\mathcal{T}^{n-1}} \Lambda_t^{\otimes (n-1)} (d\mathbf{v}_{n-1}^*) \sum_{B \in \mathcal{P}_{n-1}^+} \sum_{i \in I_{n-1}^B} \int_{\mathcal{T}_i^{[n-1], B}} \ell_t(dv_n) F(\mathcal{T}, \mathbf{v}_n) \right] \\ &= \mathbb{N}^\theta \left[\int_{\mathcal{T}^{n-1}} \Lambda_t^{\otimes (n-1)} (d\mathbf{v}_{n-1}^*) \right. \\ &\quad \left. \sum_{B \in \mathcal{P}_{n-1}^+} \sum_{i \in I_{n-1}^B} \Gamma_B \left(\mathcal{T}^{[n-1]}, H(w_B^{[n-1]}), \mathcal{M}_{B,i}^{[n-1]}, H(w_B^{[n-1]}) + h_i^{[n-1], B}, \mathcal{T}_i^{[n-1], B} \right) \right. \\ &\quad \left. \times \exp \left\{ - \sum_{A \in \mathcal{P}_{n-1}^+ \setminus \{B\}} \left\langle \mathbf{1}_{\{B \subset A\}} \Phi_{A \cup \{n\}} + \mathbf{1}_{\{A \cap B = \emptyset \text{ or } A\}} \Phi_A, \mathcal{M}_A^{[n-1]} \right\rangle \right\} \right], \end{aligned}$$

where the measure $\mathcal{M}_{B,i}^{[n-1]}$ is the measure $\mathcal{M}_B^{[n-1]}$ but for the atom at $(h_i^{[n-1], B}, \mathcal{T}_i^{[n-1], B})$:

$$\mathcal{M}_{B,i}^{[n-1]} = \mathcal{M}_B^{[n-1]} - \delta_{(h_i^{[n-1], B}, \mathcal{T}_i^{[n-1], B})},$$

and, for $(T, \mathbf{w}) \in \mathbb{T}_{\text{loc-K}}^{(n-1)}$, $(T', \varrho') \in \mathbb{T}_{\text{loc-K}}$, $\nu \in \mathbb{M}(E)$ and $h' \geq h \geq 0$:

$$\begin{aligned} \Gamma_B((T, \mathbf{w}), h, \nu, h', T') &= f(T \otimes_{\min B, h'} [0, t - h']) \exp \left\{ - \langle \Phi_{B, h' - h}, \nu \rangle \right\} \\ &\quad \times \int_{T'} \Lambda_{t-h'}(dv) \exp \left\{ - \langle \Phi_{\{n\}}, \mathcal{M}(T', (\varrho', v)) \rangle \right\}, \end{aligned}$$

with:

$$(89) \quad \Phi_{B,h''}(s, \mathbf{t}) = \mathbf{1}_{\{s \leq h''\}} \Phi_{B \cup \{n\}}(s, \mathbf{t}) + \mathbf{1}_{\{s > h''\}} \Phi_B(s - h'', \mathbf{t}).$$

For $B \in \mathcal{P}_{n-1}^+$, using the notation $\hat{\mathcal{M}}_B^{[n]} = \sum_{i \in \hat{I}_{n-1}^B} \delta_{(\hat{h}_i^{[n-1],B}, \hat{\mathcal{T}}_i^{[n-1],B})}$, we set for $i \in \hat{I}_{n-1}^B$:

$$\hat{\mathcal{M}}_{B,i}^{[n-1]} = \hat{\mathcal{M}}_B^{[n-1]} - \delta_{(\hat{h}_i^{[n-1],B}, \hat{\mathcal{T}}_i^{[n-1],B})}.$$

We deduce from the induction assumption (*i.e.* Equation (82) with $n-1$ instead of n) and the definition of Kesten tree, with $F_n = (n-1)! (\tilde{c}_t^\theta)^{2-n} e^{-2\beta\theta t} G_n$ that:

$$G_n = \mathbb{E}^\theta \left[\sum_{B \in \mathcal{P}_{n-1}^+} \sum_{i \in \hat{I}_{n-1}^B} \Gamma_B \left(\mathbf{T}_{n-1}, H(\hat{w}_{[n-1],B}), \hat{\mathcal{M}}_{B,i}^{[n-1]}, H(\hat{w}_{[n-1],B}) + \hat{h}_i^{[n-1],B}, \hat{\mathcal{T}}_i^{[n-1],B} \right) \right. \\ \left. \times \exp \left\{ - \sum_{A \in \mathcal{P}_{n-1}^+ \setminus \{B\}} \left\langle \mathbf{1}_{\{B \subset A\}} \Phi_{A \cup \{n\}} + \mathbf{1}_{\{A \cap B = \emptyset \text{ or } A\}} \Phi_A, \hat{\mathcal{M}}_A^{[n-1]} \right\rangle \right\} \right].$$

Since for $A \in \mathcal{P}_{n-1}^+$, the random measure $\mathbf{M}(T_A^*, \hat{\ell}_A^{[n-1]})(dh', d\mathcal{T}')$ is conditionally given $\hat{\ell}_A^{[n-1]}$ a Poisson point measure on $[0, \hat{\ell}_A^{[n-1]}] \times \mathbb{T}_{\text{loc-K}}$ with intensity $2\beta dh' \mathbb{N}^\theta[d\mathcal{T}']$, we deduce from Palm formula that:

$$G_n = \mathbb{E}^\theta \left[\sum_{B \in \mathcal{P}_{n-1}^+} 2\beta \int_0^{\hat{\ell}_B^{[n-1]}} dr \int \mathbb{N}^\theta[d\mathcal{T}] \Gamma_B \left(\mathbf{T}_{n-1}, H(\hat{w}_B^{[n-1]}), \hat{\mathcal{M}}_B^{[n-1]}, H(\hat{w}_B^{[n-1]}) + r, \mathcal{T} \right) \right. \\ \left. \times \exp \left\{ - \sum_{A \in \mathcal{P}_{n-1}^+ \setminus \{B_x\}} \left\langle \mathbf{1}_{\{B_x \subset A\}} \Phi_{A \cup \{n\}} + \mathbf{1}_{\{A \cap B_x = \emptyset \text{ or } A\}} \Phi_A, \hat{\mathcal{M}}_A^{[n-1]} \right\rangle \right\} \right] \\ = \mathbb{E}^\theta \left[2\beta \int_{\mathbf{T}_{n-1,t}} \mathcal{L}(dx) \int \mathbb{N}^\theta[d\mathcal{T}] \Gamma_{B_x} \left(\mathbf{T}_{n-1}, H(\hat{w}_{B_x}^{[n-1]}), \hat{\mathcal{M}}_{B_x}^{[n-1]}, H(x), \mathcal{T} \right) \right. \\ \left. \times \exp \left\{ - \sum_{A \in \mathcal{P}_{n-1}^+ \setminus \{B_x\}} \left\langle \mathbf{1}_{\{B_x \subset A\}} \Phi_{A \cup \{n\}} + \mathbf{1}_{\{A \cap B_x = \emptyset \text{ or } A\}} \Phi_A, \hat{\mathcal{M}}_A^{[n-1]} \right\rangle \right\} \right],$$

where B_x is the only element B of \mathcal{P}_{n-1}^+ such that x belongs to the branch B of \mathbf{T}_{n-1} : $x \in \llbracket \hat{w}_B^{[n-1]}, \hat{v}_B^{[n-1]} \rrbracket$, where, as \mathbf{T}_{n-1} is discrete, we recall that $\text{Split}_{n-1}(\mathbf{T}_{n-1}) = (\llbracket \hat{w}_A^{[n-1]}, \hat{v}_A^{[n-1]} \rrbracket, A \in \mathcal{P}_{n-1})$ with $\mathcal{P}_{n-1} = \mathcal{P}_{n-1}^+ \cup \{\{0\}\}$. Using (82) again for $n=1$ (or Corollary 6.9) gives:

$$\int \mathbb{N}^\theta[d\mathcal{T}] \Gamma_B(\mathbf{T}_{n-1,t}, h, \nu, h', \mathcal{T}) = f(\mathbf{T}_{n-1} \otimes_{\min B, h'} [0, t - h']) e^{-\langle \Phi_{h'-h}, \nu \rangle} \\ \times \exp \left\{ -2\beta\theta(t - h') - 2\beta \int_0^{t-h'} da \mathbb{N}^\theta \left[1 - e^{-\Phi_{\{n\}}(a, \mathcal{T})} \right] \right\}.$$

With x chosen according to the length measure $\mathcal{L}(dx)$ on \mathbf{T}_{n-1} , the tree $\mathbf{T}_{n-1} \otimes_{\min B_x, H(x)} [0, t - H(x)]$ is obtained by grafting a branch of length $t - H(x)$ at x on \mathbf{T}_{n-1} and thus will simply be denoted as $\mathbf{T}_{n-1} \otimes_x [0, t - H(x)]$ (see also Remark 6.2 for similar notation). Therefore, we

obtain:

$$\begin{aligned}
G_n = \mathbb{E}^\theta & \left[2\beta \int_{\mathbf{T}_{n-1}} \mathcal{L}(dx) f\left(\mathbf{T}_{n-1} \otimes_x [0, t - H(x)]\right) \exp\left\{-2\beta(t - H(x))\right\} \right. \\
& \times \exp\left\{-2\beta \sum_{A \in \mathcal{P}_{n-1}^+ \setminus \{B_x\}} \mathbf{1}_{\{B_x \subset A\}} \int_0^{\hat{\ell}_A^{[n-1]}} da \mathbb{N}^\theta \left[1 - e^{-\Phi_{A \cup \{n\}}(a, \mathcal{T})}\right]\right\} \\
& \times \exp\left\{-2\beta \sum_{A \in \mathcal{P}_{n-1}^+ \setminus \{B_x\}} \mathbf{1}_{\{A \cap B_x = \emptyset \text{ or } A\}} \int_0^{\hat{\ell}_A^{[n-1]}} da \mathbb{N}^\theta \left[1 - e^{-\Phi_A(a, \mathcal{T})}\right]\right\} \\
& \times \exp\left\{-2\beta \int_0^{H(x) - H(w_{B_x}^{[n-1]})} da \mathbb{N}^\theta \left[1 - e^{-\Phi_{B_x \cup \{n\}}(a, \mathcal{T})}\right]\right\} \\
& \times \exp\left\{-2\beta \int_0^{H(v_{B_x}^{[n-1]}) - H(x)} da \mathbb{N}^\theta \left[1 - e^{-\Phi_{B_x}(a, \mathcal{T})}\right]\right\} \\
& \left. \times \exp\left\{-2\beta \int_0^{t - H(x)} da \mathbb{N}^\theta \left[1 - e^{-\Phi_{\{n\}}(a, \mathcal{T})}\right]\right\} \right].
\end{aligned}$$

We deduce from Lemma 6.1 with the density:

$$f_{\text{dens}}(s) = \frac{2\beta\theta e^{2\beta\theta s}}{e^{2\beta\theta t} - 1} \mathbf{1}_{[0, t]}(s) = \tilde{c}_t^\theta \beta e^{-2\beta\theta(t-s)} \mathbf{1}_{[0, t]}(s)$$

that for a non-negative measurable function F'' defined on $\mathbb{T}_{\text{loc-K}}^{(n)}$ (or $\mathbb{T}_{\text{dis}}^{(n)}$):

$$\mathbb{E}^{\theta, t} \left[2\beta \int_{\mathbf{T}_{n-1}} \mathcal{L}(dx) F''\left(\mathbf{T}_{n-1} \otimes_x [0, t - H(x)]\right) e^{-2\beta\theta(t-H(x))} \right] = (\tilde{c}_t^\theta)^{-1} n \mathbb{E}^{\theta, t} [F''(\mathbf{T}_n)].$$

Using similar equations as (85), (86), (87) and (88) stated with \mathbf{T}_n instead of $(\mathcal{T}, \mathbf{v}_n)$ as well as an obvious choice of F'' , we obtain that:

$$G_n = (\tilde{c}_t^\theta)^{-1} n \mathbb{E}^{\theta, t} [F'(\mathbf{T}_n)],$$

where $F'(\mathbf{T}_n)$ is given by (84). Then, we deduce from (83) that:

$$G_n = (\tilde{c}_t^\theta)^{-1} n \mathbb{E}^{\theta, t} [F(\text{Graft}_n(\mathbf{T}_n, \mathcal{T}^*))].$$

This gives:

$$\begin{aligned}
\mathbb{N}^\theta \left[\int_{\mathcal{T}^n} \Lambda_t^{\otimes n}(\mathbf{dv}_n^*) F(\mathcal{T}, \mathbf{v}) \right] &= F_n = (n-1)! (\tilde{c}_t^\theta)^{2-n} e^{-2\beta\theta t} G_n \\
&= n! \left(\tilde{c}_t^\theta \right)^{1-n} e^{-2\beta\theta t} \mathbb{E}^{\theta, t} [F(\text{Graft}_n(\mathbf{T}_n, \mathcal{T}^*))].
\end{aligned}$$

Thus, Equation (82) holds for the functionals F we considered.

Recall that $\mathbb{T}_{\text{loc-K}}^{(n), 0, *}$ is the Borel subset of $\mathbb{T}_{\text{loc-K}}^{(n)}$ of the trees such that the root is not a branching vertex and the point vertices (but the root) are distinct from the root. The map:

$$(T, \mathbf{v}) \mapsto \left(\text{Span}(T, \mathbf{v}), (\mathcal{M}_A[T, \mathbf{v}], A \in \mathcal{P}_n^+) \right)$$

defined on $\mathbb{T}_{\text{loc-K}}^{(n),0,*}$ is one-to-one onto its image and bi-measurable, see Corollary 5.34. Furthermore the set $\mathbb{T}_{\text{loc-K}}^{(n),0,*}$ is of full measure with respect to the distribution of $(\mathcal{T}, \mathbf{v})$ under $\mathbb{N}^\theta[d\mathcal{T}] \Lambda_t^{\otimes n}(d\mathbf{v}^*)$, with $\mathbf{v} = (\varrho, \mathbf{v}^*)$, as \mathbb{N}^θ -a.e. the root of \mathcal{T} is not a branching vertex. Thus, $(\mathcal{T}, \mathbf{v})$ is a measurable function of $\left(\mathcal{T}^{[n]}, (\hat{\mathcal{M}}_A^{[n]}, A \in \mathcal{P}_n^+)\right)$. We then conclude by the monotone class theorem that Equation (82) holds for any non-negative measurable function F defined on $\mathbb{T}_{\text{loc-K}}^{(n)}$. \square

6.4. Main result. Let $\beta > 0$, $\theta, \alpha \in \mathbb{R}_+$ and let $S^{\alpha,\theta}$ be a Poisson point measure on $[0, \infty)$ with intensity measure $f_{\text{int}}(t) dt$, where:

$$(90) \quad f_{\text{int}}(t) = \alpha\beta e^{2\beta\theta t}, \quad t \geq 0.$$

We first consider the case $\alpha > 0$. Denote by $(\xi_i, i \in \mathbb{N}^*)$ the increasing sequence of jumping times of the inhomogeneous Poisson process $(N_t^{\alpha,\theta} = S^{\alpha,\theta}([0, t]), t \geq 0)$. We consider the $\mathbb{T}_{\text{dis}}^{(n)}$ -valued random variable \mathfrak{T}_{ξ_n} of Section 6.1.3 for $n \geq 1$ associated to f_{int} . In particular, recall that, for every $n \geq 1$, \mathfrak{T}_{ξ_n} is a discrete tree with n pointed leaves, where all of them are at height ξ_n .

For every $n \geq 1$, let $\mathcal{T}^{n,*} = (\mathcal{T}_A, A \in \mathcal{P}_n^+)$ be a family of independent Kesten trees with parameter (β, α) , independent of the tree \mathfrak{T}_{ξ_n} . We defined the random marked tree:

$$\mathcal{T}^{(n)} = \left(\Pi_n^\circ(\tilde{\mathcal{T}}^{(n)}), \text{Span}^\circ(\tilde{\mathcal{T}}^{(n)})\right) \quad \text{with} \quad \tilde{\mathcal{T}}^{(n)} = \text{Graft}_n(\mathfrak{T}_{\xi_n}, \mathcal{T}^{n,*}).$$

Thanks to Lemma 5.17 and Lemma 5.28 on the measurability of the grafting function, we deduce that $\mathcal{T}^{(n)}$ is a $\mathbb{T}_{\text{loc-K}}^{[2]}$ -valued random variable. The family of the distributions of the $\mathbb{T}_{\text{loc-K}}^{[2]}$ -valued random trees $(\mathcal{T}^{(n)}, n \geq 1)$ is consistent in the sense that, for every $n \geq 1$ and every $t \leq \xi_n$, $r_t^{[2]}(\mathcal{T}^{(n)}) \stackrel{(d)}{=} r_t^{[2]}(\mathcal{T}^{(n+1)})$. It is in particular a Cauchy sequence in $\mathbb{T}_{\text{loc-K}}^{[2]}$, and we denote by $(\mathcal{T}^{\alpha,\theta}, \mathfrak{T}^{\alpha,\theta})$ its limit which is thus a $\mathbb{T}_{\text{loc-K}}^{[2]}$ -valued random variable. By construction, $\mathfrak{T}^{\alpha,\theta}$ and $\mathfrak{T}^{\text{ske}}$ have the same distribution. This construction is a formal way to define the tree obtained by grafting on the infinite discrete tree $\mathfrak{T}^{\text{ske}}$ (which serves as a backbone) at x_i a tree \mathcal{T}_i where $((x_i, \mathcal{T}_i), i \in I)$ are the atoms of a Poisson point measure of intensity $\mathcal{L}(dx)\mathbb{N}^\theta(d\mathcal{T})$, where \mathcal{L} is the length measure on $\mathfrak{T}^{\text{ske}}$.

For $\alpha = 0$, we simply define $(\mathcal{T}^{0,\theta}, \mathfrak{T}^{0,\theta})$ as the Kesten tree with parameter (β, α) .

We then define the $\mathbb{T}_{\text{loc-K}}^{[2]}$ -valued random process $\left((\mathcal{T}_t^{\alpha,\theta}, \mathfrak{T}_t^{\alpha,\theta}), t \geq 0\right)$ by setting:

$$\mathcal{T}_t^{\alpha,\theta} = r_t(\mathcal{T}^{\alpha,\theta}) \quad \text{and} \quad \mathfrak{T}_t^{\alpha,\theta} = r_t(\mathfrak{T}^{\alpha,\theta}),$$

that is $(\mathcal{T}_t^{\alpha,\theta}, \mathfrak{T}_t^{\alpha,\theta}) = r_t^{[2]}((\mathcal{T}^{\alpha,\theta}, \mathfrak{T}^{\alpha,\theta}))$. Recall the $\mathbb{T}_{\text{loc-K}}$ -valued random variable \mathcal{T} is under \mathbb{N}^θ a $\text{L}\hat{\text{A}}\text{C}\text{v}$ tree; and we write $\mathcal{T}_t = r_t(\mathcal{T})$. We now give the main result of this section.

Proposition 6.11. *Let $\beta \in \mathbb{R}_+^*$, $\theta, \alpha \in \mathbb{R}_+$ and $t > 0$. For every non-negative measurable functional F on $\mathbb{T}_{\text{loc-K}}$ (or \mathbb{T}_K), we have:*

$$\mathbb{E} \left[F \left(\mathcal{T}_t^{\alpha,\theta} \right) \right] = \mathbb{N}^\theta \left[F(\mathcal{T}_t) M_t^{\alpha,\theta} \right].$$

Proof. We first consider the case $\alpha > 0$. Let us fix $t > 0$, and write N_t for $N_t^{\alpha,\theta}$. Recall that $\tilde{N}_t(\mathfrak{T}^{\alpha,\theta})$, the number of vertices of the tree $\mathfrak{T}^{\alpha,\theta}$ at level t is distributed as $N_t + 1$. Using Lemma 5.12 on the measurability of \tilde{N}_t , we get that $F \left(\mathcal{T}_t^{\alpha,\theta} \right) \mathbf{1}_{\{\tilde{N}_t(\mathfrak{T}^{\alpha,\theta})=n\}}$ is a well defined non-negative random variable.

Let $(\mathbf{T}_n, n \geq 0)$ be the sequence of trees defined in Section 6.1.1 associated with the function:

$$(91) \quad f_{\text{dens}}(s) = \beta \tilde{c}_t(\theta) e^{-2\beta\theta(t-s)} \mathbf{1}_{[0,t]}(s).$$

We have, with $\text{Graft}_k^\circ = \Pi_k^\circ \circ \text{Graft}_k$:

$$\begin{aligned} \mathbb{E} \left[F \left(\mathcal{T}_t^{\alpha,\theta} \right) \right] &= \sum_{n \in \mathbb{N}} \mathbb{E} \left[F \left(\mathcal{T}_t^{\alpha,\theta} \right) \mid \tilde{N}_t(\mathfrak{T}^{\alpha,\theta}) = n + 1 \right] \mathbb{P}(N_t^{\alpha,\theta} = n) \\ &= \sum_{n \in \mathbb{N}} \mathbb{E} \left[F \left(r_t(\text{Graft}_{n+1}^\circ(\mathfrak{T}_t, \mathcal{T}^{n+1,*})) \right) \mid N_t^{\alpha,\theta} = n \right] \frac{(\alpha/c_t^\theta)^n e^{-\alpha/c_t^\theta}}{n!} \\ &= \sum_{n \in \mathbb{N}} \mathbb{E} \left[F \left(r_t(\text{Graft}_{n+1}^\circ(\mathbf{T}_{n+1}, \mathcal{T}^{n+1,*})) \right) \right] \frac{(\alpha/c_t^\theta)^n e^{-\alpha/c_t^\theta}}{n!}, \end{aligned}$$

where we used that $\tilde{N}_t(\mathfrak{T}^{\alpha,\theta})$ is distributed as $N_t^{\alpha,\theta} + 1$ for the first equality, that conditionally on $\tilde{N}_t(\mathfrak{T}^{\alpha,\theta}) = n + 1$, the random variable $\mathcal{T}_t^{\alpha,\theta}$ is distributed as $r_t(\text{Graft}_{n+1}^\circ(\mathfrak{T}_t, \mathcal{T}^{n+1,*}))$ conditionally on $N_t^{\alpha,\theta} = n$ and that $N_t^{\alpha,\theta}$ is distributed as a Poisson process with intensity α at time $1/c_t^\theta$ (see Lemma 4.2) for the second one, that \mathfrak{T}_t conditionally on $N_t^{\alpha,\theta} = n$ is distributed as \mathbf{T}_{n+1} with f_{int} and f_{dens} in (73) given by (90) and (91) (see Lemma 6.6) for the last one. Recall that r_t and Π_{n+1}° is measurable. Using Theorem 6.10 and that $\nu(ds)$ in (81) is exactly $f_{\text{dens}}(s) ds$ with f_{dens} given by (91), we have:

$$\begin{aligned} \mathbb{E} \left[F \left(r_t(\text{Graft}_{n+1}^\circ(\mathbf{T}_{n+1}, \mathcal{T}^{n+1,*})) \right) \right] &= \frac{(\tilde{c}_t^\theta)^n e^{2\beta\theta t}}{(n+1)!} \mathbb{N}^\theta \left[\int_{\mathcal{T}^{n+1}} \Lambda_t^{\otimes(n+1)}(d\mathbf{v}^*) F(r_t \circ \Pi_{n+1}^\circ(\mathcal{T}, \mathbf{v})) \right] \\ &= \frac{(\tilde{c}_t^\theta)^n e^{2\beta\theta t}}{(n+1)!} \mathbb{N}^\theta \left[\int_{\mathcal{T}^{n+1}} \Lambda_t^{\otimes(n+1)}(d\mathbf{v}^*) F(\mathcal{T}_t) \right] \\ &= \frac{(\tilde{c}_t^\theta)^n e^{2\beta\theta t}}{(n+1)!} \mathbb{N}^\theta [Z_t^{n+1} F(\mathcal{T}_t)], \end{aligned}$$

as $Z_t = \Lambda_t(\mathbf{1})$ is the total local time of \mathcal{T} at level t . Thus, using the definition of $M_t^{\alpha,\theta}$ in (12), we obtain:

$$\mathbb{E} \left[F \left(\mathcal{T}_t^{\alpha,\theta} \right) \right] = \sum_{n \in \mathbb{N}} \frac{(\tilde{c}_t^\theta)^n e^{2\beta\theta t}}{(n+1)!} \mathbb{N}^\theta [Z_t^{n+1} F(\mathcal{T}_t)] \frac{(\alpha/c_t^\theta)^n e^{-\alpha/c_t^\theta}}{n!} = \mathbb{N}^\theta \left[F(\mathcal{T}_t) M_t^{\alpha,\theta} \right].$$

The simpler case $\alpha = 0$, which is left to the reader, can also be handled in a similar way. \square

As a conclusion, we deduce the following result for $\alpha > 0$.

Theorem 6.12. *Let $\alpha, \beta > 0, \theta \in \mathbb{R}$. Assume that the function a is such that as $t \rightarrow \infty$:*

$$a_t \sim \begin{cases} \alpha\beta^2 t^2, & \text{if } \theta = 0; \\ \alpha(2\theta)^{-2} e^{2\beta|\theta|t}, & \text{if } \theta \neq 0. \end{cases}$$

For every non-negative measurable function F on \mathbb{T}_K and $s > 0$, we have:

$$\lim_{t \rightarrow \infty} \mathbb{N}^\theta [F(\mathcal{T}_s) \mid Z_t = a_t] = \mathbb{E} \left[F \left(\mathcal{T}_s^{\alpha,|\theta|} \right) \right].$$

Proof. Clearly, Proposition 3.5 still holds if H_s is $\mathcal{G}_s = \sigma(r_s(\mathcal{T}))$ measurable, that is $H_s = F(\mathcal{T}_s)$ with F non-negative defined on $\mathbb{T}_{\text{loc-K}}$, and Z_t is the total local time of \mathcal{T} at level t , see Section 6.2. We deduce that:

$$\lim_{t \rightarrow \infty} \mathbb{N}^\theta [F(\mathcal{T}_s) | Z_t = a_t] = \mathbb{N}^{|\theta|} [F(\mathcal{T}_s) M_s^{\alpha, |\theta|}] = \mathbb{E} [F(\mathcal{T}_s^{\alpha, |\theta|})],$$

where we used Proposition 6.11 for the last equality. \square

Similarly, we also get the following result for $\alpha = 0$. Recall that $\mathcal{T}^{0, \theta}$ is a Kesten tree with parameter (β, θ) .

Theorem 6.13. *Let $\beta > 0$, $\theta \in \mathbb{R}$. Assume that the function a is positive such that as $t \rightarrow \infty$:*

$$a_t = \begin{cases} o(t^2), & \text{if } \theta = 0; \\ o(e^{2\beta|\theta|t}) & \text{if } \theta \neq 0. \end{cases}$$

For every non-negative measurable function F on \mathbb{T}_K and $s > 0$, we have:

$$\lim_{t \rightarrow \infty} \mathbb{N}^\theta [F(\mathcal{T}_s) | Z_t = a_t] = \mathbb{E} [F(\mathcal{T}_s^{0, |\theta|})].$$

✂

Index of notation

Trees and pointed trees

- $T, \mathbf{t}, \mathbf{T}, \mathcal{T}$: generic notations for trees (or class of equiv. trees).
- d : generic distance on a tree.
- ϱ : generic notation for the root of trees.
- $H(x) = d(\varrho, x)$: height of the vertex x .
- $H(T)$: height of the tree T .
- T_x : subtree of T above the vertex $x \in T$.
- $\llbracket x, y \rrbracket$: the branch joining the vertices x to y .
- T_0 : the rooted tree reduced to its root.
- T_1 : the rooted infinite branch.
- \mathcal{L} or \mathcal{L}^T : length measure on the tree T .
- $\mathbf{v} = (v_0 = \varrho, v_1, \dots, v_n)$: generic notation for pointed vertices of a tree.
- (T, \mathbf{v}) a (or a class of equiv. of) rooted n -pointed tree.
- $(T, S) = (T, S, d, \varrho)$ a (or a class of equiv. of) marked tree with $\varrho \in S \subset T$.

Grafting a tree on a tree

- $(T \circledast_i T', \mathbf{v} \circledast \mathbf{v}')$, also denoted by $T \circledast_i T'$, is the tree obtained by grafting T' on T at the pointed vertex $v_i \in T$ and identifying the root ϱ' of T' with v_i . The pointed vertices $\mathbf{v} \circledast \mathbf{v}'$ are the concatenation of the pointed vertices \mathbf{v} of T and the pointed vertices \mathbf{v}' (but for the root) of T' .
 - $T \circledast_{i,h} T'$, is the tree obtained by grafting T' on T at level h on the branch $\llbracket \varrho, v_i \rrbracket$.
 - $T \circledast_{i,h}^\epsilon T'$, with $\epsilon \in \{g, d\}$, same as above but for the pointed vertices of T' which are inserted on the left (if $\epsilon = g$) or on the right of v_i (if $\epsilon = d$).
-

Spanning and truncation

- $\text{Span}^\circ(T, \mathbf{v})$: the discrete rooted sub-tree of T spanned by the pointed vertices \mathbf{v} .
- $\text{Span}(T, \mathbf{v})$: the rooted tree $(\text{Span}^\circ(T, \mathbf{v}), \mathbf{v})$ with the pointed vertices \mathbf{v} .
- The map Π_n° removes the pointed vertices (but the root) from an n -pointed tree: $\Pi_n^\circ(T, \mathbf{v}) = (T, \varrho)$. Thus:

$$\Pi_n^\circ(\text{Span}(T, \mathbf{v})) = \text{Span}^\circ(T, \mathbf{v}).$$
- $r_t(T, \mathbf{v})$: the tree T truncated at level t with the spanned tree $\text{Span}^\circ(T, \mathbf{v})$, and the pointed vertices \mathbf{v} .
- $r_t^{[2]}, r_t^{[2],+}, r_t^{[2],-}, r_t^{[2]}, \tilde{r}_t^{[2],+}$: various truncation on marked trees (see Sect. 5.5 and 5.6).

Splitting and grafting

- $\mathbf{L}_n(T, \mathbf{v})$ record the lengths of all the branches of the subtree $\text{Span}(T, \mathbf{v})$ spanned by the n pointed vertices:

$$\mathbf{L}_n(T, \mathbf{v}) = (\ell_A(T, \mathbf{v}), A \in \mathcal{P}_n^+),$$
 with \mathcal{P}_n^+ the set of all subsets $A \subset \{1, \dots, n\}$ such that $A \neq \emptyset$.

- $\text{Split}_n(T, \mathbf{v})$ record the subtrees of T associated to all the branches of $\text{Span}(T, \mathbf{v})$:

$$(92) \quad \text{Split}_n(T, \mathbf{v}) = \left(\hat{T}_A(T, \mathbf{v}), A \in \mathcal{P}_n \right)$$

with $\mathcal{P}_n = \mathcal{P}_n^+ \cup \{\emptyset\}$.

- $\text{Graft}_n(T', (T_A^*, A \in \mathcal{P}_n^+))$: replace the branches labeled by A , of the discrete n -pointed tree T' by the trees T_A^* with a marked infinite branch cut at the length $\ell_A(T, \mathbf{v})$. (The discrete tree (T', \mathbf{v}') can be coded/replaced by $\mathbf{L}_n(T', \mathbf{v}')$.)

- Intuitively, we have for (T, \mathbf{v}) a n -pointed tree whose root is not a branching vertex (see (60)):

$$(T, \mathbf{v}) = \text{Graft}_n\left(\text{Span}_n(T, \mathbf{v}), \text{Split}_n(T, \mathbf{v})\right).$$

Set of (equiv. classes of) trees

- \mathbb{T}_K set of (equiv. classes of) rooted compact trees.
- $\mathbb{T}_K^{(n)}$ set of (equiv. classes of) rooted n -pointed compact trees; $\mathbb{T}_K^{(0)} = \mathbb{T}_K$.
- $d_{GH}^{(n)}$ the distance on $\mathbb{T}_K^{(n)}$; $d_{GH}^{(0)} \equiv d_{GH}$.
- \mathbb{T}_{loc-K} set of (equiv. classes of) rooted loc. compact trees.
- $\mathbb{T}_{loc-K}^* = \mathbb{T}_{loc-K} \setminus \{T_0\}$.
- \mathbb{T}_{loc-K}^0 subset of \mathbb{T}_{loc-K} of trees whose root is not a branching vertex.
- $\mathbb{T}_{loc-K}^{0,*} = \mathbb{T}_{loc-K}^0 \cap \mathbb{T}_{loc-K}^*$.
- $\mathbb{T}_{loc-K}^{(n)}$ set of (equiv. classes of) rooted n -pointed loc. compact trees; $\mathbb{T}_{loc-K}^{(0)} = \mathbb{T}_{loc-K}$.
- $d_{LGH}^{(n)}$ the distance on $\mathbb{T}_{loc-K}^{(n)}$; $d_{LGH}^{(0)} \equiv d_{LGH}$.
- $\mathbb{T}_{loc-K}^{(n),0}$ subset of $\mathbb{T}_{loc-K}^{(n)}$ of trees whose root is not a branching vertex.
- $\mathbb{T}_{loc-K}^{(n),*}$ subset of $\mathbb{T}_{loc-K}^{(n)}$ of trees whose all pointed vertices (but the root) are distinct from the root.
- $\mathbb{T}_{loc-K}^{(n),0,*} = \mathbb{T}_{loc-K}^{(n),0} \cap \mathbb{T}_{loc-K}^{(n),*}$.
- $\mathbb{T}_{dis}^{(n)}$ subset of $\mathbb{T}_K^{(n)} \subset \mathbb{T}_{loc-K}^{(n)}$ of discrete trees.
- $\mathbb{T}_{loc-K}^{[2]}$ set of (equiv. classes of) rooted loc. compact marked trees.
- $\mathbb{T}_{loc-K}^{spine}$ subset of $\mathbb{T}_{loc-K}^{[2]}$ of marked trees (T, S) such that $S = T_1$, with T_1 the infinite branch.

Trees with a marked branch and point measures

- $\mathbb{M}(E)$ set of point measures on $E = \mathbb{R}_+ \times \mathbb{T}_{loc-K}^*$ which are bounded on bounded sets of E .
- $\text{Tree} : \mathbb{M}(E) \rightarrow \mathbb{T}_{loc-K}^{spine}$ maps the measure $\mathcal{M} = \sum_{i \in I} \delta_{h_i, T_i}$ to the marked tree (T, T_1) , with the rooted tree T obtained by grafting the trees T_i on the rooted infinite branch T_1 at level h_i .
- $\mathcal{M} : \mathbb{T}_{loc-K}^{spine} \rightarrow \mathbb{M}(E)$ maps the marked tree (T, T_1) to the measure $\sum_{i \in I} \delta_{h_i, T_i}$ where $T_i \setminus \{\varrho_i\}$ are the connected component of $T \setminus T_1$ with root $\varrho_i \in T_1$ and $h_i = d(\varrho, \varrho_i)$, where ϱ is the common root of T and T_1 .
- \mathcal{M} is also defined on $\mathbb{T}_{loc-K}^{(1)}$.

Reconstruction results

- With Id the identity map:
 - $\text{Tree} \circ \mathcal{M} = \text{Id}$ on $\mathbb{T}_{loc-K}^{spine}$,
 - $\mathcal{M} \circ \text{Tree} = \text{Id}$ on $\tilde{\mathbb{M}}(E) = \text{Im}(\mathcal{M})$.
- $(T, \mathbf{v}) \in \mathbb{T}_{loc-K}^{(1),0,*}$ can be recovered in a measurable way from $(d(\varrho, v), \mathcal{M}(T, \mathbf{v}))$.
- $(T, \mathbf{v}) \in \mathbb{T}_{loc-K}^{(n),0,*}$ can be recovered in a measurable way from $(\text{Span}_n(T, \mathbf{v}), (\mathcal{M}_A[T, \mathbf{v}], A \in \mathcal{P}_n^+))$, where $\mathcal{M}_A[T, \mathbf{v}] = \mathcal{M}(\hat{T}_A(T, \mathbf{v}))$, with $\hat{T}_A(T, \mathbf{v}) \in \mathbb{T}_{loc-K}^{(1)}$ defined by the splitting operation in (92).

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