Zeta Invariants of Morse Forms

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Abstract

Given a closed real 1-form η on a closed Riemannian manifold (M, g), let d_z, δ_z and Δ_z be the induced Witten's type perturbations of the de Rham derivative and coderivative and the Laplacian on differential forms on M, parametrized by $z \in \mathbb{C}$, and let $\zeta(s, z)$ be the zeta function of $s \in \mathbb{C}$ given by $\zeta(s, z) = \mathrm{Tr}^{\mathrm{s}}(\eta \wedge \delta_z \Delta_z^{-s})$ when $\Re s \gg 0$. For a class of Morse forms η, we prove that $\zeta(s, z)$ is smooth at $s = 1$ for $|\Re z| \gg 0$, and the zeta invariant $\zeta(1, z)$ converges to some $\mathbf{z} \in \mathbb{R}$ as $\Re z \to +\infty$, uniformly on $\Im z$. We describe $\mathbf{z}$ in terms of the instantons of an auxiliary Smale gradient-like vector field X and the Mathai-Quillen current on TM defined by g. Any real cohomology class has a representative η satisfying the needed hypothesis. If n is even, we can prescribe any real value for $\mathbf{z}$ by perturbing g, η and X; if moreover M is oriented, we can also achieve the same limit as $\Re z \to -\infty$. This is used to define and describe certain tempered distributions induced by g and η. These distributions appear in another publication as the contributions from the compact leaves preserved by the flow in a trace formula for simple foliated flows on closed foliated manifolds, which gives a solution to a problem proposed by C. Deninger.

Contents

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1. Introduction

1.1 Witten's perturbed operators

Let M be a closed n-manifold. For any smooth function h on M, Witten [Wit82] introduced a perturbed de Rham differential operator $d_\mu = d + \mu\, dh\wedge$, depending on a parameter $\mu \in \mathbb{R}$. Endowing M with a Riemmanian metric g, we have a corresponding perturbed codifferential operator $\delta_\mu = \delta - \mu\, dh\lrcorner$, and a perturbed Laplacian $\Delta_\mu = d_\mu\delta_\mu + \delta_\mu d_\mu$. Since $d_\mu = e^{-\mu h}\, d\, e^{\mu h}$, it defines the same Betti numbers as d. However Δ_μ and the usual Laplacian Δ have different spectrum in general. In fact, if h is a Morse function and g is Euclidean with respect to Morse coordinates around the critical points, then the spectrum of Δ_μ develops a long gap as $\mu \to +\infty$, giving rise to the small and large spectrum. The eigenforms of the small/large eigenvalues generate the small/large subcomplex, $E_{\mu,\mathrm{sm/la}}$. When h is a Morse function, Witten gave a beautiful analytic proof of the Morse inequalities by analyzing the small spectrum. This was refined by subsequent work of Helffer and Sjöstrand [HS85] and Bismut and Zhang [BZ92, BZ94], showing that, if moreover $X := -\operatorname{grad} h$ is a Smale vector field, then the Morse complex $(\mathbf{C}^\bullet, \mathbf{d})$ of X can be considered as the limit of $(E_{\mu,\mathrm{sm}}, d_\mu)$. More precisely, for certain perturbed Morse complex $(\mathbf{C}^\bullet, \mathbf{d}_\mu)$, isomorphic to $(\mathbf{C}^\bullet, \mathbf{d})$, there is a quasi-isomorphism $\Phi_\mu : (E_{z,\mathrm{sm}}, d_\mu) \to (\mathbf{C}^\bullet, \mathbf{d}_\mu)$, defined by integration on the unstable cells of the zero points of X, which becomes an isomorphism for $\mu \gg 0$ and almost isometric as $\mu \to +\infty$ (after rescaling at every degree).

We can replace dh with any real closed 1-form η, obtaining a generalization of the Witten's perturbations, d_μ, δ_μ and Δ_μ. Now d_μ need not be gauge equivalent to d, obtaining new twisted Betti numbers β^k_μ. However the numbers β^k_μ have well defined ground values β^k_{No}, called the Novikov numbers, which depend upon the de Rham cohomology class $[\eta] \in H^1(M, \mathbb{R})$. Assume that:

(a) η is a Morse form (it has Morse type zeros), and g is Euclidean with respect to Morse coordinates around the zero points of η.

(Some concepts used in this section are recalled in Sections 3.1 and 5.1.) Then Δ_μ also develops a long gap separating a small spectrum and a large spectrum, and the analysis of the small spectrum gives Morse inequalities for the Novikov numbers. Take any auxiliary vector field X such that:

(b) X has Morse type zeros, and is gradient-like and Smale; and

(c) η is Lyapunov for X, and η and g are in standard form with respect to X.

Then the small complex approaches a perturbed Morse complex of X. We refer to work by Novikov [Nov81, Nov82], Pajitnov [Paj87], Braverman and Farber [BF97], Burghelea and Haller [BH01, BH04, BH08], and Harvey and Minervini [HM06, Min15].

We can similarly define the perturbation $d_z = d + z\,\eta\wedge$ with parameter $z = \mu + i\nu \in \mathbb{C}$ ($\mu, \nu \in \mathbb{R}$ and $i = \sqrt{-1}$). Its adjoint is $\delta_z = \delta - \bar{z}\,\eta\lrcorner$, and we have a corresponding perturbed Laplacian $\Delta_z = d_z\delta_z + \delta_z d_z$. As a first step in our study, we prove extensions of the above results to this case, taking limits as $|\mu| \to \infty$, uniformly on ν. First, assuming (a), we get the long gap in the spectrum of Δ_z separating the small and large spectrum, which depends only on μ (Theorem 3.10). Second, assuming (a)–(c), we show that the quasi-isomorphism $\Phi_z : (E_{z,\mathrm{sm}}, d_z) \to (\mathbf{C}^\bullet, \mathbf{d}_z)$ becomes an isomorphism for $|\mu| \gg 0$ and almost isometric as $|\mu| \to \infty$ (Theorem 5.3). To get that the convergence is uniform on ν, we show a version of a Sobolev inequality for a Sobolev norm defined by $\Delta_{i\nu}$, where the constant involved is independent of ν (Proposition 2.2). Then we adapt the arguments of Bismut and Zhang [BZ92, BZ94] (see also [Zha01]).

1.2 Zeta invariants of some Morse forms

Let $\Pi_z^\perp$ and Π_z^1 be the orthogonal projections to the images of Δ_z and d_z, and let w be the degree involution. We consider the zeta function $\zeta(s,z) = \zeta(s,z,\eta) := \zeta(s,\Delta_z,\eta\wedge\delta_z\mathsf{w})$ [Gil95, Section 1.12.4]. As a function of s, this is the meromorphic extension to $\mathbb{C}$ of the function $\mathrm{Tr}^{\mathrm{s}}(\eta\wedge\delta_z\Delta_z^{-s}\Pi_z^\perp) = \mathrm{Tr}^{\mathrm{s}}(\eta\wedge d_z^{-1}\Delta_z^{-s+1}\Pi_z^1)$, which is well defined and holomorphic for $\Re s \gg 0$. Then the zeta invariant $\zeta(1,z)$ would be a renormalized interpretation of the super-trace of $\eta\wedge d_z^{-1}\Pi_z^1$, which is not of trace class by the Weyl's law. However, according to the general theory of zeta functions of operators, $\zeta(s,z)$ might have a simple pole at $s=1$. To study this zeta function, we decompose it as sum of the terms defined by the contributions from the small/large spectrum, $\zeta_{\mathrm{sm/la}}(s,z) = \zeta_{\mathrm{sm/la}}(s,z,\eta)$. As a function of s, $\zeta_{\mathrm{sm}}(s,z)$ is always holomorphic on $\mathbb{C}$.

For a class of Morse forms, our first main theorem states that $\zeta(s,z)$ is smooth at $s=1$ for $|\mu| \gg 0$, and describes the asymptotic behavior of $\zeta(1,z)$ as $\mu \to \pm\infty$, uniformly on ν. In fact, since

$$\zeta(s,z,\eta) = -\zeta(s,-z,-\eta)\ , \quad \zeta_{\mathrm{sm/la}}(s,z,\eta) = -\zeta_{\mathrm{sm/la}}(s,-z,-\eta)\ , \tag{1.1}$$

it is enough to consider the case where $\mu \gg 0$ and take the limit as $\mu \to +\infty$.

We use the current $\psi(M,\nabla^M)$ of degree $n-1$ on TM constructed by Mathai and Quillen in [MQ86], depending on the Levi-Civita connection ∇^M. This current is smooth on the complement of the zero section, where it is given by the solid angle. It is also locally integrable, and its wave front set is contained in the conormal bundle in T^*TM of the zero section of TM. Since this set does not meet the conormal bundle of the map $X : M \to TM$ (assuming (b)), $X^*\psi(M,\nabla^M)$ is well defined as a current on M. Assuming (a)–(c), consider the real number

$$\mathbf{z}_{\mathrm{la}} = \mathbf{z}_{\mathrm{la}}(M,g,\eta) = \int_M \eta\wedge X^*\psi(M,\nabla^M)\ ,$$

which is known to be independent of X [BZ92, Proposition 6.1].

Now suppose also that:

(d) for every zero point p of X, the maximum value of the integrals of η along the instantons of X with α-limit $\{p\}$ only depends on the Morse index k of p.

This maximum value is denoted by $-a_k$ for some $a_k > 0$. Let m_k^1 denote the dimension of $d_z(E_{z,\mathrm{sm}}^{k-1})$ for $|\mu| \gg 0$, which is independent of z. Consider also the real number

$$\mathbf{z}_{\mathrm{sm}} = \mathbf{z}_{\mathrm{sm}}(M,g,\eta,X) = \sum_{k=1}^{n}(-1)^k\big(1-e^{a_k}\big)m_k^1\ ,$$

and let $\mathbf{z} = \mathbf{z}(M,g,\eta,X) = \mathbf{z}_{\mathrm{sm}} + \mathbf{z}_{\mathrm{la}}$.

Theorem 1.1. *Let $M \equiv (M,g)$ be a closed Riemannian manifold of dimension n. Let η be a closed real 1-form on M satisfying* (a).

(i) *For $\mu \gg 0$, $\zeta(s,z)$ is smooth for $s \neq (1-k)/2$ $(k = 0,1,\dots)$, and*

$$\zeta(1,z) = \lim_{t\downarrow 0}\mathrm{Tr}^{\mathrm{s}}\left(\eta\wedge d_z^{-1}e^{-t\Delta_z}\Pi_z^1\right)\ .$$

(ii) *Let X be a vector field on M satisfying* (b)–(c). *Then*

$$\zeta_{\mathrm{la}}(1,z) = \mathbf{z}_{\mathrm{la}} + O(\mu^{-1/2})$$

as $\mu \to +\infty$, uniformly on ν.

(iii) *If moreover* (d) *holds, then*

$$\zeta_{\mathrm{sm}}(1,z) = \mathbf{z}_{\mathrm{sm}} + O(\mu^{-1})$$

as $\mu \to +\infty$, *uniformly on* ν.

The existence of the limit of Theorem 1.1 (i) is rather surprising because $\eta \wedge d_z^{-1} e^{-t\Delta_z} \Pi_z^1$ is weakly convergent to $\eta \wedge d_z^{-1} \Pi_z^1$. An expression similar to $\mathrm{Tr}^{\mathrm{s}}(\eta \wedge d_z^{-1} e^{-t\Delta_z} \Pi_z^1)$ was used by Mrowka, Ruberman and Saveliev to define a cyclic eta invariant [MRS16]. Theorem 1.1 (iii) shows that $\mathbf{z}_{\mathrm{sm}}$ and $\mathbf{z}$ are also independent of X. Thus X will be omitted in their notation. In the notation of $\mathbf{z}_{\mathrm{sm/la}}$ and $\mathbf{z}$, we may also omit M or g if they are fixed.

By (1.1), if we take $\mu \to -\infty$ in Theorem 1.1, we have to replace $\mathbf{z}_{\mathrm{sm/la}}(\eta)$ with $-\mathbf{z}_{\mathrm{sm/la}}(-\eta)$. Descriptions of $-\mathbf{z}_{\mathrm{sm/la}}(-\eta)$ are given in (6.9) and (7.1).

Our second main theorem is about the prescription of $\mathbf{z} = \mathbf{z}(M, g, \eta)$ without changing the cohomology class of η.

Theorem 1.2. *Let M be a smooth closed manifold of dimension n.*

(i) *Let X be a vector field satisfying* (b). *For every $\xi \in H^1(M, \mathbb{R})$ and $\tau \gg 0$, there is some $\eta \in \xi$ and a Riemannian metric g satisfying* (a), (c) *and* (d) *with X such that $\mathbf{z}(M, g, \eta) = \tau$. If n is even, this property holds for all $\tau \in \mathbb{R}$.*

(ii) *Assume M is oriented and n is even. Then, for every $\xi \in H^1(M, \mathbb{R})$ and $\tau \in \mathbb{R}$, there is some $\eta \in \xi$, a Riemannian metric g and a vector field X satisfying* (a)–(d) *such that $\pm\mathbf{z}(M, g, \pm\eta) = \tau$.*

1.3 A distribution associated to some Morse forms

A trace formula for simple foliated flows on closed foliated manifolds was conjectured by C. Deninger (see e.g. [Den08]). He was motivated by analogies with a formula in Arithmetics, and previous work of Guillemin and Sternberg [Gui77]. This trace formula is an expression for a Lefschetz distribution in terms of infinitesimal data of the flow at the fixed points and closed orbits. This Lefschetz distribution should be an analogue of the Lefschetz number for the action induced by the flow on some leafwise cohomology, whose value is a distribution on $\mathbb{R}$—the precise definition of these notions is part of the problem. In [ALK02, ALK08], the first two authors proved such a trace formula when the flow has no preserved leaves; see also the contributions [Lei08, Lei14] by the third author. The general case is considerably more involved. In [ALKL20], we propose a solution to this problem using a few additional ingredients. One of them is the b-trace introduced by Melrose [Mel93]. Since the b-trace is not really a trace, it produces an extra term, denoted by Z, in the same way as the eta invariant shows up in Index Theory on manifolds with boundary. In our trace formula, the term Z is a contribution from the compact leaves preserved by the flow, which depends on the choice of a form defining the foliation and a metric on the ambient manifold. But Z may not be well defined in general; it will be proved that appropriate choices of the form and the metric guarantee its existence.

Precisely, we would like to define

$$Z = Z(M, g, \eta) = \lim_{\mu \to +\infty} Z_\mu \,, \tag{1.2}$$

in the space of tempered distributions, where $Z_\mu = Z_\mu(M, g, \eta)$ $(\mu \gg 0)$ should be a tempered distribution defined by

$$\langle Z_\mu, f \rangle = -\frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty \mathrm{Tr}^{\mathrm{s}} \left(\eta \wedge \delta_z e^{-u\Delta_z} \right) \hat{f}(\nu)\, d\nu\, du \,, \tag{1.3}$$

for any Schwartz function f, where Tr^{s} denotes the supertrace and $\hat{f}$ the Fourier transform of f.

Let δ_0 denote the Dirac distribution at 0 on $\mathbb{R}$. The problem about the definition of Z is solved in our third main theorem for the same class of Morse forms as before.

Theorem 1.3. *Let $M \equiv (M, g)$ be a closed Riemannian manifold of dimension n. Let η be a closed 1-form on M satisfying* (a), (c) *and* (d) *with some vector field satisfying* (b). *Then* (1.2) *and* (1.3) *define the tempered distribution $Z = \mathbf{z}\delta_0$.*

According to Theorems 1.2 and 1.3, we can choose η and g in the trace formula for foliated flows so that $Z(M, g, \eta) = 0$ if n is even, or $Z(M, g, \pm\eta) = 0$ if moreover M is oriented, achieving the original expression of Deninger's conjecture.

It looks clear that extensions of Theorems 1.1 to 1.3 with coefficients in flat vector bundles could be similarly proved. We only consider complex coefficients for the sake of simplicity since this is enough for our application.

1.4 Some ideas of the proofs of Theorems 1.1 to 1.3

Theorem 1.1 (i) follows by using that the derived heat trace invariants up to order n of the elliptic complex d_z are independent of z, proved by Gilkey and the first author [ALG20] (Section 4.6).

Consider the meromorphic function

$$\theta(s, z) = -\zeta(s, \Delta_z, \mathsf{Nw}) \,, \tag{1.4}$$

where N is the number operator, and write

$$\theta(s, z) = \theta_{\mathrm{sm}}(s, z) + \theta_{\mathrm{la}}(s, z) \,,$$

where

$$\theta_{\mathrm{sm/la}}(s, z) = -\zeta_{\mathrm{sm/la}}(s, \Delta_z, \mathsf{Nw}) \,, \tag{1.5}$$

using the contributions from the small/large spectrum as above. Thus $e^{\theta'(0,z)/2}$ is the factor used to define the Ray-Singer metric on $\det H_z^\bullet(M)$ [BZ92], using a prime to denote ∂_s. We obtain (Corollary 4.16)

$$\zeta_{\mathrm{la}}(1, z) = \partial_z \theta'_{\mathrm{la}}(0, z) \,. \tag{1.6}$$

This equality allows us to use the deep relation between the Ray-Singer metric and the Morse metric on $\det H_z^\bullet(M)$, proved by Bismut-Zhang [BZ92, BZ94]. In this way, using also that $\Phi_z : E_{z,\mathrm{sm}} \to \mathbf{C}^\bullet$ is an isomorphism, we obtain that $\zeta_{\mathrm{la}}(1, z)$ is asymptotic to $\mathbf{z}_{\mathrm{la}}$ as $\mu \to +\infty$ (Section 6.2). This proves Theorem 1.1 (ii).

When η is exact, we show this asymptotic expression of $\zeta_{\mathrm{la}}(1, z)$ assuming only (a) (Section 4.7), without using (1.6) and the indicated strong result of Bismut-Zhang. Instead, we apply that the index density of the elliptic complex d_z is independent of z, proved by Gilkey and the first author [ALG21] and by the authors [ALKL20].

On the other hand, given any $\xi \in H^1(M, \mathbb{R})$ and a vector field X satisfying (b), we prove that there is some $\eta \in \xi$ and a metric g satisfying (a), (c) and (d) (Theorem 7.1). This can be considered as an extension of a theorem of Smale stating the existence of nice Morse functions [Sma61, Theorem B] (the case where $\xi = 0$). Its proof is relegated to Appendix A because of its different nature.

The properties (a)–(d) are used to give an asymptotic description of $\mathbf{d}_z$ as $\mu \to +\infty$ (Section 7.2). From this asymptotic description and using that $\Phi_z : E_{z,\mathrm{sm}} \to \mathbf{C}^\bullet$ is an isomorphism for $\mu \gg 0$, we get upper and lower bounds of the nonzero small spectrum of Δ_z (Theorem 7.5), which

are independent of ν. This is a partial extension of accurate descriptions of the nonzero small eigenvalues achieved in the case where η is exact and the parameter is real [LPNV13, Mic19]. With the same procedure and using the bounds of the nonzero small spectrum, it also follows that $\zeta_{\mathrm{sm}}(1,z)$ is asymptotic to $\mathbf{z}_{\mathrm{sm}}$ as $\mu\to+\infty$ (Section 7.4), showing Theorem 1.1 (iii).

Next, by modifying η and X around its zero points of index 0 and n, without changing the cohomology class of η, we can achieve any real number as $\mathbf{z}(\eta)$, or as both $\pm\mathbf{z}(\pm\eta)$ if M is oriented and n even (Section 8). This shows Theorem 1.2.

If it is possible to switch the order of integration in (1.3),

$$\begin{aligned}\langle Z_\mu, f\rangle &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\mathrm{Tr}^{\mathrm{s}}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\,\hat f(\nu)\,du\,d\nu\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{t\downarrow 0}\mathrm{Tr}^{\mathrm{s}}\left(\eta\wedge d_z^{-1}e^{-t\Delta_z}\Pi_z^1\right)\,\hat f(\nu)\,d\nu\,,\end{aligned}\tag{1.7}$$

then Theorem 1.3 is an easy consequence of Theorem 1.1. Thus it only remains to prove that both (1.3) and (1.7) define the same tempered distribution Z_μ. This follows from the Lebesgue's dominated convergence theorem and Fubini's theorem (Section 9). The verification of the hypothesis of the Fubini's theorem requires the above lower estimate of the nonzero spectrum.

2. Witten's perturbations

2.1 Preliminaries on the Witten's perturbations

2.1.1 *Basic notation* Let M be a closed Riemannian n-manifold. For any smooth Euclidean/Hermitean vector bundle E over M, let $C^m(M;E)$, $C^\infty(M;E)$, $L^2(M;E)$, $L^\infty(M;E)$ and $H^m(M;E)$ denote the spaces of distributional sections that are C^m, C^∞, L^2, L^∞ and of Sobolev order m, respectively; as usual, E is removed from this notation if it is the trivial line bundle. Consider the induced scalar product $\langle\ ,\ \rangle$ and norm $\|\ \|$ on $L^2(M;E)$, and the induced norm $\|\ \|_{L^\infty}$ on $L^\infty(M;E)$. Fix also norms, $\|\ \|_m$ on every $H^m(M;E)$ and $\|\ \|_{C^m}$ on $C^m(M;E)$ with $\|\ \|=\|\ \|_0$ and $\|\ \|_{C^0}=\|\ \|_{L^\infty}$. If P is the orthogonal projection of $L^2(M;E)$ to some closed subspace V, then $P^\perp$ denotes the orthogonal projection to $V^\perp$.

Let $T_{\mathbb{C}}M=TM\otimes\mathbb{C}$ and $T^*_{\mathbb{C}}M=T^*M\otimes\mathbb{C}$. The exterior bundle with coefficients in $\mathbb{K}=\mathbb{R},\mathbb{C}$ is denoted by $\Lambda_{\mathbb{K}}=\Lambda_{\mathbb{K}}M$, and let $\Omega(M,\mathbb{K})=C^\infty(M;\Lambda_{\mathbb{K}})$; in particular, $C^\infty(M,\mathbb{K})=\Omega^0(M,\mathbb{K})$. The Levi-Civita connection is denoted by $\nabla=\nabla^M$. As usual, d and δ denote the de Rham derivative and coderivative, and let $D=d+\delta$ and $\Delta=D^2=d\delta+\delta d$ (the Laplacian). Let $Z(M,\mathbb{K})$ and $B(M,\mathbb{K})$ denote the kernel and image of d in $\Omega(M,\mathbb{K})$. Thus $H^\bullet(M,\mathbb{K})=Z(M,\mathbb{K})/B(M,\mathbb{K})$ is the de Rham cohomology with coefficients in $\mathbb{K}$. We typically consider complex coefficients, so we will omit $\mathbb{K}$ from all of the above notation just when $\mathbb{K}=\mathbb{C}$. Take $\|\ \|_m$ and $\|\ \|_{C^m}$ given on $\Omega(M)$ by

$$\|\alpha\|_m=\sum_{k=0}^m\|D^k\alpha\|\,,\quad \|\alpha\|_{C^m}=\sum_{k=0}^m\|\nabla^k\alpha\|_{L^\infty}\,.$$

For any homogeneous linear operator between graded vector spaces, $T:V^\bullet\to W^\bullet$, the notation T_k means its precomposition with the canonical projection of $V^\bullet$ to V^k. On any graded vector space $V^\bullet$, let w and N be the degree involution and number operator; i.e., $\mathsf{w}=(-1)^k$ and $\mathsf{N}=k$ on V^k.

For any $\eta\in\Omega^1(M,\mathbb{R})$ with $\eta^\sharp=X\in\mathfrak{X}(M):=C^\infty(M;TM)$, let $\mathcal{L}_X$ and ι_X denote the Lie derivative and interior product with respect to X, and let $\eta\lrcorner=-(\eta\wedge)^*=-\iota_X$. Using the

identity $\mathrm{Cl}(T^*M) \equiv \Lambda M$ defined by the symbol of filtered algebras, the left Clifford multiplication by η is $c(\eta) = \eta\wedge + \eta\lrcorner$, and the composition of w with the right Clifford multiplication by η is $\hat{c}(\eta) = \eta\wedge - \eta\lrcorner$; in particular, $c(\eta)^* = -c(\eta)$ and $\hat{c}(\eta)^* = \hat{c}(\eta)$. Recall that, for any $h \in C^\infty(M, \mathbb{R})$,

$$[D, h] = \hat{c}(dh) \ . \tag{2.1}$$

In the whole paper, unless otherwise indicated, we will use the following notation without further comment. We use constants $C, c > 0$ without even mentioning their existence, and their precise values may change from line to line. We may add subindices or primes to these constants if needed. We also use a complex parameter $z = \mu + i\nu \in \mathbb{C}$ ($\mu, \nu \in \mathbb{R}$ and $i = \sqrt{-1}$). Recall that $\partial_z = (\partial_\mu - i\partial_\nu)/2$ and $\partial_{\bar{z}} = (\partial_\mu + i\partial_\nu)/2$.

2.1.2 *Perturbations defined by a closed real 1-form* For any $\omega \in Z^1(M)$, we have the Witten's type perturbations d_ω, δ_ω, D_ω and Δ_ω of d, δ, D and Δ. Given $\eta \in Z^1(M, \mathbb{R})$ and $z \in \mathbb{C}$, we write $d_z = d_{z\eta}$, $\delta_z = \delta_\eta$, $D_z = D_{z\eta}$ and $\Delta_z = \Delta_{z\eta}$. These operators have the following expressions:

$$\left.\begin{aligned} &d_z = d + z\,\eta\wedge \ , \quad \delta_z = d_z^* = \delta - \bar{z}\,\eta\lrcorner \ , \\ &D_z = d_z + \delta_z = D + \mu\hat{c}(\eta) + i\nu c(\eta) = D_{i\nu} + \mu\hat{c}(\eta) \ , \\ &\Delta_z = D_z^2 = d_z\delta_z + \delta_z d_z = \Delta + \mu\mathsf{H}_\eta + i\nu\mathsf{J}_\eta + |z|^2|\eta|^2 = \Delta_{i\nu} + \mu\mathsf{H}_\eta + \mu^2|\eta|^2 \ , \end{aligned}\right\} \tag{2.2}$$

where, for $X = \eta^\sharp$,

$$\mathsf{H}_\eta = D\hat{c}(\eta) + \hat{c}(\eta)D = \mathcal{L}_X^* + \mathcal{L}_X \ , \quad \mathsf{J}_\eta = Dc(\eta) + c(\eta)D = \mathcal{L}_X^* - \mathcal{L}_X \ .$$

Note that H_η is of order zero and J_η of order one.

As families of operators, d_z and δ_z are holomorphic and anti-holomorphic functions of z, respectively. More precisely, it follows from (2.2) that

$$\left.\begin{aligned} &\partial_z d_z = \eta\wedge \ , \quad \partial_z\delta_z = 0 \ , \qquad \partial_z\Delta_z = \eta\wedge\,\delta_z + \delta_z\,\eta\wedge \ , \\ &\partial_{\bar{z}} d_z = 0 \ , \qquad \partial_{\bar{z}}\delta_z = -\eta\lrcorner \ , \quad \partial_{\bar{z}}\Delta_z = -\eta\lrcorner\, d_z - d_z\,\eta\lrcorner \ . \end{aligned}\right\} \tag{2.3}$$

The operator d_z defines an elliptic complex on $\Omega(M)$, whose cohomology is denoted by $H_z^\bullet(M)$. Since d_z has the same principal symbol as d, it is a generalized Dirac complex and Δ_z a generalized Laplacian [BGV04, Definition 2.2]. If $\theta = \eta + dh$ for some $h \in C^\infty(M, \mathbb{R})$, then the multiplication operator

$$e^{zh} : (\Omega(M), d_{z\theta}) \to (\Omega(M), d_{z\eta}) \tag{2.4}$$

is an isomorphism of differential complexes, and therefore it induces an isomorphism $H_{z\theta}^\bullet(M) \cong H_{z\eta}^\bullet(M)$. Thus the isomorphism class of $H_z^\bullet(M)$ only depends on $\xi := [\eta] \in H^1(M, \mathbb{R})$ and $z \in \mathbb{C}$. By ellipticity, D_z and Δ_z have a discrete spectrum, and there is a Hodge type decomposition

$$\Omega(M) = \ker\Delta_z \oplus \mathrm{im}\, d_z \oplus \mathrm{im}\,\delta_z \ , \tag{2.5}$$

as topological vector spaces. It induces a Hodge type isomorphism

$$H_z^\bullet(M) \cong \ker\Delta_z \ . \tag{2.6}$$

We also have

$$\ker\Delta_z = \ker D_z = \ker d_z \cap \ker\delta_z \ , \quad \mathrm{im}\,\Delta_z = \mathrm{im}\, D_z = \mathrm{im}\, d_z \oplus \mathrm{im}\,\delta_z \ .$$

The orthogonal projections of $\Omega(M)$ to $\ker\Delta_z$, $\mathrm{im}\, d_z$ and $\mathrm{im}\,\delta_z$ are denoted by $\Pi_z = \Pi_z^0$, Π_z^1 and Π_z^2, respectively; thus $\Pi_z^\perp = \Pi_z^1 + \Pi_z^2$. The restrictions $d_z : \mathrm{im}\,\delta_z \to \mathrm{im}\, d_z$, $\delta_z : \mathrm{im}\, d_z \to \mathrm{im}\,\delta_z$ and

$D_z : \operatorname{im} D_z \to \operatorname{im} D_z$ are topological isomorphisms, and therefore the compositions $d_z^{-1}\Pi^1$, $\delta_z^{-1}\Pi^2$ and $D_z^{-1}\Pi^\perp$ are defined and continuous on $\Omega(M)$. Moreover the diagram

$$\begin{array}{ccc} \operatorname{im}\delta_{z,k+1} & \xrightarrow{d_{z,k}} & \operatorname{im} d_{z,k} \\ {\scriptstyle\Delta_{z,k}}\big\downarrow & & \big\downarrow{\scriptstyle\Delta_{z,k+1}} \\ \operatorname{im}\delta_{z,k+1} & \xrightarrow{d_{z,k}} & \operatorname{im} d_{z,k} \end{array} \tag{2.7}$$

is commutative—recall that $d_{z,k}$, $\delta_{z,k}$ and $\Delta_{z,k}$ are the compositions of the projection to $\Omega^k(M)$ with d_z, δ_z and Δ_z. The alternate sum of the dimensions $\beta_z^k = \beta_z^k(M,\xi) = \dim H_z^k(M)$ ($k = 0,\dots,n$) is the Euler characteristic [Far04, Proposition 1.40],

$$\sum_k (-1)^k \beta_z^k = \chi(M)\ . \tag{2.8}$$

(This is also a consequence of the index theorem.) For every degree k, β_z^k is independent of z outside a discrete subset of $\mathbb{C}$, where β_z^k jumps (Mityagin and Novikov [Nov02, Theorem 1]). This ground value of β_z^k is called the k-th *Novikov Betti number*, and will be denoted by $\beta_{\mathrm{No}}^k = \beta_{\mathrm{No}}^k(M,\xi)$. Moreover it will be shown in Section 5.2.2 that

$$\beta_z^k = \beta_{\mathrm{No}}^k \quad \text{for} \quad |\mu| \gg 0\ . \tag{2.9}$$

(When z is real, this is proved in [Far95, Theorem 2.8], [BF97, Lemma 1.3], [BH04, Proposition 4].) Thus the discrete set of parameters $z \in \mathbb{C}$ with $\beta_z^k(M,\xi) > \beta_{\mathrm{No}}^k(M,\xi)$ for some degree k is contained in a strip $|\mu| \leqslant C$.

By (2.2) and since η is real, for all $\alpha \in \Omega(M)$,

$$\overline{d_z\alpha} = d_{\bar z}\bar\alpha\ , \quad \overline{\delta_z\alpha} = \delta_{\bar z}\bar\alpha\ , \quad \overline{D_z\alpha} = D_{\bar z}\bar\alpha\ , \quad \overline{\Delta_z\alpha} = \Delta_{\bar z}\bar\alpha\ . \tag{2.10}$$

So conjugation induces $\mathbb{C}$-antilinear $\mathbb{R}$-isomorphisms

$$H_z^k(M) \cong H_{\bar z}^k(M)\ , \quad \ker\Delta_{z,k} \cong \ker\Delta_{\bar z,k}\ ,$$

yielding $\beta_z^k = \beta_{\bar z}^k$.

2.1.3 *Case of an exact form* When $\eta = dh$ for some $h \in C^\infty(M,\mathbb{R})$, we have the original Witten's perturbations, which satisfy

$$\left.\begin{aligned} d_z = e^{-zh}\,d\,e^{zh} = e^{-i\nu h}\,d_\mu\,e^{i\nu h}\ , \quad \delta_z = e^{\bar z h}\,\delta\,e^{-\bar z h} = e^{-i\nu h}\,\delta_\mu\,e^{i\nu h}\ , \\ D_z = e^{-i\nu h}\,D_\mu\,e^{i\nu h}\ , \quad \Delta_z = e^{-i\nu h}\,\Delta_\mu\,e^{i\nu h}\ . \end{aligned}\right\} \tag{2.11}$$

Thus the multiplication operator

$$e^{zh} : (\Omega(M), d_z) \to (\Omega(M), d) \tag{2.12}$$

is an isomorphism of differential complexes, and therefore $H_z^\bullet(M) \cong H^\bullet(M)$; thus $\beta_z^k = \beta^k = \beta^k(M)$ (the kth Betti number) in this case. Moreover multiplication by $e^{i\nu h}$ defines a unitary isomorphism $\ker\Delta_z \cong \ker\Delta_\mu$.

2.1.4 *Interpretation of the closed form as a flat connection* There is a unique flat connection $\nabla^{M\times\mathbb{C}}$ on the trivial complex line bundle $M\times\mathbb{C}$ so that $\nabla^{M\times\mathbb{C}}1 = \eta$. The corresponding flat complex line bundle is denoted by $\mathcal{L} = \mathcal{L}_\eta$. Note that $\mathcal{L}_{z\eta} = \mathcal{L}^z$. Since every $\mathcal{L}^z$ is canonically trivial as line bundle, it has a canonical Hermitian structure $g^{\mathcal{L}^z}$. Let $(\Omega(M,\mathcal{L}^z) = \Omega(M), d^{\mathcal{L}^z})$ be the de Rham complex with coefficients in $\mathcal{L}^z$.

Suppose $\eta = dh$ for a while. Then the horizontal leaves of $\mathcal{L}^z$ are the graphs of the functions se^{-zh} ($s \in \mathbb{C}$). So $e^{zh} : \mathcal{L}^z \to \mathcal{L}^0$ is an isomorphism of flat bundles, which induces an isomorphism of differential complexes,

$$e^{zh} : (\Omega(M), d^{\mathcal{L}^z}) \to (\Omega(M), d) \ .$$

Comparing with the isomorphism (2.12), we get $d^{\mathcal{L}^z} = d_z$ on $\Omega(M)$. Furthermore, since $g^{\mathcal{L}^z}$ corresponds to $e^{2\mu h} g^{\mathcal{L}^0}$ via the isomorphism $e^{zh} : \mathcal{L}^z \to \mathcal{L}^0$, it follows that $\nabla^{\mathcal{L}^z} g^{\mathcal{L}^z}$ corresponds to

$$\nabla^{\mathcal{L}^0} \big(e^{2\mu h} g^{\mathcal{L}^0} \big) = 2\mu e^{2\mu h}\, dh \otimes g^{\mathcal{L}^0} \ ,$$

yielding

$$\nabla^{\mathcal{L}^z} g^{\mathcal{L}^z} = 2\mu\, dh \otimes g^{\mathcal{L}^z} \ . \tag{2.13}$$

A general η is locally exact, and therefore $\mathcal{L}^z$ can be locally described as above. Thus $d_z = d^{\mathcal{L}^z}$ on $\Omega(M) = \Omega(M, \mathcal{L}^z)$, and (2.13) gives

$$\nabla^{\mathcal{L}^z} g^{\mathcal{L}^z} = 2\mu\eta \otimes g^{\mathcal{L}^z} \ . \tag{2.14}$$

2.1.5 *Perturbed operators on oriented manifolds* Using for instance the interpretation of d_z given in Section 2.1.4, the mapping $(\alpha, \beta) \mapsto \alpha \wedge \beta$ induces a bilinear map $H_z^k(M) \times H_{-z}^l(M) \to H^{k+l}(M)$, and the mapping $(\alpha, \beta) \mapsto \alpha \wedge \bar{\beta}$ induces a sesquilinear map $H_z^k(M) \times H_{-\bar{z}}^l(M) \to H^{k+l}(M)$.

Now assume M is oriented. Then the above maps and integration on M define a nondegenerate bilinear pairing $H_z^k(M) \times H_{-z}^{n-k}(M) \to \mathbb{C}$, and a nondegenerate sesquilinear pairing $H_z^k(M) \times H_{-\bar{z}}^{n-k}(M) \to \mathbb{C}$. Thus $\beta_z^k = \beta_{-z}^{n-k} = \beta_{-\bar{z}}^{n-k} = \beta_{\bar{z}}^k$.

Let $\star$ denote the $\mathbb{C}$-linear extension to ΛM of the Hodge operator $\star$ on $\Lambda_{\mathbb{R}} M$, which is unitary, and let $\bar{\star}$ denote its $\mathbb{C}$-anti-linear extension. These operators are determined by the conditions

$$\alpha \wedge \overline{\star\beta} = g(\alpha, \beta)\ \mathrm{vol} = \alpha \wedge \bar{\star}\beta$$

for $\alpha, \beta \in \Omega(M)$, where $\mathrm{vol} = \star 1$ is the volume form. The following equalities on $\Omega^k(M)$ follow from (2.2) and the usual equalities relating $\star$, d, δ, $\eta\wedge$ and $\eta\lrcorner$ (see e.g. [Roe98, Chapters 1 and 3], [Gil95, Section 1.5.2], [BGV04, Section 3.6]):

$$\left.\begin{aligned} d_z \star &= (-1)^k \star \delta_{-\bar{z}} \ , \quad \delta_z \star = (-1)^{k+1} \star d_{-\bar{z}} \ , \quad \Delta_z \star = \star \Delta_{-\bar{z}} \ , \\ d_z \bar{\star} &= (-1)^k \bar{\star}\, \delta_{-z} \ , \quad \delta_z \bar{\star} = (-1)^{k+1} \bar{\star}\, d_{-z} \ , \quad \Delta_z \bar{\star} = \bar{\star}\, \Delta_{-z} \ . \end{aligned}\right\} \tag{2.15}$$

Then we get a linear isomorphism $\star : \ker \Delta_z \to \ker \Delta_{-\bar{z}}$ and an anti-linear isomorphism $\bar{\star} : \ker \Delta_z \to \ker \Delta_{-z}$, inducing a linear isomorphism $H_z^k(M) \cong H_{-\bar{z}}^{n-k}(M)$ and an anti-linear isomorphism $H_z^k(M) \cong H_{-z}^{n-k}(M)$ by (2.6).

2.2 Perturbation of the Sobolev norms

For $m \in \mathbb{N}_0$ and $\omega \in Z^1(M)$, define the norm $\| \ \|_{m,\omega}$ on $H^m(M; \Lambda)$ by

$$\|\alpha\|_{m,\omega} = \sum_{k=0}^{m} \big\| D_\omega^k \alpha \big\| \ .$$

Proposition 2.1. *For all $\omega \in Z^1(M)$ and $\alpha \in H^m(M; \Lambda)$,*

$$\|\alpha\|_{m,\omega} \leqslant C_m \sum_{k=0}^{m} \|\omega\|_{C^k}^{m-k} \|\alpha\|_k \ , \quad \|\alpha\|_m \leqslant C_m \sum_{k=0}^{m} \|\omega\|_{C^k}^{m-k} \|\alpha\|_{k,\omega} \ .$$

Proof. We proceed by induction on m. We have $\| \ \|_{0,\omega} = \| \ \|$. Now take $m > 0$ and assume these inequalities hold for $m-1$. For $\eta \in Z^1(M,\mathbb{R})$ and $\alpha \in \Omega(M)$, we have

$$\|\hat{c}(\eta)\alpha\|_m, \|c(\eta)\alpha\|_m \leqslant C'_m \|\eta\|_{C^m} \|\alpha\|_m \ . \tag{2.16}$$

Applying these inequalities to the real and imaginary parts of ω, and using the induction hypothesis and (2.2), we get

$$\begin{aligned}
\|\alpha\|_{m,\omega} = \|\alpha\| + \|D_\omega \alpha\|_{m-1,\omega} &\leqslant \|\alpha\| + C_{m-1} \sum_{k=0}^{m-1} \|\omega\|_{C^k}^{m-1-k} \|D_\omega \alpha\|_k \\
&\leqslant \|\alpha\| + C_{m-1} \sum_{k=0}^{m-1} \|\omega\|_{C^k}^{m-1-k} \big(\|D\alpha\|_k + C'_k \|\omega\|_{C^k} \|\alpha\|_k \big) \\
&\leqslant \|\alpha\| + C_{m-1} \sum_{k=0}^{m-1} \|\omega\|_{C^k}^{m-1-k} \big(\|\alpha\|_{k+1} + C'_k \|\omega\|_{C^k} \|\alpha\|_k \big) \\
&\leqslant C_m \sum_{l=0}^{m} \|\eta\|_{C^l}^{m-l} \|\alpha\|_l \ ,
\end{aligned}$$

and

$$\begin{aligned}
\|\alpha\|_m = \|\alpha\| + \|D\alpha\|_{m-1} &\leqslant \|\alpha\| + \|D_\omega \alpha\|_{m-1} + C'_{m-1} \|\omega\|_{C^{m-1}} \|\alpha\|_{m-1} \\
&\leqslant \|\alpha\| + C_{m-1} \sum_{k=0}^{m-1} \big(\|\omega\|_{C^k}^{m-1-k} \|D_\omega \alpha\|_{k,\omega} + C'_{m-1} \|\omega\|_{C^k}^{m-k} \|\alpha\|_{k,\omega} \big) \\
&\leqslant \|\alpha\| + C_{m-1} \sum_{k=0}^{m-1} \big(\|\omega\|_{C^k}^{m-1-k} \|\alpha\|_{k+1,\omega} + C'_{m-1} \|\eta\|_{C^k}^{m-k} \|\alpha\|_{k,\omega} \big) \\
&\leqslant C_m \sum_{l=0}^{m} \|\omega\|_{C^l}^{m-l} \|\alpha\|_{l,\omega} \ . \ \square
\end{aligned}$$

Let $Z(M,\mathbb{Z}) \subset Z(M,\mathbb{R})$ denote the graded subspace of forms that represent cohomology classes in the image of the canonical homomorphism $H^\bullet(M,\mathbb{Z}) \to H^\bullet(M,\mathbb{R})$. Recall that we can consider $H^1(M,\mathbb{Z})$ as a lattice in $H^1(M,\mathbb{R})$ by the universal coefficient theorem for cohomology. Let θ be the multivalued angle function on $\mathbb{S}^1$. Then $d\theta$ is the angular form on $\mathbb{S}^1$ with $\int_{\mathbb{S}^1} d\theta = 2\pi$. For $\eta \in Z^1(M,\mathbb{R})$, we have $\eta \in 2\pi Z^1(M,\mathbb{Z})$ if and only if there is some smooth map $h : M \to \mathbb{S}^1$ such that $\eta = h^* d\theta$ (see e.g. [Far04, Lemma 2.1]).

In Proposition 2.1, the dependence of the constants on ω cannot be avoided. For instance, for $M = \mathbb{S}^1$ with the standard metric $g = (d\theta)^2$, we have $\|1\|_m = \sqrt{2\pi}$, whereas $\|1\|_{m,i\eta} = \sqrt{2\pi} \sum_{k=0}^m |\nu|^k$ for $\eta = \nu \, d\theta$ ($\nu \in \mathbb{R}$). However, the following version of a Sobolev inequality for $\| \ \|_{m,i\eta}$ involves a constant independent of η.

Proposition 2.2. *If $m > n/2$, for all $\eta \in Z^1(M,\mathbb{R})$ and $\alpha \in H^m(M;\Lambda)$,*

$$\|\alpha\|_{L^\infty} \leqslant C_m \|\alpha\|_{m,i\eta} \ .$$

Proof. By the Sobolev embedding theorem, we have

$$C_{m,i\eta} := \sup_{0 \neq \alpha \in \Omega(M)} \frac{\|\alpha\|_{L^\infty}}{\|\alpha\|_{m,i\eta}} > 0 \ .$$

Take any $\eta \in Z^1(M,\mathbb{R})$ and $\omega \in 2\pi Z^1(M,\mathbb{Z})$, and let $\eta' = \eta + \omega$. Then $\omega = h^*d\theta$ for some smooth function $h : M \to \mathbb{S}^1$. Since the difference between the multiple values of θ at every point of $\mathbb{S}^1$ are in $2\pi\mathbb{Z}$, the functions $e^{\pm ih^*\theta}$ are well defined and smooth on M. Moreover, applying (2.11) locally, we get $D_{i\eta'} = e^{-ih^*\theta}\, D_{i\eta}\, e^{ih^*\theta}$. So, for $0 \neq \alpha \in \Omega(M)$,

$$\begin{aligned}\|\alpha\|_{L^\infty} &= \|e^{i\,h^*\theta}\alpha\|_{L^\infty} \leqslant C_{m,i\eta}\|e^{ih^*\theta}\alpha\|_{m,i\eta}\\ &= C_{m,i\eta}\sum_{k=0}^{m}\|D_{i\eta}^k\, e^{ih^*\theta}\alpha\| = C_{m,i\eta}\sum_{k=0}^{m}\|e^{-ih^*\theta}\, D_{i\eta}^k\, e^{ih^*\theta}\alpha\|\\ &= C_{m,i\eta}\sum_{k=0}^{m}\|D_{i\eta'}^k\alpha\| = C_{m,i\eta}\|\alpha\|_{m,i\eta'}\ .\end{aligned}$$

This shows that

$$\sup_{\eta'\in\eta+2\pi Z^1(M,\mathbb{Z})} C_{m,i\eta'} \leqslant C_{m,i\eta}\ . \tag{2.17}$$

Since $2\pi H^1(M,\mathbb{Z})$ is a lattice in $H^1(M,\mathbb{R})$, there is a compact subset $K \subset H^1(M,\mathbb{R})$ such that

$$K + 2\pi H^1(M,\mathbb{Z}) = H^1(M,\mathbb{R})\ . \tag{2.18}$$

Take a linear subspace $V \subset Z^1(M,\mathbb{R})$ such that the canonical projection $V \to H^1(M,\mathbb{R})$ is an isomorphism, and let $L \subset V$ be the compact subset that corresponds to K. By (2.18),

$$L + 2\pi Z^1(M,\mathbb{Z}) = Z^1(M,\mathbb{R})\ . \tag{2.19}$$

Moreover L is bounded with respect to $\|\ \|_{C^m}$. Therefore, by Proposition 2.1, for all $\eta \in L$ and $\alpha \in \Omega(M)$,

$$\|\alpha\|_{L^\infty} \leqslant C_{m,0}\|\alpha\|_m \leqslant C_m\|\alpha\|_{m,i\eta}\ ,$$

yielding

$$\sup_{\eta\in L} C_{m,i\eta} \leqslant C_m\ . \tag{2.20}$$

The result follows from (2.17), (2.19) and (2.20). □

Given $\eta \in Z^1(M,\mathbb{R})$, we write $\|\ \|_{m,z} = \|\ \|_{m,z\eta}$. Proposition 2.1 has the following direct consequence.

Corollary 2.3. *For all $\alpha \in H^m(M;\Lambda)$,*

$$\|\alpha\|_{m,z} \leqslant C_m\sum_{k=0}^{m}|z|^{m-k}\|\alpha\|_k\ ,\quad \|\alpha\|_m \leqslant C_m\sum_{k=0}^{m}|z|^{m-k}\|\alpha\|_{k,z}\ .$$

Proposition 2.4. *For all $\alpha \in H^1(M;\Lambda)$,*

$$\|\alpha\|_{1,z} \leqslant C\big(\|\alpha\|_{1,i\nu} + |\mu|\|\alpha\|\big)\ ,\quad \|\alpha\|_{1,i\nu} \leqslant C\big(\|\alpha\|_{1,z} + |\mu|\|\alpha\|\big)\ .$$

Proof. By (2.2) and (2.16),

$$\begin{aligned}\|\alpha\|_{1,z} = \|\alpha\| + \|D_z\alpha\| &\leqslant \|\alpha\| + \|D_{i\nu}\alpha\| + C'|\mu|\|\alpha\| \leqslant C\big(\|\alpha\|_{1,i\nu} + |\mu|\|\alpha\|\big)\ ,\\ \|\alpha\|_{1,i\nu} = \|\alpha\| + \|D_{i\nu}\alpha\| &\leqslant \|\alpha\| + \|D_z\alpha\| + C'|\mu|\|\alpha\| \leqslant C\big(\|\alpha\|_{1,z} + |\mu|\|\alpha\|\big)\ .\ \square\end{aligned}$$

3. Small and large complexes

3.1 Preliminaries on Morse forms

Recall that a critical point p of any $h \in C^\infty(M,\mathbb{R})$ is called *nondegenerate* if $\mathrm{Hess}_p\, h$ is a nondegenerate symmetric bilinear form on T_pM. The index of $\mathrm{Hess}_p\, h$, denoted by $\mathrm{ind}(p)$, is called the *index* of h at p. By the Morse lemma [Mil63, Lemma 2.2], this means that

$$h - h(p) = \frac{1}{2}\sum_{j=1}^{n} \epsilon_{p,j}(x_p^j)^2 \tag{3.1}$$

on the domain U_p of some coordinates $x_p = (x_p^1, \dots, x_p^n)$ (centered) at p, called *Morse coordinates*, where

$$\epsilon_{p,j} = \begin{cases} -1 & \text{if } j \leqslant \mathrm{ind}(p) \\ 1 & \text{if } j > \mathrm{ind}(p)\ . \end{cases} \tag{3.2}$$

Recall that h is called a *Morse function* when all of its critical points are nondegenerate. In this case, its critical points form a finite set denoted by $\mathrm{Crit}(h)$. The Morse functions form an open and dense subset of $C^\infty(M,\mathbb{R})$ [Hir76, Theorem 6.1.2]. On every U_p, we can assume the metric is Euclidean with respect to Morse coordinates:

$$g = \sum_{j=1}^{n} (dx_p^j)^2\ . \tag{3.3}$$

Now take any $\eta \in Z^1(M,\mathbb{R})$. A zero p of η is said to be *nondegenerate* if $\eta = dh_{\eta,p}$ around p for some local Morse function $h_{\eta,p}$, which is chosen so that $h_{\eta,p}(p) = 0$. The index $\mathrm{ind}(p)$ of $h_{\eta,p}$ at p is also called the *index* of η at p. On the domain U_p of Morse coordinates $x_p = (x_p^1, \dots, x_p^n)$ for $h_{\eta,p}$ at p, $h_{\eta,p}$ is given by the right-hand sides of (3.1), and

$$\eta = \sum_{j=1}^{n} \epsilon_{p,j} x_p^j\, dx_p^j\ . \tag{3.4}$$

It is also said that $x_p = (x_p^1, \dots, x_p^n)$ are *Morse coordinates* for η at p. With the notation

$$x_p^- = (x_p^1, \dots, x_p^k) : U_p \to \mathbb{R}^k\ , \quad x_p^+ = (x_p^{k+1}, \dots, x_p^n) : U_p \to \mathbb{R}^{n-k}\ ,$$

for $k = \mathrm{ind}(p)$, we can write

$$h_{\eta,p} = \frac{1}{2}\big(|x_p^+|^2 - |x_p^-|^2\big)\ . \tag{3.5}$$

If all zeros are nondegenerate, then η is called a *Morse form*. In this case, its zeros form a finite set, $\mathcal{X} = \mathrm{Zero}(\eta)$; subsets of $\mathcal{X}$ defined by conditions on the index are denoted by writing the conditions as subscripts; for instance, $\mathcal{X}_k$, $\mathcal{X}_+$ and $\mathcal{X}_{<k}$ are the subsets of zeros of index k, of positive index, and of index $< k$, respectively. For any $\xi \in H^1(M,\mathbb{R})$, the Morse representatives of ξ form a dense open subset of ξ, considered as a subspace of $\Omega^1(M,\mathbb{R})$ with the C^∞ topology (see e.g. [Paj06, Theorem 2.1.25]).

From now on, unless otherwise stated, we will use some $\eta \in Z^1(M,\mathbb{R})$ and a Riemannian metric g on M such that η is a Morse form, and g is Euclidean with respect to Morse coordinates around zero points of η; i.e., η and g satisfy (a) (Section 1.1).

The Hopf index of $-\eta^\sharp$ at every $p \in \mathcal{X}_k$ is $(-1)^k$ (see Section 5.1.1). Thus, by the Hopf index

theorem for $-\eta^\sharp$,

$$\sum_{k=0}^{n}(-1)^k|\mathcal{X}_k| = \chi(M)\ . \tag{3.6}$$

3.2 The small and large spectrum

Consider the perturbed operators (2.2) defined by η and g. Recall that $\mathcal{X}$ denotes the zero set of η. We can suppose the closures of the domains of Morse coordinates, $\overline{U_p}$ ($p \in \mathcal{X}$), are disjoint from each other, and $x_p(U_p) = (-4r, 4r)^n$ for some $r > 0$ independent of p. Let $U = \bigcup_{p\in\mathcal{X}} U_p$.

Denoting also the coordinates of $\mathbb{R}^n$ by $(x_p^1, \ldots, x_p^n)$, consider the function $h_p \in C^\infty(\mathbb{R}^n)$ defined by the right-hand side of (3.1) (or (3.5)). Let $d'_{p,z}$, $\delta'_{p,z}$, $D'_{p,z}$ and $\Delta'_{p,z}$ ($z \in \mathbb{C}$) denote the corresponding Witten's operators on $\mathbb{R}^n$, whose restrictions to $(-4r, 4r)^n$ agree via x_p with the restrictions of d_μ, δ_μ, D_μ and Δ_μ to U_p.

Proposition 3.1 See e.g. [Roe98, Chapters 9 and 14], [Zha01, Sections 4.5 and 4.7]. *The following holds for $\mu \in \mathbb{R}$:*

(i) *We have*

$$\Delta'_{p,\mu} = \sum_{j=1}^{n}\Big(-\Big(\frac{\partial}{\partial x_p^j}\Big)^2 + \mu^2(x_p^j)^2 + \mu\epsilon_{p,j}[dx_p^j\lrcorner, dx_p^j\wedge]\Big)\ . \tag{3.7}$$

Moreover, using multi-index notation,

$$[dx_p^j\lrcorner, dx_p^j\wedge]dx_p^J = \begin{cases} dx_p^J & \text{if } j \in J \\ -dx_p^J & \text{if } j \notin J\ . \end{cases}$$

(ii) *$\Delta'_{p,\mu}$ is a non-negative selfadjoint operator in $L^2(\mathbb{R}^n;\Lambda)$ with a discrete spectrum, which consists of the eigenvalues*

$$\mu\sum_{j=1}^{n}(1 + 2u_j + \epsilon_{p,j}v_j)\ , \tag{3.8}$$

where $u_j \in \mathbb{N}_0$ and $v_j = \pm 1$. For the restriction of $\Delta'_{p,\mu}$ to k-forms, the spectrum has the additional requirement that exactly k of the numbers v_j are equal to 1. In particular, 0 is an eigenvalue of $\Delta'_{p,\mu}$ with multiplicity 1 (choosing $u_j = 0$ and $v_j = -\epsilon_{p,j}$ for all j), and the nonzero eigenvalues are of order μ as $\mu \to +\infty$. $D'_{p,\mu}$ is also a selfadjoint operator in $L^2(\mathbb{R}^n;\Lambda)$ with a discrete spectrum, which consists of the positive and negative square roots of (3.8).

(iii) *The kernel of $D'_{p,\mu}$ and $\Delta'_{p,\mu}$ is generated by the normalized form*

$$e'_{p,\mu} = \Big(\frac{\mu}{\pi}\Big)^{n/4} e^{-\mu|x_p|^2/2}\, dx_p^1 \wedge \cdots \wedge dx_p^{\mathrm{ind}(p)}\ .$$

For any $z \in \mathbb{C}$ with $\mu > 0$, let $\Delta'_{p,z} = e^{-i\nu h_p}\Delta'_{p,\mu}e^{i\nu h_p}$. Since the operator of multiplication by $e^{-i\nu h_p}$ is unitary, $\Delta'_{p,z}$ is also selfadjoint and non-negative in $L^2(\mathbb{R}^n;\Lambda)$, it has a discrete spectrum with the same eigenvalues and multiplicities as $\Delta'_{p,\mu}$, and its kernel is generated by the normalized form $e'_{p,z} := e^{-i\nu h_p}e'_{p,\mu}$. We will also use the notation

$$e'_{p,z} = x_p^* e'_{p,z} \in C^\infty\big(U_p;\Lambda^{\mathrm{ind}(p)}\big)\ .$$

The function $x_p^* h_p \in C^\infty(U_p)$ agrees with $h_{\eta,p}$, which is also denoted by h_p in this section.

Fix an even C^∞ function $\rho : \mathbb{R} \to [0,1]$ such that $\rho = 1$ on $[-r,r]$ and $\operatorname{supp}\rho \subset [-2r,2r]$. For every $p \in \mathcal{X}$, let

$$\rho_p = \rho(x_p^1)\cdots\rho(x_p^n) \in C_c^\infty(U_p)\ , \tag{3.9}$$

$$e_{p,\mu} = \frac{\rho_p}{a_\mu} e'_{p,\mu} \in C_c^\infty\big(U_p; \Lambda^{\mathrm{ind}(p)}\big)\ , \tag{3.10}$$

$$e_{p,z} = e^{-i\nu h_p} e_{p,\mu} = \frac{\rho_p}{a_\mu} e'_{p,z} \in C_c^\infty\big(U_p; \Lambda^{\mathrm{ind}(p)}\big)\ , \tag{3.11}$$

where

$$a_\mu = \bigg(\int_{-2r}^{2r} \rho(x)^2 e^{-\mu x^2}\, dx\bigg)^{n/2} = \Big(\frac{\pi}{\mu}\Big)^{n/4} + O(e^{-c\mu})\ , \tag{3.12}$$

as $\mu \to +\infty$. The extensions by zero of the forms $e_{p,z}$ to M are also denoted by $e_{p,z}$. They form an orthonormal basis of a graded subspace $E_z \subset \Omega(M)$ with $\dim E_z = |\mathcal{X}|$. Let P_z be the orthogonal projection of $L^2(M;\Lambda)$ to E_z; thus $P_z^\perp$ is the orthogonal projection to $E_z^\perp$.

Remark 3.2. For the sake of simplicity, most of our results are stated for $\mu \gg 0$ or as $\mu \to +\infty$, but they have obvious versions for $\mu \ll 0$ or as $\mu \to -\infty$, as follows by considering $-\eta$.

Proposition 3.3. *If $\mu \gg 0$ and $\beta \in H^1(M;\Lambda)$ with $\operatorname{supp}\beta \subset M \setminus U$, then*

$$\|D_z\beta\| \geqslant C\mu\,\|\beta\|\ .$$

Proof. This follows like [Zha01, Proposition 4.7], using that H_η is of order zero in (2.2). Actually, according to the statement of [Zha01, Proposition 4.7], this inequality would hold with $\sqrt{\mu}$ instead of μ, but its proof clearly shows that using μ is fine. □

Proposition 3.4. *The following properties hold:*

(i) $P_z D_z P_z = 0$.

(ii) *If $\mu \gg 0$, $\alpha \in E_z$ and $\beta \in E_z^\perp \cap H^1(M;\Lambda)$, then*

$$\|P_z^\perp D_z\alpha\| \leqslant e^{-c\mu}\|\alpha\|\ ,\quad \|P_z D_z\beta\| \leqslant e^{-c\mu}\|\beta\|\ .$$

(iii) *If $\mu \gg 0$ and $\beta \in E_z^\perp \cap H^1(M;\Lambda)$, then*

$$\|P_z^\perp D_z\beta\| \geqslant C\sqrt{\mu}\,\|\beta\|\ .$$

Proof. This follows like [Zha01, Propositions 4.11, 4.12 and 5.6]. Property (i) is true because every $D_z e_{p,z}$ is supported in U_p and has homogeneous components of degree different from $\mathrm{ind}(p)$; therefore it is orthogonal to $\ker\Delta_z$. The other properties are consequences of Propositions 3.1 and 3.3 and (3.9)–(3.12). According to [Zha01, Proposition 4.11], the inequalities of (ii) hold with $1/\mu$ instead of $e^{-c\mu}$, but its proof shows that indeed $e^{-c\mu}$ can be achieved. □

Proposition 3.5. *For all $m \in \mathbb{N}_0$, if $\mu \gg 0$, then*

$$\|D_z e_{p,z}\|_m, \|D_z e_{p,z}\|_{m,i\nu} \leqslant e^{-c_m\mu}\ .$$

Proof. From Proposition 3.1 (iii), (2.1), (3.10) and (3.11), we get

$$D_z e_{p,z} = D_z\Big(\frac{\rho_p}{a_\mu} e'_{p,z}\Big) = e^{-i\nu h_p}\frac{1}{a_\mu}\Big(\frac{\pi}{4}\Big)^{n/4}\hat{c}(d\rho_p)e'_{p,\mu}\ . \tag{3.13}$$

Thus the stated estimate of $\|D_z e_{p,z}\|_m$ is true by (3.10) and (3.12), and since $d\rho_p = 0$ around p. (When $\nu = 0$, this is indicated in [Zha01, Eq. (6.17)].)

By (2.11), for all $k \in \mathbb{N}_0$ and $p \in \mathcal{X}$, the form $D_{i\nu}^k D_z e_{p,z}$ is the extension by zero of the form $e^{-i\nu h_p} D^k D_\mu e_{p,\mu}$ on U_p. Then the stated estimate of $\|D_z e_{p,z}\|_{m,i\nu}$ follows from the case $\nu = 0$. □

Corollary 3.6. *If $\mu \gg 0$, then*

$$\|D_z e_{p,z}\|_{L^\infty} \leqslant e^{-c\mu} .$$

Proof. Apply Propositions 2.2 and 3.5. □

Consider the partition of $\operatorname{spec} \Delta_z$ into its intersections with $[0,1]$ and $(1,\infty)$, called the *small* and *large spectrum*; the term *small/large eigenvalues* may be also used. Let $E_{z,\mathrm{sm}} \subset \Omega(M)$ denote the graded finite dimensional subspace generated by the eigenforms of the small eigenvalues, let $E_{z,\mathrm{la}} = E_{z,\mathrm{sm}}^{\perp}$ in $L^2(M;\Lambda)$, and let $P_{z,\mathrm{sm/la}}$ be the orthogonal projections to $E_{z,\mathrm{sm/la}}$. Note that $E_{z,\mathrm{sm}}$ and $E_{z,\mathrm{la}} \cap \Omega(M)$ are subcomplexes with d_z, called the *small* and *large complexes*, and the term *small/large projection* will be used for $P_{z,\mathrm{sm/la}}$. Thus $(\Omega(M), d_z)$ splits into a topological direct sum of the subcomplexes $E_{z,\mathrm{sm}}$ and $E_{z,\mathrm{la}} \cap \Omega(M)$, and (2.5) gives

$$H^\bullet(E_{z,\mathrm{sm}}, d_z) \equiv H_z^\bullet(M) , \quad H^\bullet(E_{z,\mathrm{la}} \cap \Omega(M), d_z) = 0 . \tag{3.14}$$

For any operator B defined on $\Omega(M)$ or $L^2(M;\Lambda)$, let $B_{z,\mathrm{sm/la}} = BP_{z,\mathrm{sm/la}}$.

Proposition 3.7. *For all $m \in \mathbb{N}_0$, $\mu \gg 0$ and $\alpha \in E_z$,*

$$\|\alpha - P_{z,\mathrm{sm}}\alpha\|_{m,i\nu} \leqslant e^{-c_m\mu}\|\alpha\| .$$

Proof. This follows like [Zha01, Lemma 5.8 and Theorem 6.7], using $\| \|_{m,i\nu}$ instead of $\| \|_m$. The following are the main steps of the proof.

Let $\mathbb{S}^1 = \{\, \omega \in \mathbb{C} \mid |\omega| = 1 \,\}$. With the argument of the proof of [Zha01, Eq. (5.27)], using Proposition 3.4, we get that, for all $\alpha \in H^1(M;\Lambda)$, $w \in \mathbb{S}^1$ and $\mu \gg 0$,

$$\|(w - D_z)\alpha\| \geqslant C\|\alpha\| .$$

Thus $w - D_z : H^1(M;\Lambda) \to L^2(M;\Lambda)$ has a bounded inverse and, for all $\beta \in L^2(M;\Lambda)$, $w \in \mathbb{S}^1$ and $\mu \gg 0$,

$$\big\|(w - D_z)^{-1}\beta\big\| \leqslant C^{-1}\|\beta\| . \tag{3.15}$$

On the other hand, arguing like in the proof of [Zha01, Eq. (6.18)], it follows that, for all $\gamma \in H^m(M;\Lambda)$, $w \in \mathbb{S}^1$ and $\mu \gg 0$,

$$\|\gamma\|_{m,i\nu} \leqslant C_m\big(\big\|(w - D_z)\gamma\big\|_{m-1,i\nu} + \mu\|\gamma\|_{m-1,i\nu} + \|\gamma\|\big) .$$

Continuing by induction on $m \in \mathbb{N}_0$, we obtain

$$\|\gamma\|_{m,i\nu} \leqslant C_m\Big(\mu^m\|\gamma\| + \sum_{k=1}^{m} \mu^{k-1}\big\|(w - D_z)\gamma\big\|_{m-k,i\nu}\Big) .$$

In other words, for all $\beta \in H^{m-1}(M;\Lambda)$,

$$\big\|(w - D_z)^{-1}\beta\big\|_{m,i\nu} \leqslant C_m\Big(\mu^m\big\|(w - D_z)^{-1}\beta\big\| + \sum_{k=1}^{m} \mu^{k-1}\|\beta\|_{m-k,i\nu}\Big) .$$

Applying (3.15) to this inequality, we get

$$\big\|(w - D_z)^{-1}\beta\big\|_{m,i\nu} \leqslant C_m\mu^m\|\beta\|_{m-1,i\nu} . \tag{3.16}$$

Then, by Proposition 3.5,

$$\big\|(w - D_z)^{-1}D_z e_{p,z}\big\|_{m,i\nu} = O\big(e^{-c_m\mu}\big) \tag{3.17}$$

as $\mu \to +\infty$, uniformly on $w \in \mathbb{S}^1$. But, endowing $\mathbb{S}^1$ with the counter-clockwise orientation, basic spectral theory gives (see e.g. [DS88a, Section VII.3])

$$\begin{aligned} P_{z,\mathrm{sm}} e_{p,z} - e_{p,z} &= \frac{1}{2\pi i} \int_{\mathbb{S}^1} \big((w - D_z)^{-1} - w^{-1}\big) e_{p,z}\, dw \\ &= \frac{1}{2\pi i} \int_{\mathbb{S}^1} w^{-1} (w - D_z)^{-1} D_z e_{p,z}\, dw \,. \end{aligned} \tag{3.18}$$

The result follows using (3.17) in (3.18). □

Corollary 3.8. *For $\mu \gg 0$ and $\alpha \in E_z$,*

$$\|\alpha - P_{z,\mathrm{sm}}\alpha\|_{L^\infty} \leqslant e^{-c\mu} \|\alpha\| \,.$$

Proof. Apply Propositions 2.2 and 3.7.

Alternatively, the proof of Proposition 3.7 can be modified as follows to get this result (some step of this alternative argument will be used later). Iterating (3.16), we get

$$\big\|(w - D_z)^{-1}\beta\big\|_{m,i\nu} \leqslant C'_m \mu^{(m+1)m/2} \|\beta\| \,,$$

for all $\beta \in L^2(M;\Lambda)$. Then, by Proposition 2.2,

$$\big\|(w - D_z)^{-1}\beta\big\|_{L^\infty} \leqslant C \mu^{(m+1)m/2} \|\beta\| \,. \tag{3.19}$$

Thus, by Proposition 3.5,

$$\big\|(w - D_z)^{-1} D_z e_{p,z}\big\|_{L^\infty} = O\big(e^{-c_m \mu}\big)$$

as $\mu \to +\infty$. Finally, apply this expression in (3.18). □

Corollary 3.9. *If $\mu \gg 0$, then $P_{z,\mathrm{sm}} : E_z \to E_{z,\mathrm{sm}}$ is an isomorphism; in particular, $\dim E_{z,\mathrm{sm}} = |\mathcal{X}|$ and $\dim E^k_{z,\mathrm{sm}} = |\mathcal{X}_k|$.*

Proof. The result follows from Proposition 3.7 for $m = 0$ like [Zha01, Proposition 5.5]. □

When $\mu \gg 0$, (3.6) also follows from Corollary 3.9, (2.8) and (3.14).

Theorem 3.10 Cf. [BH01, Theorem 3]. *We have*

$$\operatorname{spec} \Delta_z \subset \big[0, e^{-c|\mu|}\big] \cup \big[C|\mu|, \infty\big) \,.$$

Proof. We can assume $\mu \geqslant 0$ according to Remark 3.2. By Propositions 2.4, 3.4 and 3.7, for all $\alpha \in E_z$,

$$\begin{aligned} \|D_z P_{z,\mathrm{sm}}\alpha\| &\leqslant \|D_z \alpha\| + \|D_z(\alpha - P_{z,\mathrm{sm}}\alpha)\| \leqslant \|D_z\alpha\| + \|\alpha - P_{z,\mathrm{sm}}\alpha\|_{1,z} \\ &\leqslant \|P_z^\perp D_z \alpha\| + C(\mu\|\alpha - P_{z,\mathrm{sm}}\alpha\| + \|\alpha - P_{z,\mathrm{sm}}\alpha\|_{1,i\nu}) \\ &\leqslant \big(e^{-c\mu} + C\big(\mu e^{-c_0\mu} + e^{-c_1\mu}\big)\big)\|\alpha\| \,. \end{aligned}$$

Hence, by Corollary 3.9, for all $\beta \in E_{z,\mathrm{sm}}$,

$$0 \leqslant \langle \Delta_z \beta, \beta\rangle = \|D_z\beta\|^2 \leqslant e^{-c\mu}\, \|\beta\|^2 \,.$$

This shows that

$$\operatorname{spec} \Delta_z \cap [0,1] \subset \big[0, e^{-c\mu}\big] \,. \tag{3.20}$$

Now let $\phi \in E_{z,\mathrm{la}} \cap H^1(M;\Lambda)$, and write $\alpha = P_z\phi \in E_z$ and $\beta = P_z^\perp \phi \in E_z^\perp \cap H^1(M;\Lambda)$. By Proposition 3.7,

$$\|\alpha\|^2 = \langle \alpha, \phi\rangle = \langle \alpha - P_{z,\mathrm{sm}}\alpha, \phi\rangle \leqslant \|\alpha - P_{z,\mathrm{sm}}\alpha\|\|\phi\| \leqslant e^{-c_0\mu}\|\alpha\|\|\phi\| \,,$$

yielding

$$\|\alpha\| \leqslant e^{-c_0\mu}\|\phi\| \ .$$

So

$$\|\beta\| = \|\phi - \alpha\| \geqslant \|\phi\| - \|\alpha\| \geqslant \big(1 - e^{-c_0\mu}\big)\|\phi\| \ .$$

Then, by Proposition 3.4,

$$\begin{aligned}\|D_z\phi\| &\geqslant \|D_z\beta\| - \|D_z\alpha\| \geqslant \|P_z^\perp D_z\beta\| - e^{-c\mu}\|\alpha\| \\ &\geqslant C\sqrt{\mu}\,\|\beta\| - e^{-c\mu}\|\phi\| \geqslant \big(C\sqrt{\mu}\big(1 - e^{-c_0\mu}\big) - e^{-c\mu}\big)\|\phi\| \ .\end{aligned}$$

Therefore, for all $\phi \in E_{z,\mathrm{la}} \cap H^1(M;\Lambda)$,

$$\langle \Delta_z\phi, \phi\rangle = \|D_z\phi\|^2 \geqslant C\mu\|\phi\|^2 \ .$$

This proves that

$$\operatorname{spec}\Delta_z \cap (1,\infty) \subset [C\mu,\infty) \ . \tag{3.21}$$

The inclusions (3.20) and (3.21) give the result for $\mu \gg 0$. But, in those inclusions, we can take c and C so small that, if one of them is not true for some $\mu \geqslant 0$, then $C\mu \leqslant e^{-c\mu}$. □

3.3 Ranks of some projections in the small complex

Recall that $(\Pi_z^\perp)_{\mathrm{sm},k}$, $\Pi^1_{z,\mathrm{sm},k}$ and $\Pi^2_{z,\mathrm{sm},k}$ denote the orthogonal projections to the images of $\Delta_{z,\mathrm{sm},k}$, $d_{z,\mathrm{sm},k-1}$ and $\delta_{z,\mathrm{sm},k+1}$, respectively. Let $m_{z,k}$, $m^1_{z,k}$ and $m^2_{z,k}$ be the corresponding ranks (or traces) of these projections. They satisfy

$$m_{z,k} = m^1_{z,k} + m^2_{z,k} \ , \quad m^1_{z,0} = m^2_{z,n} = 0 \ , \quad m^2_{z,k} = m^1_{z,k+1} \ , \tag{3.22}$$

where the last equality is true because $d_z : \operatorname{im}\delta_z \to \operatorname{im} d_z$ is an isomorphism. For $\mu \gg 0$, we have $m_{z,k}, m^j_{z,k} \leqslant |\mathcal{X}_k|$ by Corollary 3.9 and (3.22).

Lemma 3.11. *The numbers $m^j_{z,k}$ are determined by the numbers $m_{z,k}$.*

Proof. This follows from (3.22) with an easy induction argument on k. □

Lemma 3.12. *We have $m_{z,k} = |\mathcal{X}_k| - \beta^k_z$.*

Proof. This is a consequence of (2.5), (3.14) and Corollary 3.9. □

Corollary 3.13. $\mathrm{Tr}^{\mathrm{s}}((\Pi_z^\perp)_{\mathrm{sm}}) = 0$.

Proof. By (2.8), (3.6) and Lemma 3.12,

$$\mathrm{Tr}^{\mathrm{s}}\left((\Pi_z^\perp)_{\mathrm{sm}}\right) = \sum_k (-1)^k |\mathcal{X}_k| - \sum_k (-1)^k \beta^k_z = \chi(M) - \chi(M) = 0 \ .$$ □

Lemma 3.14. *If M is oriented, then, for $k = 0, \dots, n$,*

$$m_{z,k} = m_{-\bar z,n-k} = m_{-z,n-k} \ , \quad m^1_{z,k} = m^2_{-\bar z,n-k} = m^2_{-z,n-k} \ .$$

Proof. This is true because, by (2.15),

$$\begin{aligned}&(\Pi_z^\perp)_{\mathrm{sm},k} \star = \star\,(\Pi^\perp_{-\bar z})_{\mathrm{sm},n-k} \ , \quad \Pi^1_{z,\mathrm{sm},k} \star = \star\,\Pi^2_{-\bar z,\mathrm{sm},n-k} \ , \\ &(\Pi_z^\perp)_{\mathrm{sm},k}\,\bar\star = \bar\star\,(\Pi^\perp_{-z})_{\mathrm{sm},n-k} \ , \quad \Pi^1_{z,\mathrm{sm},k}\,\bar\star = \bar\star\,\Pi^2_{-z,\mathrm{sm},n-k} \ .\end{aligned}$$ □

Corollary 3.15. *For $|\mu| \gg 0$, $m_{z,k}$ and $m^j_{z,k}$ only depend on $|\mathcal{X}_k|$ and the class $\xi = [\eta] \in H^1(M,\mathbb{R})$.*

Proof. Apply (2.9) and Lemmas 3.11 and 3.12. □

By Corollary 3.15, we write $m_k = m_{z,k}$ and $m_k^j = m_{z,k}^j$ for $|\mu| \gg 0$.

Corollary 3.16. $\mathrm{Tr}^{\mathrm{s}}(\Pi_{z,\mathrm{sm}}^j) = 0$ *if* M *is oriented,* n *is even and* $\mu \gg 0$.

Proof. By Corollaries 3.13 and 3.15 and Lemma 3.14, and since n is even,

$$\mathrm{Tr}^{\mathrm{s}}\left(\Pi_{z,\mathrm{sm}}^1\right) = -\,\mathrm{Tr}^{\mathrm{s}}\left(\Pi_{z,\mathrm{sm}}^2\right) = -\,\mathrm{Tr}^{\mathrm{s}}\left(\Pi_{z,\mathrm{sm}}^1\right) .$$ □

3.4 Asymptotic properties of the small projection

Notation 3.17. Consider a function $f(x) > 0$ $(x > 0)$. When referring to vectors in Banach spaces, the order notation $O(f(|\mu|))$ $(\mu \to \pm\infty)$ will be used for a family of vectors $v = v(z)$ $(z \in \mathbb{C})$ with $\|v(z)\| = O(f(|\mu|))$. This notation applies e.g. to bounded operators between Banach spaces. We may also consider this notation when the Banach spaces depend on z.

Proposition 3.18. *For every* $\tau \in \mathbb{R}$, *on* $L^2(M;\Lambda)$, *as* $\mu \to +\infty$,

$$P_{z,\mathrm{sm}} = P_z + O\big(e^{-c\mu}\big) = P_{z,\mathrm{sm}} P_{z+\tau,\mathrm{sm}} P_{z,\mathrm{sm}} + O\big(\mu^{-2}\big) = P_{z+\tau,\mathrm{sm}} + O\big(\mu^{-1}\big) .$$

Proof. By Corollary 3.9, for $\mu \gg 0$, the elements $P_{z,\mathrm{sm}} e_{p,z}$ $(p \in \mathcal{X})$ form a base of $E_{z,\mathrm{sm}}$. Applying the Gram-Schmidt process to this base, we get an orthonormal base $\tilde{e}_{p,z}$. By Proposition 3.7,

$$\tilde{e}_{p,z} = e_{p,z} + O\big(e^{-c\mu}\big) . \tag{3.23}$$

This gives the first equality of the statement: for any $\alpha \in L^2(M;\Lambda)$,

$$P_z\alpha = \sum_{p\in\mathcal{X}} \langle \alpha, e_{p,z}\rangle e_{p,z} = \sum_{p\in\mathcal{X}} \langle \alpha, \tilde{e}_{p,z}\rangle \tilde{e}_{p,z} + O\big(e^{-c\mu}\big)\|\alpha\| = P_{z,\mathrm{sm}}\alpha + O\big(e^{-c\mu}\big)\|\alpha\| .$$

Since the sets U_p $(p \in \mathcal{X})$ are disjoint one another, for $p \neq q$ in $\mathcal{X}$,

$$\langle e_{p,z}, e_{q,z+\tau}\rangle = 0 . \tag{3.24}$$

On the other hand, by (3.9)–(3.12), we can also assume

$$\begin{aligned}
\langle e_{p,z}, e_{p,z+\tau}\rangle &= \langle e^{-i\nu h_p} e_{p,\mu}, e^{-i\nu h_p} e_{p,\mu+\tau}\rangle = \langle e_{p,\mu}, e_{p,\mu+\tau}\rangle \\
&= \frac{(\mu(\mu+\tau))^{n/4}}{\pi^{n/2}} \big\langle \rho_p e^{-\mu|x_p|^2/2}, \rho_p e^{-(\mu+\tau)|x_p|^2/2}\big\rangle + O\big(e^{-c\mu}\big) \\
&= \frac{(\mu(\mu+\tau))^{n/4}}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-(\mu+\tau/2)|x_p|^2}\, dx_p + O\big(e^{-c\mu}\big) \\
&= \frac{(\mu(\mu+\tau))^{n/4}}{(\mu+\tau/2)^{n/2}} + O\big(e^{-c\mu}\big) = 1 + O\big(\mu^{-2}\big) ,
\end{aligned} \tag{3.25}$$

where $dx_p = dx_p^1 \dots dx_p^n = \mathrm{vol}(x_p)$. Combining (3.23) for z and $z+\tau$ with (3.24) and (3.25), we obtain

$$\begin{aligned}
P_{z+\tau,\mathrm{sm}}\tilde{e}_{p,z} &= \sum_{q\in\mathcal{X}} \langle \tilde{e}_{p,z}, \tilde{e}_{q,z+\tau}\rangle \tilde{e}_{q,z+\tau} = \sum_{q\in\mathcal{X}} \langle e_{p,z}, e_{q,z+\tau}\rangle e_{q,z+\tau} + O\big(e^{-c\mu}\big) \\
&= e_{p,z+\tau} + O\big(\mu^{-2}\big) = \tilde{e}_{p,z+\tau} + O\big(\mu^{-2}\big) .
\end{aligned} \tag{3.26}$$

Repeating (3.26) interchanging the roles of z and $z+\tau$, we get

$$P_{z,\mathrm{sm}} P_{z+\tau,\mathrm{sm}} \tilde{e}_{p,z} = P_{z,\mathrm{sm}} \tilde{e}_{p,z+\tau} + O\big(\mu^{-2}\big) = \tilde{e}_{p,z} + O\big(\mu^{-2}\big) .$$

This gives the second equality of the statement: for any $\alpha \in L^2(M;\Lambda)$,

$$\begin{aligned} P_{z,\mathrm{sm}}\alpha &= \sum_{p\in\mathcal{X}} \langle \alpha, \tilde{e}_{p,z}\rangle \tilde{e}_{p,z} = P_{z,\mathrm{sm}} P_{z+\tau,\mathrm{sm}} \sum_{p\in\mathcal{X}} \langle \alpha, \tilde{e}_{p,z}\rangle \tilde{e}_{p,z} + O\big(\mu^{-2}\big)\|\alpha\| \\ &= P_{z,\mathrm{sm}}\alpha + O\big(\mu^{-2}\big)\|\alpha\| \,. \end{aligned}$$

By (3.26),

$$\begin{aligned} \big\|\tilde{e}_{p,z} - \tilde{e}_{p,z+\tau}\big\|^2 &= \|\tilde{e}_{p,z}\|^2 - 2\Re\langle \tilde{e}_{p,z}, \tilde{e}_{p,z+\tau}\rangle + \|\tilde{e}_{p,z+\tau}\|^2 = 2 - 2\Re\langle P_{z+\tau,\mathrm{sm}}\tilde{e}_{p,z}, \tilde{e}_{p,z+\tau}\rangle \\ &= 2 - 2\Re\langle \tilde{e}_{p,z+\tau}, \tilde{e}_{p,z+\tau}\rangle + O\big(\mu^{-2}\big) = O\big(\mu^{-2}\big) \,, \end{aligned}$$

which means

$$\tilde{e}_{p,z} = \tilde{e}_{p,z+\tau} + O\big(\mu^{-1}\big) \,. \tag{3.27}$$

The last stated equality follows from (3.26) and (3.27): for any $\alpha \in L^2(M;\Lambda)$,

$$P_{z,\mathrm{sm}}\alpha = \sum_{p\in\mathcal{X}} \langle \alpha, \tilde{e}_{p,z}\rangle \tilde{e}_{p,z} = \sum_{p\in\mathcal{X}} \langle \alpha, \tilde{e}_{p,z+\tau}\rangle \tilde{e}_{p,z+\tau} + O\big(\mu^{-1}\big)\alpha = P_{z+\tau,\mathrm{sm}}\alpha + O\big(\mu^{-1}\big)\alpha \,. \qquad \square$$

Corollary 3.19. *For every $\tau \in \mathbb{R}$, on $L^2(M;\Lambda)$,*

$$d_{z+\tau,\mathrm{sm}} - d_{z+\tau}P_{z,\mathrm{sm}} = O\big(\mu^{-1}\big) \quad (\mu \to +\infty) \,.$$

Proof. Since $d_{z+\tau} = d_z + \tau\,\eta\wedge$, it follows from Theorem 3.10 that $d_{z+\tau}$ is bounded on $E_{z,\mathrm{sm}} + E_{z+\tau,\mathrm{sm}}$, uniformly on $\mu \gg 0$. Hence, by Proposition 3.18,

$$d_{z+\tau,\mathrm{sm}} - d_{z+\tau}P_{z,\mathrm{sm}} = d_{z+\tau}(P_{z+\tau,\mathrm{sm}} - P_{z,\mathrm{sm}}) = O\big(\mu^{-1}\big) \,. \qquad \square$$

Proposition 3.20. *On $L^2(M;\Lambda)$,*

$$P_{z,\mathrm{sm}}\,\eta\wedge, \eta\wedge P_{z,\mathrm{sm}} = O\big(\mu^{-1/2}\big) \quad (\mu \to +\infty) \,.$$

Proof. Let $\alpha \in L^2(M;\Lambda^k)$ and $p \in \mathcal{X}_{k+1}$. Using multi-index notation, write $\alpha = \sum_{|J|=k} \alpha_J\, dx_p^J$ on U_p with $\alpha_J \in L^2(U_p)$. Let $J_j = \{1,\dots,k+1\}\backslash\{j\}$ $(j = 1,\dots,k+1)$. By Proposition 3.1 (iii), (3.10) and (3.11),

$$\begin{aligned} \langle \eta\wedge\alpha, e_{p,z}\rangle &= -\frac{1}{a_\mu}\sum_{j=1}^{k+1} \big\langle \alpha_{J_j} x_p^j\, dx_p^j \wedge dx_p^{J_j}, e^{-i\nu h_p}\rho_p e^{-\mu|x_p|^2/2}\, dx_p^1\wedge\cdots\wedge dx_p^{k+1}\big\rangle \\ &= (-1)^j \frac{1}{a_\mu}\sum_{j=1}^{k+1} \big\langle \alpha_{J_j} x_p^j, e^{-i\nu h_p}\rho_p e^{-\mu|x_p|^2/2}\big\rangle \,. \end{aligned}$$

Hence

$$|\langle \eta\wedge\alpha, e_{p,z}\rangle| \leqslant \frac{1}{a_\mu}\sum_{j=1}^{k+1} \big|\big\langle \alpha_{J_j}, e^{-i\nu h_p}\rho_p x_p^j e^{-\mu|x_p|^2/2}\big\rangle\big| \leqslant \|\alpha\| \frac{1}{a_\mu}\sum_{j=1}^{k+1} \big\|\rho_p x_p^j e^{-\mu|x_p|^2/2}\big\| \,, \tag{3.28}$$

and

$$\begin{aligned}\big\|\rho_p x_p^j e^{-\mu|x_p|^2/2}\big\| &= \Big(\int_{U_p} (\rho_p(x_p)x_p^j)^2 e^{-\mu|x_p|^2}\ \mathrm{vol}(x_p)\Big)^{1/2}\\ &= \Big(\int_{-2r}^{2r} \rho(x)^2 x^2 e^{-\mu x^2}\,dx\Big)^{\frac{1}{2}}\Big(\int_{-2r}^{2r} \rho(y)^2 e^{-\mu y^2}\,dy\Big)^{\frac{n-1}{2}}\\ &= \Big(\int_{-2r}^{2r} x^2 e^{-\mu x^2}\,dx\Big)^{\frac{1}{2}}\Big(\int_{-2r}^{2r} e^{-\mu y^2}\,dy\Big)^{\frac{n-1}{2}} + O\big(e^{-c\mu}\big)\\ &= \frac{1}{(2\mu)^{1/2}}\Big(\frac{\pi}{\mu}\Big)^{n/4} + O\big(e^{-c\mu}\big)\ . \end{aligned} \tag{3.29}$$

By (3.12), (3.28) and (3.29),

$$|\langle \eta\wedge\alpha, e_{p,z}\rangle| = \|\alpha\|\, O\big(\mu^{-1/2}\big)\ .$$

Since

$$P_z(\eta\wedge\alpha) = \sum_{p\in\mathcal{X}_{k+1}} \langle \eta\wedge\alpha, e_{p,z}\rangle e_{p,z}\ ,$$

it follows that

$$\|P_z(\eta\wedge\alpha)\| = \|\alpha\|\, O\big(\mu^{-1/2}\big)\ .$$

This shows that $P_z\,\eta\wedge = O\big(\mu^{-1/2}\big)$, yielding $P_{z,\mathrm{sm}}\,\eta\wedge = O\big(\mu^{-1/2}\big)$ by Proposition 3.18.

On the other hand, for $q\in\mathcal{X}_k$,

$$\eta\wedge e_{q,z} = \frac{\rho_q}{a_\mu} e^{-\mu|x_p|^2/2} \sum_{j=k+1}^{n} x_q^j\, dx_q^j\wedge dx_q^1\wedge\cdots\wedge dx_q^k\ .$$

So

$$\|\eta\wedge e_{q,z}\| = \frac{1}{a_\mu}\Bigg(\sum_{j=k+1}^{n}\int_{U_q} (\rho_q(x_q)x_q^j)^2 e^{-\mu|x_p|^2}\ \mathrm{vol}(x_q)\Bigg)^{1/2}\ ,$$

which becomes $O\big(\mu^{-1/2}\big)$ like in the previous part of the proof. Since

$$\eta\wedge P_z\alpha = \sum_{q\in\mathcal{X}_k} \langle \alpha, e_{q,z}\rangle\, \eta\wedge e_{q,z}\ ,$$

we get $\eta\wedge P_z = O\big(\mu^{-1/2}\big)$, yielding $\eta\wedge P_{z,\mathrm{sm}} = O\big(\mu^{-1/2}\big)$ as before. □

3.5 Derivatives of the small projection

Remark 3.21. For reasons of brevity, most of the results about derivatives are stated for ∂_z, which may be simply denoted with a dot. But there are obvious versions of those results for $\partial_{\bar z}$ with analogous proofs.

Proposition 3.22. *We have*

$$\operatorname{rank}\partial_z P_{z,\mathrm{sm}} \leqslant 2|\mathcal{X}| \quad (\mu\gg 0)\ , \quad \partial_z P_{z,\mathrm{sm}} = O\big(\mu^{-1/2}\big) \quad (\mu\to+\infty)\ .$$

Proof. By (2.3) and Theorem 3.10, for $\mu\gg 0$ and every $\omega\in\mathbb{S}^1$, a standard computation gives

$$\partial_z\big((w-D_z)^{-1}\big) = (w-D_z)^{-1}\,\eta\wedge\,(w-D_z)^{-1}\ . \tag{3.30}$$

Then, by (3.15), $\partial_z\big((w-D_z)^{-1}\big)$ defines an operator on $L^2(M;\Lambda)$, bounded uniformly on $w\in\mathbb{S}^1$ and $z\in\mathbb{C}$. By (3.15) and Proposition 3.20, we also get

$$P_{z,\mathrm{la/sm}}\partial_z\big((w-D_z)^{-1}\big)\,P_{z,\mathrm{sm/la}}=(w-D_z)^{-1}P_{z,\mathrm{la/sm}}\,\eta\wedge P_{z,\mathrm{sm/la}}(w-D_z)^{-1}=O\big(\mu^{-1/2}\big)\ ,$$

uniformly on $w\in\mathbb{S}^1$.

On the other hand, applying again basic spectral theory, we obtain

$$P_{z,\mathrm{sm}}=\frac{1}{2\pi i}\int_{\mathbb{S}^1}(w-D_z)^{-1}\,dw$$

for $\mu\gg 0$, yielding

$$\dot{P}_{z,\mathrm{sm}}=\frac{1}{2\pi i}\int_{\mathbb{S}^1}\partial_z\big((w-D_z)^{-1}\big)\,dw\ , \tag{3.31}$$

which defines an operator on $L^2(M;\Lambda)$, bounded uniformly on z.

Using that $P_{z,\mathrm{sm}}$ is an orthogonal projection, the argument of the proof of [BGV04, Proposition 9.37] shows that

$$\dot{P}_{z,\mathrm{sm}}=P_{z,\mathrm{la}}\dot{P}_{z,\mathrm{sm}}P_{z,\mathrm{sm}}+P_{z,\mathrm{sm}}\dot{P}_{z,\mathrm{sm}}P_{z,\mathrm{la}}\ . \tag{3.32}$$

So $\operatorname{rank}\dot{P}_{z,\mathrm{sm}}\leqslant 2\operatorname{rank}P_{z,\mathrm{sm}}\leqslant 2|\mathcal{X}|$ by Corollary 3.9, and

$$\begin{aligned}\dot{P}_{z,\mathrm{sm}}&=\frac{1}{2\pi i}\int_{\mathbb{S}^1}P_{z,\mathrm{la}}\partial_z\big((w-D_z)^{-1}\big)\,P_{z,\mathrm{sm}}\,dw\\&\quad+\frac{1}{2\pi i}\int_{\mathbb{S}^1}P_{z,\mathrm{sm}}\partial_z\big((w-D_z)^{-1}\big)\,P_{z,\mathrm{la}}\,dw=O\big(\mu^{-1/2}\big)\ .\ \square\end{aligned}$$

Lemma 3.23. *For all $p\in\mathcal{X}$,*

$$\partial_z e_{p,z}=\left(\frac{n}{8\mu}-\frac{|x_p^+|^2}{2}+O(e^{-c\mu})\right)e_{p,z}\quad(\mu\to+\infty)\ .$$

Proof. Using integration by parts, and since ρ is an even function and ρ' vanishes on $[-r,r]$, we obtain

$$\begin{aligned}\int_{-2r}^{2r}\rho(x)^2x^2e^{-\mu x^2}\,dx&=\frac{1}{2\mu}\int_{-2r}^{2r}(2\rho(x)\rho'(x)x+\rho(x)^2)e^{-\mu x^2}\,dx\\&=\frac{1}{2\mu}\Big(\frac{\pi}{\mu}\Big)^{\frac{1}{2}}+O(e^{-c\mu})\ .\end{aligned}\tag{3.33}$$

So

$$\begin{aligned}\partial_\mu a_\mu&=\partial_\mu\bigg(\bigg(\int_{-2r}^{2r}\rho(x)^2e^{-\mu x^2}\,dx\bigg)^{\frac{n}{2}}\bigg)\\&=-\frac{n}{2}\bigg(\int_{-2r}^{2r}\rho(x)^2e^{-\mu x^2}\,dx\bigg)^{\frac{n}{2}-1}\int_{-2r}^{2r}\rho(x)^2x^2e^{-\mu x^2}\,dx\\&=-\frac{n}{2}\Big(\frac{\pi}{\mu}\Big)^{\frac{n}{4}-\frac{1}{2}}\frac{1}{2\mu}\Big(\frac{\pi}{\mu}\Big)^{\frac{1}{2}}+O(e^{-c\mu})=-\frac{n}{4\mu}\Big(\frac{\pi}{\mu}\Big)^{\frac{n}{4}}+O(e^{-c\mu})\ .\end{aligned}$$

Hence, by (3.12),

$$\partial_\mu\Big(\frac{1}{a_\mu}\Big)=-\frac{\partial_\mu a_\mu}{a_\mu^2}=\frac{n}{4\mu}\Big(\frac{\pi}{\mu}\Big)^{\frac{n}{4}}\Big(\frac{\mu}{\pi}\Big)^{\frac{n}{2}}+O(e^{-c\mu})=\frac{n}{4\mu}\Big(\frac{\mu}{\pi}\Big)^{\frac{n}{4}}+O(e^{-c\mu})\ . \tag{3.34}$$

It also follows from Proposition 3.1 (iii), (3.10), (3.12) and (3.34) that

$$\partial_\mu e_{p,\mu} = \partial_\mu\Big(\frac{\rho_p}{a_\mu} e^{-\mu|x_p|^2/2}\, dx_p^1 \wedge \cdots \wedge dx_p^{\mathrm{ind}(p)}\Big) = \Big(\partial_\mu\Big(\frac{1}{a_\mu}\Big) a_\mu - \frac{|x_p|^2}{2}\Big) e_{p,\mu}$$
$$= \Big(\frac{n}{4\mu} - \frac{|x_p|^2}{2} + O(e^{-c\mu})\Big) e_{p,\mu}\ . \tag{3.35}$$

So, by (3.11),

$$\partial_\mu e_{p,z} = \Big(\frac{n}{4\mu} - \frac{|x_p|^2}{2} + O(e^{-c\mu})\Big) e_{p,z}\ , \quad \partial_\nu e_{p,z} = -i h_p e_{p,z}\ . \tag{3.36}$$

Then the result follows using (3.5). □

Proposition 3.24. *For all $p \in \mathcal{X}$,*

$$\|\partial_z(D_z e_{p,z})\|_{L^\infty} = O(e^{-c\mu}) \quad (\mu \to +\infty)\ .$$

Proof. From (3.13), we get

$$\partial_z(D_z e_{p,z}) = \frac{1}{2}\Big(e^{-i\nu h_p}\partial_\mu\Big(\frac{1}{a_\mu}\Big(\frac{\pi}{\mu}\Big)^{n/4}\Big)\hat{c}(d\rho_p) e_{p,\mu} + e^{-i\nu h_p}\frac{1}{a_\mu}\Big(\frac{\pi}{\mu}\Big)^{n/4}\hat{c}(d\rho_p)\partial_\mu e_{p,\mu}$$
$$- h_p e^{-i\nu h_p}\frac{1}{a_\mu}\Big(\frac{\pi}{\mu}\Big)^{n/4}\hat{c}(d\rho_p) e_{p,\mu}\Big)\ . \tag{3.37}$$

By (3.12) and (3.34),

$$\partial_\mu\Big(\frac{1}{a_\mu}\Big(\frac{\pi}{\mu}\Big)^{\frac{n}{4}}\Big) = \partial_\mu\Big(\frac{1}{a_\mu}\Big)\Big(\frac{\pi}{\mu}\Big)^{\frac{n}{4}} - \frac{n\pi}{4a_\mu\mu^2}\Big(\frac{\pi}{\mu}\Big)^{\frac{n}{4}-1}$$
$$= \frac{n}{4\mu}\Big(\frac{\mu}{\pi}\Big)^{\frac{n}{4}}\Big(\frac{\pi}{\mu}\Big)^{\frac{n}{4}} - \frac{n\pi}{4\mu^2}\Big(\frac{\pi}{\mu}\Big)^{\frac{n}{4}-1}\Big(\frac{\mu}{\pi}\Big)^{\frac{n}{4}} + O(e^{-c\mu}) = O(e^{-c\mu})\ . \tag{3.38}$$

The result follows applying Proposition 3.1 (iii), (3.10), (3.12), (3.35) and (3.38) to (3.37), and using that $d\rho_p = 0$ around p. □

Proposition 3.25. *For every $p \in \mathcal{X}$,*

$$\|\partial_z(P_{z,\mathrm{sm}} e_{p,z} - e_{p,z})\|_{L^\infty} = O(e^{-c\mu}) \quad (\mu \to +\infty)\ .$$

Proof. By (3.18),

$$\partial_z(P_{z,\mathrm{sm}} e_{p,z} - e_{p,z}) = \frac{1}{2\pi i}\int_{\mathbb{S}^1} w^{-1}\partial_z\big((w - D_z)^{-1}\big) D_z e_{p,z}\, dw$$
$$+ \frac{1}{2\pi i}\int_{\mathbb{S}^1} w^{-1}(w - D_z)^{-1}\partial_z(D_z e_{p,z})\, dw\ .$$

Now apply (3.19), (3.30), Corollary 3.6 and Proposition 3.24. □

4. Zeta invariants of Morse forms

4.1 Preliminaries on asymptotic expansions of heat kernels

Let A be a positive semi-definite symmetric elliptic differential operator of order a, and B a differential operator of order b; both of them are defined in $C^\infty(M; E)$ for some Hermitian vector bundle E over M. Then Be^{-tA} is a smoothing operator with Schwartz kernel $K_t(x, y)$ in

$C^\infty(M^2; E \boxtimes E^*)$ (omitting the Riemannian density $\mathrm{vol}(y)$ of the second factor). On the diagonal, there is an asymptotic expansion (as $t \downarrow 0$) [Gil95, Lemma 1.9.1], [BGV04, Theorem 2.30],

$$K_t(x,x) \sim \sum_{l=0}^{\infty} e_l(x) t^{(l-n-b)/a} \,, \tag{4.1}$$

where $e_l \in C^\infty(M; E \otimes E^*)$ is locally computable in terms of the jets of the local coefficients of A and B. Hence the function

$$h(t) = \mathrm{Tr}\left(Be^{-tA}\right) = \int_M \mathrm{tr}\, K_t(x,x)\ \mathrm{vol}(x)$$

has an asymptotic expansion

$$h(t) \sim \sum_{l=0}^{\infty} a_l t^{(l-n-b)/a} \,, \tag{4.2}$$

where

$$a_l = \int_M \mathrm{tr}\, e_l(x)\ \mathrm{vol}(x) \,. \tag{4.3}$$

Given any $\lambda \geqslant 0$, let $P_{A,\lambda}$ be the spectral projection of A corresponding to $[0,\lambda]$; thus $P^\perp_{A,\lambda}$ is the spectral projection corresponding to (λ,∞). By ellipticity, $P_{A,\lambda}$ is of finite rank, and $Be^{-tA}P_{A,\lambda}$ is a smoothing operator defined for all $t \in \mathbb{R}$. Take any orthonormal frame $\phi_1, \ldots, \phi_\kappa$ of $\mathrm{im}\, P_{A,\lambda}$, consisting of eigensections with corresponding eigenvalues $0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_\kappa \leqslant \lambda$. Then the Schwartz kernel $H_t(x,y)$ of $Be^{-tA}P_{A,\lambda}$ ($t \geqslant 0$) is given by

$$H_t(x,y) = \sum_{j=1}^{\kappa} e^{-t\lambda_j} (B\phi_j)(x) \otimes \phi_j(y) \,,$$

using the identity $E \equiv E^*$ given by the Hermitian structure. Thus $H_t(x,y)$ is defined for all $t \in \mathbb{R}$ and smooth. So

$$\mathrm{Tr}(Be^{-tA}P_{A,\lambda}) = \int_M \mathrm{tr}\, H_t(x,x)\ \mathrm{vol}(x) \,.$$

In particular, for $t = 0$, we have

$$H_0(x,x) = \sum_{j=1}^{\kappa} (B\phi_j)(x) \otimes \phi_j(x) \,, \tag{4.4}$$

$$\mathrm{Tr}(BP_{A,\lambda}) = \int_M \mathrm{tr}\, H_0(x,x)\ \mathrm{vol}(x) \,. \tag{4.5}$$

The Schwartz kernel of $Be^{-tA}P^\perp_{A,\lambda}$ is $\widetilde{K}_t(x,y) = K_t(x,y) - H_t(x,y)$ ($t > 0$), which has an asymptotic expansion

$$\widetilde{K}_t(x,x) \sim \sum_{l=0}^{\infty} \tilde{e}_l(x) t^{(l-n-b)/a} \,, \tag{4.6}$$

where the first $n+b$ sections $\tilde{e}_l$ are given by

$$\tilde{e}_l(x) = \begin{cases} e_l(x) & \text{if } l < n+b \\ e_l(x) - H_0(x,x) & \text{if } l = n+b \,. \end{cases}$$

Then the function

$$\tilde{h}_\lambda(t) = \mathrm{Tr}\left(Be^{-tA}P^\perp_{A,\lambda}\right) = \mathrm{Tr}\left(Be^{-tA}\right) - \mathrm{Tr}(Be^{-tA}P_{A,\lambda}) \tag{4.7}$$

has an asymptotic expansion

$$\tilde{h}_\lambda(t) = \int_M \widetilde{K}_t(x,x)\ \mathrm{vol}(x) \sim \sum_{l=0}^{\infty} \tilde{a}_{\lambda,l} t^{(l-n-b)/a}\ , \tag{4.8}$$

where the first $n+b$ coefficients $\tilde{a}_l$ are given by

$$\tilde{a}_{\lambda,l} = \begin{cases} a_l & \text{if } l < n+b \\ a_l - \mathrm{Tr}(BP_{A,\lambda}) & \text{if } l = n+b\ . \end{cases} \tag{4.9}$$

Consider also smooth families of such operators, $\{A_\epsilon\}$ and $\{B_\epsilon\}$, for ϵ in some parameter space. Then $\mathrm{Tr}(B_\epsilon e^{-tA_\epsilon})$ is smooth in (t,ϵ), and we add ϵ to the above notation, writing for instance $K_t(x,y,\epsilon)$, $e_l(x,\epsilon)$, $h(t,\epsilon)$, $a_l(\epsilon)$, $\widetilde{K}_t(x,y,\epsilon)$, $\tilde{e}_{\lambda,l}(x,\epsilon)$, $\tilde{h}_\lambda(t,\epsilon)$ and $\tilde{a}_{\lambda,l}(\epsilon)$ in (4.1), (4.2), (4.6) and (4.8). The operator $B_\epsilon P_{A_\epsilon,\lambda}$ may not be smooth in ϵ when some non-constant spectral branch of $\{A_\epsilon\}$ reaches the value λ. If the values of the non-constant spectral branches of $\{A_\epsilon\}$ stay uniformly away from λ, then $\tilde{h}_\lambda(t,\epsilon)$ is smooth in (t,ϵ).

4.2 Preliminaries on zeta functions of operators

Proposition 4.1 See e.g. [Gil95, Theorems 1.12.2 and 1.12.5], [BGV04, Propositions 9.35–9.37]. *The following holds:*

(i) *For every $\lambda \geqslant 0$, there is a meromorphic function $\zeta(s,A,B,\lambda)$ on $\mathbb{C}$ such that, for $\Re s \gg 0$,*

$$\zeta(s,A,B,\lambda) = \mathrm{Tr}\left(BA^{-s}P_{A,\lambda}^{\perp}\right) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \tilde{h}_\lambda(t)\, dt\ . \tag{4.10}$$

(ii) *The meromorphic function $\Gamma(s)\zeta(s,A,B,\lambda)$ has simple poles at the points $s=(n+b-l)/a$, for $l \in \mathbb{N}_0$ with $\tilde{a}_l \neq 0$. The corresponding residues are $\tilde{a}_l$, and $\zeta(s,A,B,\lambda)$ is smooth away from these exceptional values of s.*

(iii) *For $\mu > \lambda \geqslant 0$, let $\lambda_1 \leqslant \cdots \leqslant \lambda_k$ denote the eigenvalues of A in $(\lambda,\mu]$, taking multiplicities into account, and let $\psi_1,\ldots,\psi_k$ be corresponding orthonormal eigensections. Then, for all s,*

$$\zeta(s,A,B,\mu) - \zeta(s,A,B,\lambda) = \sum_{j=1}^{k} \lambda_k^{-s} \langle B\psi_j, \psi_j\rangle\ .$$

(iv) *For smooth families $\{A_\epsilon\}$ and $\{B_\epsilon\}$ of such operators, if the non-constant branches of eigenvalues of $\{A_\epsilon\}$ stay uniformly away from the value λ, then $\zeta(s,A_\epsilon,B_\epsilon,\lambda)$ is smooth in (s,ϵ) away from the exceptional values of s given in* (ii).

(v) *Consider the conditions of* (iv) *for ϵ in some open neighborhood of 0 in $\mathbb{R}$. If A_0 and B_0 commute, then*

$$\partial_\epsilon \zeta(s,A_\epsilon,B_\epsilon,\lambda)\big|_{\epsilon=0} = \zeta(s,A_0,\dot{B}_0,\lambda) - s\zeta(s+1,A_0,\dot{A}_0B_0,\lambda)\ ,$$

where the dot denotes ∂_ϵ.

The last expression of (4.10) is the Mellin transform of the function $\tilde{h}_\lambda(t)$ divided by $\Gamma(s)$. This function $\zeta(s,A,B,\lambda)$ is called the *zeta function* of (A,B,λ). If $B=1$ or $\lambda=0$, they may be omitted from the notation.

We will also use $\zeta(s,A,B,\lambda)$ when B is not a differential operator, with the same definition. Then the asymptotic expansion (4.8) and the properties stated in Proposition 4.1 need to be

checked. With this generality, we can write

$$\zeta(s, A, B, \lambda) = \zeta(s, A, BP^{\perp}_{A,\lambda}) = \zeta(s, A, P^{\perp}_{A,\lambda}B) \ ,$$

and we have

$$\zeta(s, A, B) = \zeta(s, A, BP_{A,\lambda}) + \zeta(s, A, B, \lambda) \ .$$

The function $\zeta(s, A, BP_{A,\lambda})$ is always defined and holomorphic on $\mathbb{C}$ because $P_{A,\lambda}$ is of finite rank.

In particular, when A is Δ_z or $\Delta_{z,k}$ $(k = 0, \ldots, n)$ and B is another operator in $L^2(M; \Lambda)$, we will use $\zeta(s, A, B, \lambda)$ if it is defined. We get

$$\zeta(s, \Delta_z, B) = \zeta_{\mathrm{sm}}(s, \Delta_z, B) + \zeta_{\mathrm{la}}(s, \Delta_z, B) \ ,$$

where, with the notation of Section 3.2,

$$\zeta_{\mathrm{sm/la}}(s, \Delta_z, B) = \zeta(s, \Delta_z, B_{z,\mathrm{sm/la}}) \ .$$

These are the contributions from the small/large spectrum to $\zeta(s, \Delta_z, B)$, which are called the *small/large zeta functions* of (Δ_z, B).

4.3 Zeta invariants of Morse forms

By Proposition 4.1 (i), as a function of s, $\zeta(s, \Delta_z, \eta\wedge D_z\mathsf{w})$ is meromorphic on $\mathbb{C}$. Moreover, for $\Re s \gg 0$,

$$\begin{aligned}\zeta(s, \Delta_z, \eta\wedge D_z\mathsf{w}) &= \mathrm{Tr}^{\mathrm{s}}\left(\eta\wedge D_z\Delta_z^{-s}\Pi^{\perp}_z\right) = \mathrm{Tr}^{\mathrm{s}}\left(\eta\wedge \delta_z\Delta_z^{-s}\Pi^{1}_z\right)\\ &= \mathrm{Tr}^{\mathrm{s}}\left(\eta\wedge D_z^{-1}\Delta_z^{-s+1}\Pi^{\perp}_z\right) = \mathrm{Tr}^{\mathrm{s}}\left(\eta\wedge d_z^{-1}\Delta_z^{-s+1}\Pi^{1}_z\right) \ ,\end{aligned}$$

using that $\eta\wedge d_z$ and $\eta\wedge\delta_z^{-1}$ change the degree of homogeneous forms. So, when $\zeta(s, \Delta_z, \eta\wedge D_z\mathsf{w})$ is regular at $s = 1$, the value $\zeta(1, \Delta_z, \eta\wedge D_z\mathsf{w})$ is a renormalized version of the super-trace of $\eta\wedge d_z^{-1}\Pi^1_z$, which is called the *zeta invariant* of (M, g, η) for the scope of this paper. Unfortunately, according to Proposition 4.1 (ii) and since $\Gamma(s)$ is regular at $s = 1$, $\zeta(s, \Delta_z, \eta\wedge D_z\mathsf{w})$ might have a simple pole at $s = 1$. It will be shown that we can choose η in the given class $\xi \in H^1(M, \mathbb{R})$ such that $\zeta(s, \Delta_z, \eta\wedge D_z\mathsf{w})$ is regular at $s = 1$ for $\mu \gg 0$. To achieve this task, we consider its decomposition into small/large zeta functions (Section 4.2),

$$\zeta(s, \Delta_z, \eta\wedge D_z\mathsf{w}) = \zeta_{\mathrm{sm}}(s, \Delta_z, \eta\wedge D_z\mathsf{w}) + \zeta_{\mathrm{la}}(s, \Delta_z, \eta\wedge D_z\mathsf{w}) \ .$$

The values $\zeta_{\mathrm{sm/la}}(1, \Delta_z, \eta\wedge D_z\mathsf{w})$ will be called the *small/large zeta invariant* of (M, g, η), if they are defined—$\zeta_{\mathrm{sm}}(1, \Delta_z, \eta\wedge D_z\mathsf{w})$ is always defined.

4.4 Heat invariants of perturbed operators

For $k = 0, \ldots, n$, the Schwartz kernels of $e^{-t\Delta_{z,k}}$ and $e^{-t\Delta_{z,k}}P_{z,\mathrm{la},k}$ are denoted by $K_{z,k,t}(x, y)$ and $\widetilde{K}_{z,k,t}(x, y)$, respectively. Their restrictions to the diagonal have asymptotic expansions (as $t \downarrow 0$),

$$\begin{aligned}K_{z,k,t}(x, x) &\sim \sum_{l=0}^{\infty} e_{k,l}(x, z)t^{(l-n)/2} \ ,\\ \widetilde{K}_{z,k,t}(x, x) &\sim \sum_{l=0}^{\infty} \tilde{e}_{k,l}(x, z)t^{(l-n)/2} \ ,\end{aligned} \tag{4.11}$$

where $e_{k,l}(x,z)$ is locally given by smooth expressions involving z and the jets of the local coefficients of g and η. Take $\mu \gg 0$ so that $P_{z,\mathrm{la},k}$ depends smoothly on z (Theorem 3.10) and $\dim E^k_{z,\mathrm{sm}} = |\mathcal{X}_k|$ (Corollary 3.9). Thus $\tilde{e}_{k,l}(x,z)$ also depends smoothly on z (Section 4.1). Then

$$\tilde{e}_{k,l}(x,z) = \begin{cases} e_{k,l}(x,z) & \text{if } l < n \\ e_{k,n}(x,z) - H_{z,k,0}(x,x) & \text{if } l = n \,, \end{cases} \tag{4.12}$$

where $H_{z,k,t}(x,y)$ is the Schwartz kernel of $e^{-t\Delta_{z,k}} P_{z,\mathrm{sm},k}$, which is defined for all $t \in \mathbb{R}$ and smooth. According to Section 4.2, the corresponding functions

$$h_k(t,z) = \mathrm{Tr}\left(e^{-t\Delta_{z,k}}\right) , \quad \tilde{h}_k(t,z) = \mathrm{Tr}\left(e^{-t\Delta_{z,k}} P_{z,\mathrm{la},k}\right) ,$$

have asymptotic expansions

$$h_k(t,z) \sim \sum_{l=0}^{\infty} a_{k,l}(z) t^{(l-n)/2} , \quad \tilde{h}_k(t,z) \sim \sum_{l=0}^{\infty} \tilde{a}_{k,l}(z) t^{(l-n)/2} , \tag{4.13}$$

where, by (4.3)–(4.5) and (4.9),

$$a_{k,l}(z) = \int_M \mathrm{tr}^{\mathrm{s}}\, e_{k,l}(x,z)\ \mathrm{vol}(x) , \tag{4.14}$$

$$\tilde{a}_{k,l}(z) = \int_M \mathrm{tr}^{\mathrm{s}}\, \tilde{e}_{k,l}(x,z)\ \mathrm{vol}(x) = \begin{cases} a_{k,l}(z) & \text{if } l < n \\ a_{k,l}(z) - |\mathcal{X}_k| & \text{if } l = n \,. \end{cases} \tag{4.15}$$

Consider the operators $e^{-t\Delta_z}\mathsf{w}$ and $e^{-t\Delta_z} P_{z,\mathrm{la}}\mathsf{w}$, whose respective Schwartz kernels are

$$K_{z,t}(x,y) = \sum_{k=0}^{n} (-1)^k K_{z,k,t}(x,y) , \quad \widetilde{K}_{z,t}(x,y) = \sum_{k=0}^{n} (-1)^k \widetilde{K}_{z,k,t}(x,y) .$$

We have induced asymptotic expansions

$$K_{z,t}(x,x) \sim \sum_{l=0}^{\infty} e_l(x,z) t^{(l-n)/2} , \quad \widetilde{K}_{z,t}(x,x) \sim \sum_{l=0}^{\infty} \tilde{e}_l(x,z) t^{(l-n)/2} ,$$

where

$$e_l(x,z) = \sum_{k=0}^{n} (-1)^k e_{k,l}(x,z) , \quad \tilde{e}_l(x,z) = \sum_{k=0}^{n} (-1)^k \tilde{e}_{k,l}(x,z) .$$

The corresponding functions,

$$h(t,z) = \mathrm{Tr}^{\mathrm{s}}\left(e^{-t\Delta_z}\right) = \sum_{k=0}^{n} (-1)^k h_k(t,z) ,$$

$$\tilde{h}(t,z) = \mathrm{Tr}^{\mathrm{s}}\left(e^{-t\Delta_z} P_{z,\mathrm{la}}\right) = \sum_{k=0}^{n} (-1)^k \tilde{h}_k(t,z) ,$$

have asymptotic expansions

$$h(t,z) \sim \sum_{l=0}^{\infty} a_l(z) t^{(l-n)/2} , \quad \tilde{h}(t,z) \sim \sum_{l=0}^{\infty} \tilde{a}_l(z) t^{(l-n)/2} ,$$

where

$$a_l(z) = \sum_{k=0}^{n} (-1)^k a_{k,l}(z) , \quad \tilde{a}_l(z) = \sum_{k=0}^{n} (-1)^k \tilde{a}_{k,l}(z) .$$

Theorem 4.2 [ALG21, Theorem 1.5], [ALKL20]. *We have:*

(i) $e_l(x,z)=0$ *for* $l<n$; *and,*

(ii) *if* n *is even, then* $e_n(x,z)=e(M,\nabla^M)(x)$.

Remark 4.3. Actually, [ALG21, Theorem 1.5] gives Theorem 4.2 when z is real. But, since the functions $e_l(x,z)$ have local expressions, we can assume η is exact. Then the result can be extended to non-real z using (2.11).

4.5 Derived heat invariants of perturbed operators

For $k=0,\dots,n$ and $j=1,2$, let

$$h_k^j(t,z)=\mathrm{Tr}\left(e^{-t\Delta_{z,k}}\Pi_{z,k}^j\right)\ ,\quad \tilde{h}_k^j(t,z)=\mathrm{Tr}\left(e^{-t\Delta_{z,k}}\Pi_{z,\mathrm{la},k}^j\right)\ .$$

Lemma 4.4. *We have*

$$h_{k+1}^1(t,z)=h_k^2(t,z)=\sum_{p=0}^{k}(-1)^{k-p}h_p(t,z)=\sum_{q=k+1}^{n}(-1)^{q-k-1}h_q(t,z)\ .$$

Proof. This follows by induction on k, using that

$$h_0^1(t,z)=h_n^2(t,z)=0\ ,\quad h_k(t,z)=h_k^1(t,z)+h_k^2(t,z)\ ,\quad h_k^2(t,z)=h_{k+1}^1(t,z)\ .$$

The last equality holds because (2.7) is commutative. □

Let

$$h^j(t,z)=\sum_{k=0}^{n}(-1)^k h_k^j(t,z)=\mathrm{Tr}^{\mathrm{s}}\left(e^{-t\Delta_z}\Pi_z^j\right)\ ,$$
$$\tilde{h}^j(t,z)=\sum_{k=0}^{n}(-1)^k \tilde{h}_k^j(t,z)=\mathrm{Tr}^{\mathrm{s}}\left(e^{-t\Delta_z}\Pi_{z,\mathrm{la}}^j\right)\ .$$

Thus

$$h(t,z)=h^1(t,z)+h^2(t,z)\ ,\quad \tilde{h}(t,z)=\tilde{h}^1(t,z)+\tilde{h}^2(t,z)\ . \tag{4.16}$$

Corollary 4.5. *We have* $h(t,z)=0$.

Proof. This is a direct consequence of Lemma 4.4 and (4.16). □

Corollary 4.6. *We have*

$$-(-1)^j h^j(t,z)=\sum_{k=0}^{n}(-1)^k k h_k(t,z)=\mathrm{Tr}^{\mathrm{s}}\left(\mathsf{N}e^{-t\Delta_z}\Pi_z^{\perp}\right)\ .$$

Proof. By Lemma 4.4 and Corollary 4.5,

$$\begin{aligned}h^1(t,z)&=\sum_{k=0}^{n}(-1)^k\sum_{q=k}^{n}(-1)^{q-k}h_q(t,z)=\sum_{q=0}^{n}(-1)^q(q+1)h_q(t,z)\\&=h(t,z)+\sum_{q=0}^{n}(-1)^q q h_q(t,z)=\sum_{q=0}^{n}(-1)^q q h_q(t,z)\ .\ \square\end{aligned}$$

Applying (4.13) and Lemma 4.4, we get

$$h_k^j(t,z) \sim \sum_{l=0}^{\infty} a_{k,l}^j(z) t^{(l-n)/2} , \quad h^j(t,z) \sim \sum_{l=0}^{\infty} a_l^j(z) t^{(l-n)/2} , \tag{4.17}$$

where

$$a_{k+1,l}^1(z) = a_{k,l}^2(z) = \sum_{p=0}^{k} (-1)^{k-p} a_{p,l}(t,z) = \sum_{q=k+1}^{n} (-1)^{q-k-1} a_{q,l}(t,z) ,$$

$$-(-1)^j a_l^j(z) = \sum_{k=0}^{n} (-1)^k k a_{k,l}(z) .$$

This $a_l^1(z)$ is sometimes called the *derived heat invariant* of Δ_z [GS77], [RS71], [Gil95, page 181], [ALG20].

For $\mu \gg 0$, Lemma 4.4, Corollary 4.5 and (4.17) have obvious versions for $\tilde{h}_k^j(t,z)$ and $\tilde{h}^j(t,z)$, with the similar proofs. The coefficients of the corresponding asymptotic expansions are denoted by $\tilde{a}_{k,l}^j(z)$ and $\tilde{a}_l^j(z)$.

Theorem 4.7 [ALG20, Theorem 1.3 (1b)]. *For all $l \leqslant n$, $a_l^1(z)$ and $\tilde{a}_l^1(z)$ are independent of z.*

Remark 4.8. Again, [ALG20, Theorem 1.3 (1b)] gives Theorem 4.7 in case of $a_l^1(z)$ for real z, but this can be extended for non-real z like in Remark 4.3. Then the case of $\tilde{a}_l^1(z)$ follows from (4.15). In Theorem 4.7, the case of $a_l^1(z)$ is true for any $\eta \in Z^1(M,\mathbb{R})$.

Remark 4.9. Theorem 4.2 and the case of $a_l^1(z)$ in Theorem 4.7 hold for any $\eta \in Z^1(M,\mathbb{R})$ (there is no need of other conditions). This is also true for all properties of Sections 4.4 and 4.5 concerning $K_{z,k,t}(x,y)$, $e_{k,l}(x,z)$, $h_k(t,z)$, $h(t,z)$, $h^j(t,z)$, $h_k^j(t,z)$, $a_{k,l}(z)$, $a_l(z)$, $a_{k,l}^j(z)$, $\zeta(s,\Delta_z,\Pi_z^j\mathsf{w})$ and $\theta(s,z)$.

4.6 Regularity

By (4.8), we have an asymptotic expansion of the form

$$\mathrm{Tr}^{\mathrm{s}}\left(\eta\wedge D_z e^{-t\Delta_z} P_{z,\mathrm{la}}\right) \sim \sum_{l=0}^{\infty} \tilde{b}_l(z) t^{(l-n-1)/2} . \tag{4.18}$$

Lemma 4.10. *The function $\zeta_{\mathrm{la}}(s,\Delta_z,\eta\wedge D_z\mathsf{w})$ has a simple pole at every $s=(n+1-l)/2$ with $\tilde{b}_l(z)\neq 0$, for $1\leqslant l\leqslant n$ and for even/odd $l\geqslant n+2$ if n is even/odd, whose residue is $\tilde{b}_l(z)$, and it is smooth away from these values of s. Moreover the value of $\zeta_{\mathrm{la}}(s,\Delta_z,\eta\wedge D_z\mathsf{w})$ at every regular point $s=(n+1-l)/2$, for odd $l\geqslant n+1$, is $(l-n-1)!\,\tilde{b}_l(z)$.*

Proof. By Proposition 4.1 (ii), the product $\Gamma(s)\zeta_{\mathrm{la}}(s,\Delta_z,\eta\wedge D_z\mathsf{w})$ has a simple pole at every $s=(n+1-l)/2$ with $\tilde{b}_l(z)\neq 0$ ($l\in\mathbb{N}_0$), whose residue is $\tilde{b}_l(z)$, and $\zeta_{\mathrm{la}}(s,\Delta_z,\eta\wedge D_z\mathsf{w})$ is smooth away from these exceptional values of s. Then the result follows because $\Gamma(s)$ has a simple pole at every point $s=-k$ ($k\in\mathbb{N}_0$), whose residue is $(-1)^k/k!$, and it is smooth on $\mathbb{C}\setminus(-\mathbb{N}_0)$. □

By Corollary 4.6 and using the notation (1.4) and (1.5), we get

$$(-1)^j\zeta(s,\Delta_z,\Pi_z^j\mathsf{w}) = \theta(s,z) ,$$
$$(-1)^j\zeta_{\mathrm{sm/la}}(s,\Delta_z,\Pi_z^j\mathsf{w}) = \theta_{\mathrm{sm/la}}(s,z) . \tag{4.19}$$

Thus the following result follows like Lemma 4.10.

Lemma 4.11. *The function $\zeta_{\rm la}(s,\Delta_z,\Pi_z^1\mathsf{w})$ has a simple pole at every $s=(n-l)/2$ with $\tilde{a}_l^1(z)\neq 0$, for $1\leqslant l\leqslant n-1$ and for odd/even $l\geqslant n+1$ if n is even/odd, whose residue is $\tilde{a}_l^1(z)$, and it is smooth away from these values of s. Moreover the value of $\zeta_{\rm la}(s,\Delta_z,\Pi_z^1\mathsf{w})$ at every regular point $s=(n-l)/2$, for even $l\geqslant n$, is $(l-n)!\,\tilde{a}_l^1(z)$.*

According to Lemmas 4.10 and 4.11, $\zeta_{\rm la}(s,\Delta_z,\eta\wedge D_z\mathsf{w})$ and $\zeta_{\rm la}(s,\Delta_z,\Pi_z^1\mathsf{w})$ are smooth at $s=0$ [See67], but they might have a simple pole at $s=1$.

Proposition 4.12. *We have*

$$\partial_z\zeta_{\rm la}(s,\Delta_z,\Pi_z^1\mathsf{w})=-s\zeta_{\rm la}(s+1,\Delta_z,\eta\wedge D_z\mathsf{w})\;.$$

Proof. Recall that a dot may be used to denote ∂_z. Like in (3.32),

$$\dot{\Pi}^1_{z,\rm la}=\big(\Pi^1_{z,\rm la}\big)^{\perp}\dot{\Pi}^1_{z,\rm la}\Pi^1_{z,\rm la}+\Pi^1_{z,\rm la}\dot{\Pi}^1_{z,\rm la}\big(\Pi^1_{z,\rm la}\big)^{\perp}\;. \tag{4.20}$$

Therefore, for $\Re s\gg 0$,

$$\zeta_{\rm la}(s,\Delta_z,\dot{\Pi}_z^1\mathsf{w})=\mathrm{Tr}^{\rm s}\big(\dot{\Pi}^1_{z,\rm la}\Delta_z^{-s}\Pi^1_{z,\rm la}\big)=0\;,$$

yielding $\zeta_{\rm la}(s,\Delta_z,\dot{\Pi}_z^1\mathsf{w})=0$ for all s because this is a meromorphic function of s. Hence, since Δ_z and $\Pi^1_{z,\rm la}\mathsf{w}$ commute, Proposition 4.1 (i),(v) give

$$\partial_z\zeta_{\rm la}(s,\Delta_z,\Pi_z^1\mathsf{w})=-s\zeta_{\rm la}(s+1,\Delta_z,\dot{\Delta}_z\Pi_z^1\mathsf{w})=-s\,\mathrm{Tr}^{\rm s}\big(\dot{\Delta}_z\Delta_z^{-s-1}\Pi^1_{z,\rm la}\big)\;.$$

Next, by (2.3),

$$\dot{\Delta}_z\Pi^1_{z,\rm la}=(\eta\wedge\delta_z+\delta_z\,\eta\wedge)\Pi^1_{z,\rm la}=\eta\wedge\delta_z\Pi^1_{z,\rm la}+\delta_z\,\eta\wedge\Pi^1_{z,\rm la}\;. \tag{4.21}$$

But, since $\Pi_z^1\delta_z=0$,

$$\mathrm{Tr}^{\rm s}\big(\delta_z\,\eta\wedge\Delta_z^{-s-1}\Pi^1_{z,\rm la}\big)=-\,\mathrm{Tr}^{\rm s}\big(\eta\wedge\Delta_z^{-s-1}\Pi^1_{z,\rm la}\delta_z\big)=0\;. \tag{4.22}$$

Combining (4.6)–(4.22) and Proposition 4.1 (i), we get

$$\begin{aligned}\partial_z\zeta_{\rm la}(s,\Delta_z,\Pi_z^1\mathsf{w})&=-s\,\mathrm{Tr}^{\rm s}\big(\eta\wedge\delta_z\Delta_z^{-s-1}\Pi^1_{z,\rm la}\big)=-s\,\mathrm{Tr}^{\rm s}\big(\eta\wedge D_z\Delta_z^{-s-1}\Pi^1_{z,\rm la}\big)\\&=-s\zeta_{\rm la}(s+1,\Delta_z,\eta\wedge D_z\mathsf{w})\;.\;\square\end{aligned}$$

Theorem 4.13. *For $\mu\gg 0$, $\zeta_{\rm la}(s,\Delta_z,\eta\wedge D_z\mathsf{w})$ is smooth on $\mathbb{C}\setminus((1-\mathbb{N}_0)/2)$.*

Proof. By Lemma 4.10, it is enough to prove that $\zeta_{\rm la}(s,\Delta_z,\eta\wedge D_z\mathsf{w})$ is regular at $s=s_l=(n+1-l)/2$ for $l\leqslant n-1$, which means $\tilde{b}_l(z)=0$.

First, take $l\leqslant n-2$. By Lemmas 4.10 and 4.11, Theorem 4.7, and Proposition 4.12, for $\mu\gg 0$,

$$0=\partial_z\tilde{a}^1_{l+1}(z)=-(s_l-1)\tilde{b}_l(z)\;.$$

Thus $\tilde{b}_l(z)=0$ because $s_l-1=(n-1-l)/2>0$.

Next, take $l=n-1$. (The previous argument does not apply in this case because $s_{n-1}-1=0$.) By Theorem 4.7 and Lemma 4.11, for $\mu\gg 0$,

$$\partial_z\zeta_{\rm la}(0,\Delta_z,\Pi_z^1\mathsf{w})=\partial_z\tilde{a}^1_n(z)=0\;.$$

So there is a holomorphic function $\phi_z(s)$, defined around $s=0$ and depending smoothly on z, such that

$$\partial_z\zeta_{\rm la}(s,\Delta_z,\Pi_z^1\mathsf{w})=s\phi_z(s)\;.$$

Hence, by Proposition 4.12,

$$\phi_z(s) = -\zeta_{\mathrm{la}}(s+1, \Delta_z, \eta\wedge D_z\mathsf{w}) \ ,$$

obtaining that $\zeta_{\mathrm{la}}(s, \Delta_z, \eta\wedge D_z\mathsf{w})$ is regular at $s = s_{n-1} = 1$. □

From now on, we use the simpler notation

$$\zeta(s,z) = \zeta(s, \Delta_z, \eta\wedge D_z\mathsf{w}) \ , \quad \zeta_{\mathrm{sm/la}}(s,z) = \zeta_{\mathrm{sm/la}}(s, \Delta_z, \eta\wedge D_z\mathsf{w}) \ .$$

We may also add η to this notation, writing $\zeta(s,z,\eta)$ and $\zeta_{\mathrm{sm/la}}(s,z,\eta)$.

Corollary 4.14. *If $\Re s > 1/2$ and $\mu \gg 0$, then*

$$\zeta_{\mathrm{la}}(s,z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}^{\mathrm{s}} \left(\eta\wedge D_z e^{-t\Delta_z} P_{z,\mathrm{la}}\right) dt \ ,$$

where the integral is absolutely convergent.

Proof. By Lemma 4.10, Theorem 4.13 and (4.18), for $\mu \gg 0$,

$$\operatorname{Tr}^{\mathrm{s}} \left(\eta\wedge D_z e^{-t\Delta_z} P_{z,\mathrm{la}}\right) = O\big(t^{-1/2}\big) \quad (t \to 0) \ . \tag{4.23}$$

Moreover there is some $c > 0$ such that

$$\operatorname{Tr}^{\mathrm{s}} \left(\eta\wedge D_z e^{-t\Delta_z} P_{z,\mathrm{la}}\right) = O(e^{-ct}) \quad (t \to \infty) \ . \tag{4.24}$$

So the stated integral is absolutely convergent for $\Re s > 1/2$, defining a holomorphic function of s on this half-plane. Then the stated equality is true because it holds for $\Re s \gg 0$. □

Corollary 4.15. *For $\mu \gg 0$,*

$$\begin{aligned}
\zeta_{\mathrm{sm}}(1,z) &= \operatorname{Tr}^{\mathrm{s}}(\eta\wedge D_z^{-1}(\Pi_z^\perp)_{\mathrm{sm}}) \ , \\
\zeta_{\mathrm{la}}(1,z) &= \lim_{t\downarrow 0} \operatorname{Tr}^{\mathrm{s}} \left(\eta\wedge D_z^{-1} e^{-t\Delta_z} P_{z,\mathrm{la}}\right) \ , \\
\zeta(1,z) &= \lim_{t\downarrow 0} \operatorname{Tr}^{\mathrm{s}} \left(\eta\wedge D_z^{-1} e^{-t\Delta_z} \Pi_z^\perp\right) \ .
\end{aligned}$$

Proof. By Corollary 4.14, (4.23) and (4.24), and since

$$\operatorname{Tr}^{\mathrm{s}} \left(\eta\wedge D_z^{-1} e^{-t\Delta_z} P_{z,\mathrm{la}}\right) = O(e^{-ct}) \quad (t \to \infty) \ ,$$

we get

$$\begin{aligned}
\zeta_{\mathrm{la}}(1,z) &= \int_0^\infty \operatorname{Tr}^{\mathrm{s}} \left(\eta\wedge D_z e^{-u\Delta_z} P_{z,\mathrm{la}}\right) du = \lim_{t\downarrow 0} \int_t^\infty \operatorname{Tr}^{\mathrm{s}} \left(\eta\wedge D_z e^{-u\Delta_z} P_{z,\mathrm{la}}\right) du \\
&= \lim_{t\downarrow 0} \operatorname{Tr}^{\mathrm{s}} \left(\eta\wedge D_z^{-1} e^{-t\Delta_z} P_{z,\mathrm{la}}\right) \ .
\end{aligned}$$

The expressions of $\zeta(1,z)$ and $\zeta_{\mathrm{sm}}(1,z)$ follow from the expression of $\zeta_{\mathrm{la}}(1,z)$ and Proposition 4.1 (iii). □

Theorem 4.13 and Corollary 4.15 give Theorem 1.1 (i).

Corollary 4.16. *If $\mu \gg 0$, then* (1.6) *is true.*

Proof. Apply Proposition 4.12 and Theorem 4.13. □

4.7 The case of exact forms

Let us consider the special case where $\eta = dh$ for a Morse function h. These results will be extended later, but the arguments are much simpler in this case, with weaker conditions and less ingredients.

Lemma 4.17. *For $\mu \gg 0$,*

$$\begin{aligned}\operatorname{Tr}^{\mathrm{s}}\left(\eta\wedge d_z^{-1}\Pi^1_{z,\mathrm{sm}}\right) &= -\operatorname{Tr}^{\mathrm{s}}\left(h\,(\Pi^\perp_z)_{\mathrm{sm}}\right),\\ \operatorname{Tr}^{\mathrm{s}}\left(\eta\wedge d_z^{-1}e^{-t\Delta_z}\Pi^1_{z,\mathrm{la}}\right) &= -\operatorname{Tr}^{\mathrm{s}}\left(h\,e^{-t\Delta_z}P_{z,\mathrm{la}}\right),\\ \operatorname{Tr}^{\mathrm{s}}\left(\eta\wedge d_z^{-1}e^{-t\Delta_z}\Pi^1_z\right) &= -\operatorname{Tr}^{\mathrm{s}}\left(h\,e^{-t\Delta_z}\Pi^\perp_z\right).\end{aligned}$$

Proof. We prove the first equality, the other ones being analogous. Since $\eta\wedge = [d,h]$,

$$\begin{aligned}\operatorname{Tr}^{\mathrm{s}}\left(\eta\wedge d_z^{-1}\Pi^1_{z,\mathrm{sm}}\right) &= \operatorname{Tr}^{\mathrm{s}}\left([d_z,h]\,d_z^{-1}\Pi^1_{z,\mathrm{sm}}\right) = \operatorname{Tr}^{\mathrm{s}}\left(d_z\,h\,d_z^{-1}\Pi^1_{z,\mathrm{sm}}\right) - \operatorname{Tr}^{\mathrm{s}}\left(h\,d_z d_z^{-1}\Pi^1_{z,\mathrm{sm}}\right)\\ &= -\operatorname{Tr}^{\mathrm{s}}\left(h\,d_z^{-1}\Pi^1_{z,\mathrm{sm}}d_z\right) - \operatorname{Tr}^{\mathrm{s}}\left(h\,\Pi^1_{z,\mathrm{sm}}\right)\\ &= -\operatorname{Tr}^{\mathrm{s}}\left(h\,d_z^{-1}d_z\Pi^2_{z,\mathrm{sm}}\right) - \operatorname{Tr}^{\mathrm{s}}\left(h\,\Pi^1_{z,\mathrm{sm}}\right)\\ &= -\operatorname{Tr}^{\mathrm{s}}\left(h\,\Pi^2_{z,\mathrm{sm}}\right) - \operatorname{Tr}^{\mathrm{s}}\left(h\,\Pi^1_{z,\mathrm{sm}}\right) = -\operatorname{Tr}^{\mathrm{s}}\left(h\,(\Pi^\perp_z)_{\mathrm{sm}}\right). \square\end{aligned}$$

Remark 4.18. The last equality of Lemma 4.17 does not require any condition on η or g (h can be any smooth real-valued function).

Corollary 4.19. *For $\mu \gg 0$,*

$$\begin{aligned}\zeta_{\mathrm{sm}}(1,z) &= -\operatorname{Tr}^{\mathrm{s}}\left(h\,(\Pi^\perp_z)_{\mathrm{sm}}\right),\\ \zeta_{\mathrm{la}}(1,z) &= -\lim_{t\downarrow 0}\operatorname{Tr}^{\mathrm{s}}\left(h\,e^{-t\Delta_z}P_{z,\mathrm{la}}\right),\\ \zeta(1,z) &= -\lim_{t\downarrow 0}\operatorname{Tr}^{\mathrm{s}}\left(h\,e^{-t\Delta_z}\Pi^\perp_z\right).\end{aligned}$$

Proof. Apply Corollary 4.15 and Lemma 4.17. □

Corollary 4.20. *The value $\zeta_{\mathrm{sm}}(1,z)$ is uniformly bounded on z for $\mu \gg 0$.*

Proof. The operator $h\,(\Pi^\perp_z)_{\mathrm{sm}}$ is uniformly bounded and, for $\mu \gg 0$, has uniformly bounded rank. So $\operatorname{Tr}^{\mathrm{s}}(h\,(\Pi^\perp_z)_{\mathrm{sm}})$ is uniformly bounded on z for $\mu \gg 0$, and therefore the result follows from Corollary 4.19. □

Corollary 4.21. *If $\mu \gg 0$, then $\zeta(1,z), \zeta_{\mathrm{sm/la}}(1,z) \in \mathbb{R}$.*

Proof. We consider the case of $\zeta(1,z)$, the other cases being similar. By Corollary 4.19, it is enough to prove that $\operatorname{Tr}^{\mathrm{s}}(h\,e^{-t\Delta_z}\Pi^\perp_z) \in \mathbb{R}$. This is true because, taking adjoints,

$$\operatorname{Tr}^{\mathrm{s}}\left(h\,e^{-t\Delta_z}\Pi^\perp_z\right) = \overline{\operatorname{Tr}^{\mathrm{s}}\left(\Pi^\perp_z e^{-t\Delta_z}\,h\right)} = \overline{\operatorname{Tr}^{\mathrm{s}}\left(h\,\Pi^\perp_z e^{-t\Delta_z}\right)} = \overline{\operatorname{Tr}^{\mathrm{s}}\left(h\,e^{-t\Delta_z}\Pi^\perp_z\right)}. \qquad \square$$

Corollary 4.22. *If M is oriented, n is even and $|\mu| \gg 0$, then*

$$\begin{aligned}\zeta(1,z) &= \zeta(1,-\bar z) = \zeta(1,-z) = \zeta(1,\bar z),\\ \zeta_{\mathrm{sm/la}}(1,z) &= \zeta_{\mathrm{sm/la}}(1,-\bar z) = \zeta_{\mathrm{sm/la}}(1,-z) = \zeta_{\mathrm{sm/la}}(1,\bar z).\end{aligned}$$

Proof. We prove the case of $\zeta_{\mathrm{sm}}(1,z)$, the other cases being similar. By (2.15) and since n is even,

$$\begin{aligned}\operatorname{Tr}^{\mathrm{s}}\left(h\,(\Pi^\perp_z)_{\mathrm{sm}}\right) &= \operatorname{Tr}^{\mathrm{s}}\left(\star\star^{-1}h\,(\Pi^\perp_z)_{\mathrm{sm}}\right) = \operatorname{Tr}^{\mathrm{s}}\left(\star^{-1}\,h\,(\Pi^\perp_z)_{\mathrm{sm}}\,\star\right)\\ &= \operatorname{Tr}^{\mathrm{s}}\left(\star^{-1}\star h\,(\Pi^\perp_{-\bar z})_{\mathrm{sm}}\right) = \operatorname{Tr}^{\mathrm{s}}\left(h\,(\Pi^\perp_{-\bar z})_{\mathrm{sm}}\right).\end{aligned}$$

Thus the first equality of the statement follows by Corollary 4.19. The second equality follows with a similar argument, using $\bar\star$ instead of $\star$. The third equality is a consequence of the other ones. □

Theorem 4.23. *The following limit holds uniformly on ν:*

$$\lim_{\mu\to+\infty}\zeta_{\mathrm{la}}(1,z)=-\int_M h\,e(M,\nabla^M)\ \mathrm{vol}+\sum_{p\in\mathcal{X}}(-1)^{\mathrm{ind}(p)}h(p)\ .$$

Proof. By (4.11), (4.12), Theorem 4.2 and Corollary 4.19, for $\mu\gg 0$,

$$\begin{aligned}\zeta_{\mathrm{la}}(1,z)&=-\lim_{t\downarrow 0}\mathrm{Tr}^{\mathrm{s}}\left(h\,e^{-t\Delta_z}P_{z,\mathrm{la}}\right)=-\int_M h(x)\ \mathrm{tr}^{\mathrm{s}}\,\tilde e_n(x,z)\ \mathrm{vol}(x)\\&=-\int_M h(x)\ \mathrm{tr}^{\mathrm{s}}\,e_n(x,z)\ \mathrm{vol}(x)+\mathrm{Tr}^{\mathrm{s}}(hP_{z,\mathrm{sm}})\\&=-\int_M h\,e(M,\nabla^M)\ \mathrm{vol}+\mathrm{Tr}^{\mathrm{s}}(hP_{z,\mathrm{sm}})\ .\end{aligned}$$

According to Corollary 3.9, the elements $P_{z,\mathrm{sm}}e_{p,z}$ $(p\in\mathcal{X})$ form a base of $E^k_{z,\mathrm{sm}}$ when $\mu\gg 0$. Applying the Gram-Schmidt process to this base, we get an orthonormal frame $\tilde e_{p,z}$ $(p\in\mathcal{X})$ of $E_{z,\mathrm{sm}}$. By Proposition 3.7 for $m=0$ and (3.9)–(3.12), we get

$$\lim_{\mu\to+\infty}\langle h\,\tilde e_{p,z},\tilde e_{q,z}\rangle=\lim_{\mu\to+\infty}\langle he_{p,z},e_{q,z}\rangle=h(p)\delta_{pq}\ .$$

Hence

$$\lim_{\mu\to+\infty}\mathrm{Tr}^{\mathrm{s}}(hP_{z,\mathrm{sm}})=\sum_{k=0}^n(-1)^k\sum_{p\in\mathcal{X}_k}h(p)\ .$$ □

5. The small complex vs the Morse complex

5.1 Preliminaries on Smale vector fields

5.1.1 *Vector fields with Morse type zeros* Let X be a real smooth vector field on M with flow $\phi=\{\phi^t\}$. Let $\mathcal{X}=\mathrm{Zero}(X)$ denote the set of zeros of X (or rest points ϕ). It is said that a zero p of X is of *Morse type* with (*Morse*) *index* of $\mathrm{ind}(p)$ if, using the notation (3.2),

$$X=-\sum_{j=1}^n\epsilon_{p,j}x_p^j\,\frac{\partial}{\partial x_p^j}\tag{5.1}$$

on the domain U_p of some coordinates $x_p=(x_p^1,\dots,x_p^n)$ at p, also called *Morse coordinates*. This condition means that $X=-\operatorname{grad}_g h_{X,p}$ on U_p, where $h_{X,p}$ and g are given on U_p by the right-hand side of (3.1) and (3.3). The coordinates x_p used in (5.1) are not unique; that expression is invariant by taking positive multiples of the coordinates (contrary to the expressions (3.1), (3.3) and (3.4)). But $\mathrm{ind}(p)$ is independent of x_p. If (3.3), (3.4) and (5.1) hold with the same coordinates, then η and g are said to be in *standard form* with respect to X around p. In this case, $C\eta$ and Cg $(C>0)$ are also in standard form with respect to X around p; indeed, $C\eta$, X and Cg satisfy (3.3), (3.4) and (5.1) with the coordinates $\sqrt{C}x_p$. When g is defined on M, if η and g are in standard form with respect to X around every $p\in\mathcal{X}$, then η and g are said to be in *standard form* with respect to X. This concept is also applied to any Morse function h on M referring to dh and g. The reference to g may be omitted in this terminology.

The vector field X is fixed in most of the paper and plays an auxiliary role. Unless otherwise indicated, we assume from now on that X has Morse type zeros. Then $\mathcal{X}$ is finite, and the sets $\mathcal{X}_k$, $\mathcal{X}_+$ and $\mathcal{X}_{<k}$ are defined like in Section 3.1.

5.1.2 *Stable and unstable manifolds* For $k = 0, \dots, n$ and $p \in \mathcal{X}_k$, the *stable/unstable manifolds* of p are smooth injective immersions, $\iota_p^\pm : W_p^\pm \to M$, where the images $\iota_p^\pm(W_p^\pm)$ consist of the points satisfying $\phi^t(x) \to p$ as $t \to \pm\infty$, and the manifolds W_p^+ and W_p^- are diffeomorphic to $\mathbb{R}^{n-k}$ and $\mathbb{R}^k$, respectively [Sma63, Theorem 9.1]. In particular, $p \in \iota_p^\pm(W_p^\pm)$, and the maps ι_p^+ and ι_p^- meet transversely at p. Let $p^\pm = (\iota_p^\pm)^{-1}(p)$. Assume every U_p is connected, and let $U_p^\pm$ be the connected component of $(\iota_p^\pm)^{-1}(U_p)$ that contains $p^\pm$. The restriction $\iota_p^\pm : U_p^\pm \to (x_p^\pm)^{-1}(0)$ is a diffeomorphism, and therefore $(U_p^\pm, x_p^\pm \iota_p^\pm)$ is a coordinate system of $W_p^\pm$ centered at $p^\pm$.

5.1.3 *Gradient-like vector fields* Given a Morse function h on M in standard form with respect to X, we have $X = -\operatorname{grad}_g h$ on M for some Riemannian metric g if and only if $Xh < 0$ on $M \setminus \mathcal{X}$ [BFK10, Lemma 2.1], [Lau12, Section 6.1.3]; in this case, X is said to be *gradient-like* (with respect to h). If X is gradient-like, then the maps $\iota_p^\pm$ are embeddings [Sma60b, Lemma 3.8], [BFK10, Lemma 2.2], and their images cover M [Sma61, Lemma 1.1], [BFK10, Corollary 2.5]. Thus, in this case, we will write $W_p^\pm = \iota_p^\pm(W_p^\pm)$ and $p^\pm = p$, and $\iota_p^\pm$ becomes the inclusion map.

Unless otherwise indicated, we also assume in the rest of the paper that X is gradient-like.

5.1.4 *Smale vector fields* X is said to be *Smale* if $W_p^+ \pitchfork W_q^-$ for all $p, q \in \mathcal{X}$. In this case, $\mathcal{M}(p,q) := W_p^+ \cap W_q^-$ is a ϕ-saturated smooth submanifold of dimension $\operatorname{ind}(p) - \operatorname{ind}(q)$. In particular, $\mathcal{M}(p,p) = \{p\}$, and define $\mathcal{T}(p,p) = \emptyset$. If $p \neq q$, then the induced $\mathbb{R}$-action on $\mathcal{M}(p,q)$ is free and proper, and therefore the orbit space $\mathcal{T}(p,q) := \mathcal{M}(p,q)/\mathbb{R}$ is a smooth manifold of dimension $\operatorname{ind}(p) - \operatorname{ind}(q) - 1$. The elements of $\mathcal{T}(p,q)$ are the (unparameterized) trajectories with α-limit $\{p\}$ and ω-limit $\{q\}$, which are oriented by X. If $\operatorname{ind}(p) \leqslant \operatorname{ind}(q)$, then $\mathcal{T}(p,q) = \emptyset$. If $\operatorname{ind}(p) - \operatorname{ind}(q) = 1$, then $\mathcal{T}(p,q)$ consists of isolated points, each of them representing a trajectory in M. Let $\mathcal{T} = \bigcup_{p,q \in \mathcal{X}} \mathcal{T}(p,q)$, and

$$\mathcal{T}_p^1 = \bigcup_{q \in \mathcal{X}_{\operatorname{ind}(p)-1}} \mathcal{T}(p,q)\,, \quad \mathcal{T}_k^1 = \bigcup_{p \in \mathcal{X}_k} \mathcal{T}_p^1\,, \quad \mathcal{T}^1 = \bigcup_{k=0}^{n} \mathcal{T}_k^1\,.$$

The elements of $\mathcal{T}^1$ are called *instantons*.[1]

Unless otherwise indicated, besides the above conditions, we assume from now on that X is Smale; i.e., we assume (b) (Section 1.1). Thus the α- and ω-limits of the orbits of X are zero points [Sma61, Theorem B and Lemma 1.1].

5.1.5 *Lyapunov forms* Any $\eta \in Z^1(M, \mathbb{R})$ is said to be *Lyapunov* for X if $\eta(X) < 0$ on $M \setminus \mathcal{X}$ [BH08, Definition 2.3]. By (b), every class in $H^1(M, \mathbb{R})$ has a representative η which is Lyapunov for X and $\eta^\sharp = -X$ for some Riemannian metric g on M, with η and g in standard form with respect to X [BH04, Proposition 16 (i)], [BH08, Observations 2.5 and 2.6], [HM06, Lemma 3.7], [Lau12, Section 6.1.3]. Moreover X can be C^∞-approximated by gradient-like Smale vector fields that agree with X around $\mathcal{X}$ [BH08, Proposition 2.4] (this follows from [Sma61, Theorem A]). A well known consequence is that, for any Morse function h, there is a C^∞-dense set of Riemannian metrics g on M such that $-\operatorname{grad}_g h$ is Smale; this density is also true in the subspace of metrics

[1] Sometimes, the elements of $\mathcal{T}$ are called *instantons*, and the elements of $\mathcal{T}^1$ *proper instantons* [Bot88].

that are Euclidean with respect to Morse coordinates on given neighborhoods of the critical points.

5.1.6 *Completion of the unstable manifolds*

Proposition 5.1 [BZ92, Appendix by F. Laudenbach, Proposition 2], [Lat94, Chapter 2], [Bur97, Theorem 2.1], [BH01, Theorem 1], [BFK10, Theorem 4.4], [Lau12, Sections A.2 and A.8], [Min15, Corollary 2.3.2]. *The following holds for every $p \in \mathcal{X}_k$ $(k = 0, \dots, n)$:*

(i) *$\overline{W_p^-}$ is a C^1 submanifold with conic singularities (in the sense of [BZ92, Appendix by F. Laudenbach, Section a)] and [Lau12, Appendix A.1]) and a Whitney stratified subspace. Its strata are the submanifolds W_q^- for $q \in \mathcal{X}_{<k}$ with $\mathcal{T}(p,q) \neq \emptyset$. As a consequence, W_p^- has finite volume, and*

$$\overline{W_q^-} \cap \overline{W_p^-} \subset \bigcup_{x \in \mathcal{X}_{<k}} W_x^-$$

if $q \neq p$ in $\mathcal{X}_k$; in particular, $p \notin \overline{W_q^-}$.

(ii) *There is a compact k-manifold with corners[2] $\widehat{W}_p^-$ whose l-corner[3] $(0 \leqslant l \leqslant k)$ is*

$$\partial_l \widehat{W}_p^- = \bigsqcup_{(q_0,\dots,q_l) \in \{p\} \times \mathcal{X}^l} \left(\prod_{j=1}^{l} \mathcal{T}(q_{j-1}, q_j) \right) \times W_{q_l}^- \ .$$

In particular, the interior of $\widehat{W}_p^-$ is $\partial_0 \widehat{W}_p^- = W_p^-$, and the set $\mathcal{T}(p,q)$ is finite if $q \in \mathcal{X}_{k-1}$.

(iii) *There is a smooth map $\hat{\imath}_p^- : \widehat{W}_p^- \to M$ whose restriction to every component of $\partial_l \widehat{W}_p^-$ in* (ii) *is given by the factor projection to $W_{q_l}^-$; in particular, $\hat{\imath}_p^- = \imath_p^-$ on W_p^-. Moreover $\hat{\imath}_p^- : \widehat{W}_p^- \to \overline{W_p^-}$ is a stratified map.*

By Proposition 5.1 (i), we can choose the open sets U_p $(p \in \mathcal{X}_k,\ k = 0, \dots, n)$ so small that $U_p \cap \overline{W_q^-} = \emptyset$ if $q \neq p$ in $\mathcal{X}_k$.

For every $q \in \mathcal{X}_{k-1}$ and $\gamma \in \mathcal{T}(p,q)$, the closure $\bar{\gamma}$ in M is a compact oriented submanifold with boundary of dimension one, and $\partial \bar{\gamma} = \{p, q\}$. We may also consider $\bar{\gamma}$ as the closure of γ in $\widehat{W}_p^-$.

5.2 Preliminaries on the Morse complex

5.2.1 *The Morse complex* Fix an orientation $\mathcal{O}_p^-$ of every unstable manifold W_p^- $(p \in \mathcal{X}_k,\ k = 0, \dots, n)$, which can be also considered as an orientation of $\widehat{W}_p^-$. Then $W_p^- \equiv (W_p^-, \mathcal{O}_p^-)$ defines a current of dimension k on M, also denoted by W_p^-; namely, for $\alpha \in \Omega^k(M)$,

$$\langle W_p^-, \alpha \rangle = \int_{W_p^-} \alpha = \int_{\widehat{W}_p^-} (\hat{\imath}_p^-)^* \alpha \ .$$

Let $\partial_1 \mathcal{O}_p^-$ be the orientation of $\partial_1 \widehat{W}_p^-$ induced by $\mathcal{O}_p^-$ like in the Stokes' theorem. The restriction of $\partial_1 \mathcal{O}_p^-$ to every component $\mathcal{T}(p,q) \times W_q^-$ $(q \in \mathcal{X}_{k'})$ of $\partial_1 \widehat{W}_p^-$ is of the form $\mathcal{O}_{p,q} \otimes \mathcal{O}_q^-$ for a unique orientation $\mathcal{O}_{p,q}$ of $\mathcal{T}(p,q)$. If $k' = k-1$, then $\mathcal{O}_{p,q}$ can be represented by a unique function $\epsilon_{p,q} : \mathcal{T}(p,q) \to \{\pm 1\}$; combining these functions, we get a map $\epsilon : \mathcal{T}^1 \to \{\pm 1\}$. By

[2] In the sense of [Mel96, Section 1.1.8].
[3] The union of the interiors of the boundary faces of codimension l.

the descriptions of $\partial_1\widehat{W}_p^-$ and $\hat\iota_p^- : \partial_1\widehat{W}_p^- \to M$, and by the Stokes' theorem for manifolds with corners, we have [BZ92, Appendix by F. Laudenbach], [HM06, Remark 1.9], [BFK10, Theorem 3.6 and Proposition 5.3], [Lau12, Section 6.5.3]

$$\partial W_p^- = \sum_{q\in\mathcal{X}_{k-1},\ \gamma\in\mathcal{T}(p,q)} \epsilon(\gamma)\, W_q^- \ . \tag{5.2}$$

Thus the currents W_p^- ($p\in\mathcal{X}$) generate over $\mathbb{C}$ a finite dimensional subcomplex $(C_\bullet(X,W^-),\partial)$ of the complex $(\Omega(M)',\partial)$ of currents on M, called the *Morse complex*. The simpler notation $\mathbf{C}_\bullet = \mathbf{C}_\bullet(X) = C_\bullet(X,W^-)$ may be also used. Moreover $\mathbf{C}_\bullet \hookrightarrow \Omega(M)'$ induces an isomorphism[4] $H_\bullet(\mathbf{C}_\bullet,\partial)\cong H_\bullet(M,\mathbb{C})$ [Tho49, Sma60a, Mil65] (see also [Flo89, Sch93, Sch99], [HS85, Theorem 0.1], [BZ92, Appendix by F. Laudenbach, Proposition 7], [Lau12, Section 6.6.5]).

The *dual Morse complex* is the dual $(C^\bullet(X,W^-),\mathbf{d})$ of $(\mathbf{C}_\bullet,\partial)$; namely, $C^k(X,W^-) = (\mathbf{C}_k)^* \equiv \mathbb{C}^{\mathcal{X}_k}$ ($k=0,\dots,n$). We will usually denote $\mathbf{C}^\bullet = \mathbf{C}^\bullet(X) = C^\bullet(X,W^-)$. Moreover boldface notation will be used for elements of $\mathbf{C}^\bullet$ and operators on $\mathbf{C}^\bullet$. Let $\mathbf{e}_p$ ($p\in\mathcal{X}$) denote the elements of the canonical base of $\mathbf{C}^\bullet$, determined by $\mathbf{e}_p(q)=\delta_{pq}$, using the Kronecker delta. Then, for $q\in\mathcal{X}_{k-1}$,

$$\mathbf{d}\mathbf{e}_q = \sum_{p\in\mathcal{X}_k,\ \gamma\in\mathcal{T}(p,q)} \epsilon(\gamma)\, \mathbf{e}_p \ . \tag{5.3}$$

5.2.2 *The perturbed Morse complex* Take any $\eta\in Z^1(M,\mathbb{R})$ defining a class $\xi\in H^1(M,\mathbb{R})$ (there is no need of any condition on η or g in Sections 5.2.2 to 5.2.4). For reasons of brevity, write $\eta(\gamma)=\int_\gamma\eta$ for every $\gamma\in\mathcal{T}^1$. According to [BH01, BH04, BH08], $(\mathbf{C}^\bullet,\mathbf{d})$ has an analog of the Witten's perturbation, $(\mathbf{C}^\bullet,\mathbf{d}_z=\mathbf{d}_{z\eta})$ ($z\in\mathbb{C}$), where, for $q\in\mathcal{X}_{k-1}$ ($k=1,\dots,n$),

$$\mathbf{d}_z\mathbf{e}_q = \sum_{p\in\mathcal{X}_k,\ \gamma\in\mathcal{T}(p,q)} \epsilon(\gamma) e^{z\eta(\gamma)}\mathbf{e}_p \ . \tag{5.4}$$

If $\eta=dh$ for some $h\in C^\infty(M,\mathbb{R})$, then $\mathbf{d}_z = e^{-zh}\mathbf{d}e^{zh}$ on $\mathbf{C}^\bullet$ because $\eta(\gamma)=h(q)-h(p)$ for $p\in\mathcal{X}_k$, $q\in\mathcal{X}_{k-1}$ and $\gamma\in\mathcal{T}(p,q)$; here, $e^{\pm zh}$ also denotes the operator of multiplication by the restriction of this function to $\mathcal{X}$. It will be said that $(\mathbf{C}^\bullet,\mathbf{d}_z)$ ($z\in\mathbb{C}$) is the *perturbed dual Morse complex* defined by X and η. A perturbation $(\mathbf{C}_\bullet,\partial^z)$ is similarly defined, adding the factor $e^{z\eta(\gamma)}$ to the terms of the right-hand side of (5.2).

Since W_p^- ($p\in\mathcal{X}_k$, $k=0,\dots,n$) is diffeomorphic to $\mathbb{R}^k$, there is a unique $h_{\eta,p}^-\in C^\infty(W_p^-,\mathbb{R})$ such that $h_{\eta,p}^-(\hat p^-)=0$ and $dh_{\eta,p}^- = (\iota_p^-)^*\eta$. Indeed $h_{\eta,p}^-$ has a smooth extension $\hat h_{\eta,p}^-$ to $\widehat{W}_p^-$ because $\widehat{W}_p^-$ is contractile. By Proposition 5.1 (ii), for all $q\in\mathcal{X}_{k-1}$ and $\gamma\in\mathcal{T}(p,q)$, we have $\hat h_{\eta,p}^-(\gamma,\hat q^-)=\eta(\gamma)$, yielding

$$\hat h_{\eta,q}^- \equiv \hat h_{\eta,p}^- - \eta(\gamma) \quad\text{on}\quad \widehat{W}_q^- \equiv \{\gamma\}\times\widehat{W}_q^- \subset \partial_1\widehat{W}_p^- \ .$$

According to [BH01, Proposition 4], [BH04, Proposition 10], [BH08, Propositions 2.15 and 2.16 and Section 6.2], a sujective homomorphism of complexes,

$$\Phi_z : (\Omega(M),d_z)\to(\mathbf{C}^\bullet,\mathbf{d}_z) \ ,$$

[4] Actually, $H_\bullet(M,\mathbb{Z})$ is isomorphic to the homology of the complex of free Abelian groups generated by the currents W_p^-.

is defined by

$$\Phi_z(\omega)(p) = \int_{W_p^-} e^{zh^-_{\eta,p}}\omega = \int_{\widehat{W}_p^-} e^{z\hat{h}^-_{\eta,p}}(\hat{\imath}_p^-)^*\omega \ .$$

Moreover Φ_z is a quasi-isomorphism for all $z \in \mathbb{C}$ [BZ92, Proposition 1 in the Appendix by F. Laudenbach] (see also [BZ92, Theorem 2.9], [BZ94, Theorem 1.6], [BH08, Proposition 2.17 and Section 6.2]). Then, by (3.14), the same is true for

$$\Phi_z : (E_{z,\mathrm{sm}}, d_z) \to (\mathbf{C}^\bullet, \mathbf{d}_z) \ .$$

Since direct adaptation of [BH04, Appendix A] shows that, for $k = 0, \dots, n$, $\dim H^k(\mathbf{C}^\bullet, \mathbf{d}_z)$ is independent of $z \in \mathbb{C}$ with $|\mu| \gg 0$, we get (2.9).

5.2.3 *Morse complex with coefficients in a flat vector bundle* With more generality [BZ92, Section 1c)], for a flat vector bundle F, we may consider $(C^\bullet(X, W^-, F), \mathbf{d}^F)$, where $C^k(X, W^-, F) = \bigoplus_{p\in\mathcal{X}_k} F_p$, and $\mathbf{d}^F\mathbf{e}$ ($\mathbf{e} \in F_q$, $q \in \mathcal{X}_{k-1}$) is defined like in the right-hand side of (5.3), replacing $\mathbf{e}_p$ with the parallel transport of $\mathbf{e}$ along $\bar\gamma^{-1}$. This is the dual of the complex $(C_\bullet(X, W^-, F^*), \partial^{F^*})$, where $C_k(X, W^-, F^*) = \bigoplus_{p\in\mathcal{X}_k} F_p^*$, and $\partial^F f$ ($f \in F_p^*$, $p \in \mathcal{X}_k$) is defined like in the right-hand side of (5.2), replacing W_q^- with the parallel transport of f along $\bar\gamma$. A homomorphism

$$\Phi^F = \Phi^{X,F} : (\Omega(M, F), d) \to \big(C^\bullet(X, W^-, F), \mathbf{d}^F\big)$$

can be defined like Φ_z, using the isomorphism

$$\Omega^\bullet\big(\widehat{W}_p^-, (\hat{\imath}_p^-)^*F\big) \cong \Omega^\bullet\big(\widehat{W}_p^-\big) \otimes F_p$$

given by the parallel transport of $(\hat{\imath}_p^-)^*F$. With this generality, Φ^F is also induces a quasi-isomorphism [BZ92, Theorem 2.9]. If $F = \mathcal{L}^z$ (Section 2.1.4), then

$$\big(C^\bullet(X, W^-, \mathcal{L}^z), \mathbf{d}^{\mathcal{L}^z}\big) \equiv (\mathbf{C}^\bullet, \mathbf{d}_z) \ , \quad \Phi^{\mathcal{L}^z} \equiv \Phi_z \ .$$

5.2.4 *Hodge theory of the Morse complex* Consider the Hermitian scalar product on $\mathbf{C}^\bullet$ so that the canonical base $\mathbf{e}_p$ ($p \in \mathcal{X}$) is orthonormal. All operators induced by $\mathbf{d}_z$ and this Hermitian structure are called *perturbed Morse operators*. For instance, besides $\mathbf{d}_z$, we have the perturbed Morse operators

$$\boldsymbol{\delta}_z = \mathbf{d}_z^* \ , \quad \mathbf{D}_z = \mathbf{d}_z + \boldsymbol{\delta}_z \ , \quad \boldsymbol{\Delta}_z = \mathbf{D}_z^2 = \mathbf{d}_z\boldsymbol{\delta}_z + \boldsymbol{\delta}_z\mathbf{d}_z \ .$$

In particular, it will be said that $\boldsymbol{\Delta}_z$ is the *perturbed Morse Laplacian*, and its eigenvalues will be called *perturbed Morse eigenvalues*. If $z = 0$, we omit the subscript "z" and the word "perturbed". From (5.4), we easily get

$$\boldsymbol{\delta}_z\mathbf{e}_p = \sum_{q\in\mathcal{X}_{k-1},\ \gamma\in\mathcal{T}(p,q)} e^{\bar z\eta(\gamma)}\epsilon(\gamma)\,\mathbf{e}_q \ , \tag{5.5}$$

for $p \in \mathcal{X}_k$. We also have

$$\mathbf{C}^\bullet = \ker\boldsymbol{\Delta}_z \oplus \operatorname{im}\mathbf{d}_z \oplus \operatorname{im}\boldsymbol{\delta}_z \ ,$$
$$\ker\boldsymbol{\Delta}_z = \ker\mathbf{D}_z = \ker\mathbf{d}_z \cap \ker\boldsymbol{\delta}_z \ , \quad \operatorname{im}\boldsymbol{\Delta}_z = \operatorname{im}\mathbf{D}_z = \operatorname{im}\mathbf{d}_z \oplus \operatorname{im}\boldsymbol{\delta}_z \ .$$

The orthogonal projections of $\mathbf{C}^\bullet$ to $\ker\boldsymbol{\Delta}_z$, $\operatorname{im}\mathbf{d}_z$ and $\operatorname{im}\boldsymbol{\delta}_z$ are denoted by $\boldsymbol{\Pi}_z = \boldsymbol{\Pi}_z^0$, $\boldsymbol{\Pi}_z^1$ and $\boldsymbol{\Pi}_z^2$, respectively. The compositions $\mathbf{d}_z^{-1}\boldsymbol{\Pi}^1$, $\boldsymbol{\delta}_z^{-1}\boldsymbol{\Pi}^2$ and $\mathbf{D}_z^{-1}\boldsymbol{\Pi}^\perp$ are defined like in Section 2.1.2, and there is an obvious version of the commutative diagram (2.7).

5.3 The small complex vs the Morse complex

Our main objects of interest are the form $\eta \in Z^1(M;\mathbb{R})$ and the Riemannian metric g; X plays an auxiliary role. Unless otherwise indicated, assume from now on that η is Lyapunov for X, and η and g are in standard form with respect to X; i.e., we assume (c) besides (a) and (b). Since every $\xi \in H^1(M,\mathbb{R})$ is Lyapunov for X by (b), we can choose some $\eta \in \xi$ and g satisfying (a) and (c).

For every $p \in \mathcal{X}$, consider the functions $h_{\eta,p}$, $h_{X,p}$, $h^-_{\eta,p}$ and $\hat{h}^-_{\eta,p}$ defined in Sections 3.1, 5.1.1 and 5.2.2. Since η and g are in standard form with respect to X, we have $h_{\eta,p} = h_{X,p}$ on U_p, and

$$h^-_{\eta,p} = h_{\eta,p} = -\frac{1}{2}|x^-_p|^2 \tag{5.6}$$

on U^-_p. From now on, the simpler notation $h_p = h_{\eta,p} = h_{X,p}$, $h^-_p = h^-_{\eta,p}$ and $\hat{h}^-_p = \hat{h}^-_{\eta,p}$ will be used. Since η is Lyapunov for X,

$$h^-_p < 0 \quad \text{on} \quad W^-_p \setminus \{p\}\,. \tag{5.7}$$

Consider the notation of Section 5.2.2. Let $J_z : \mathbf{C}^\bullet \to E_z$ be the $\mathbb{C}$-linear isometry given by $J_z(\mathbf{e}_p) = e_{p,z}$, and let $\Psi_z = P_{z,\mathrm{sm}} J_z : \mathbf{C}^\bullet \to E_{z,\mathrm{sm}}$, which is an isomorphism for $\mu \gg 0$ (Corollary 3.9). By Proposition 3.7,

$$\|\Psi_z \mathbf{e}\| = \big(1 + O\big(e^{-c\mu}\big)\big)\|\mathbf{e}\| \quad (\mu \to +\infty)$$

for all $\mathbf{e} \in \mathbf{C}^\bullet$. Using polarization (see e.g. [Kat95, Section I.6.2]) and conjugation, this means that, as $\mu \to +\infty$,

$$\Psi^*_z \Psi_z = 1 + O\big(e^{-c\mu}\big)\,, \quad \Psi_z \Psi^*_z = 1 + O\big(e^{-c\mu}\big)\,. \tag{5.8}$$

Notation 5.2. Consider functions $u(z)$ and $v(z)$ ($z \in \mathbb{C}$) with values in Banach spaces. The notation $u(z) \asymp_0 v(z)$ ($\mu \to \pm\infty$) means

$$u(z) = v(z) + O\big(e^{-c|\mu|}\big) \quad (\mu \to \pm\infty)\,.$$

We may also consider this asymptotic relation when the Banach spaces also depend on z.

Theorem 5.3 Cf. [BZ94, Theorem 6.11], [Zha01, Theorem 6.9], [BH01, Theorem 4]. *For every $\tau \in \mathbb{R}$, as $\mu \to +\infty$,*

$$\Phi_{z+\tau}\Psi_z \asymp_0 \Big(\frac{\pi}{\mu + \tau/2}\Big)^{\mathsf{N}/2} \Big(\frac{\mu}{\pi}\Big)^{n/4}\,.$$

Proof. We adapt the proof of [Zha01, Theorem 6.9] to the case of complex parameter. For every $p \in \mathcal{X}_k$,

$$\Phi_{z+\tau}\Psi_z \mathbf{e}_p = \sum_{q \in \mathcal{X}_k} \mathbf{e}_q \int_{\widehat{W}^-_q} e^{(z+\tau)\hat{h}^-_q}\, (\hat{\imath}^-_q)^* P_{z,\mathrm{sm}} e_{p,z}\,. \tag{5.9}$$

Then the result follows by checking the asymptotics of these integrals.

In the case $q = p$, by (5.7) and Corollary 3.8,

$$\int_{\widehat{W}^-_p} e^{(z+\tau)\hat{h}^-_p}\, (\hat{\imath}^-_p)^* (P_{z,\mathrm{sm}} - 1) e_{p,z} \asymp_0 0\,.$$

But, by Proposition 3.1 (iii), (3.9)–(3.12) and (5.6),

$$\begin{aligned}\int_{\widehat{W}_p^-} e^{(z+\tau)\hat{h}_p^-}(\hat{\imath}_p^-)^* e_{p,z} &= \int_{\widehat{W}_p^-} e^{(z+\tau)\hat{h}_p^-}(\hat{\imath}_p^-)^*\big(e^{-i\nu h_p} e_{p,\mu}\big) = \int_{\widehat{W}_p^-} e^{(\mu+\tau)\hat{h}_p^-}(\hat{\imath}_p^-)^* e_{p,\mu} \\ &= \frac{1}{a_\mu}\Big(\int_{-2r}^{2r} \rho(x) e^{-(2\mu+\tau)x^2/2}\,dx\Big)^k \\ &= \Big(\frac{\pi}{\mu+\tau/2}\Big)^{k/2}\Big(\frac{\mu}{\pi}\Big)^{n/4}\big(1+O\big(e^{-c\mu}\big)\big)\,. \end{aligned} \tag{5.10}$$

(When $\tau = 0$, the last equality is the same as [Zha01, Eq. (6.30)].)

For $q \neq p$ in $\mathcal{X}_k$, since $e_{p,z} = 0$ on $\overline{W_q^-}$ because $U_p \cap \overline{W_q^-} = \emptyset$ (Section 5.1.6), like in the previous case, we get

$$\int_{\widehat{W}_q^-} e^{(z+\tau)\hat{h}_q^-}(\hat{\imath}_q^-)^* P_{z,\mathrm{sm}} e_{p,z} \asymp_0 0\,. \qquad \square$$

Corollary 5.4. *For every $\tau \in \mathbb{R}$, if $\mu \gg 0$, then $\Phi_{z+\tau} : E_{z,\mathrm{sm}} \to \mathbf{C}^\bullet$ is an isomorphism.*

Proof. Apply Theorem 5.3 and Corollary 3.9. $\square$

Remark 5.5. The argument of the proof of Theorem 5.3 shows that

$$\Phi_z J_z = \Big(\frac{\pi}{\mu}\Big)^{\mathsf{N}/2-n/4} + O\big(e^{-c\mu}\big) \quad (\mu \to +\infty)\,.$$

So $\Phi_z : E_z \to \mathbf{C}^\bullet$ is an isomorphism for $\mu \gg 0$ (see also [BH08, Lemma 5.2]).

Let

$$\widetilde{\Psi}_z = \Big(\frac{\mu}{\pi}\Big)^{\mathsf{N}/2-n/4}\Psi_z : \mathbf{C}^\bullet \to E_{z,\mathrm{sm}}\,.$$

Corollary 5.6. *Consider $\widetilde{\Psi}_z^* : E_{z,\mathrm{sm}} \to \mathbf{C}^\bullet$. As $\mu \to +\infty$,*

$$\widetilde{\Psi}_z^*\widetilde{\Psi}_z = \Big(\frac{\mu}{\pi}\Big)^{\mathsf{N}-n/2} + O\big(e^{-c\mu}\big)\,, \quad \widetilde{\Psi}_z\widetilde{\Psi}_z^* = \Big(\frac{\mu}{\pi}\Big)^{\mathsf{N}-n/2} + O\big(e^{-c\mu}\big)\,.$$

Proof. This is a direct consequence of (5.8). $\square$

Corollary 5.7. *For any $\tau \in \mathbb{R}$, consider $\Phi_{z+\tau} : E_{z,\mathrm{sm}} \to \mathbf{C}^\bullet$. As $\mu \to +\infty$,*

$$\Phi_{z+\tau}\widetilde{\Psi}_z \asymp_0 \Big(\frac{\mu}{\mu+\tau/2}\Big)^{\mathsf{N}/2}\,, \quad \widetilde{\Psi}_z\Phi_{z+\tau} \asymp_0 \Big(\frac{\mu}{\mu+\tau/2}\Big)^{\mathsf{N}/2}\,.$$

Proof. The first relation is a restatement of Theorem 5.3. The second relation follows by conjugating the first one by $\widetilde{\Psi}_z$ and using Corollary 5.6. $\square$

Corollary 5.8. *As $\mu \to +\infty$, $\widetilde{\Psi}_z^{-1} \asymp_0 \Phi_z$ on $E_{z,\mathrm{sm}}$.*

Proof. By Corollaries 5.6 and 5.7, on $E_{z,\mathrm{sm}}$,

$$\widetilde{\Psi}_z^{-1} \asymp_0 \widetilde{\Psi}_z^{-1}\widetilde{\Psi}_z\Phi_z = \Phi_z\,. \qquad \square$$

In the rest of this section, we consider $\Phi_z : E_{z,\mathrm{sm}} \to \mathbf{C}^\bullet$ unless otherwise indicated.

Corollary 5.9. *As $\mu \to +\infty$,*

$$\Phi_z^*\Phi_z \asymp_0 \Big(\frac{\pi}{\mu}\Big)^{\mathsf{N}-n/2}\,, \quad \Phi_z\Phi_z^* \asymp_0 \Big(\frac{\pi}{\mu}\Big)^{\mathsf{N}-n/2}\,.$$

Proof. We show the first relation, the other one being similar. By Corollaries 5.6 and 5.8, on $E_{z,\mathrm{sm}}$,

$$\Phi_z^*\Phi_z \asymp_0 \big(\widetilde{\Psi}_z^{-1}\big)^*\widetilde{\Psi}_z^{-1} = \big(\widetilde{\Psi}_z^*\big)^{-1}\widetilde{\Psi}_z^{-1} = \big(\widetilde{\Psi}_z\widetilde{\Psi}_z^*\big)^{-1} \asymp_0 \Big(\frac{\pi}{\mu}\Big)^{\mathsf{N}-n/2} . \qquad \square$$

Corollary 5.10. *As $\mu \to +\infty$,*

$$\widetilde{\Psi}_z \asymp_0 \Big(\frac{\mu}{\pi}\Big)^{\mathsf{N}-n/2}\Phi_z^* .$$

Proof. By Corollaries 5.7 and 5.9,

$$\widetilde{\Psi}_z \asymp_0 \Big(\frac{\mu}{\pi}\Big)^{\mathsf{N}-n/2}\widetilde{\Psi}_z\Phi_z\Phi_z^* \asymp_0 \Big(\frac{\mu}{\pi}\Big)^{\mathsf{N}-n/2}\Phi_z^* . \qquad \square$$

Corollary 5.11. *For every $\tau \in \mathbb{R}$, as $\mu \to +\infty$,*

$$\Phi_{z+\tau}P_{z+\tau,\mathrm{sm}}\widetilde{\Psi}_z \asymp_0 \Big(\frac{\mu}{\mu+\tau/2}\Big)^{\mathsf{N}/2} + O\big(\mu^{-1}\big) .$$

Proof. By Corollaries 5.6, 5.7 and 5.9 and Proposition 3.18,

$$\Phi_{z+\tau}P_{z+\tau,\mathrm{sm}}\widetilde{\Psi}_z = \Phi_{z+\tau}(P_{z+\tau,\mathrm{sm}} - P_{z,\mathrm{sm}})\widetilde{\Psi}_z + \Phi_{z+\tau}\widetilde{\Psi}_z \asymp_0 O\big(\mu^{-1}\big) + \Big(\frac{\mu}{\mu+\tau/2}\Big)^{\mathsf{N}/2} . \qquad \square$$

Corollary 5.12. *As $\mu \to +\infty$,*

$$d_{z,\mathrm{sm}} \asymp_0 \widetilde{\Psi}_z\mathbf{d}_z\Phi_z , \quad \delta_{z,\mathrm{sm}} \asymp_0 \widetilde{\Psi}_z\boldsymbol{\delta}_z\Phi_z .$$

Proof. By Theorem 3.10 and Corollary 5.7,

$$d_{z,\mathrm{sm}} \asymp_0 \widetilde{\Psi}_z\Phi_z d_{z,\mathrm{sm}} = \widetilde{\Psi}_z\mathbf{d}_z\Phi_z .$$

Now, taking adjoints and using Corollaries 5.6, 5.9 and 5.10, we obtain

$$\delta_{z,\mathrm{sm}} = \Phi_{z,\mathrm{sm}}^*\boldsymbol{\delta}_z\widetilde{\Psi}_z^* \asymp_0 \widetilde{\Psi}_z\boldsymbol{\delta}_z\Phi_z . \qquad \square$$

The orthogonal projections of $\mathbf{C}^\bullet$ to the subspaces $\Phi_z(\ker\Delta_{z,\mathrm{sm}})$, $\Phi_z(\operatorname{im} d_{z,\mathrm{sm}})$ and $\Phi_z(\operatorname{im}\delta_{z,\mathrm{sm}})$ are denoted by $\widetilde{\boldsymbol{\Pi}}_z = \widetilde{\boldsymbol{\Pi}}_z^0$, $\widetilde{\boldsymbol{\Pi}}_z^1$ and $\widetilde{\boldsymbol{\Pi}}_z^2$, respectively. The inclusion $\Phi_z(\operatorname{im} d_{z,\mathrm{sm}}) \subset \operatorname{im}\mathbf{d}_z$

$$\widetilde{\boldsymbol{\Pi}}_z^1 = \widetilde{\boldsymbol{\Pi}}_z^1\boldsymbol{\Pi}_z^1 . \tag{5.11}$$

Corollary 5.13. *For $j = 0, 1, 2$, as $\mu \to +\infty$,*

$$\Phi_z\Pi_{z,\mathrm{sm}}^j \asymp_0 \widetilde{\boldsymbol{\Pi}}_z^j\Phi_{z,\mathrm{sm}} , \quad \Pi_{z,\mathrm{sm}}^j \asymp_0 \widetilde{\Psi}_z\widetilde{\boldsymbol{\Pi}}_z^j\Phi_{z,\mathrm{sm}} , \quad \Pi_{z,\mathrm{sm}}^j\widetilde{\Psi}_z \asymp_0 \widetilde{\Psi}_z\widetilde{\boldsymbol{\Pi}}_{z,\mathrm{sm}}^j .$$

Proof. We only prove the case of $\widetilde{\boldsymbol{\Pi}}_z^2$, the other cases being similar. Let $\alpha_{z,1}, \ldots, \alpha_{z,p_z}$ be an orthonormal frame of $\delta_z(E_{z,\mathrm{sm}}^{k+1})$. So $\Phi_z\alpha_{z,1}, \ldots, \Phi_z\alpha_{z,p_z}$ is a base of $\Phi_z\delta_z(E_{z,\mathrm{sm}}^{k+1})$ for $\mu \gg 0$ by Corollary 5.4. Applying the Gram-Schmidt process to this base, we get an orthonormal base $\mathbf{f}_{z,1}, \ldots, \mathbf{f}_{z,p_z}$ of $\Phi_z\delta_z(E_{z,\mathrm{sm}}^{k+1})$. By Corollary 5.9,

$$\langle\Phi_z\alpha_{z,a}, \Phi_z\alpha_{z,b}\rangle \asymp_0 \Big(\frac{\pi}{\mu}\Big)^{k-n/2}\delta_{ab} ,$$

for $1 \leqslant a, b \leqslant p_z$. So

$$\mathbf{f}_{z,a} \asymp_0 \Big(\frac{\mu}{\pi}\Big)^{k/2-n/4}\Phi_z\alpha_{z,a} .$$

Hence, by Corollary 5.9, for any $\beta \in E^k_{z,\mathrm{sm}}$,

$$\widetilde{\mathbf{\Pi}}^2_z \Phi_z \beta = \sum_{a=1}^{p_z} \langle \Phi_z \beta, \mathbf{f}_{z,a} \rangle \mathbf{f}_{z,a} \asymp_0 \left(\frac{\mu}{\pi}\right)^{k-n/2} \sum_{a=1}^{p_z} \langle \Phi_z \beta, \Phi_z \alpha_{z,a} \rangle \Phi_z \alpha_{z,a}$$
$$\asymp_0 \sum_{a=1}^{m} \langle \beta, \alpha_{z,a} \rangle \Phi_z \alpha_{z,a} = \Phi_z \Pi^2_{z,\mathrm{sm}} \beta \, .$$

This shows the first relation of the statement because $\dim E^k_{z,\mathrm{sm}} < \infty$. Then the other stated relations follow using Corollaries 5.6, 5.7 and 5.9. □

According to Corollary 5.4, in the following corollaries, we take $\mu \gg 0$ so that $\Phi_z : E_{z,\mathrm{sm}} \to \mathbf{C}^\bullet$ is an isomorphism.

Corollary 5.14. *As $\mu \to +\infty$,*

$$(\Phi_z^{-1})^* \Phi_z^{-1} \asymp_0 \left(\frac{\mu}{\pi}\right)^{\mathsf{N}-n/2} , \quad \Phi_z^{-1} (\Phi_z^{-1})^* \asymp_0 \left(\frac{\mu}{\pi}\right)^{\mathsf{N}-n/2} .$$

Proof. By Corollary 5.9, for $\mathbf{e} \in \mathbf{C}^k$ with $\|\mathbf{e}\| = 1$,

$$\left\|\Phi_z^{-1}\mathbf{e}\right\| \asymp_0 \left(\frac{\mu}{\pi}\right)^{k/2-n/4} \left\|\Phi_z \Phi_z^{-1}\mathbf{e}\right\| = \left(\frac{\mu}{\pi}\right)^{k/2-n/4} ,$$

yielding the first stated relation. The second one has a similar proof. □

Corollary 5.15. *As $\mu \to +\infty$,*

$$\Phi_z^* \asymp_0 \left(\frac{\pi}{\mu}\right)^{\mathsf{N}-n/2} \Phi_z^{-1} , \quad \widetilde{\Psi}_z \asymp_0 \Phi_z^{-1} .$$

Proof. By Corollaries 5.9 and 5.14,

$$\Phi_z^* = \Phi_z^* \Phi_z \Phi_z^{-1} \asymp_0 \left(\frac{\pi}{\mu}\right)^{\mathsf{N}-n/2} \Phi_z^{-1} , \quad \widetilde{\Psi}_z = \widetilde{\Psi}_z \Phi_z \Phi_z^{-1} \asymp_0 \Phi_z^{-1} . \qquad \square$$

Corollary 5.16. *We have $\widetilde{\mathbf{\Pi}}^1_z = \mathbf{\Pi}^1_z$ for $\mu \gg 0$, and $\widetilde{\mathbf{\Pi}}^2_z \asymp_0 \mathbf{\Pi}^2_z$ as $\mu \to +\infty$.*

Proof. Since $\Phi_z(\operatorname{im} d_{z,\mathrm{sm}}) = \operatorname{im} \mathbf{d}_z$ for $\mu \gg 0$, we get $\widetilde{\mathbf{\Pi}}^1_z = \mathbf{\Pi}^1_z$.

To prove $\widetilde{\mathbf{\Pi}}^2_z \asymp_0 \mathbf{\Pi}^2_z$ as $\mu \to +\infty$, consider the notation of the proof of Corollary 5.13. We have $\alpha_{z,a} = \delta_z \beta_{z,a}$ $(a = 1, \ldots, p_z)$ for some base $\beta_{z,1}, \ldots, \beta_{z,p_z}$ of $\operatorname{im} d_{z,\mathrm{sm},k}$. Hence, by Corollaries 5.7, 5.9 and 5.12,

$$\Phi_z \alpha_{z,a} = \Phi_z \delta_z \beta_{z,a} \asymp_0 \Phi_z \widetilde{\Psi}_z \boldsymbol{\delta}_z \Phi_z \beta_{z,a} \asymp_0 \boldsymbol{\delta}_z \Phi_z \beta_{z,a} , \tag{5.12}$$

and $\boldsymbol{\delta}_z \Phi_z \beta_{z,1}, \ldots, \boldsymbol{\delta}_z \Phi_z \beta_{z,p_z}$ is a base of $\operatorname{im} \boldsymbol{\delta}_{z,k+1}$. Applying the Gram-Schmidt process to this base, we get an orthonormal base $\mathbf{g}_{z,1}, \ldots, \mathbf{g}_{z,p_z}$ of $\operatorname{im} \boldsymbol{\delta}_{z,k+1}$ satisfying $\mathbf{g}_{z,a} \asymp_0 \mathbf{f}_{z,a}$ by (5.12). Then, for any $\mathbf{e} \in \mathbf{C}^k$ with $\|\mathbf{e}\| = 1$,

$$\widetilde{\mathbf{\Pi}}^2_z \mathbf{e} = \sum_{a=1}^{p_z} \langle \mathbf{e}, \mathbf{g}_{z,a} \rangle \mathbf{g}_{z,a} \asymp_0 \sum_{a=1}^{p_z} \langle \mathbf{e}, \mathbf{f}_{z,a} \rangle \mathbf{f}_{z,a} = \mathbf{\Pi}^2_z \mathbf{e} \, . \qquad \square$$

Corollary 5.17. *We have*

$$d_{z,\mathrm{sm}} = \Phi_z^{-1} \mathbf{d}_z \Phi_{z,\mathrm{sm}} , \quad d^{-1}_{z,\mathrm{sm}} \Pi^1_{z,\mathrm{sm}} = \Pi^2_{z,\mathrm{sm}} \Phi_z^{-1} \mathbf{d}_z^{-1} \Phi_z \Pi^1_{z,\mathrm{sm}} .$$

Proof. The first equality follows like the first relation of Corollary 5.12, using Φ_z^{-1} instead of $\widetilde{\Psi}_z$. To prove the second one, take any $\alpha \in \operatorname{im} d_{z,\mathrm{sm}}$. Since

$$d_z \Pi_{z,\mathrm{sm}}^2 \Phi_z^{-1} \mathbf{d}_z^{-1} \Phi_z \alpha = d_z \Phi_z^{-1} \mathbf{d}_z^{-1} \Phi_z \alpha = \Phi_z^{-1} \mathbf{d}_z \mathbf{d}_z^{-1} \Phi_z \alpha = \alpha$$

with $\Pi_{z,\mathrm{sm}}^2 \Phi_z^{-1} \mathbf{d}_z^{-1} \Phi_z \alpha \in \operatorname{im} \delta_{z,\mathrm{sm}}$, we obtain

$$\Pi_{z,\mathrm{sm}}^2 \Phi_z^{-1} \mathbf{d}_z^{-1} \Phi_z \alpha = d_{z,\mathrm{sm}}^{-1} \alpha \, . \qquad \square$$

5.4 Derivatives of some homomorphisms

Theorem 5.18. *As $\mu \to +\infty$,*

$$\partial_z(\Phi_z \Psi_z), \partial_{\bar{z}}(\Phi_z \Psi_z) \asymp_0 \Big(\frac{n}{8\mu} - \frac{\mathsf{N}}{4\mu}\Big)\Big(\frac{\pi}{\mu}\Big)^{\mathsf{N}/2 - n/4} \, .$$

Proof. By (5.9),

$$\partial_z(\Phi_z \Psi_z \mathbf{e}_p) = \sum_{q \in \mathcal{X}_k} \mathbf{e}_q \bigg(\int_{\widehat{W}_q^-} \hat{h}_q^- e^{z\hat{h}_q^-} (\hat{\imath}_q^-)^* P_{z,\mathrm{sm}} e_{p,z} + \int_{\widehat{W}_q^-} e^{z\hat{h}_q^-} (\hat{\imath}_q^-)^* \partial_z (P_{z,\mathrm{sm}} e_{p,z}) \bigg) \, , \tag{5.13}$$

for every $p \in \mathcal{X}_k$ $(k = 0, \ldots, n)$. We estimate each of these integrals.

Like in the proof of Theorem 5.3, we get, for any $q \neq p$ in $\mathcal{X}_k$,

$$\int_{\widehat{W}_p^-} \hat{h}_p^- e^{z\hat{h}_p^-} (\hat{\imath}_p^-)^* (P_{z,\mathrm{sm}} - 1) e_{p,z} \asymp_0 0 \, , \tag{5.14}$$

$$\int_{\widehat{W}_q^-} \hat{h}_q^- e^{z\hat{h}_q^-} (\hat{\imath}_q^-)^* P_{z,\mathrm{sm}} e_{p,z} \asymp_0 0 \, . \tag{5.15}$$

Moreover, by Proposition 3.1 (iii), (3.9)–(3.12) and (3.33),

$$\begin{aligned} \int_{\widehat{W}_p^-} \hat{h}_p^- e^{z\hat{h}_p^-} (\hat{\imath}_p^-)^* e_{p,z} &= -\frac{k}{2a_\mu} \bigg(\int_{-2r}^{2r} \rho(x) e^{-\mu x^2/2} \, dx \bigg)^{k-1} \int_{-2r}^{2r} \rho(x) x^2 e^{-\mu x^2/2} \, dx \\ &= -\frac{k}{4\mu} \Big(\frac{\pi}{\mu}\Big)^{\frac{k}{2} - \frac{n}{4}} + O(e^{-c\mu}) \, . \end{aligned} \tag{5.16}$$

On the other hand, by (5.7) and Proposition 3.25 ,

$$\int_{\widehat{W}_q^-} e^{z\hat{h}_q^-} (\hat{\imath}_q^-)^* \partial_z (P_{z,\mathrm{sm}} e_{p,z} - e_{p,z}) \asymp_0 0 \, ,$$

for all $q \in \mathcal{X}_k$. In the case $q = p$, by (5.10) and Lemma 3.23,

$$\begin{aligned} \int_{\widehat{W}_p^-} e^{z\hat{h}_p^-} (\hat{\imath}_p^-)^* \partial_z e_{p,z} &= \Big(\frac{n}{8\mu} + O(e^{-c\mu})\Big) \int_{\widehat{W}_p^-} e^{z\hat{h}_p^-} (\hat{\imath}_p^-)^* e_{p,z} \\ &= \Big(\frac{n}{8\mu} + O(e^{-c\mu})\Big) \bigg(\Big(\frac{\pi}{\mu}\Big)^{\frac{k}{2} - \frac{n}{4}} + O(e^{-c\mu}) \bigg) \\ &= \frac{n}{8\mu} \Big(\frac{\pi}{\mu}\Big)^{\frac{k}{2} - \frac{n}{4}} + O(e^{-c\mu}) \, . \end{aligned} \tag{5.17}$$

In the case $q \neq p$, using Lemma 3.23 and arguing again like in the proof of Theorem 5.3, we get

$$\int_{\widehat{W}_q^-} e^{z\hat{h}_q^-} (\hat{\imath}_q^-)^* \partial_z e_{p,z} \asymp_0 0 \quad (\mu \to +\infty) \, . \tag{5.18}$$

Now the result for ∂_z follows from (5.13)–(5.16), (5.17) and (5.18).

If we consider $\partial_{\bar z}$, the proof has to be modified as follows. In the expression analogous to (5.13), the first term of the right-hand side must be removed. In the analogue of Lemma 3.23, we get $|x_p^-|$ instead of $|x_p^+|^2$ by (3.5) and (3.36). So $\partial_{\bar z}(\Phi_z\Psi_z)$ has the same final expression as $\partial_z(\Phi_z\Psi_z)$ by (5.16). □

Theorem 5.19. *As $\mu\to+\infty$,*

$$\partial_z\big((\Psi_z^*\Psi_z)^{\pm1}\big),\partial_{\bar z}\big((\Psi_z^*\Psi_z)^{\pm1}\big)=O\big(\mu^{-1/2}\big)\ .$$

Proof. We only show the case of ∂_z. Consider $P_{z,\mathrm{sm}}:E_z\to E_{z,\mathrm{sm}}$, whose adjoint is $P_z:E_{z,\mathrm{sm}}\to E_z$. Then, since $J_z:\mathbf{C}^\bullet\to E_z$ is an isometry,

$$\Psi_z^*\Psi_z=(P_{z,\mathrm{sm}}J_z)^*P_{z,\mathrm{sm}}J_z=J_z^{-1}P_zP_{z,\mathrm{sm}}J_z\ .$$

It follows that, for every $p\in\mathcal{X}_k$ $(k=0,\dots,n)$,

$$\Psi_z^*\Psi_z\mathbf{e}_p=\sum_{q\in\mathcal{X}_k}\langle P_{z,\mathrm{sm}}e_{p,z},e_{q,z}\rangle\mathbf{e}_q\ .$$

Therefore

$$\partial_z(\Psi_z^*\Psi_z)\mathbf{e}_p=\sum_{q\in\mathcal{X}_k}\big(\langle\partial_z(P_{z,\mathrm{sm}})e_{p,z},e_{q,z}\rangle+\langle P_{z,\mathrm{sm}}\partial_z(e_{p,z}),e_{q,z}\rangle+\langle P_{z,\mathrm{sm}}e_{p,z},\partial_{\bar z}(e_{q,z})\rangle\big)\mathbf{e}_q\ .$$

Then, by Propositions 3.18 and 3.22, Lemma 3.23 and its analogue for $\partial_{\bar z}$,

$$\begin{aligned}\partial_z(\Psi_z^*\Psi_z)\mathbf{e}_p&=O\big(\mu^{-1/2}\big)+\Big(\frac{n}{8\mu}-\frac{1}{2}\big\langle|x_p^+|^2e_{p,z},e_{p,z}\big\rangle\Big)\mathbf{e}_p+O\big(e^{-c\mu}\big)\\&=\Big(\frac{n}{8\mu}-\frac{1}{2}\big\langle|x_p^+|^2e_{p,z},e_{p,z}\big\rangle\Big)\mathbf{e}_p+O\big(\mu^{-1/2}\big)\ .\end{aligned}$$

But, by (3.12) and (3.33),

$$\begin{aligned}\big\langle|x_p|^2e_{p,z},e_{p,z}\big\rangle&=\Big(\int_{-2r}^{2r}\rho(x)^2e^{-\mu x^2}\,dx\Big)^{n-1}(n-k)\int_{-2r}^{2r}y^2\rho(y)^2e^{-\mu y^2}\,dy\\&=\frac{n-k}{2\mu}\Big(\frac{\pi}{\mu}\Big)^{\frac{n}{2}}+O\big(e^{-c\mu}\big)\ .\end{aligned}$$

Hence

$$\partial_z(\Psi_z^*\Psi_z)\mathbf{e}_p=\Big(\frac{n}{8\mu}-\frac{n-k}{4\mu}\Big(\frac{\pi}{\mu}\Big)^{\frac{n}{2}}\Big)\mathbf{e}_p+O\big(\mu^{-1/2}\big)=O\big(\mu^{-1/2}\big)\ ,$$

yielding the stated expression for $\partial_z\big(\Psi_z^*\Psi_z\big)$.

Now, arguing like in the proof of (3.30) and using (5.8), we get

$$\begin{aligned}\partial_z\big((\Psi_z^*\Psi_z)^{-1}\big)&=-(\Psi_z^*\Psi_z)^{-1}\partial_z(\Psi_z^*\Psi_z)(\Psi_z^*\Psi_z)^{-1}\\&=-\big(1+O\big(e^{-c\mu}\big)\big)O\big(\mu^{-1/2}\big)\big(1+O\big(e^{-c\mu}\big)\big)=O\big(\mu^{-1/2}\big)\ .\ \square\end{aligned}$$

6. Asymptotics of the large zeta invariant

6.1 Preliminaries on Quillen metrics

6.1.1 *Case of a finite dimensional complex* All vector spaces considered here are over $\mathbb{C}$. For a line λ, its dual λ^* is also denoted by λ^{-1}. For a vector space V of finite dimension, let $\det V=\bigwedge^{\dim V}V$. For a graded vector space $V^\bullet$ of finite dimension, let $\det V^\bullet=\bigotimes_k(\det V^k)^{(-1)^k}$.

Now consider a finite dimensional cochain complex $(V^\bullet, \partial)$, whose cohomology is denoted by $H^\bullet(V)$. Then there is a canonical isomorphism [KM76], [BGS88, Section 1 a)]

$$\det V^\bullet \cong \det H^\bullet(V) \ . \tag{6.1}$$

Given a Hermitian metric on $V^\bullet$ so that the homogeneous components V^k are orthogonal one another, the corresponding norm $\| \ \|_{V^\bullet}$ on $V^\bullet$ induces a metric $\| \ \|_{\det V^\bullet}$ on $\det V^\bullet$, which corresponds to a metric $\| \ \|_{\det H^\bullet(V)}$ on $\det H^\bullet(V)$ via (6.1).

On the other hand, consider the induced Laplacian, $\Box = (\partial + \partial^*)^2 = \partial\partial^* + \partial^*\partial$, whose kernel is a graded vector subspace $\mathcal{H}^\bullet$. Then finite dimensional Hodge theory gives an isomorphism $H^\bullet(V) \cong \mathcal{H}^\bullet$, which induces an isomorphism

$$\det H^\bullet(V) \cong \det \mathcal{H}^\bullet \ . \tag{6.2}$$

The restriction of $\| \ \|_{V^\bullet}$ to $\mathcal{H}^\bullet$ induces a metric $\| \ \|_{\det \mathcal{H}^\bullet}$ on $\det \mathcal{H}^\bullet$, which corresponds to another metric $| \ |_{\det H^\bullet(V)}$ on $\det H^\bullet(V)$ via (6.2).

Let $\Box'$ denote the restriction $\Box : \operatorname{im}\Box \to \operatorname{im}\Box$. For $s \in \mathbb{C}$, let

$$\theta(s) = \theta(s, \Box) = -\operatorname{Tr^s}(\mathsf{N}(\Box')^{-s}) \ . \tag{6.3}$$

(Do not confuse the superscript "s" of the supertrace with the complex variable s.) This defines a holomorphic function on $\mathbb{C}$. Then the above metrics on $\det H^\bullet(V)$ satisfy [BGS88, Proposition 1.5], [BZ92, Theorem 1.1], [BZ94, Theorem 1.4]

$$\| \ \|_{\det H^\bullet(V)} = | \ |_{\det H^\bullet(V)} e^{\theta'(0)/2} \ . \tag{6.4}$$

If $H^\bullet(V) = 0$, then $\det H^\bullet(V) \equiv \mathbb{C}$ is canonically generated by 1, and we have $\|1\|_{\det H^\bullet(V)} = e^{\theta'(0)/2}$. Using the orthogonal projection $\Pi^1 : V \to \operatorname{im}\partial$, we can write (6.3) as

$$\theta(s) = -\operatorname{Tr^s}\left((\Box')^{-s}\Pi^1\right) \ . \tag{6.5}$$

Let $(\widetilde{V}^\bullet, \tilde{\partial})$ be another finite dimensional cochain complex, endowed with a Hermitian metric so that the homogeneous components are orthogonal to each other, and let $\phi : (V, \partial) \to (\widetilde{V}^\bullet, \tilde{\partial})$ be an isomorphism of cochain complexes, which may not be unitary. Then (see the proof of [BZ94, Theorem 6.17])

$$\log\left(\frac{\| \ \|_{\det H^\bullet(\widetilde{V})}}{\| \ \|_{\det H^\bullet(V)}}\right)^2 = \operatorname{Tr^s}(\log(\phi^*\phi)) \ . \tag{6.6}$$

6.1.2 *Case of an elliptic complex* Some of the concepts of Section 6.1.1 extend to the case where $V^\bullet = C^\infty(M; E^\bullet)$, for some graded Hermitian vector bundle $E^\bullet$ over M, and ∂ is an elliptic differential complex of order one. Then $\det H^\bullet(V)$ is defined because $\dim H^\bullet(V) < \infty$. Moreover Hodge theory for the Laplacian $\Box$ gives the isomorphism (6.2). Thus at least the norm $| \ |_{\det H^\bullet(V)}$ is defined in this setting. Now the expression (6.3) only defines $\theta(s) = \theta(s, \Box)$ when $\Re s > n/2$, but it has a meromorphic extension to $\mathbb{C}$, denoted in the same way; indeed, (6.3) becomes

$$\theta(s) = \theta(s, \Box) = -\zeta(s, \Box, \mathsf{Nw}) \ ,$$

for $\Re s > n/2$, and therefore this equality also holds for the meromorphic extensions. Furthermore $\theta(s)$ is smooth at $s = 0$ [See67], and $\theta'(0)$ can be considered as a renormalized version of the super-trace of the operator $\mathsf{N}\log(\Box')$, which is not of trace class. Thus the right-hand side of (6.4) is defined in this way and plays the role of an analytic version of the metric $\| \ \|_{\det H^\bullet(V)}$, which is not directly defined. This kind of metrics were introduced by D. Quillen [Qui85] for the case

of the Dolbeault complex. The expression (6.5) also holds in this case for $\Re s \gg 0$; in fact, it becomes

$$\theta(s) = -\zeta\big(s, \Box, \Pi^1 \mathsf{w}\big) ,$$

where this zeta function can be shown to define a meromorphic function on $\mathbb{C}$, even though Π^1 is not a differential operator, and this equality holds as meromorphic functions.

6.1.3 *Reidemeister, Milnor and Ray-Singer metrics* Let F be a flat vector bundle over M, defined by a representation ρ of $\pi_1 M$, and let ∇^F denote its covariant derivative. Consider a smooth triangulation K of M and the corresponding cochain complex $C^\bullet(K, F)$ with coefficients in F, whose cohomology is isomorphic to $H^\bullet(M, F)$ via the quasi-isomorphism

$$\Omega(M; F) \to C^\bullet(K, F) = C_\bullet(K, F^*)^*$$

defined by integration of differential forms on smooth simplices. Given a Hermitian structure g^F on F, we get an induced metric on $C^\bullet(K, F)$, and the concepts of Section 6.1.1 can be applied. In this case, the left-hand side of (6.4) is called the *Reidemeister metric*, denoted by $\| \ \|^{\mathrm{R}}_{\det H^\bullet(M,F)}$.

If $\nabla^F g^F = 0$ (ρ is unitary) and $H^\bullet(M, F) = 0$, then the Reidemeister torsion $\tau_M(\rho)$ is defined using K, and it is a topological invariant of M [Fra35, Rei35, dR50]. Moreover $\tau_M(\rho) = \|1\|^{\mathrm{R}}_{\det H^\bullet(M,F)}$ is the exponential factor of the right-hand side of (6.4) [RS71, Proposition 1.7]. If we only assume $\nabla^F g^F = 0$, then $\| \ \|^{\mathrm{R}}_{\det H^\bullet(M,F)}$ is still a topological invariant of M.

Next, given a vector field X on M satisfying (b), consider the complex $(C^\bullet(-X, W^-, F), \mathbf{d}^F)$, whose cohomology is also isomorphic to $H^\bullet(M, F)$ via the quasi-isomorphism

$$\Phi^{-X,F} : \Omega(M, F) \to C^\bullet(-X, W^-, F) = C_\bullet(-X, W^-, F^*)^* .$$

This complex has a metric induced by g^F, like in Section 5.2.2, and the concepts of Section 6.1.1 can be also applied. In this case, the left-hand side of (6.4) is called the *Milnor metric*, denoted by $\| \ \|^{\mathrm{M},X}_{\det H^\bullet(M,F)}$, and the metric factor of the right-hand side of (6.4) is denoted by $| \ |^{\mathrm{M},X}_{\det H^\bullet(M,F)}$. If $\nabla^F g^F = 0$, then $\| \ \|^{\mathrm{M},X}_{\det H^\bullet(M,F)} = \| \ \|^{\mathrm{R}}_{\det H^\bullet(M,F)}$ [Mil66, Theorem 9.3].

Finally, the concepts of Section 6.1.2 can be applied to $(\Omega(M, F), d^F)$, whose cohomology is again $H^\bullet(M, F)$. In this case, the right-hand side of (6.4) is called the *Ray-Singer metric*, denoted by $\| \ \|^{\mathrm{RS}}_{\det H^\bullet(M,F)}$, and the metric factor of the right-hand side of (6.4) is denoted by $| \ |^{\mathrm{RS}}_{\det H^\bullet(M,F)}$. If $H^\bullet(M, F) = 0$, then the exponential factor of the right-hand side of (6.4) is called the *analytic torsion* or *Ray-Singer torsion*, denoted by $T_M(\rho)$. These concepts were introduced by Ray and Singer [RS71], who conjectured that $T_M(\rho) = \tau_M(\rho)$ if $\nabla^F g^F = 0$ and $H^\bullet(M, F) = 0$, and $\| \ \|^{\mathrm{RS}}_{\det H^\bullet(M,F)} = \| \ \|^{\mathrm{R}}_{\det H^\bullet(M,F)}$ assuming only that $\nabla^F g^F = 0$. Independent proofs of this conjecture were given by Cheeger [Che79] and Müller [M$\ddot{\text{}}$78]. Actually, this equality still holds if the induced Hermitian structure $g^{\det F}$ on $\det F$ is flat [M$\ddot{\text{}}$93].

In the case where $g^{\det F}$ is not assumed to be flat, Bismut and Zhang [BZ92, BZ94] extended the above results by introducing an additional term. The first ingredient of this extra term is the 1-form

$$\theta(F, g^F) = \operatorname{tr}\left((g^F)^{-1}\nabla^F g^F\right) , \tag{6.7}$$

which vanishes if and only if $g^{\det F}$ is flat. Moreover $\theta(F, g^F)$ is closed and its cohomology class of $\theta(F, g^F)$ is independent of the choice of g^F [BZ92, Proposition 4.6]; this class measures the obstruction to the existence of a flat Hermitian structure on $\det F$.

Let $e(M, \nabla^M)$ be the representative of the Euler class of M given by the Chern-Weil theory

using g^M, and let $\psi(M,\nabla^M)$ be the current of degree $n-1$ on TM constructed in [MQ86] (see also [BGS90, Section 3], [BZ92, Section 3], [BH06, Section 2], [BH08, Section 4]). Identify the image of the zero section of TM with M, and identify the conormal bundle of M in TM with T^*M. Let δ_M be the current on TM defined by integration on M, and let $\pi : TM \to M$ be the vector bundle projection. Since M may not be oriented, the form $e(M,\nabla^M)$ is valued in the orientation line bundle $o(M)$ of M, and the currents are the elements of $\Omega(M,o(M))'$.

Proposition 6.1 Bismut-Zhang [BZ92, Theorem 3.7]. *The following holds:*

(i) *For any smooth function $\lambda : TM \to \mathbb{R}^{\pm}$, under the mapping $v \mapsto \lambda v$, $\psi(M,\nabla^M)$ is changed into $(\pm 1)^n \psi(M,\nabla^M)$.*

(ii) *The current $\psi(M,\nabla^M)$ is locally integrable, and its wave front set is contained in T^*M (thus $\psi(M,\nabla^M)$ is smooth on $TM \setminus M$).*

(iii) *The restriction of $-\psi(M,\nabla^M)$ to the fibers of $TM \setminus M$ coincides with the solid angle defined by g^M.*

(iv) *We have*

$$d\psi(M,\nabla^M) = \pi^* e(M,\nabla^M) - \delta_M \ .$$

Remark 6.2. In Proposition 6.1, observe that (i) and (iv) are compatible because $e(M,\nabla^M) = 0$ if n is odd. By (ii)–(iv), the restriction of $\psi(M,\nabla^M)$ to $TM \setminus M$ is induced by a smooth differential form on the sphere bundle which transgresses $e(M,\nabla^M)$ (such a differential form was already defined and used in [Che44]).

Theorem 6.3 Bismut-Zhang [BZ92, Theorem 0.2], [BZ94, Theorem 0.2]. *We have*

$$\log\left(\frac{\| \ \|^{\mathrm{RS}}_{\det H^\bullet(M,F)}}{\| \ \|^{\mathrm{M},X}_{\det H^\bullet(M,F)}}\right)^2 = -\int_M \theta(F,g^F) \wedge X^* \psi(M,\nabla^M) \ .$$

Remark 6.4. By (b), $X = -\operatorname{grad}_{g'} h$ for some Morse function h and some Riemannian metric g' on M, which may not be the given metric g^M. If we fix h, the right-hand side of the equality in Theorem 6.3 is independent of the choice of X satisfying $X = -\operatorname{grad}_{g'} h$ for some g' [BZ92, Proposition 6.1].

Theorem 6.3 will be applied to the case of the flat complex line bundle $\mathcal{L}^z$ with a Hermitian structure $g^{\mathcal{L}^z}$ (Section 2.1.2). By (2.14) and (6.7),

$$\theta(\mathcal{L}^z, g^{\mathcal{L}^z}) = 2\mu\eta \ . \tag{6.8}$$

6.2 Asymptotics of the large zeta invariant

We prove Theorem 1.1 (ii) here. With the notation of Section 6.1.2, consider the meromorphic function $\theta(s,z) = \theta(s,\Delta_z)$, also defined in (1.4), as well as its components $\theta_{\mathrm{sm/la}}(s,z)$ defined in (1.5). Consider also the current $\psi(M,\nabla^M)$ of degree $n-1$ on TM (Section 6.1.3). By Proposition 6.1 (i),

$$-\mathbf{z}_{\mathrm{la}}(-\eta) = (-1)^n \mathbf{z}_{\mathrm{la}}(\eta) \ . \tag{6.9}$$

Notation 6.5. Let $\asymp_1$ be defined like $\asymp_0$ in Notation 5.2, using $O(|\mu|^{-1/2})$ instead of $O(e^{-c|\mu|})$.

Take some Morse function h on M such that $Xh < 0$ on $M \setminus \mathcal{X}$, and h is in standard form with respect to X. Then $X = -\operatorname{grad}_{g'} h$ for some Riemannian metric g' (Section 5.1.3), which

may not be the given metric g. Consider the flat complex line bundle $\mathcal{L}_{dh-z\eta}$ with the Hermitian structure $g^{F_{dh-z\eta}}$ (Section 2.1.2). Note that $\mathbf{d}_{-dh}^{\mathcal{L}_{dh-z\eta}} \equiv \mathbf{d}_{-z\eta}$ on $C^\bullet(-X, W^-, \mathcal{L}_{dh-z\eta}) \equiv \mathbf{C}^\bullet(-X)$. So, by (6.8), Theorem 6.3 and Remark 6.4,

$$\log \frac{\| \|^{\mathrm{RS}}_{\det H^\bullet_{-z}(M)}}{\| \|^{\mathrm{M},X}_{\det H^\bullet_{-z}(M)}} = -\int_M (dh - \mu\eta) \wedge X^* \psi(M, \nabla^M) \,, \tag{6.10}$$

where $H^\bullet_{-z}(M) = H^\bullet_{-z\eta}(M)$. With the notation of Section 6.1.3, let

$$\| \|^{\mathrm{RS,sm}}_{\det H^\bullet_{-z}(M)} = | \, |^{\mathrm{RS}}_{\det H^\bullet_{-z}(M)} e^{\theta'_{\mathrm{sm}}(0,-z)/2} \,.$$

By (6.4),

$$\log \frac{\| \|^{\mathrm{RS}}_{\det H^\bullet_{-z}(M)}}{\| \|^{\mathrm{M},X}_{\det H^\bullet_{-z}(M)}} = \log \frac{\| \|^{\mathrm{RS,sm}}_{\det H^\bullet_{-z}(M)}}{\| \|^{\mathrm{M},X}_{\det H^\bullet_{-z}(M)}} + \frac{\theta'_{\mathrm{la}}(0,-z)}{2} \,. \tag{6.11}$$

By (6.6) and Corollary 5.4, for $\mu \gg 0$,

$$\begin{aligned} \log \left(\frac{\| \|^{\mathrm{RS,sm}}_{\det H^\bullet_{-z}(M)}}{\| \|^{\mathrm{M},X}_{\det H^\bullet_{-z}(M)}} \right)^2 &= -\operatorname{Tr}^{\mathrm{s}}(\log(\Phi^*_{-z}\Phi_{-z})) = -\operatorname{Tr}^{\mathrm{s}}\big(\log\big(\Psi_{-z}^{-1}\Phi^*_{-z}\Phi_{-z}\Psi_{-z}\big)\big) \\ &= -\operatorname{Tr}^{\mathrm{s}}\big(\log\big((\Psi^*_{-z}\Psi_{-z})^{-1}(\Phi_{-z}\Psi_{-z})^*\Phi_{-z}\Psi_{-z}\big)\big) \,. \end{aligned} \tag{6.12}$$

From (5.8) and Theorems 5.3, 5.18 and 5.19, we obtain

$$\big((\Psi^*_{-z}\Psi_{-z})^{-1}(\Phi_{-z}\Psi_{-z})^*\Phi_{-z}\Psi_{-z}\big)^{-1} = \Big(\frac{\pi}{\mu}\Big)^{\frac{n}{2}-\mathsf{N}} + O\big(e^{-c\mu}\big)$$

and

$$\begin{aligned} &\partial_z\big((\Psi^*_{-z}\Psi_{-z})^{-1}(\Phi_{-z}\Psi_{-z})^*\Phi_{-z}\Psi_{-z}\big) \\ &\quad= \partial_z\big((\Psi^*_{-z}\Psi_{-z})^{-1}\big)(\Phi_{-z}\Psi_{-z})^*\Phi_{-z}\Psi_{-z} + (\Psi^*_{-z}\Psi_{-z})^{-1}(\partial_{\bar z}(\Phi_{-z}\Psi_{-z}))^*\Phi_{-z}\Psi_{-z} \\ &\qquad + (\Psi^*_{-z}\Psi_{-z})^{-1}(\Phi_{-z}\Psi_{-z})^*\partial_z(\Phi_{-z}\Psi_{-z}) \\ &\quad= \Big(O\big(\mu^{-1/2}\big) + \big(1 + O\big(e^{-c\mu}\big)\big)\Big(\frac{n}{4\mu} - \frac{\mathsf{N}}{2\mu}\Big)\Big)\Big(\frac{\pi}{\mu}\Big)^{\mathsf{N}-\frac{n}{2}} + O\big(e^{-c\mu}\big) \\ &\quad= \Big(O\big(\mu^{-1/2}\big) + \Big(\frac{n}{4\mu} - \frac{\mathsf{N}}{2\mu}\Big)\Big)\Big(\frac{\pi}{\mu}\Big)^{\mathsf{N}-\frac{n}{2}} + O\big(e^{-c\mu}\big) \,. \end{aligned}$$

So

$$\begin{aligned} &\partial_z \operatorname{Tr}^{\mathrm{s}}\big(\log\big((\Psi^*_{-z}\Psi_{-z})^{-1}(\Phi_{-z}\Psi_{-z})^*\Phi_{-z}\Psi_{-z}\big)\big) \\ &\quad= \operatorname{Tr}^{\mathrm{s}}\big((\Psi^*_{-z}\Psi_{-z})^{-1}(\Phi_{-z}\Psi_{-z})^*\Phi_{-z}\Psi_{-z}\big)^{-1}\partial_z\big((\Psi^*_{-z}\Psi_{-z})^{-1}(\Phi_{-z}\Psi_z)^*\Phi_{-z}\Psi_{-z}\big) \\ &\quad= O\big(\mu^{-1/2}\big) + \operatorname{Tr}^{\mathrm{s}}\Big(\frac{n}{4\mu} - \frac{\mathsf{N}}{2\mu}\Big) + O\big(e^{-c\mu}\big) = O\big(\mu^{-1/2}\big) \,. \end{aligned}$$

Then, by (6.12),

$$\partial_z \log \frac{\| \|^{\mathrm{RS,sm}}_{\det H^\bullet_{-z}(M)}}{\| \|^{\mathrm{M},X}_{\det H^\bullet_{-z}(M)}} = O\big(\mu^{-1/2}\big) \,. \tag{6.13}$$

By taking the derivative with respect to z of both sides of (6.10), and using (6.11), (6.13) and Corollary 4.16, we get $\zeta_{\mathrm{la}}(1,-z) \asymp_1 \mathbf{z}_{\mathrm{la}}$. Then Theorem 1.1 (ii) follows because $\mathbf{z}_{\mathrm{la}}$ is independent

of z.

Remark 6.6. In the case where $\eta = dh$, Theorem 1.1 (ii) agrees with Theorem 4.23. In fact, by Proposition 6.1 (iv), Theorem 1.1 (ii) and the Stokes formula,

$$\begin{aligned}\zeta_{\mathrm{la}}(1,z) &\asymp_1 -\int_M h\, X^* d\psi(M,\nabla^M) = -\int_M h\, X^*(\pi^* e(M,\nabla^M) - \delta_M) \\ &= -\int_M h\, e(M,\nabla^M) + \sum_{p\in\mathcal{X}} (-1)^{\mathrm{ind}(p)} h(p)\ .\end{aligned}$$

7. Asymptotics of the small zeta-invariant

7.1 Condition on the integrals along instantons

Let

$$\begin{aligned}\mathcal{M}_p &= \mathcal{M}_p(\eta, X) = -\max\{\, \eta(\gamma) \mid \gamma \in \mathcal{T}_p^1 \,\} \quad \big(p \in \mathcal{X}_+\big)\ , \\ \mathcal{M}_k &= \mathcal{M}_k(\eta, X) = \min_{p\in\mathcal{X}_k} \mathcal{M}_p \quad (k = 1,\dots,n)\ .\end{aligned}$$

Thus (d) means that $\mathcal{M}_p = \mathcal{M}_k$ for all $k = 1,\dots,n$ and $p \in \mathcal{X}_k$. The following result will be proved in Appendix A.

Theorem 7.1. *For every $\xi \in H^1(M,\mathbb{R})$ and numbers $a_n \geqslant \cdots \geqslant a_1 \gg 0$ or $a_1 \geqslant \cdots \geqslant a_n \gg 0$, there is some $\eta \in \xi$, satisfying* (a) *and* (c) *with the given X and some metric g, such that $\mathcal{M}_p(\eta, X) = a_k$ for all $k = 1,\dots,n$ and $p \in \mathcal{X}_k$.*

Remark 7.2. If $\xi \neq 0$, for $p \in \mathcal{X}_k$, $q \in \mathcal{X}_{k-1}$ and $\gamma, \delta \in \mathcal{T}(p,q) \subset \mathcal{T}_p^1$, the period $\langle \xi, \bar\gamma\bar\delta^{-1}\rangle = \eta(\gamma) - \eta(\delta)$ may not be zero. Hence it may not be possible to get $\eta(\gamma) = -a_k$ for all $\gamma \in \mathcal{T}_p^1$, contrary to the case where $\xi = 0$.

From now on, we assume η satisfies (d), besides (a) and (c). By Theorem 7.1, this is possible for any prescription of the class $\xi = [\eta] \in H^1(M,\mathbb{R})$. Let $a_k = \mathcal{M}_k(\eta, X)$ $(k = 1,\dots,n)$. Then $-\eta$ also satisfies (a), (c) and (d) with $-X$ and g, and $\mathcal{M}_k(-\eta, -X) = a_{n-k+1}$. So, by Corollary 3.15,

$$-\,\mathbf{z}_{\mathrm{sm}}(-\eta) = -\sum_{k=1}^{n} (-1)^k \big(1 - e^{a_{n-k+1}}\big) m_k^1\ . \tag{7.1}$$

Lemma 7.3. *Suppose M is oriented and n is even. Then*

$$\mathbf{z}_{\mathrm{sm}} = \sum_{k=1}^{n} (-1)^k e^{a_k} m_k^1\ .$$

If moreover all numbers a_k are equal one another, then $\mathbf{z}_{\mathrm{sm}} = 0$.

Proof. Use Lemma 3.14 and Corollary 3.16. □

7.2 Asymptotics of the perturbed Morse operators

Consider the notation of Section 5.2.2. By (5.4),

$$\mathbf{d}_{z,k-1} = e^{-a_k z}(\mathbf{d}'_{k-1} + \mathbf{d}''_{z,k-1})\ , \tag{7.2}$$

for $k = 1, \dots, n$, where

$$\mathbf{d}'_{k-1}\mathbf{e}_q = \sum_{p\in\mathcal{X}_k,\ \gamma\in\mathcal{T}(p,q),\ \eta(\gamma)=-a_k} \epsilon(\gamma)\mathbf{e}_p\ , \tag{7.3}$$

$$\mathbf{d}''_{z,k-1}\mathbf{e}_q = \sum_{p\in\mathcal{X}_k,\ \gamma\in\mathcal{T}(p,q),\ \eta(\gamma)<-a_k} e^{z(a_k+\eta(\gamma))}\epsilon(\gamma)\mathbf{e}_p\ , \tag{7.4}$$

for $q \in \mathcal{X}_{k-1}$. Observe that

$$e^{a_k z}\mathbf{d}_{z,k-1} = \mathbf{d}'_{k-1} + O(e^{-c\mu}) \quad (\mu \to +\infty)\ . \tag{7.5}$$

So

$$\mathbf{d}'_k\mathbf{d}'_{k-1} = \lim_{\mu\to+\infty} e^{(a_{k+1}+a_k)z}\mathbf{d}_{z,k}\mathbf{d}_{z,k-1} = 0\ .$$

Hence the operator $\mathbf{d}' = \sum_k \mathbf{d}'_k$ on $\mathbf{C}^\bullet$ satisfies $(\mathbf{d}')^2 = 0$. Taking adjoints in (7.2)–(7.4), or using (5.5), we also get

$$\boldsymbol{\delta}_{z,k} = e^{-a_k\bar{z}}(\boldsymbol{\delta}'_k + \boldsymbol{\delta}''_{z,k})\ , \tag{7.6}$$

for $k = 1, \dots, n$, where

$$\boldsymbol{\delta}'_k\mathbf{e}_p = \sum_{q\in\mathcal{X}_{k-1},\ \gamma\in\mathcal{T}(p,q),\ \eta(\gamma)=-a_k} \epsilon(\gamma)\,\mathbf{e}_q\ , \tag{7.7}$$

$$\boldsymbol{\delta}''_{z,k}\mathbf{e}_p = \sum_{q\in\mathcal{X}_{k-1},\ \gamma\in\mathcal{T}(p,q),\ \eta(\gamma)=-a_k} e^{\bar{z}(a_k+\eta(\gamma))}\epsilon(\gamma)\mathbf{e}_q\ , \tag{7.8}$$

for $p \in \mathcal{X}_k$. Moreover (7.5) yields

$$e^{a_k\bar{z}}\boldsymbol{\delta}_{z,k} = \boldsymbol{\delta}'_k + O(e^{-c\mu}) \quad (\mu \to +\infty)\ . \tag{7.9}$$

Let $\boldsymbol{\delta}' = \sum_k \boldsymbol{\delta}'_k = (\mathbf{d}')^*$, and let

$$\mathbf{D}' = \mathbf{d}' + \boldsymbol{\delta}'\ , \quad \boldsymbol{\Delta}' = (\mathbf{D}')^2 = \mathbf{d}'\boldsymbol{\delta}' + \boldsymbol{\delta}'\mathbf{d}'\ .$$

We have

$$\mathbf{C}^\bullet = \ker\boldsymbol{\Delta}' \oplus \operatorname{im}\mathbf{d}' \oplus \operatorname{im}\boldsymbol{\delta}'\ ,$$
$$\operatorname{im}\boldsymbol{\Delta}' = \operatorname{im}\mathbf{D}' = \operatorname{im}\mathbf{d}' \oplus \operatorname{im}\boldsymbol{\delta}'\ , \quad \ker\boldsymbol{\Delta}' = \ker\mathbf{D}' = \ker\mathbf{d}' \cap \ker\boldsymbol{\delta}'\ .$$

The orthogonal projections of $\mathbf{C}^\bullet$ to $\ker\boldsymbol{\Delta}'$, $\operatorname{im}\mathbf{d}'$ and $\operatorname{im}\boldsymbol{\delta}'$ are denoted by $\boldsymbol{\Pi}' = \boldsymbol{\Pi}'^0$, $\boldsymbol{\Pi}'^1$ and $\boldsymbol{\Pi}'^2$, respectively. Like in Sections 2.1.2 and 5.2.4, the composition $(\mathbf{d}')^{-1}\boldsymbol{\Pi}'^1$ is defined on $\mathbf{C}^\bullet$. From (7.5) and (7.9), we easily get that, as $\mu \to +\infty$,

$$\boldsymbol{\Pi}^j_{z,k} = \boldsymbol{\Pi}'^j_k + O(e^{-c\mu}) \quad (j = 0, 1, 2)\ , \tag{7.10}$$

$$e^{-a_k z}(\mathbf{d}_{z,k-1})^{-1}\boldsymbol{\Pi}^1_{z,k} = (\mathbf{d}'_{k-1})^{-1}\boldsymbol{\Pi}'^1_k + O(e^{-c\mu})\ . \tag{7.11}$$

By (7.5) and (7.9), on $\operatorname{im}\boldsymbol{\delta}_{z,k} + \operatorname{im}\mathbf{d}_{z,k-1}$,

$$\boldsymbol{\Delta}_z = e^{-2a_k\mu}\boldsymbol{\Delta}' + O(e^{-(2a_k+c)\mu}) \quad (\mu \to +\infty)\ . \tag{7.12}$$

Proposition 7.4. *For $k = 0, \dots, n$ and $\mu \gg 0$, the spectrum of $\boldsymbol{\Delta}_z$ on $\operatorname{im}\boldsymbol{\delta}_{z,k} + \operatorname{im}\mathbf{d}_{z,k-1}$ is contained in an interval of the form*

$$\left[Ce^{-2a_k\mu}, C'e^{-2a_k\mu}\right] \quad (C' \geqslant C)\ .$$

Proof. The positive eigenvalues of $\boldsymbol{\Delta}'$ are contained in an interval $[C_0, C'_0]$ $(C'_0 \geqslant C_0 > 0)$.

By (7.12), for $\mu \gg 0$ and $\mathbf{e} \in \operatorname{im} \boldsymbol{\delta}_{z,k} + \operatorname{im} \mathbf{d}_{z,k-1}$,

$$\langle \boldsymbol{\Delta}_z \mathbf{e}, \mathbf{e} \rangle \geqslant e^{2a_k\mu} \langle \boldsymbol{\Delta}' \mathbf{e}, \mathbf{e} \rangle - C_1 e^{-(2a_k+c)\mu} \|\mathbf{e}\|^2 \geqslant \big(C_0 e^{-2a_k\mu} - C_1 e^{-(2a_k+c)\mu}\big) \|\mathbf{e}\|^2 \,,$$
$$\langle \boldsymbol{\Delta}_z \mathbf{e}, \mathbf{e} \rangle \leqslant e^{2a_k\mu} \langle \boldsymbol{\Delta}' \mathbf{e}, \mathbf{e} \rangle + C_1 e^{-(2a_k+c)\mu} \|\mathbf{e}\|^2 \leqslant \big(C_0' e^{-2a_k\mu} + C_1 e^{-(2a_k+c)\mu}\big) \|\mathbf{e}\|^2 \,.$$

Then result follows taking $0 < C < C_0$ and $C' > C_0'$. □

7.3 Estimates of the nonzero small spectrum

Theorem 7.5. *If $\mu \gg 0$, the spectrum of $\Delta_{z,\mathrm{sm}}$ on $\operatorname{im} \delta_{z,\mathrm{sm},k} + \operatorname{im} d_{z,\mathrm{sm},k-1}$ is contained in an interval of the form*

$$[C\mu e^{-2a_k\mu}, C'\mu e^{-2a_k\mu}] \quad (C' \geqslant C) \,.$$

Proof. By the commutativity of (2.7), for every eigenvalue λ of $\Delta_{z,\mathrm{sm}}$ on $\operatorname{im} \delta_{z,\mathrm{sm},k} + \operatorname{im} d_{z,\mathrm{sm},k-1}$, there are normalized λ-eigenforms, $e \in \operatorname{im} \delta_{z,\mathrm{sm},k}$ and $e' \in \operatorname{im} d_{z,\mathrm{sm},k-1}$, so that $d_z e = \lambda^{1/2} e'$ and $\delta_z e' = \lambda^{1/2} e$. Hence the maximum and minimum of the spectrum of $\Delta_{z,\mathrm{sm}}$ on $\operatorname{im} \delta_{z,\mathrm{sm},k} + \operatorname{im} d_{z,\mathrm{sm},k-1}$ is $\|d_{z,\mathrm{sm},k-1}\|^2$ and $\|d^{-1}_{z,\mathrm{sm},k-1} \Pi^1_{z,\mathrm{sm},k}\|^{-2}$, respectively. Similarly, the maximum and minimum of the spectrum of $\boldsymbol{\Delta}_z$ on $\operatorname{im} \boldsymbol{\delta}_{z,k} + \operatorname{im} \mathbf{d}_{z,k-1}$ is $\|\mathbf{d}_{z,k-1}\|^2$ and $\|\mathbf{d}^{-1}_{z,k-1} \boldsymbol{\Pi}^1_{z,k}\|^{-2}$, respectively. Then the result follows from Corollaries 5.9, 5.14 and 5.17 and Proposition 7.4:

$$\begin{aligned}
\|d_{z,\mathrm{sm},k-1}\|^2 &\leqslant \|\Phi^{-1}_{z,k}\|^2 \|\mathbf{d}_{z,k-1}\|^{-2} \|\Phi_{z,\mathrm{sm},k-1}\|^{-2} \\
&\leqslant \Big(\Big(\frac{\mu}{\pi}\Big)^{k-n/2} + O\big(e^{-c\mu}\big)\Big) C_0' e^{-2a_k\mu} \Big(\Big(\frac{\pi}{\mu}\Big)^{k-1-n/2} + O\big(e^{-c\mu}\big)\Big) \\
&\leqslant C' \mu e^{-2a_k\mu} \,, \\
\|d^{-1}_{z,\mathrm{sm},k-1} \Pi^1_{z,\mathrm{sm},k-1}\|^{-2} &\geqslant \|\Phi^{-1}_{z,k-1}\|^{-2} \|\mathbf{d}^{-1}_{z,k-1} \boldsymbol{\Pi}^1_{z,k}\|^{-2} \|\Phi_{z,\mathrm{sm},k}\|^{-2} \\
&\geqslant \Big(\Big(\frac{\pi}{\mu}\Big)^{k-1-n/2} + O\big(e^{-c\mu}\big)\Big) C_0 e^{-2a_k\mu} \Big(\Big(\frac{\mu}{\pi}\Big)^{k-n/2} + O\big(e^{-c\mu}\big)\Big) \\
&\geqslant C\mu e^{-2a_k\mu} \,. \qquad \square
\end{aligned}$$

7.4 Asymptotics of the small zeta invariant

Theorem 1.1 (iii) is proved here.

Notation 7.6. Let $\asymp_2$ be defined like $\asymp_0$ in Notation 5.2, using $O(|\mu|^{-1})$ instead of $O(e^{-c|\mu|})$.

Theorem 7.7. *As $\mu \to +\infty$,*

$$\eta \wedge d_z^{-1} \Pi^1_{z,\mathrm{sm},k} \asymp_2 \pm\big(1 - e^{a_k}\big) \Pi^1_{z,\mathrm{sm},k} \,.$$

Proof. Consider the notation of Sections 5.3 and 7.2. By Corollary 5.13 and (5.11),

$$\Pi^1_{z,\mathrm{sm}} \asymp_0 \widetilde{\Psi}_z \widetilde{\boldsymbol{\Pi}}^1_z \Phi_{z,\mathrm{sm}} = \widetilde{\Psi}_z \widetilde{\boldsymbol{\Pi}}^1_z \boldsymbol{\Pi}^1_z \Phi_{z,\mathrm{sm}} = \widetilde{\Psi}_z \boldsymbol{\Pi}^1_z \Phi_{z,\mathrm{sm}} - \widetilde{\Psi}_z \big(\widetilde{\boldsymbol{\Pi}}^1_z\big)^{\perp} \boldsymbol{\Pi}^1_z \Phi_{z,\mathrm{sm}} \,.$$

But $\big(\widetilde{\boldsymbol{\Pi}}^1_z\big)^{\perp} \boldsymbol{\Pi}^1_z = 0$ if $\mu \gg 0$ by Corollary 5.16. Hence

$$\Pi^1_{z,\mathrm{sm}} \asymp_0 \widetilde{\Psi}_z \boldsymbol{\Pi}^1_z \Phi_{z,\mathrm{sm}} \,. \tag{7.13}$$

For brevity, let $R_z = P_{z-1,\mathrm{sm}} P_{z,\mathrm{sm}}$ on $L^2(M;\Lambda)$, and $S_z = \Phi_z \widetilde{\Psi}_{z-1}$ and $T_z = \Phi_{z-1} P_{z-1,\mathrm{sm}} \widetilde{\Psi}_z$ on $\mathbf{C}^\bullet$. By (7.5)–(7.11), (7.13), Propositions 3.18 and 7.4, Corollaries 3.19, 5.6, 5.7, 5.9, 5.11,

5.13, 5.15 and 5.17, and Theorem 7.5,

$$\begin{aligned}
e^{a_k}\Pi^1_{z,\mathrm{sm},k} &\asymp_0 e^{a_k}\widetilde{\Psi}_z\mathbf{\Pi}^1_{z,k}\Phi_{z,\mathrm{sm}} \asymp_2 e^{a_k}\widetilde{\Psi}_z\mathbf{\Pi}'^1_k\Phi_{z,\mathrm{sm}} = e^{a_k}\widetilde{\Psi}_z\mathbf{d}'_{k-1}(\mathbf{d}'_{k-1})^{-1}\mathbf{\Pi}'^1_k\Phi_{z,\mathrm{sm}}\\
&\asymp_2 e^{a_k}R_z\widetilde{\Psi}_zS_z\mathbf{d}'_{k-1}T_z(\mathbf{d}'_{k-1})^{-1}\mathbf{\Pi}'^1_k\Phi_{z,\mathrm{sm}}\\
&\asymp_2 e^{a_k}R_z\widetilde{\Psi}_{z-1}\mathbf{d}'_{k-1}T_z(\mathbf{d}'_{k-1})^{-1}\mathbf{\Pi}'^1_k\Phi_{z,\mathrm{sm}}\\
&\asymp_2 e^{a_k}\widetilde{\Psi}_{z-1}\mathbf{d}'_{k-1}T_z(\mathbf{d}'_{k-1})^{-1}\mathbf{\Pi}'^1_k\Phi_{z,\mathrm{sm}}\\
&\asymp_2 e^{a_k}\widetilde{\Psi}_{z-1}e^{a_k(z-1)}\mathbf{d}_{z-1,k-1}T_ze^{-a_kz}\mathbf{d}^{-1}_{z,k-1}\mathbf{\Pi}^1_{z,k}\Phi_{z,\mathrm{sm}}\\
&= \widetilde{\Psi}_{z-1}\mathbf{d}_{z-1,k-1}T_z\mathbf{d}^{-1}_{z,k-1}\mathbf{\Pi}^1_{z,k}\Phi_{z,\mathrm{sm}}\\
&\asymp_0 \widetilde{\Psi}_{z-1}\mathbf{d}_{z-1,k-1}T_z\widetilde{\mathbf{\Pi}}^2_{z,k-1}\mathbf{d}^{-1}_{z,k-1}\widetilde{\mathbf{\Pi}}^1_{z,k}\Phi_{z,\mathrm{sm}}\\
&\asymp_0 \widetilde{\Psi}_{z-1}\mathbf{d}_{z-1,k-1}\Phi_{z-1}P_{z-1,\mathrm{sm}}\Pi^2_{z,k-1}\widetilde{\Psi}_z\mathbf{d}^{-1}_{z,k-1}\Phi_z\Pi^1_{z,\mathrm{sm}}\\
&\asymp_0 \widetilde{\Psi}_{z-1}\mathbf{d}_{z-1,k-1}\Phi_{z-1}P_{z-1,\mathrm{sm}}\Pi^2_{z,k-1}\Phi_z^{-1}\mathbf{d}^{-1}_{z,k-1}\Phi_z\Pi^1_{z,\mathrm{sm}}\\
&= \widetilde{\Psi}_{z-1}\Phi_{z-1}d_{z-1,\mathrm{sm},k-1}d^{-1}_{z,\mathrm{sm},k-1}\Pi^1_{z,\mathrm{sm},k} \asymp_0 d_{z-1}d_z^{-1}\Pi^1_{z,\mathrm{sm},k}\,.
\end{aligned}$$

Therefore

$$\eta\wedge d_z^{-1}\Pi^1_{z,\mathrm{sm},k} = (d_z - d_{z-1})d_z^{-1}\Pi^1_{z,\mathrm{sm},k} \asymp_2 (1-e^{a_k})\Pi^1_{z,\mathrm{sm},k}\,. \qquad \square$$

Theorem 1.1 (iii) follows from Corollaries 3.9 and 4.15 and Theorem 7.7.

Remark 7.8. Theorem 1.1 (iii) agrees with Corollaries 4.20 to 4.22 by (7.1) and Lemma 7.3.

8. Prescription of the asymptotics of the zeta invariant

We prove Theorem 1.2 here. By Theorem 7.1, given $a \gg 0$, there is some $\eta_0 \in \xi$ and some metric g satisfying (a) and (d) with the given X, and so that $\mathcal{M}_k(\eta_0, X) = a$ for all $k = 1, \ldots, n$. Using the notation of Section 3.1, we are going to modify η_0 only in every U_p for $p \in \mathcal{X}_0 \cup \mathcal{X}_n$.

Fix any $\epsilon > 0$ such that, for every $p \in \mathcal{X}_0 \cup \mathcal{X}_n$, the open ball $B(p, 3\epsilon)$ is contained in U_p. Let

$$V = \bigcup_{p\in\mathcal{X}_0\cup\mathcal{X}_n} B(p,\epsilon)\,, \quad V' = \bigcup_{p\in\mathcal{X}_0\cup\mathcal{X}_n} B(p,2\epsilon)\,.$$

Take a smooth function $\sigma : [0, 3\epsilon] \to [0, 1]$ so that

$$\sigma' \leqslant 0\,, \quad \sigma([0,\epsilon]) = 1\,, \quad \sigma([2\epsilon, 3\epsilon]) = 0\,.$$

Let $f_j \in C^\infty(M, \mathbb{R})$ $(j = 0, n)$ be the extension by zero of the combination of the functions $\sigma(|x_p|) \in C^\infty_{\mathrm{c}}(B(p, 3\epsilon), \mathbb{R})$ $(p \in \mathcal{X}_j)$. We have

$$df_j \subset V'_j \setminus V_j\,, \quad f_j(V_j) = 1\,, \quad f_j(M \setminus V'_j) = 0\,, \quad Xf_0 \geqslant 0\,, \quad Xf_n \leqslant 0\,.$$

For any $c_0, c_n \geqslant 0$, let $\eta = \eta(c_0, c_n) = \eta_0 - c_0\, df_0 + c_n\, df_n$. This closed 1-form satisfies (a) and (d) with X and g, and we have

$$\mathcal{M}_1(\eta, X) = a + c_0\,, \quad \mathcal{M}_n(\eta, X) = a + c_n\,, \quad \mathcal{M}_k(\eta_1, X) = a \quad (1 < k < n)\,.$$

Hence, by Corollary 3.15,

$$\mathbf{z}_{\mathrm{sm}}(\eta) - \mathbf{z}_{\mathrm{sm}}(\eta_0) = e^a(e^{c_0} - 1)m^1_1 + (-1)^n e^a(1 - e^{c_n})m^1_n\,. \tag{8.1}$$

By (a), $e(M,\nabla^M)=0$ on every U_p ($p\in\mathcal{X}$). So, using the Stokes formula,

$$\begin{aligned}\mathbf{z}_{\mathrm{la}}(\eta)-\mathbf{z}_{\mathrm{la}}(\eta_0)&=\int_M (c_0\,df_0-c_n\,df_n)\wedge X^*\psi(M,\nabla^M)\\&=\int_M (c_nf_n-c_0f_0)\,X^*d\psi(M,\nabla^M)\\&=\int_M (c_nf_n-c_0f_0)\,e(M,\nabla^M)-\sum_{p\in\mathcal{X}}(-1)^{\mathrm{ind}(p)}(c_nf_n-c_0f_0)(p)\\&=c_0|\mathcal{X}_0|-(-1)^nc_n|\mathcal{X}_n|\;,\end{aligned}\tag{8.2}$$

Combining (8.1) and (8.2), we obtain

$$\mathbf{z}(\eta)-\mathbf{z}(\eta_0)=e^a(e^{c_0}-1)m_1^1+(-1)^ne^a(1-e^{c_n})m_n^1+c_0|\mathcal{X}_0|-(-1)^nc_n|\mathcal{X}_n|\;.\tag{8.3}$$

So, if n is even (respectively, odd), given any $\tau\in\mathbb{R}$ (respectively, $\tau\gg 0$), we get $\mathbf{z}(\eta(c_0,c_n))=\tau$ for some $c_0,c_n\geqslant 0$ since $|\mathcal{X}_0|,|\mathcal{X}_n|>0$ by (b).

Now assume M is oriented and n is even. Then $\mathbf{z}(\eta_0)=-\mathbf{z}(-\eta_0)$ by (6.9) and Lemma 7.3. Using local changes of X and applying [Sma61, Lemmas 1.1 and 1.2], we can increase $|\mathcal{X}_0|$ or $|\mathcal{X}_n|$ as much as desired. So we can assume $|\mathcal{X}_0|-|\mathcal{X}_n|\in(\tau-\mathbf{z}(\eta_0))\cdot\mathbb{R}^+$ and $m_0,m_n>0$ by Lemma 3.12, yielding $m_1^1,m_n^1>0$ by (3.22). Moreover $m_1^1=m_n^1$ by (3.22) and Lemma 3.14. Thus, taking $c_0=c_n=:c$, the expression (8.3) becomes

$$\mathbf{z}(\eta)-\mathbf{z}(\eta_0)=c(|\mathcal{X}_0|-|\mathcal{X}_n|)\;.$$

Hence $\mathbf{z}(\eta)=\tau$ for some $c\geqslant 0$.

9. The switch of the order of integration

The proof of Theorem 1.3 is given in this section. Let $\mathcal{S}$ be the Schwartz space on $\mathbb{R}$. Recall that the space of tempered distributions is the continuous dual space $\mathcal{S}'$, with the topology of uniform convergence on bounded sets. Suppose first that (1.7) is used as definition of Z_μ. By Theorem 1.1, the expression (1.7) defines a tempered distribution Z_μ for $\mu\gg 0$. Moreover, using also the formula of the inverse Fourier transform, we get, for $f\in\mathcal{S}$,

$$\langle Z_\mu,f\rangle=\frac{1}{2\pi}\int_{-\infty}^{\infty}\zeta(1,z)\,\hat f(\nu)\,d\nu\to\frac{\mathbf{z}}{2\pi}\int_{-\infty}^{\infty}\hat f(\nu)\,d\nu=\mathbf{z}f(0)\;,$$

as $\mu\to+\infty$, uniformly on ν. For every $C>0$, this convergence is also uniform on $f\in\mathcal{S}$ with $|\hat f(\nu)|,|\nu^2\hat f(\nu)|\leqslant C$. So $Z_\mu\to\mathbf{z}\delta_0$ in $\mathcal{S}'$ as $\mu\to+\infty$. To get Theorem 1.3, it only remains to prove the following.

Theorem 9.1. *Both* (1.3) *and* (1.7) *define the same tempered distribution Z_μ for $\mu\gg 0$.*

Proposition 9.2. *For $\mu\gg 0$, $t>0$ and $f\in\mathcal{S}$,*

$$\int_{-\infty}^{\infty}\int_t^{\infty}\left|\,\mathrm{Tr}^{\mathrm{s}}\left(\eta\wedge\delta_ze^{-u\Delta_z}\right)\right||\hat f(\nu)|\,du\,d\nu<\infty\;.$$

Proof. By [DS88b, Corollary XI.9.8 and Lemma XI.9.9 (d)],

$$\begin{aligned}\left|\,\mathrm{Tr}^{\mathrm{s}}\left(\eta\wedge\delta_ze^{-u\Delta_z}\right)\right|&\leqslant\left|\eta\wedge\delta_ze^{-u\Delta_z}\right|_1\leqslant\|\eta\wedge\|\left|\delta_ze^{-u\Delta_z}\right|_1\\&=\|\eta\|_{L^\infty}\,\mathrm{Tr}\left((d_z\delta_z)^{1/2}e^{-u\Delta_z}\right)\leqslant\|\eta\|_{L^\infty}\,\mathrm{Tr}\left(\Delta_z^{1/2}e^{-u\Delta_z}\right)\;,\end{aligned}$$

where $|\ |_1$ denotes the trace norm. Hence

$$\int_t^\infty \left|\operatorname{Tr^s}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\right| du \leqslant \|\eta\|_{L^\infty}\int_t^\infty \operatorname{Tr}\left(\Delta_z^{1/2}e^{-u\Delta_z}\right) du = \|\eta\|_{L^\infty}\operatorname{Tr}\left(\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\right).$$

The operator $(I+D^2)^{-N}$ is of trace class for any $N>n$. Therefore

$$\operatorname{Tr}\left(\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\right) \leqslant \left|(I+D^2)^{-N}\right|_1\left\|(I+D^2)^N\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\right\|.$$

By Corollary 2.3 and Theorem 7.5, for $\mu\gg 0$ and $\alpha\in L^2(M;\Lambda)$,

$$\begin{aligned}
&\left\|(I+D^2)^N\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\alpha\right\| \\
&\qquad\leqslant C_0\left\|\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\alpha\right\|_{2N} \leqslant C_1|z|^{2N}\left\|\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\alpha\right\|_{2N,z} \\
&\qquad= C_2|z|^{2N}\sum_{k=0}^{2N}\left\|D_z^k\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\alpha\right\| \leqslant C_3|z|^{2N}\sum_{k=0}^{2N}\frac{1}{t^{k/2}}\left\|\Delta_z^{-1/2}\Pi_z^\perp\alpha\right\| \\
&\qquad\leqslant C|z|^{2N}\left(1+t^{-N}\right)e^{c\mu}\|\alpha\|.
\end{aligned}$$

Thus, since $f\in\mathcal{S}$,

$$\begin{aligned}
&\int_{-\infty}^\infty\int_t^\infty \left|\operatorname{Tr^s}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\right| |\hat f(\nu)|\, du\, d\nu \\
&\qquad\leqslant C\|\eta\|_{L^\infty}\left|(I+D^2)^{-N}\right|_1\left(1+t^{-N}\right)e^{c\mu}\int_{-\infty}^\infty |z|^{2N}|\hat f(\nu)|\, d\nu < \infty. \quad \square
\end{aligned}$$

Proof of Theorem 9.1. Using Theorem 1.1 and Proposition 9.2 to apply the Lebesgue's dominated convergence theorem and Fubini's theorem, we compute

$$\begin{aligned}
-\frac{1}{2\pi}\int_{-\infty}^\infty \lim_{t\downarrow 0}\operatorname{Tr^s}\left(\eta\wedge d_z^{-1}e^{-t\Delta_z}\Pi_z^1\right)\hat f(\nu)\, d\nu &= -\frac{1}{2\pi}\lim_{t\downarrow 0}\int_{-\infty}^\infty \operatorname{Tr^s}\left(\eta\wedge d_z^{-1}e^{-t\Delta_z}\Pi_z^1\right)\hat f(\nu)\, d\nu \\
&= \frac{1}{2\pi}\lim_{t\downarrow 0}\int_{-\infty}^\infty\int_t^\infty \operatorname{Tr^s}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\hat f(\nu)\, du\, d\nu \\
&= \frac{1}{2\pi}\lim_{t\downarrow 0}\int_t^\infty\int_{-\infty}^\infty \operatorname{Tr^s}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\hat f(\nu)\, d\nu\, du \\
&= \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty \operatorname{Tr^s}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\hat f(\nu)\, d\nu\, du\ . \square
\end{aligned}$$

Appendix A. Integrals along instantons

Theorem 7.1 is proved here. We show the case where $a_n\geqslant\cdots\geqslant a_1\gg 0$. Then the case where $a_1\geqslant\cdots\geqslant a_n\gg 0$ follows by using $-X$ and $-\xi$.

By [Sma61, Theorem B], there is some Morse function h on M such that $h(\mathcal{X}_k)=\{k\}$ $(k=0,\dots,n)$, $Xh<0$ on $M\setminus\mathcal{X}$, and h is in standard form with respect to X; in particular, $\operatorname{Crit}_k(h)=\mathcal{X}_k$. Now we proceed like in the proof of [BH04, Proposition 16 (i)]. Since $\mathcal{X}$ is finite, there is some $\eta'\in\xi$ such that $\eta'=0$ on some open neighborhood U_p of every $p\in\mathcal{X}$. Let $U_k=\bigcup_{p\in\mathcal{X}_k}U_p$ and $U=\bigcup_k U_k$. We can assume $h(U_k)\subset(k-1/4,k+1/4)$ for all $k=0,\dots,n$. If $C\gg 0$, then the representative $\eta'':=\eta'+C\,dh$ of ξ satisfies $\eta''(X)<0$ on $M\setminus\mathcal{X}$.

For $k=0,\dots,n$, let $I_k^\pm\subset\mathbb{R}$ be the closed interval with boundary points $k\pm 1/4$ and $k\pm 1/2$. Since there are no critical values of h in $I_k^\pm$, every $T_k^\pm:=h^{-1}(I_k^\pm)$ is compact submanifold with boundary of dimension n, every $\Sigma_k^\pm:=h^{-1}(k\pm 1/2)$ is a closed submanifold of codimension 1,

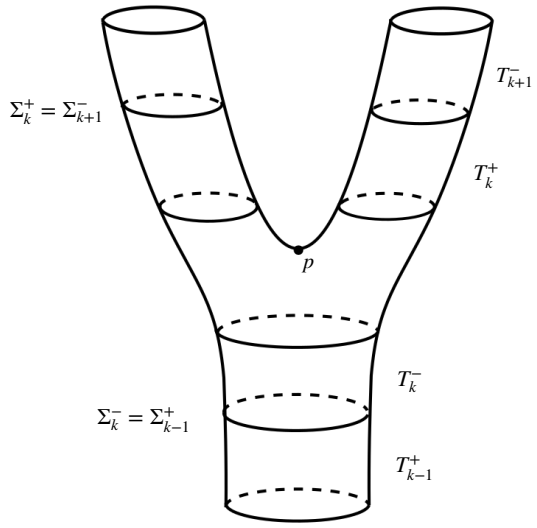

FIGURE 1. A representation of the sets $T_k^\pm$, $\Sigma_k^\pm$, T_{k-1}^+ and T_{k+1}^-, taking $\mathcal{X}_k = \{p\}$.

and there are identities $T_k^\pm \equiv \Sigma_k^\pm \times I_k^\pm$ given by $x \equiv (\pi_k^\pm(x), h(x))$ $(x \in T_k^\pm)$, where $\pi_k^\pm(x)$ is the unique point of $\Sigma_k^\pm$ that meets the ϕ-orbit of x. Of course, $\Sigma_k^- = \Sigma_{k-1}^+$ $(k = 1, \dots, n)$ and $T_0^- = \Sigma_0^- = T_n^+ = \Sigma_n^+ = \emptyset$. (See Figure 1.)

We have $\Sigma_k^\pm \pitchfork \iota_p^\pm(W_p^\pm)$ for $p \in \mathcal{X}_k$. Let $K_p^\pm = \Sigma_k^\pm \cap \iota_p^\pm(W_p^\pm)$ and $K_k^\pm = \bigcup_{p\in\mathcal{X}_k} K_p^\pm$, which are closed submanifolds of $\Sigma_k^\pm$; K_k^- is of codimension k in Σ_k^-, and K_k^+ of codimension $n-k$ in Σ_k^+. Since the α- and ω-limits of the orbits of X are zero points, the orbit of ϕ through every point $x \in \Sigma_k^+ \setminus K_k^+$ meets $\Sigma_k^- \setminus K_k^-$ at a unique point $\psi_k(x) := \phi^{\tau_k(x)}(x)$ $(\tau_k(x) > 0)$. This defines a diffeomorphism $\psi_k : \Sigma_k^+ \setminus K_k^+ \to \Sigma_k^- \setminus K_k^-$ and a smooth function $\tau_k : \Sigma_k^+ \setminus K_k^+ \to \mathbb{R}^+$. Moreover the sets $K_p^\pm$ $(p \in \mathcal{X}_k)$ have corresponding open neighborhoods $V_p^\pm$ in $\Sigma_k^\pm$, with disjoint closures, such that $\psi_k(V_p^+ \setminus K_p^+) = V_p^- \setminus K_p^-$. Take smooth functions $\lambda_p^\pm$ $(p \in \mathcal{X}_k)$ on $\Sigma_k^\pm$ so that $0 \leqslant \lambda_p^\pm \leqslant 1$, $\operatorname{supp}\lambda_p^\pm \subset V_p^\pm$, $\lambda_p^\pm = 1$ on $K_p^\pm$, and $\lambda_p^+ = \psi_k^*\lambda_p^-$ on $\Sigma_k^+ \setminus K_k^+$. Moreover let

$$\begin{aligned}
\widetilde{T}_k &= h^{-1}([k-1/2, k+1/2])\,, \quad \widetilde{K}_p = \widetilde{T}_k \cap \big(\iota_p^+(W_p^+) \cup \iota_p^-(W_p^-)\big)\,,\\
\widetilde{V}_p &= \{\, \phi^t(x) \mid x \in V_p^+ \setminus K_p^+,\ 0 \leqslant t \leqslant \tau_k(x) \,\} \cup \widetilde{K}_p\,,\\
\widetilde{K}_k &= \bigcup_{p\in\mathcal{X}_k} \widetilde{K}_p\,, \quad \widetilde{V}_k = \bigcup_{p\in\mathcal{X}_k} \widetilde{V}_p\,, \quad M_k = h^{-1}((-\infty, k+1/2])\,.
\end{aligned}$$

Thus $M_k = \widetilde{T}_0 \cup \cdots \cup \widetilde{T}_k$. Note that $\widetilde{T}_k$ and M_k are compact submanifolds with boundary of dimension n, and every $\widetilde{V}_p$ (respectively, $\widetilde{K}_p$) is open (respectively, closed) in $\widetilde{T}_k$. We also get smooth functions $\tilde{\lambda}_p$ $(p \in \mathcal{X}_k)$ on $\widetilde{T}_k$ determined by the condition $\tilde{\lambda}_p(\phi^t(x)) = \lambda_p^+(x)$ for all $x \in \Sigma_k^+ \setminus K_k^+$ and $0 \leqslant t \leqslant \tau_k(x)$. They satisfy $0 \leqslant \tilde{\lambda}_p \leqslant 1$, $\operatorname{supp}\tilde{\lambda}_p \subset \widetilde{V}_p$, and $\tilde{\lambda}_p = 1$ on $\widetilde{K}_p$.

Let

$$A_p = \max\{\, |\eta'(\gamma)| \mid \gamma \in \mathcal{T}^1_p \,\} \quad \big(p \in \mathcal{X}_+\big)\ ,$$
$$A_k = \max_{p\in\mathcal{X}_k} A_p \quad (k = 1, \dots, n)\ , \quad A = \max\{A_1, \dots, A_n\}\ .$$

We can suppose $C > A$ and $a_1 > C + A > 0$. For $p \in \mathcal{X}_k$, $q \in \mathcal{X}_{k-1}$ and $\gamma \in \mathcal{T}(p, q)$, we have

$$dh(\gamma) = h(q) - h(p) = -1\ .$$

Therefore

$$0 > \eta''(\gamma) = \eta'(\gamma) + C\, dh(\gamma) \geqslant -A - C > -a_1 \quad (\gamma \in \mathcal{T}^1)\ . \tag{A.1}$$

Claim 1. For $k = 0, \dots, n$, there is a smooth function f_k on M such that

$$df_k(X) \leqslant 0\ , \tag{A.2}$$
$$\operatorname{supp} df_k \subset \mathring{M}_k\ , \tag{A.3}$$
$$\max\{\, (\eta'' + df_k)(\gamma) \mid \gamma \in \mathcal{T}^1_p \,\} = -a_l \quad (p \in \mathcal{X}_l,\ 1 \leqslant l \leqslant k)\ , \tag{A.4}$$
$$(\eta'' + df_k)(\delta) > -a_k \quad (\delta \in \mathcal{T}^1_{k+1})\ . \tag{A.5}$$

The statement follows directly from Claim 1 taking $\eta = \eta'' + df_n$. So we only have to prove this assertion.

We proceed by induction on k. For $k = 0$, we choose $f_0 = 0$. Then (A.4) is vacuous, (A.2) and (A.3) are trivial, and (A.5) is given by (A.1).

Now take any $k \geqslant 1$ and assume f_{k-1} is defined and satisfies (A.2)–(A.5). Let

$$b_p = -\max\{\, (\eta'' + df_{k-1})(\gamma) \mid \gamma \in \mathcal{T}^1_p \,\} \quad (p \in \mathcal{X}_k)\ , \tag{A.6}$$
$$b_k = \min\{\, b_p \mid p \in \mathcal{X}_k \,\}\ .$$

For every $p \in \mathcal{X}_k$, we have $b_p < a_{k-1} \leqslant a_k$ because f_{k-1} satisfies (A.5). So there is a smooth function h^-_p on I^-_k such that $(h^-_p)' \geqslant 0$, $h^-_p = 0$ around $k - 1/2$, and $h^-_p = a_k - b_p$ around $k - 1/4$. Let $\tilde{h}^-_p$ be the function on $V^-_p \times I^-_k \subset \Sigma^-_k \times I^-_k \equiv T^-_k$ given by $\tilde{h}^-_p(x, s) = h^-_p(s)$. We have $\tilde{h}^-_p = 0$ around $V^-_p \times \{k - 1/2\}$ and $\tilde{h}^-_p = a_k - b_p$ around $V^-_p \times \{k - 1/4\}$. Thus $\tilde{h}^-_p$ has a smooth extension to $\widetilde{V}_p$, also denoted by $\tilde{h}^-_p$, which is equal to $a_k - b_p$ on $\widetilde{V}_p \setminus T^-_k$. The function $\tilde{\lambda}_p\tilde{h}^-_p$ on $\widetilde{V}_p$ can be extended by zero to get a smooth function on $\widetilde{T}_k$, also denoted by $\tilde{\lambda}_p\tilde{h}^-_p$. Let $\tilde{h}^-_k = \sum_{p\in\mathcal{X}_k} \tilde{\lambda}_p\tilde{h}^-_p$ on $\widetilde{T}_k$.

On the other hand, let ρ_k be a smooth function on I^+_k such that $\rho'_k \geqslant 0$, $\rho_k = 0$ around $k + 1/4$, and $\rho_k = 1$ around $k + 1/2$. Let $\tilde{\rho}_k$ be the smooth function on $T^+_k \equiv \Sigma^+_k \times I^+_k$ given by $\tilde{\rho}_k(x, s) = \rho_k(s)$, and let

$$\tilde{h}^+_k = \tilde{h}^-_k(1 - \tilde{\rho}_k) + (a_k - b_k)\tilde{\rho}_k$$

on T^+_k. This smooth function is equal to $\tilde{h}^-_k$ around $\Sigma^+_k \times \{k + 1/4\}$, and is equal to $a_k - b_k$ around $\Sigma^+_k \times \{k + 1/2\} \equiv \Sigma^+_k$. So the functions, $\tilde{h}^-_k$ on $\widetilde{T}_k \setminus T^+_k$ and $\tilde{h}^+_k$ on T^+_k, can be combined to produce a smooth function $\tilde{h}_k$ on $\widetilde{T}_k$. Since $\tilde{h}_k = 0$ around Σ^-_k and $\tilde{h}_k = a_k - b_k$ around Σ^+_k, there is a smooth extension of $\tilde{h}_k$ to M, also denoted by $\tilde{h}_k$, which is constant on $M \setminus \widetilde{T}_k$.

Let $f_k = f_{k-1} + \tilde{h}_k$ on M. This smooth function satisfies (A.2) because f_{k-1} satisfies (A.2), and X induces the opposite of the standard orientation on every fiber $\{x\} \times I^\pm_k \equiv I^\pm_k$ of $T^\pm_k$ ($x \in \Sigma^\pm_k$). It also satisfies (A.3) and (A.4) for $p \in \mathcal{X}_l$ with $1 \leqslant l < k$ because f_{k-1} satisfies these properties and $d\tilde{h}_k$ is supported in the interior of $\widetilde{T}_k$.

Next, take any $p \in \mathcal{X}_k$, $q \in \mathcal{X}_{k-1}$ and $\gamma \in \mathcal{T}(p,q) \subset \mathcal{T}_p^1$. We have $\gamma \cap T_k^- \equiv \{x\} \times I_k^-$ for some $x \in K_p^- \cap K_q^+ \subset \Sigma_k^- = \Sigma_{k-1}^+$, and the orientation of $\gamma \cap T_k^-$ agrees with the opposite of the standard orientation of $\{x\} \times I_k^- \equiv I_k^-$. Then

$$\begin{aligned}(\eta'' + df_k)(\gamma) &= (\eta'' + df_{k-1} + d\tilde{h}_k)(\gamma) \leqslant -b_p + \lambda_p^-(x) d\tilde{h}_p^-(\gamma) \\ &= -b_p - \int_{I_k^-} dh_p^- = -b_p - (a_k - b_p) = -a_k \ .\end{aligned}$$

Here, the equality holds when the maximum of (A.6) is achieved at γ. Hence f_k also satisfies (A.4) for $p \in \mathcal{X}_k$.

Finally, take any $p \in \mathcal{X}_k$, $u \in \mathcal{X}_{k+1}$ and $\delta \in \mathcal{T}(u,p) \subset \mathcal{T}_u^1 \subset \mathcal{T}_{k+1}^1$. Thus $\delta \cap T_k^+ \equiv \{y\} \times I_k^+$ for some $y \in K_p^+ \cap K_u^- \subset \Sigma_k^+ = \Sigma_{k+1}^-$, and the orientation of $\delta \cap T_k^+$ agrees with the opposite of the standard orientation of $\{y\} \times I_k^+ \equiv I_k^+$. Then

$$\begin{aligned}(\eta'' + df_k)(\delta) &= (\eta'' + df_{k-1} + d\tilde{h}_k)(\delta) = \eta''(\delta) + d\tilde{h}_k^+(\delta) \\ &= \eta''(\delta) + \big(\tilde{h}_k^-(y) - (a_k - b_k)\big) \int_{I_k^-} d\rho_k = \eta''(\delta) + \tilde{\lambda}_p(y)\tilde{h}_p^-(y) + b_k - a_k \\ &= \eta''(\delta) + a_k - b_p + b_k - a_k = \eta''(\delta) + b_k - b_p \geqslant \eta''(\delta) > -a_k \ ,\end{aligned}$$

where the second equality is true because f_{k-1} satisfies (A.3), and the last inequality holds by (A.1). So f_k satisfies (A.5).

References

ALG20 J.A. Álvarez López and P. Gilkey, *Derived heat trace asymptotics for the de Rham and Dolbeualt complexes*, Pure Appl. Funct. Anal., to appear, arXiv:2010.01146, 2020.

ALG21 ———, *The local index density of the perturbed de Rham complex*, Czechoslovak Math. J. **71** (2021), no. 3, 901–932.

ALK02 J.A. Álvarez López and Y.A. Kordyukov, *Distributional Betti numbers of transitive foliations of codimension one*, Foliations: geometry and dynamics. Proceedings of the Euroworkshop, Warsaw, Poland, May 29–June 9, 2000 (Singapore), World Sci. Publ., 2002, pp. 159–183. MR 1882768

ALK08 ———, *Lefschetz distribution of Lie foliations*, C^*-algebras and elliptic theory II (Basel), Trends Math., Birkhäuser, 2008, `http://dx.doi.org/10.1007/978-3-7643-8604-7_1`, pp. 1–40. MR 2408134

ALKL20 J.A. Álvarez López, Y.A. Kordyukov, and E. Leichtnam, *A trace formula for foliated flows*, in preparation, 2020.

BF97 M. Braverman and M. Farber, *Novikov type inequalities for differential forms with non-isolated zeros*, Math. Proc. Cambridge Philos. Soc. **122** (1997), 357–375. MR 1458239

BFK10 D. Burghelea, L. Friedlander, and T. Kappeler, *On the space of trajectories of a generic gradient like vector field*, An. Univ. Vest Timiş. Ser. Mat.-Inform. (2010), no. 1-2, 45–126. MR 2849328

BGS88 J.-M. Bismut, H. Gillet, and C. Soulé, *Analytic torsion and holomorphic determinant bundles. I. Bott-Chern forms and analytic torsion*, Comm. Math. Phys. **115** (1988), no. 1, 49–78. MR 929146

BGS90 ———, *Complex immersions and Arakelov geometry*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990. MR 1086887

BGV04 N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2004, Corrected reprint of the 1992 original. MR 2273508

BH01 D. Burghelea and S. Haller, *On the topology and analysis of a closed one form. I. (Novikov's theory revisited)*, Essays on geometry and related topics, Vol. 1, 2, Monogr. Enseign. Math., vol. 38, Enseignement Math., Geneva, 2001, pp. 133–175. MR 1929325

BH04 ———, *Laplace transform, dynamics, and spectral geometry*, arXiv:math/0405037, 2004.

BH06 ———, *Euler structures, the variety of representations and the Milnor-Turaev torsion*, Geom. Topol. **10** (2006), 1185–1238. MR 2255496

BH08 ———, *Dynamics, Laplace transform and spectral geometry*, J. Topol. **1** (2008), no. 1, 115–151. MR 2365654

Bot88 R. Bott, *Morse theory indomitable*, Publ. Math. Inst. Hautes Études Sci. **68** (1988), 99–114. MR 1001450

Bur97 D. Burghelea, *Lectures on Witten-Helffer-Sjöstrand theory*, Proceedings of the Third International Workshop on Differential Geometry and its Applications and the First German-Romanian Seminar on Geometry (Sibiu, 1997), vol. 5, 1997, pp. 85–99. MR 1723597

BZ92 J.-M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller*, Astérisque **205** (1992), 235 pp., with an appendix by F. Laudenbach. MR 1185803

BZ94 ———, *Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle*, Geom. Funct. Anal. **4** (1994), 136–212. MR 1262703

Che44 S. Chern, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. Math. (2) **45** (1944), 741–752. MR 1458239

Che79 J. Cheeger, *Analytic torsion and the heat equation*, Ann. of Math. (2) **109** (1979), no. 2, 259–322. MR 528965

Den08 C. Deninger, *Analogies between analysis on foliated spaces and arithmetic geometry*, Groups and analysis, London Math. Soc. Lecture Note Ser., vol. 354, Cambridge Univ. Press, Cambridge, 2008, `https://doi.org/10.1017/CBO9780511721410.010`, pp. 174–190. MR 2528467

dR50 G. de Rham, *Complexes à automorphismes et homéomorphie différentiable*, Ann. Inst. Fourier (Grenoble) **2** (1950), 51–67. MR 43468

DS88a N. Dunford and J. T. Schwartz, *Linear operators. Part I: General theory*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1988. MR 1009162

DS88b ———, *Linear operators. Part II: Spectral theory. Selfadjoint operators in Hilbert space*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1988. MR 1009163

Far95 M. Farber, *Singularities of the analytic torsion*, J. Differential Geom. **41** (1995), no. 3, 528–572. MR 1338482

Far04 ———, *Topology of closed one-forms*, Mathematical Surveys and Monographs, vol. 108, Amer. Math. Soc., Providence, RI, 2004. MR 2034601

Flo89 A. Floer, *Witten's complex and infinite dimensional Morse theory*, J. Differential Geom. **30** (1989), no. 1, 207–221. MR 1001276

Fra35 W. Franz, *Über die Torsion einer Überdeckung*, J. Reine Angew. Math. **173** (1935), 245–254. MR 1581473

Gil95 P.B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, second ed., Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR 1396308

GS77 P. Günther and R. Schimming, *Curvature and spectrum of compact Riemannian manifolds*, J. Differential Geom. **12** (1977), no. 4, 599–618. MR 512929

Gui77 V. Guillemin, *Lectures on spectral theory of elliptic operators*, Duke Math. J. **44** (1977), no. 3, 485–517, `http://projecteuclid.org/euclid.dmj/1077312384`. MR 0448452

Hir76 M.W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York, Heidelberg, Berlin, 1976.

HM06 F.R. Harvey and G. Minervini, *Morse Novikov theory and cohomology with forward supports*, Math. Ann. **335** (2006), no. 4, 787–818. MR 2232017

HS85 B. Helffer and J. Sjöstrand, *Puits multiples en mécanique semi-classique. IV. étude du complexe de Witten*, Comm. Partial Differential Equations **10** (1985), 245–340. MR 780068

Kat95 T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition. MR 1335452

KM76 F.F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"*, Math. Scand. **39** (1976), no. 1, 19–55. MR 437541

Lat94 F. Latour, *Existence de* 1*-formes fermées non singulières dans une classe de cohomologie de de Rham*, Inst. Hautes Études Sci. Publ. Math. **80** (1994), 135–194. MR 1320607

Lau12 F. Laudenbach, *Transversalité, courants et théorie de Morse*, Les Éditions de l'École Polytechnique, Palaiseau, 2012, Un cours de topologie différentielle. Exercises proposed by François Labourie. MR 3088239

Lei08 E. Leichtnam, *On the analogy between arithmetic geometry and foliated spaces*, Rend. Mat. Appl. (7) **28** (2008), no. 2, 163–188. MR 2463936

Lei14 ———, *On the analogy between L-functions and Atiyah-Bott-Lefschetz trace formulas for foliated spaces*, Rend. Mat. Appl. (7) **35** (2014), no. 1-2, 1–34. MR 3241361

LPNV13 D. Le Peutrec, F. Nier, and C. Viterbo, *Precise Arrhenius law for p-forms: the Witten Laplacian and Morse-Barannikov complex*, Ann. Henri Poincaré **14** (2013), no. 3, 567–610. MR 3035640

M$\ddot{\text{7}}$8 W Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. in Math. **28** (1978), no. 3, 233–305. MR 498252

M$\ddot{\text{9}}$3 W. Müller, *Analytic torsion and R-torsion for unimodular representations*, J. Amer. Math. Soc. **6** (1993), no. 3, 721–753. MR 1189689

Mel93 R.B. Melrose, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, vol. 4, A.K. Peters, Ltd., Wellesley, MA, 1993. MR 1348401

Mel96 ———, *Differential analysis on manifolds with corners*, `http://www-math.mit.edu/~rbm/book.html`, 1996.

Mic19 L. Michel, *About small eigenvalues of the Witten Laplacian*, Pure Appl. Anal. **1** (2019), no. 2, 149–206. MR 3949372

Mil63 J. Milnor, *Morse theory*, Annals of Mathematics Studies, vol. 51, Princeton University Press, Princeton, New Jersey, 1963, Based on lecture notes by M. Spivak and R. Wells. MR 0163331

Mil65 J.W. Milnor, *Lectures on the h-cobordism theorem. Notes by L. Siebenmann and J. Sondow*, Princeton University Press, Princeton, N.J., 1965. MR 0190942

Mil66 J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966), 358–426. MR 196736

Min15 G. Minervini, *A current approach to Morse and Novikov theories*, Rend. Mat. Appl. (7) **36** (2015), no. 3-4, 95–195. MR 3533253

MQ86 V. Mathai and D. Quillen, *Superconnections, Thom classes, and equivariant differential forms*, Topology **25** (1986), no. 1, 85–110. MR 836726

MRS16 T. Mrowka, D. Ruberman, and N. Saveliev, *An index theorem for end-periodic operators*, Compos. Math. **152** (2016), no. 2, 399–444. MR 3462557

Nov81 S.P. Novikov, *Multivalued functions and functionals. An analog of the Morse theory*, Soviet. Math., Dokl. **24** (1981), 222–226. MR 630459

Nov82 ———, *The Hamiltonian formalism and a multivalued analogue of Morse theory*, Russian Math. Surveys **37** (1982), 1–56. MR 676612

Nov02 ———, *On the exotic De-Rham cohomology. Perturbation theory as a spectral sequence*, arXiv:math-ph/0201019, 2002.

Paj87 A.V. Pajitnov, *An analytic proof of the real part of Novikov's inequalities*, Soviet Math., Dokl. **35** (1987), 456–457. MR 891557

Paj06 ———, *Circle-valued Morse theory*, de Gruyter Studies in Mathematics, vol. 32, Walter de Gruyter & Co., Berlin, 2006. MR 2319639

Qui85 D. Quillen, *Determinants of Cauchy-Riemann operators on Riemann surfaces*, Funct. Anal. Appl. **19** (1985), no. 1, 37–41. MR 783704

Rei35 K. Reidemeister, *Homotopieringe und Linsenräume*, Abh. Math. Sem. Univ. Hamburg **11** (1935), no. 1, 102–109. MR 3069647

Roe98 J. Roe, *Elliptic operators, topology and asymptotic methods*, second ed., Pitman Research Notes in Mathematics, vol. 395, Longman, Harlow, 1998. MR 1670907

RS71 D.B. Ray and I.M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. **7** (1971), 145–210. MR 295381

Sch93 M. Schwarz, *Morse homology*, Progress in Mathematics, vol. 111, Birkhäuser Verlag, Basel, 1993. MR 1239174

Sch99 ———, *Equivalences for Morse homology*, Geometry and Topology in Dynamics (Winston-Salem, NC, 1998/San Antonio, TX, 1999) (Providence, RI), Contemporary Mathematics, vol. 246, American Mathematical Society, 1999, pp. 197–216. MR 1732382

See67 R.T. Seeley, *Complex powers of an elliptic operator*, Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966) (Providence, R.I.), vol. 10, Amer. Math. Soc., 1967, pp. 288–307. MR 0237943

Sma60a S. Smale, *The generalized Poincaré conjecture in higher dimensions*, Bull. Amer. Math. Soc. **66** (1960), 373–375. MR 124912

Sma60b ———, *Morse inequalities for a dynamical system*, Bull. Amer. Math. Soc. **66** (1960), 43–49. MR 117745

Sma61 ———, *On gradient dynamical systems*, Ann. of Math. (2) **74** (1961), 199–206. MR 0133139

Sma63 ———, *Stable manifolds for differential equations and diffeomorphisms*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), no. 1–2, 97–116. MR 0165537

Tho49 R. Thom, *Sur une partition en cellules associée à une fonction sur une variété*, C.R. Acad. Sci. Paris **228** (1949), 973–975. MR 29160

Wit82 E. Witten, *Supersymmetry and Morse theory*, J. Differ. Geom. **17** (1982), 661–692. MR 683171

Zha01 W. Zhang, *Lectures on Chern-Weil theory and Witten deformations*, Nankai Tracts in Mathematics, vol. 4, World Scientific Publishing Co., Inc., River Edge, NJ, 2001. MR 1864735

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