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# GLOBAL SMOOTH SOLUTIONS FOR TRIANGULAR REACTION-CROSS DIFFUSION SYSTEMS

JESSICA GUERAND, ANGELIKI MENEGAKI, AND ARIANE TRESCASES

ABSTRACT. For a class of reaction cross-diffusion systems of two equations with a cross-diffusion term in the first equation and with self-diffusion terms, we prove that the unique local smooth solution given by Amann theorem is actually global. This class of systems arises in Population dynamics, and extends the triangular Shigesada-Kawasaki-Teramoto system when general power-laws growth are considered in the reaction and diffusion rates.

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## 1. INTRODUCTION

1.1. **The system.** The purpose of this article is to study the global existence of a smooth solution of the following system,

$$\begin{aligned}
 \partial_t u - \Delta \left( (d_u + d_\alpha \langle u \rangle^\alpha + d_\beta \langle v \rangle^\beta) u \right) &= u(r_u - r_a \langle u \rangle^a - r_b \langle v \rangle^b), & \text{in } [0, \infty) \times \Omega, \\
 \partial_t v - \Delta \left( (d_v + d_\gamma \langle v \rangle^\gamma) v \right) &= v(r_v - r_c \langle v \rangle^c - r_d \langle u \rangle^d), & \text{in } [0, \infty) \times \Omega, \\
 \nabla u \cdot n = \nabla v \cdot n &= 0, & \text{in } [0, \infty) \times \partial\Omega, \\
 u(0, \cdot) = u_0, \quad v(0, \cdot) &= v_0, & \text{in } \Omega,
 \end{aligned} \tag{1.1}$$

with the bracket  $\langle \cdot \rangle$  defined by

$$\langle z \rangle := (1 + z^2)^{1/2}, \quad z \geq 0, \tag{1.2}$$

and where  $u = u(t, x) \geq 0$  and  $v = v(t, x) \geq 0$  are the unknowns,  $\Omega$  is a smooth open bounded domain of  $\mathbb{R}^m$  for  $m \geq 2$  and  $n(x)$  is the outward normal vector at point  $x \in \partial\Omega$ . The functions  $u_0, v_0 : \Omega \rightarrow \mathbb{R}$  are nonnegative initial data belonging to  $C^{2+\nu}$  for some  $\nu > 0$ . The constant parameters are defined in the set  $D$ ,

$$D := \{d_u, d_v, d_\alpha, d_\beta, d_\gamma, r_u, r_v, r_a, r_b, r_c, r_d, a, b, c, d, \alpha, \beta, \gamma\} \in ((0, \infty))^{17} \times [0, \infty), \tag{1.3}$$

and are assumed to satisfy

$$d < \frac{1}{2} \max\{1 + a, 2 + \alpha, a + \alpha, 2\alpha + 2/m\} \quad \text{and} \quad d < \max\left(\frac{8\alpha(m+1)}{(m^2-4)_+}, \frac{4\alpha+2a}{m+2}\right), \quad (1.4)$$

where we recall that  $m$  is the space dimension. This system is a prototypical extension of the triangular Shigasada-Kawasaki-Teramoto system arising in Population dynamics, [SKT79]:

$$\begin{aligned} \partial_t u - \Delta\left((d_u + d_\alpha u + d_\beta v)u\right) &= u(r_u - r_a u - r_b v), \quad \text{in } [0, \infty) \times \Omega, \\ \partial_t v - \Delta\left((d_v + d_\gamma v)v\right) &= v(r_v - r_c v - r_d u), \quad \text{in } [0, \infty) \times \Omega, \end{aligned} \quad (1.5)$$

when the linear terms in the reaction and diffusion rates are replaced with functionals with a power law growth at infinity. System (1.5) and its extension System (1.1) model the evolution of the space densities of the populations of two living species interacting through their movement. The terms  $d_\alpha \langle u \rangle^\alpha u$  and  $d_\gamma \langle v \rangle^\gamma v$  are the self-diffusion terms, and  $d_\beta \langle v \rangle^\beta u$  is the cross-diffusion term. In the diffusion rates, we use the bracket  $\langle \cdot \rangle$  taken at some powers  $\alpha$ ,  $\beta$  or  $\gamma$  to guarantee a smooth behaviour close to zero even for powers  $\alpha$ ,  $\beta$  or  $\gamma$  less than one. When the power  $\alpha$ ,  $\beta$  or  $\gamma$  respectively is greater or equal to one, one could replace the term  $\langle u \rangle^\alpha$ ,  $\langle v \rangle^\beta$  or  $\langle v \rangle^\gamma$  respectively, by the simpler power law term  $u^\alpha$ ,  $v^\beta$  or  $v^\gamma$  respectively, and our results would still apply. The same applies for the reaction rates  $\langle u \rangle^a$ ,  $\langle v \rangle^b$ ,  $\langle v \rangle^c$  and  $\langle u \rangle^d$ .

By considering System (1.1) instead of the original system (1.5) we follow the idea of replacing linear interaction rates by more general nonlinear terms. This idea goes back to the studies of [GJ72, GA73] where some non-linear functions were proved to be more appropriate than linear ones to model the competitive interaction between two species (in their study, two species of drosophila), modelled in that case with an ordinary differential system. We use power laws for simplicity, though a wide range of functions could be considered. Such generalized (triangular) Shigasada-Kawasaki-Teramoto systems have been studied for example in [PT90, Wan05, Yam95, DT15, Tre16].

**1.2. Main result.** Our main result is the following.

**Theorem 1** (Global smooth solutions). *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^m$  such that  $\partial\Omega$  is smooth. Let the coefficients of (1.1) satisfy (1.3)–(1.4) and  $u_0, v_0$  be nonnegative functions in  $C^{2+\nu}(\overline{\Omega})$  for some  $\nu > 0$  satisfying the Neumann boundary condition*

$$\nabla u_0 \cdot n = \nabla v_0 \cdot n = 0 \quad \text{on } \partial\Omega.$$

*Then the system possesses a unique, nonnegative, global solution  $(u, v)$  such that*

$$u, v \in C^{\frac{2+\nu}{2}, 2+\nu}([0, \infty) \times \overline{\Omega}) \cap C^\infty((0, \infty) \times \overline{\Omega}).$$

To prove this theorem, we rely on Amann's results which gives the local existence of a unique smooth solution together with an extension criteria (for a large class of system including (1.1), see [Ama90, Ama89, Ama93]). More precisely, for System (1.1), Amann's result reads as follows.

**Theorem 2** (Local smooth solutions [Ama90, Ama89, Ama93]). *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^m$  such that  $\partial\Omega$  is smooth. Let the coefficients of (1.1) satisfy (1.3) and  $u_0, v_0$  be nonnegative functions in  $W^{1,p_0}(\Omega)$  with  $p_0 > m$ . Then there exists a maximal time  $t_{\max} \in (0, \infty]$  such that the system (1.1) has a unique nonnegative solution  $(u, v)$  in  $(0, t_{\max}) \times \Omega$  such that*

$$u, v \in C([0, t_{\max}), W^{1,p_0}(\Omega)) \cap C^\infty((0, t_{\max}) \times \bar{\Omega}),$$

Moreover, if  $t_{\max} < \infty$ , then

$$\lim_{t \rightarrow t_{\max}^-} \left[ \|u(t, \cdot)\|_{W^{1,p_0}(\Omega)} + \|v(t, \cdot)\|_{W^{1,p_0}(\Omega)} \right] = \infty.$$

**1.3. State of the art.** The local existence of smooth solutions together with an extension criteria for a wide class of parabolic systems including (1.1) was established by Amann in [Ama90, Ama89, Ama93], as stated in Theorem 2.

For global existence, many works focus on the original (quadratic) triangular Shigesada-Kawasaki-Teramoto system (1.5). When there is self-diffusion in the first equation, that is when  $d_\alpha > 0$ , the existence of global smooth solutions was first established in dimensions  $m = 1$  and  $m = 2$ , [LNW98, Yag93], then in dimension  $m \leq 5$  in [CLY04, LNN03], then  $m \leq 9$  in [VT08]. Finally, Hoang *et al.* recently extended these results to any dimension  $m \geq 1$  in [HNP15]. Their method rely on De Giorgi techniques and on the proof of a Sobolev regularity result for nonlinear parabolic scalar equations with self-diffusion. We also mention the interesting work of [TW19] where the authors show that for the case  $d_\alpha > 0$ , in convex domains of dimension  $m \leq 9$ , the global solutions are uniformly bounded and the long-time behaviour is studied. The situation without self-diffusion in the first equation ( $d_\alpha = 0$ ) is more delicate, and global smooth solutions have been obtained only in dimension  $m = 2$  in [LNW98], with the exception of the case with no self diffusion at all ( $d_\alpha = d_\gamma = 0$ ) which is treated in any dimension  $m \geq 1$  in [CLY04] and [DT15]. However, global weak solutions exist in any dimension  $m$  even for the case  $d_\alpha = 0$  and  $d_\gamma > 0$ : see [Tre16].

Let us now present works on more generic forms of triangular cross-diffusion systems allowing for power law growth in the spirit of (1.1). In absence of self-diffusion terms, that is  $\alpha = \gamma = 0$ , global smooth solutions were obtained in [PT90] with assumptions on the reaction part that amount to assume a fast decay in  $u$  of the reaction terms and in [Yam95] when  $a > d$ . Global weak solutions were furthermore obtained in [DT15] in the same case  $a > d$  and when  $a \leq 1$  and  $d \leq 2$  (without the brackets  $\langle \cdot \rangle$ ). Allowing the presence of self-diffusion, global existence has been obtained in [Wan05] in the case  $\alpha > 0$ ,  $\gamma = 0$  under a (dimension-dependent) condition of smallness of the parameter  $d$  w.r.t. the parameter  $a$ . Then, global weak solutions were obtained in [Tre16] when the powers satisfy:  $(\alpha > 0, d < 2 + \alpha, a < 1 + \alpha)$  or  $(\alpha = 0, d \leq 2, a \leq 1)$ , with no condition on  $\gamma$  other than nonnegativity (and again, without the need of the brackets  $\langle \cdot \rangle$ ). We also mention here the recent paper [Le21] where regularity criteria leading to uniqueness of a specific weak-strong notion of solutions are provided for a class of cross-diffusion systems.

In the general case where  $\gamma$  can be positive, smooth solutions were not studied, which is the purpose of the present work. The case  $\gamma > 0$  is the most delicate case, as one can not

rely on the properties of the heat operator: see Remark 4. In particular, for the original SKT system (1.5), we observe from the literature that the presence of self-diffusion in the first equation  $\alpha > 0$  is crucial to obtain regularity. Here, we want to investigate how much self-diffusion, quantified by the parameter  $\alpha$ , is actually enough to obtain smooth solutions for the more general system (1.1). A first answer is given by the assumption (1.4). However, we give in the following subsection some possible relaxations of these assumptions.

**1.4. Strategy of the proof and possible extensions.** Our strategy of proof relies on the following main ingredients: first we apply Amann's theorem to obtain local smooth solutions on  $[0, t_{\max})$ , and the objective is to extend them globally. Let  $T \in (0, \infty)$  with  $T \leq t_{\max}$ . Our objective is to prove that the local solution satisfy

$$\sup_{t \in (0, T)} \|u(t, \cdot)\|_{W^{1,p_0}(\Omega)} + \|v(t, \cdot)\|_{W^{1,p_0}(\Omega)} < \infty, \quad (1.6)$$

so that necessarily by Theorem 2, the unique local solution in Theorem 2 is global, which gives Theorem 1.

Our first basic estimates are the maximum principle for  $v$  and a combination of a duality estimate and an energy-type estimate coming from the logistic-type growth for  $u$ . Then, we iterate some higher-order energy-type estimates for  $u$  and classical parabolic regularity estimates for  $v$  in Sobolev spaces in order to obtain that both  $u$  and  $\nabla v$  are in  $L^p((0, T) \times \Omega)$  for all  $p > 1$ . We then obtain that  $u$  is bounded by defining a class of energy estimate that allows us to use De Giorgi method. And finally we conclude using the classical Hölder and Schauder estimates.

We list in the following remarks some possible extensions of this result and perspectives.

*Remark 3* (The case  $\alpha = 0$ ). When  $\alpha > 0$ , the self-diffusion in the first equation allows to prove more regularity for  $u$ . This appears in our assumption (1.4), and can also be seen directly for example in the duality estimate which gives a bound for  $u$  in  $L^{2+\alpha}$ . The case where  $\alpha = 0$  is an unfavorable case. It is actually covered by our method when the reaction parameters  $a$  and  $d$  satisfy  $d(m+2) < 2a$ .

*Remark 4* (The case  $\gamma = 0$ ). The case where there is no self-diffusion in the second equation, that is  $\gamma = 0$ , is a very favorable case. In this case, the properties of the heat operator allow to obtain regularity for the first and second space derivatives of  $v$  without any dimension-dependent assumption, and this even in the unfavorable case where there is no self-diffusion in the first equation. For example, the results hold under the condition  $\gamma = 0$ ,  $\alpha = 0$ ,  $d < a$ , see [Yam95] or [DT15].

*Remark 5* (Dimension-dependent assumptions). We expect that our result holds for larger range for  $d$  rather than  $d < \max\left(\frac{8\alpha(m+1)}{(m^2-4)_+}, \frac{4\alpha+2a}{m+2}\right)$ . We use this assumption in the proof of Lemma 16 in order to obtain the continuity of  $v$  (thanks to dimension-dependent embeddings), which allows to apply classical results of parabolic regularity in Sobolev spaces: see the result due to Ladyzenskaya *et al.* stated later in Lemma 14. In fact, for the original system (1.5) we recover the results in [VT08] with the same restriction on the dimension, see also [TW19]. However, some recent results for the original system (1.5) due to Hoang

*et al.*, [HNP15], suggest that we can bypass this classical tool and obtain directly estimates on the gradient of  $v$  in any  $L^p$  space, conditionally to the  $L^p$  regularity of the right hand side, without requiring the continuity of  $v$ . The result of Hoang *et al.* could be extended to system (1.1) with a generic  $\gamma > 0$ , and as a consequence we could remove the space-dependent assumption. This is the purpose of an ongoing work.

*Remark 6* (Weak solutions). Using the same techniques we could also prove that the weak solutions obtained in [Tre16] with initial conditions in  $W^{2,\infty}$ , are actually smooth. In that case one can deduce that the two solutions in [Ama93] and in [Tre16] must coincide.

**1.5. Notations.** We end this section by giving some notations and definitions.

Recall that  $T \in (0, \infty)$  is defined in Sec. 1.4, with  $T \leq t_{\max}$ , and  $(u, v)$  are nonnegative solutions of (1.1) on  $[0, T) \times \Omega$  given by Theorem 2.

We define the domain  $\Omega_T = (0, T) \times \Omega$ . Throughout this paper,  $g \in W_p^2(\Omega)$  means  $g, \partial_{x_i}g, \partial_{x_i x_j}g$  are in  $L^p(\Omega)$ , and  $g \in W_p^{1,2}(\Omega_T)$  means  $g, \partial_{x_i}g, \partial_{x_i x_j}g$  and  $\partial_t g$  are in  $L^p(\Omega_T)$  for  $i, j = 1, \dots, m$ . The space  $W_p^{1,2}(\Omega_T)$  is endowed with the norm

$$\|g\|_{W_p^{1,2}(\Omega_T)} = \|g\|_{L^p(\Omega_T)} + \|\partial_t g\|_{L^p(\Omega_T)} + \|\nabla g\|_{L^p(\Omega_T)} + \sum_{i,j=1}^m \|\partial_{x_i x_j} g\|_{L^p(\Omega_T)}.$$

Here  $\nabla g = (\partial_{x_1}g, \dots, \partial_{x_m}g)$ . We define also the positive part of a function  $g_+ = \max(g, 0)$ . We occasionally write  $f \lesssim g$  in order to say that  $f \leq Cg$  for some constant  $C$  which depends on time  $T$ , domain  $\Omega$  and the parameters  $D$ .

## 2. GAIN OF INTEGRABILITY OF $\nabla_x v$ AND $u$

In this section we prove that  $u$  and  $\nabla_x v$  are in any  $L^p$  space with  $p \in (0, \infty)$ . To get this result, we use an iteration process to transfer of regularity between  $u$  and  $\nabla v$ . We first give some basic estimates to initialize the iteration process. Then we give some conditional properties of transfer of regularity between  $u$  and  $\nabla_x v$  and we conclude by iteration.

**2.1. Basic estimates.** We start with the following basic estimates.

**Lemma 7** (Basic estimates). *The solution  $(u, v)$  of (1.1) satisfies the following bounds*

$$\sup_{t \in (0, T)} \int_{\Omega} u(t, x) dx \leq C, \quad \iint_{\Omega_T} u^{1+a}(t, x) dx dt \leq C_1(T), \quad (2.1)$$

$$\sup_{t \in (0, T)} \|v(t, \cdot)\|_{L^\infty(\Omega)} \leq C, \quad (2.2)$$

where  $C$  depends only on the initial data, the domain  $\Omega$  and the parameters  $D$ , while  $C_1(T)$  depends linearly on time.

*Proof.* Let us integrate the first equation of (1.1) on  $\Omega_t$  for  $t \in (0, T]$

$$\int_{\Omega} u(t, x) dx = \int_{\Omega} u(0, x) dx + \iint_{\Omega_T} u (d_u - d_a \langle u \rangle^a - d_b \langle v \rangle^b) dx ds. \quad (2.3)$$

Using the nonnegativity of  $u$  and  $v$  and the fact that  $d_u z - d_a z \langle z \rangle^a + \frac{d_a}{2} z^{a+1} \leq C$  for all  $z \geq 0$ , we get the estimate using twice (2.3), once for time  $t$  and once for time  $T$ ,

$$\int_{\Omega} u(t, x) dx + \iint_{\Omega_T} \frac{d_a}{2} u^{1+a} dx dt \leq \int_{\Omega} u(0, x) dx + C|\Omega|T, \quad (2.4)$$

which concludes the two first estimates. The last estimate is a direct consequence of the maximum principle.  $\square$

**Lemma 8** (Interpolation Inequality). *Let  $w$  be the function  $w = (d_v + d_\gamma \langle v \rangle^\gamma)v$  and  $p \geq 1$ . It holds that*

$$\|w\|_{L^{2p}((0,T),W^{1,2p}(\Omega))}^{2p} \leq C \|w\|_{L^p((0,T),W^{2,p}(\Omega))}^p.$$

*Proof.* To prove this we use  $W^{2,p}(\Omega) \cap L^\infty(\Omega) \subset W^{1,2p}(\Omega)$  of [BL12, Theorem 6.4.5 p.153] or [BM19, Theorem 1] which gives

$$\|w\|_{W^{1,2p}(\Omega)} \leq C \|w\|_{L^\infty(\Omega)}^\theta \|w\|_{W^{2,p}(\Omega)}^{1-\theta},$$

so using  $\theta = \frac{1}{2}$  we have

$$\begin{aligned} \int_0^T \|w\|_{W^{1,2p}(\Omega)}^{2p} dt &\leq C \int_0^T \|w\|_{L^\infty(\Omega)}^{2p\theta} \|w\|_{W^{2,p}(\Omega)}^{2p(1-\theta)} dt \\ &\leq C \|w\|_{L^\infty(\Omega_T)}^p \int_0^T \|w\|_{W^{2,p}(\Omega)}^p dt \\ &\leq C \|w\|_{L^p((0,T),W^{2,p}(\Omega))}^p, \end{aligned}$$

where we have used Lemma 7 to bound the  $L^\infty$ -norm in the right-hand side.  $\square$

The next basic estimate is a consequence of the following lemma from [DLMT15], Lemma 2.11, which is obtained by duality techniques.

**Lemma 9** (Duality estimate). *Let  $M : [0, T] \times \Omega \rightarrow \mathbb{R}_+$  be a positive continuous function lower bounded by a positive constant and let  $r_u > 0$ . Any smooth nonnegative solution of the differential inequality*

$$\begin{aligned} \partial_t u - \Delta(Mu) &\leq r_u u \text{ on } \Omega, \\ \partial_n(Mu) &= 0, \text{ on } \partial\Omega, \end{aligned}$$

*satisfies the following bound*

$$\iint_{\Omega_T} Mu^2 \leq \exp(2r_u T) \times \left( C_\Omega^2 \|u(0, \cdot)\|_{H_m^{-1}(\Omega)}^2 + \overline{u(0, \cdot)}^2 \iint_{\Omega_T} M \right),$$

*with  $\overline{u(0, \cdot)}$  the mean value of  $u(0, \cdot)$  on  $\Omega$  and  $C_\Omega$  the Poincaré-Wirtinger constant.*

As a consequence, we get the following duality estimate.

**Corollary 10** (Duality estimate). *Let  $(u, v)$  be the solution of (1.1). Then  $u$  satisfies the following bound,*

$$\iint_{\Omega_T} u^{2+\alpha}(t, x) dx dt \leq C, \quad (2.5)$$

*where  $C$  depends only on the initial data, the domain  $\Omega$ , the time  $T$  and the parameters  $D$ .*

*Proof.* Applying Lemma 9 with  $M := d_u + d_\alpha \langle u \rangle^\alpha + d_\beta \langle v \rangle^\beta$  yields the estimate

$$\iint_{\Omega_T} (u^2 + u^{2+\alpha} + u^2 v^\beta)(t, x) dx dt \leq C \left( 1 + \iint_{\Omega_T} (u^\alpha + v^\beta)(t, x) dx dt \right) \quad (2.6)$$

where we used the fact that  $u^\alpha \leq \langle u \rangle^\alpha \leq C(1 + u^\alpha)$ . Using now the  $L^\infty$ -bound of  $v$  from Lemma 7 and the fact that  $Cz^\alpha \leq C' + \frac{1}{2}z^{2+\alpha}$  for all  $z \geq 0$ , we conclude the proof.  $\square$

## 2.2. Transfer of integrability from $\nabla_x v$ to $u$ .

**Lemma 11** (Power energy estimate for  $u$ ). *Let  $\rho \geq \max(1, \alpha)$  and  $p > 2$ . Considering the system of cross-diffusion equations given by (1.1) with  $\alpha > 0$  and assuming that  $\nabla v \in L^p(\Omega_T)$  and  $u \in L^\rho(\Omega_T) \cap L^{\frac{p}{p-2}(\rho-\alpha)}(\Omega_T)$ , the following energy estimate holds true for all  $s < t \in [0, T]$ ,*

$$\begin{aligned} & \int_{\Omega} u^\rho(t, x) dx + \iint_{(s,t) \times \Omega} |\nabla u^{\frac{\rho}{2} + \frac{\alpha}{2}}|^2(\tau, x) dx d\tau + \iint_{(s,t) \times \Omega} u^{\alpha+\rho}(\tau, x) dx d\tau \\ & \lesssim \int_{\Omega} u^\rho(s, x) dx + \left( \iint_{(s,t) \times \Omega} |\nabla v|^p(\tau, x) dx d\tau \right)^{\frac{2}{p}} \left( \iint_{(s,t) \times \Omega} u^{\frac{p}{p-2}(\rho-\alpha)}(\tau, x) dx d\tau \right)^{\frac{p-2}{p}} + 1, \end{aligned} \quad (2.7)$$

where the constant depends on the initial data,  $D$ ,  $\Omega$ ,  $T$  and  $\rho$ .

*Proof.* We multiply the equation for  $u$  by  $u^{\rho-1}$  and integrate on  $(s, t) \times \Omega$

$$\begin{aligned} & \iint_{(s,t) \times \Omega} \frac{\partial_\tau u^\rho}{\rho} dx d\tau + 2 \frac{\rho-1}{\rho} \iint_{(s,t) \times \Omega} \beta d_\beta \frac{v}{\langle v \rangle^{2-\beta}} u^{\frac{\rho}{2}} \nabla v \cdot \nabla u^{\frac{\rho}{2}} dx d\tau \\ & + 4 \frac{\rho-1}{\rho^2} \iint_{(s,t) \times \Omega} \left( \left( d_u + d_\alpha \left( 1 + \alpha \frac{u^2}{\langle u \rangle^2} \right) \langle u \rangle^\alpha + d_\beta \langle v \rangle^\beta \right) |\nabla u^{\frac{\rho}{2}}|^2 \right) dx d\tau \\ & = \iint_{(s,t) \times \Omega} u^\rho (r_u - r_a \langle u \rangle^a - r_b \langle v \rangle^b) dx d\tau. \end{aligned}$$

Using the fact that  $z^\rho (r_u - \frac{r_a}{2} z^a) \leq C$  for all  $z \geq 0$  we get

$$\begin{aligned} & \int_{\Omega} u^\rho(t, x) dx + C_1 \iint_{(s,t) \times \Omega} |\nabla u^{\frac{\rho}{2} + \frac{\alpha}{2}}|^2 dx d\tau + C_2 \iint_{(s,t) \times \Omega} \frac{v}{\langle v \rangle^{2-\beta}} u^{\frac{\rho}{2} - \frac{\alpha}{2}} \nabla v \cdot \nabla u^{\frac{\rho}{2} + \frac{\alpha}{2}} dx d\tau \\ & + \frac{r_a}{2} \iint_{(s,t) \times \Omega} u^{\alpha+\rho}(\tau, x) dx d\tau \\ & \leq \iint_{(s,t) \times \Omega} u^\rho \left( r_u - \frac{r_a}{2} \langle u \rangle^a - r_b \langle v \rangle^b \right) dx dt + \int_{\Omega} u^\rho(s, x) dx \\ & \leq \int_{\Omega} u^\rho(s, x) dx + C. \end{aligned} \quad (2.8)$$

Applying Young's inequality and then a Hölder inequality to the third term of the left hand side yields

$$\begin{aligned} & \iint_{(s,t) \times \Omega} \frac{v}{\langle v \rangle^{2-\beta}} u^{\frac{\rho}{2} - \frac{\alpha}{2}} \nabla v \cdot \nabla u^{\frac{\rho}{2} + \frac{\alpha}{2}} dx d\tau \\ & \leq \frac{1}{2} \iint_{(s,t) \times \Omega} |\nabla u^{\frac{\rho}{2} + \frac{\alpha}{2}}|^2 dx d\tau + \frac{1}{2} \iint_{(s,t) \times \Omega} \frac{v^2}{\langle v \rangle^{4-2\beta}} u^{\rho-\alpha} |\nabla v|^2 dx d\tau \\ & \leq \frac{1}{2} \iint_{(s,t) \times \Omega} |\nabla u^{\frac{\rho}{2} + \frac{\alpha}{2}}|^2 dx d\tau + \end{aligned} \quad (2.9)$$



$$+C(1 + \max v^{2(\beta-1)_+}) \left( \iint_{(s,t) \times \Omega} u^{\frac{p}{p-2}(\rho-\alpha)} dx d\tau \right)^{\frac{p-2}{p}} \left( \iint_{(s,t) \times \Omega} |\nabla v|^p dx d\tau \right)^{\frac{2}{p}}.$$

Combining this with (2.8) gives the result.  $\square$

We finally have the following bound on the gradient of  $v$  thanks to our assumptions on the parameters.

**Proposition 12** (Integrability of  $\nabla v$ ). *Let  $(u, v)$  be a solution of (1.1). Suppose (1.3)–(1.4), in particular that  $2d < \max\{1 + a, 2 + \alpha, a + \alpha, 2\alpha + 2/m\}$ . Then*

$$\|\nabla v\|_{L^4(\Omega_T)} \leq C$$

where  $C$  depends only on the initial data, the domain  $\Omega$ , the time  $T$  and the parameters  $D$ .

Furthermore there exists  $\rho > 1$  such that

$$\int_0^T \int_{\Omega} \left| \nabla u^{\frac{\rho+\alpha}{2}} \right|^2 (s, x) dx ds \leq C(T). \quad (2.10)$$

*Proof.* We first prove that  $\|\nabla v\|_{L^2(\Omega_T)} \leq C$ . By multiplying the equation of  $v$  in (1.1) by  $v$  and integrating on  $\Omega_T$ , one gets

$$\begin{aligned} \int_{\Omega} \frac{v^2}{2}(T, x) dx + \iint_{\Omega_T} \left( d_v + d_{\gamma} \left( 1 + \gamma \frac{v^2}{\langle v \rangle^2} \right) \langle v \rangle^{\gamma} \right) |\nabla v|^2 dx ds \\ \leq \int_{\Omega} \frac{v^2}{2}(0, x) dx + \iint_{\Omega_T} v (r_v - r_c \langle v \rangle^c - r_d \langle u \rangle^d) dx ds \\ \leq \int_{\Omega} \frac{v^2}{2}(0, x) dx + C. \end{aligned} \quad (2.11)$$

Then, we recall that from Lemma 11 we have

$$\begin{aligned} \int_{\Omega} u^{\rho}(T, x) dx + \iint_{\Omega_T} |\nabla u^{\frac{\rho+\alpha}{2}}|^2(t, x) dx dt + \iint_{\Omega_T} u^{a+\rho}(t, x) dx dt \\ \lesssim \int_{\Omega} u^{\rho}(0, x) dx + \iint_{\Omega_T} u^{\rho-\alpha} |\nabla v|^2(t, x) dx dt + \iint_{\Omega_T} u^{\rho} dx dt. \end{aligned} \quad (2.12)$$

For the first term in the right-hand side we apply Young's inequality with a constant  $\eta$ , to get

$$\iint_{\Omega_T} u^{\rho-\alpha} |\nabla v|^2(t, x) dx dt \leq \eta \iint_{\Omega_T} u^{2\rho-2\alpha}(t, x) dx dt + \frac{1}{4\eta} \iint_{\Omega_T} |\nabla v|^4(t, x) dx dt. \quad (2.13)$$

We recall  $w := (d_v + d_{\gamma} \langle v \rangle^{\gamma})v$ . From Lemma 7 we have that  $w$  satisfies a bound in  $L^{\infty}$ . Furthermore, by integration by part and using the Neumann boundary condition, we know that

$$\|\Delta w\|_{L^2((0,T) \times \Omega)}^2 = \sum_{\substack{i=1, \dots, m \\ j=1, \dots, m}} \left\| \frac{\partial^2 w}{\partial x_i \partial x_j} \right\|_{L^2(\Omega_T)}^2.$$

By the interpolation from Lemma 8 applied for  $p = 2$ , i.e. between  $L^{\infty}$  and  $H^2$ , we can write

$$\iint_{\Omega_T} |\nabla v|^4(t, x) dx dt \lesssim \iint_{\Omega_T} |\Delta w|^2(t, x) dx dt. \quad (2.14)$$

Therefore (2.12) together with (2.13) gives,

$$\begin{aligned} & \int_{\Omega} u^{\rho}(T, x) dx + \iint_{\Omega_T} |\nabla u^{\frac{\rho}{2} + \frac{\alpha}{2}}|^2(t, x) dx dt + \iint_{\Omega_T} u^{a+\rho}(t, x) dx dt \\ & \lesssim \int_{\Omega} u^{\rho}(0, x) dx + \eta \iint_{\Omega_T} u^{2\rho-2\alpha}(t, x) dx dt + \frac{1}{4\eta} \iint_{\Omega_T} |\Delta w|^2(t, x) dx dt \\ & \quad + \iint_{\Omega_T} u^{\rho}(t, x) dx dt. \end{aligned} \quad (2.15)$$

Moreover  $w$  satisfies the following equation

$$\partial_t w - \left( d_u + d_{\gamma} \langle v \rangle^{\gamma} \left( 1 + \frac{\gamma v^2}{\langle v \rangle^2} \right) \right) \Delta w = \left( d_u + d_{\gamma} \langle v \rangle^{\gamma} \left( 1 + \frac{\gamma v^2}{\langle v \rangle^2} \right) \right) v (r_v - r_c \langle v \rangle^c - r_d \langle u \rangle^d), \quad (2.16)$$

and also  $\nabla w \cdot n = 0$  on  $\partial\Omega$  and  $w(0, x) = (d_v + d_{\gamma} \langle v_0 \rangle^{\gamma}) v_0$ . The estimates we get for  $\nabla w$  then are directly transferred to  $\nabla v$  through their relation  $\nabla v = \left( d_u + d_{\gamma} \langle v \rangle^{\gamma} \left( 1 + \frac{\gamma v^2}{\langle v \rangle^2} \right) \right)^{-1} \nabla w$ .

By multiplying (2.16) by  $\Delta w$  and integrating in  $\Omega_T$  we obtain,

$$\begin{aligned} & \int_{\Omega} \frac{|\nabla w|^2}{2}(T, x) dx + \iint_{\Omega_T} \left( d_u + d_{\gamma} \langle v \rangle^{\gamma} + \frac{\gamma d_{\gamma} v^2}{\langle v \rangle^{2-\gamma}} \right) |\Delta w|^2 dt dx \\ & \leq \int_{\Omega} \frac{|\nabla w|^2}{2}(0, x) dx + \iint_{\Omega_T} \left( d_u + d_{\gamma} \langle v \rangle^{\gamma} + \frac{\gamma d_{\gamma} v^2}{\langle v \rangle^{2-\gamma}} \right) v |r_v - r_c \langle v \rangle^c - r_d \langle u \rangle^d| |\Delta w| dt dx. \end{aligned} \quad (2.17)$$

We use next Young's inequality for the last integral, the fact that  $v$  is upper bounded from Lemma 7 and the inequality  $\langle u \rangle^{2d} \leq C(1 + u^{2d})$  for some  $C$ , to get

$$\begin{aligned} & \int_{\Omega} \frac{|\nabla w|^2}{2}(T, x) dx + \iint_{\Omega_T} \left( d_u + d_{\gamma} \langle v \rangle^{\gamma} + \frac{\gamma d_{\gamma} v^2}{\langle v \rangle^{2-\gamma}} \right) |\Delta w|^2 dt dx \\ & \leq \int_{\Omega} \frac{|\nabla w|^2}{2}(0, x) dx + C \iint_{\Omega_T} (1 + u^{2d})(t, x) dt dx, \end{aligned} \quad (2.18)$$

Gathering therefore the relations (2.12), (2.13) and (2.18) together and neglecting the (time-independent) constants, one has the following inequality

$$\begin{aligned} & \int_{\Omega} u^{\rho}(T, x) dx + \frac{1}{\eta} \int_{\Omega} \frac{|\nabla w|^2}{2}(T, x) dx + \iint_{\Omega_T} |\nabla u^{\frac{\rho}{2} + \frac{\alpha}{2}}|^2(t, x) dx dt + \iint_{\Omega_T} u^{a+\rho}(t, x) dx dt \\ & \quad + \frac{1}{\eta} \iint_{\Omega_T} \left( d_u + d_{\gamma} \langle v \rangle^{\gamma} + \frac{\gamma d_{\gamma} v^2}{\langle v \rangle^{2-\gamma}} \right) |\Delta w|^2 dx dt \\ & \lesssim \eta \iint_{\Omega_T} u^{2\rho-2\alpha}(t, x) dx dt + \frac{1}{\eta} \iint_{\Omega_T} (1 + u^{2d}) dx dt \\ & \quad + \iint_{\Omega_T} u^{\rho} dx dt + \int_{\Omega} u^{\rho}(0, x) dx + \frac{1}{\eta} \int_{\Omega} \frac{|\nabla w|^2}{2}(0, x) dx. \end{aligned} \quad (2.19)$$

We continue by considering three different cases regarding the values of  $a, \alpha, m$  and defining an appropriate  $\rho > 1$  in each case that satisfies  $\rho > 2\alpha$  and  $d < \rho - \alpha$  as well. We then estimate

the above right-hand side by

$$\begin{aligned} & \eta \iint_{\Omega_T} u^{2\rho-2\alpha}(t, x) dx dt + \frac{1}{\eta} \iint_{\Omega_T} (1 + u^{2d}) dx dt + \iint_{\Omega_T} u^\rho(t, x) dx dt + \\ & \quad + \int_{\Omega} u^\rho(0, x) dx + \frac{1}{\eta} \int_{\Omega} \frac{|\nabla w|^2}{2}(0, x) dx \quad (2.20) \\ & \lesssim \left( \eta \iint_{\Omega_T} u^{2\rho-2\alpha}(t, x) dx dt + C_\eta \right) \end{aligned}$$

for some positive constant  $C_\eta$ .

**Case 1:**  $\max(1 + a, 2 + \alpha, a + \alpha, 4\alpha + 4/m) = \max(1 + a, 2 + \alpha)$ .

Let  $\rho := \alpha + \max(1 + a, 2 + \alpha)/2 > 1$ , so that  $2\rho - 2\alpha \leq \max(1 + a, 2 + \alpha)$ . Then by equation (2.5), we have that  $u^{2\rho-2\alpha}$  is in  $L^1(\Omega_T)$ . Therefore, after an integration in time, taking  $\eta = 1$  and using the initial regularity,

$$\begin{aligned} & \int_{\Omega} u^\rho(T, x) dx + \int_{\Omega} \frac{|\nabla w|^2}{2}(T, x) dx + \int_0^T \int_{\Omega} |\nabla u^{\frac{\rho}{2} + \frac{\alpha}{2}}|^2(t, x) dx dt + \int_0^T \int_{\Omega} u^{a+\rho}(t, x) dx dt \\ & \quad + \int_0^T \int_{\Omega} \left( d_u + d_\gamma \langle v \rangle^\gamma + \frac{\gamma d_\gamma v^2}{\langle v \rangle^{2-\gamma}} \right) |\Delta w|^2 dx dt \leq C(T). \quad (2.21) \end{aligned}$$

**Case 2:**  $\max(1 + a, 2 + \alpha, a + \alpha, 4\alpha + 4/m) = a + \alpha$ . Let  $\rho := 2\alpha + a > 1$ , and therefore,  $2\rho - 2\alpha = \rho + a$ , and then, taking  $\eta$  small enough, one recovers the same bound as in (2.21).

**Case 3:**  $\max(1 + a, 2 + \alpha, a + \alpha) < 4\alpha + 4/m$ . Let  $\rho := 3\alpha + 2/m$ . Then  $\rho > 1$  since

$$\rho = 3\alpha + 2/m = 4\alpha + 4/m - \alpha - 2/m > 2 + \alpha - \alpha - 2/m = 2 - 2/m \geq 1.$$

Then in order to bound the right-hand side, we apply Gagliardo-Nirenberg inequality for  $1 \leq q = \frac{4(\rho-\alpha)}{\rho+\alpha}$ :

$$\begin{aligned} & \int_{\Omega} u^{2\rho-2\alpha}(t, x) dx = \|u^{\frac{\rho+\alpha}{2}}(t, \cdot)\|_{L^q(\Omega)}^q \\ & \leq C \|\nabla u^{\frac{\rho+\alpha}{2}}(t, \cdot)\|_{L^2(\Omega)}^2 \|u^{\frac{\rho+\alpha}{2}}(t, \cdot)\|_{L^{\frac{2}{\rho+\alpha}}}^{q-2} + C \|u^{\frac{\rho+\alpha}{2}}(t, \cdot)\|_{L^{\frac{2}{\rho+\alpha}}(\Omega)}^q. \quad (2.22) \end{aligned}$$

for a constant  $C$ . One then has that

$$\|u^{\frac{\rho+\alpha}{2}}(t, \cdot)\|_{L^{\frac{2}{\rho+\alpha}}(\Omega)} = \int_{\Omega} u dx \leq C \quad \text{for all } t \in (0, T) \quad (2.23)$$

from Lemma 7. We integrate the Gagliardo-Nirenberg inequality in time and we choose  $\eta$  small enough so that we can absorb the term  $\iint_{\Omega_T} |\nabla u^{\frac{\rho}{2} + \frac{\alpha}{2}}|^2(t, x) dx dt$  in the left-hand side. Then the inequality in (2.21) holds.

Combining (2.21) and (2.14) concludes the proof.  $\square$

We continue by observing that the integrability of  $\nabla v$  allows us to obtain integrability for powers of  $u$ .

**Corollary 13.** *Let  $D$  satisfy (1.3)–(1.4) and let  $u, v$  be solutions of (1.1). If  $\nabla v \in L^p(\Omega_T)$  with  $p > 2$  then  $u^r \in L^1(\Omega_T)$  for all  $r \in \left(0, \max \left\{ \frac{p\alpha(m+1)}{(m+2-p)_+}, \frac{p}{2}\alpha + \frac{p-2}{2}a \right\} \right)$  where  $m$  is the space dimension.*

*Proof.* First, if  $r \in (0, 1]$ , it is a direct consequence of the mass estimate in Lemma 7 and Jensen's inequality.

Then, for  $r > 1$  such that  $r < \frac{p}{2}\alpha + \frac{p-2}{2}a$ , we use the energy inequality (2.7) from Lemma 11 with  $\rho := r$ . Since  $\rho + a > \frac{p}{p-2}(\rho - \alpha)$ , one can absorb the right-hand side by the last term of the left-hand side and conclude that  $u^r \in L^1(\Omega_T)$ .

Then, for  $r > 1$  such that  $r < \frac{p\alpha(m+1)}{(m+2-p)_+}$ , the idea here is to use a Gagliardo-Nirenberg inequality combined with the previous lemma to deduce a sequence of  $L^p$  spaces where  $u$  belongs. Let  $\rho \geq \alpha$ . We define  $q = 2 + \frac{4\rho}{m(\rho+\alpha)}$ . By the Gagliardo-Nirenberg inequality [Nir59], there exists  $C > 0$  depending on  $\rho, m, \alpha$  and  $\Omega$  such that

$$\|u^{\frac{\rho+\alpha}{2}}(t, \cdot)\|_{L^q(\Omega)} \leq C \|\nabla u^{\frac{\rho+\alpha}{2}}(t, \cdot)\|_{L^2(\Omega)}^{\frac{2}{q}} \|u^{\frac{\rho+\alpha}{2}}(t, \cdot)\|_{L^{\frac{2\rho}{\rho+\alpha}}}^{\frac{4\rho}{mq(\rho+\alpha)}} + C \|u^{\frac{\rho+\alpha}{2}}(t, \cdot)\|_{L^{\frac{2\rho}{\rho+\alpha}}(\Omega)}. \quad (2.24)$$

Putting (2.24) to the exponent  $q$ , integrating it in time gives

$$\|u\|_{L^{\rho+\alpha+\frac{2\rho}{m}}(\Omega_T)}^{\rho+\alpha+\frac{2\rho}{m}} \leq C \left( \sup_{t \in (0, T)} \|u(t, \cdot)\|_{L^\rho}^{\frac{2\rho}{m}} + \sup_{t \in (0, T)} \|u(t, \cdot)\|_{L^\rho}^{q\frac{\rho+\alpha}{2}} \right) \left( \|\nabla u^{\frac{\rho+\alpha}{2}}\|_{L^2(\Omega_T)}^2 + 1 \right). \quad (2.25)$$

Now using Lemma 11 with  $s = 0$  together with the assumption that  $\nabla v \in L^p(\Omega_T)$ , we know that

$$\sup_{t \in (0, T)} \|u(t, \cdot)\|_{L^\rho(\Omega)}^\rho \leq C \left( \|u_0\|_{L^\rho(\Omega)}^\rho + \|u\|_{L^{\frac{p}{p-2}(\rho-\alpha)}(\Omega_T)}^{\rho-\alpha} + 1 \right) \quad (2.26)$$

and

$$\|\nabla u^{\frac{\rho+\alpha}{2}}\|_{L^2(\Omega_T)}^2 \leq C \left( \|u_0\|_{L^\rho(\Omega)}^\rho + \|u\|_{L^{\frac{p}{p-2}(\rho-\alpha)}(\Omega_T)}^{\rho-\alpha} + 1 \right). \quad (2.27)$$

Then reinserting (2.26) and (2.27) in (2.25) we get (using the regularity of the initial data)

$$\|u\|_{L^{\rho+\alpha+\frac{2\rho}{m}}(\Omega_T)}^{\rho+\alpha+\frac{2\rho}{m}} \leq C \left( 1 + \|u\|_{L^{\frac{p}{p-2}(\rho-\alpha)}(\Omega_T)}^{(\rho-\alpha)(1+q\frac{\rho+\alpha}{2\rho})} \right). \quad (2.28)$$

For  $\eta_0 > 1$ , let us define  $\rho_0 > \alpha$  by  $\eta_0 = \frac{p}{p-2}(\rho_0 - \alpha)$ , and let us define the sequences  $(\eta_k)_k$  and  $(\rho_k)_k$  recursively as

$$\eta_{k+1} = \rho_k + \alpha + \frac{2\rho_k}{m}, \quad \eta_{k+1} = \frac{p}{p-2}(\rho_{k+1} - \alpha), \quad k \geq 0. \quad (2.29)$$

By definition of these sequences, (2.28) gives that if  $u \in L^{\eta_k}(\Omega_T)$  then  $u \in L^{\eta_{k+1}}(\Omega_T)$ .

We can write, for all  $k \geq 0$ ,

$$\eta_{k+1} = 2 \left( 1 + \frac{1}{m} \right) \alpha + \frac{p-2}{p} \left( 1 + \frac{2}{m} \right) \eta_k, \quad (2.30)$$

so that studying the limit we get that in the limit  $k \rightarrow \infty$

$$\eta_k \rightarrow \frac{\alpha p(1+m)}{2+m-p} \quad \text{if } p < m+2, \quad (2.31)$$

$$\eta_k \rightarrow \infty \quad \text{if } p \geq m+2, \quad (2.32)$$

and that concludes the proof.  $\square$

**2.3. Transfer of integrability from  $u$  to  $\nabla_x v$ .** We prove that we can transfer the integrability of  $u$  to  $\nabla_x v$ . To get this result we make use of  $W_p^{1,2}$  estimates that we recall from [LSU68, Theorem 9.1 Chapter 5 and remark p.351] and [VT08, Lemma 2.2].

**Lemma 14** ( $W_p^{1,2}$  estimates). *Let  $3 < q < \infty$ ,  $w_0 \in W_q^2(\Omega)$  and  $w$  be the unique solution in  $W_q^{1,2}(\Omega_T)$  of the equation*

$$\begin{cases} w_t - a(t, x)\Delta w = f(t, x) & (t, x) \in \Omega_T, \\ \nabla w \cdot n = 0 & x \in \partial\Omega, t > 0, \\ w(t_0, x) = w_0 & x \in \Omega \end{cases} \quad (2.33)$$

where  $a(t, x)$  is a continuous function on  $\overline{\Omega_T}$  satisfying

$$0 < \lambda \leq a(t, x) \leq \Lambda, \quad \forall (t, x) \in \overline{\Omega_T},$$

where  $\lambda$  and  $\Lambda$  are positive constants. We assume that  $f \in L^q(\Omega_T)$ . Then there exists a constant  $c_q > 0$  depending on  $q, T, \lambda$  and  $\Lambda$  such that

$$\|w\|_{W_q^{1,2}(\Omega_T)} \leq c_q \left( \|f\|_{L^q(\Omega_T)} + \|w_0\|_{W_q^2(\Omega_T)} \right),$$

where  $w_0$  satisfies  $\nabla w_0 \cdot n = 0$  on  $\partial\Omega$ .

**Corollary 15** (Transfer of integrability from  $u$  to  $\nabla v$ ). *Let  $D$  satisfy (1.3)–(1.4) and let  $u, v$  be solutions of (1.1) with  $v$  uniformly continuous on  $\Omega_T$ . If  $u^d \in L^p(\Omega_T)$  for  $p \in [2, +\infty)$  then  $\nabla v \in L^{2p}(\Omega_T)$ .*

*Proof of Corollary 15.* Recall the function  $w = (d_v + d_\gamma \langle v \rangle^\gamma)v$  and its equation in (2.16). See that  $v$  is uniformly continuous in  $\Omega_T$  so continuous in  $\overline{\Omega_T}$ . Combining the Lemmas 14 and 8, both  $\nabla w$  and  $\nabla v$  are in  $L^{2p}(\Omega_T)$  since

$$\nabla w = \left( d_u + d_\gamma \langle v \rangle^\gamma + \frac{\gamma d_\gamma v^2}{\langle v \rangle^{2-\gamma}} \right) \nabla v.$$

$\square$

**2.4. Iteration and conclusion.** We deduce using the previous subsections that  $u$  and  $\nabla v$  are in any  $L^p$  space.

**Lemma 16.** *Let  $u$  and  $v$  be solutions of (1.1) where  $D$  satisfies (1.4). Then  $u$  and  $\nabla v$  are in  $L^p(\Omega_T)$  for all  $p \in (1, \infty)$  with a bound only depending on  $p$ , the initial data,  $D, T, \Omega$  and  $m$ .*

*Proof.* Let us prove that  $v$  is uniformly continuous in  $\Omega$  to be able to apply Corollary 15. By Proposition 12 we have that  $\nabla v \in L^4(\Omega_T)$ , so we can apply Corollary 13 with  $p = 4$  and deduce that  $u \in L^r(\Omega_T)$  for any  $r < \max\left(\frac{4\alpha(m+1)}{(m-2)_+}, 2\alpha + a\right)$ . Therefore  $u^d \in L^{r'}(\Omega_T)$  for any  $r' < \left(\frac{4\alpha(m+1)}{d(m-2)_+}, \frac{2\alpha+a}{d}\right)$  where thanks to the condition  $d < \max\left(\frac{8\alpha(m+1)}{(m^2-4)_+}, \frac{4\alpha+2a}{m+2}\right)$  in (1.4), we can choose  $r' > \frac{m+2}{2}$ . We know that  $v$  is solution of the following equation in divergence form

$$\partial_t v - \nabla \cdot \left( \left( d_v + d_\gamma \left( 1 + \gamma \frac{v^2}{\langle v \rangle^2} \langle v \rangle^\gamma \right) \right) \nabla v \right) = v(r_v - r_c \langle v \rangle^c - r_d \langle u \rangle^d),$$

where the diffusion coefficient  $\left( d_v + d_\gamma \left( 1 + \gamma \frac{v^2}{\langle v \rangle^2} \langle v \rangle^\gamma \right) \right)$  is bounded and the right-hand side  $v(r_v - r_c \langle v \rangle^c - r_d \langle u \rangle^d)$  in  $L^{r'}(\Omega_T)$  for some  $r' > \frac{m+2}{2}$ . Applying [DiB93, Theorem 1.3, p. 43] or [LSU68, Theorem 10.1, p. 204], we deduce that  $v$  is Hölder continuous. So in particular uniformly continuous in  $\Omega_T$ .

Notice now that Corollaries 13 and 15 give the following sequence of implications

$$\nabla v \in L^p(\Omega_T) \Rightarrow u^d \in L^p(\Omega_T) \Rightarrow \nabla v \in L^{2p}(\Omega_T) \Rightarrow u^d \in L^{2p}(\Omega_T). \quad (2.34)$$

Therefore  $\nabla v \in L^4(\Omega_T)$  yields immediately the result.  $\square$

*Remark 17.* We remark here that one can obtain an  $L^p(\Omega)$ -estimate for the  $\nabla u$  as in equation (2.10) in Lemma 12 uniformly in time following the proof proposed in [TW19, Prop. 3.4]. If one further assumes convexity of the domain, following [TW19] one can make the estimates in Lemma 16 uniform in time under additional assumptions on the parameters.

### 3. $L^\infty$ ESTIMATE FOR $u$ AND PROOF OF THE MAIN THEOREM

We now prove our main theorem. We first prove that the solution  $u$  of (1.1) is bounded on  $\Omega_T$ , and then conclude thanks to classical Hölder and Schauder estimates.

**Lemma 18.** *For any solution  $u : \Omega_T \rightarrow \mathbb{R}$  of (1.1) with  $D$  satisfying (1.4), the following estimate holds true,*

$$\|u\|_{L^\infty(\Omega_T)} \leq C(T),$$

where  $C(T)$  only depends on the initial data,  $D$ ,  $T$ ,  $\Omega$  and  $m$ .

*Proof.* The result can be obtained with the Moser iteration technique, see [TW19], [TW11, Appendix A]. For the paper to be self-contained we present our proof: it relies on the De-Giorgi technique. The details are presented in the Appendix A. We prove separately the boundedness on  $(0, \frac{T}{2}) \times \Omega$  and  $(\frac{T}{2}, T) \times \Omega$ .

First by a Sobolev embedding (see for example [AF03, Theorem p.85]) since by Theorem 2,  $u \in C([0, T], W^{1,p_0}(\Omega))$ , we deduce

$$\sup_{t \in [0, \frac{T}{2}]} \|u(t, \cdot)\|_{L^\infty(\Omega)} \leq C \sup_{t \in [0, \frac{T}{2}]} \|u(t, \cdot)\|_{W^{1,p_0}(\Omega)} \leq C(T).$$

Then apply Lemma 20 with  $\tau_1 = 0$  and  $\tau_2 = \frac{T}{2}$  to  $\frac{\delta u}{\|u\|_{L^{m+3}(\Omega_T)} + \delta}$  which still satisfies the energy estimate (A.1) for  $\rho = m + 3$ , to deduce that  $u$  is bounded in  $(\frac{T}{2}, T) \times \Omega$  which ends the proof.  $\square$

We now conclude the proof of Theorem 1 using the classical Hölder and Schauder estimates.

*Proof.* Let us define  $A(t, x)$ ,  $F(t, x)$  and  $f(t, x)$ ,  $B(t, x)$  and  $g(t, x)$  as follows,

$$A(t, x) = d_u + d_\alpha \left(1 + \alpha \frac{u^2}{\langle u \rangle^2}\right) \langle u \rangle^\alpha + d_\beta \langle v \rangle^\beta,$$

$$F(t, x) = \beta d_\beta \frac{uv}{\langle v \rangle^{2-\beta}} \nabla v = G(t, x) \nabla v,$$

$$f(t, x) = u(r_u - r_a \langle u \rangle^a - r_b \langle v \rangle^b).$$

Then  $u$  is a solution of

$$\partial_t u - \nabla \cdot (A(t, x) \nabla u) + \nabla \cdot F = f.$$

Let us define  $B(t, x)$  and  $g(t, x)$  as follows,

$$B(t, x) = d_v + d_\gamma \left(1 + \gamma \frac{v^2}{\langle v \rangle^2}\right) \langle v \rangle^\gamma,$$

$$g(t, x) = v(r_v - r_c \langle v \rangle^c - r_d \langle u \rangle^d).$$

Then  $v$  is a solution of

$$\partial_t v - \nabla \cdot (B(t, x) \nabla v) = g.$$

From Lemmas 7, 16 and 20, there are constants  $\Lambda > 0$  and  $C > 0$  such that

$$\Lambda^{-1} \leq A(t, x) \leq \Lambda \quad \text{and} \quad \Lambda^{-1} \leq B(t, x) \leq \Lambda \quad \forall (t, x) \in \Omega_T,$$

$$\|f\|_{L^\infty(\Omega_T)} + \|g\|_{L^\infty(\Omega_T)} \leq C,$$

$$\|F\|_{L^p(\Omega_T)} \leq C \quad \text{for any } p \in (2, \infty).$$

Using for both  $u$  and  $v$  the classical Hölder regularity theory [LSU68, Theorem 10.1, p.204] or [DiB93, Theorem 1.3, Remark 1.1 p.43], there exists constants  $\varpi \in (0, 1)$  and  $C(T) > 0$  such that

$$\|u\|_{C^{\varpi, \frac{\varpi}{2}}(\overline{\Omega_T})} \leq C(T) \quad \text{and} \quad \|v\|_{C^{\varpi, \frac{\varpi}{2}}(\overline{\Omega_T})} \leq C(T).$$

We define  $w = (d_v + d_\gamma \langle v \rangle^\gamma) v$  and  $\bar{w} = (d_u + d_\alpha \langle u \rangle^\alpha + d_\beta \langle v \rangle^\beta) u$  which solves

$$\partial_t w = B(t, x) \Delta w + B(t, x) g(t, x),$$

and

$$\partial_t \bar{w} = A(t, x) \Delta \bar{w} + A(t, x) f(t, x) + G(t, x) \Delta w + G(t, x) g(t, x),$$

with homogeneous Neumann boundary conditions. We apply the Schauder estimate [LSU68, Theorem 5.3, pp.320-321] first to  $w$  since the coefficients in the equation of  $w$  are Hölder continuous, to obtain

$$\|v\|_{C^{\frac{2+\kappa}{2}, 2+\kappa}(\overline{\Omega_T})} \leq C(T),$$

where  $\kappa \in (0, \nu]$ . Using the last estimate and the Schauder estimate for the equation of  $\bar{w}$ , we get

$$\|u\|_{C^{\frac{2+\mu}{2}, 2+\mu}(\overline{\Omega_T})} \leq C(T),$$

where  $\mu \in (0, \nu]$ . Iterating again, it gives  $u, v \in C^{\frac{2+\nu}{2}, 2+\nu}(\overline{\Omega_T})$  regarding to the initial data. So  $\nabla u$  and  $\nabla v$  are Hölder continuous on  $\overline{\Omega_T}$  so necessarily (1.6) is satisfied. Consequently, by Theorem 2 the maximal time of the local solutions is  $t_{\max} = \infty$  which ends the proof.  $\square$

#### APPENDIX A. BOUNDEDNESS OF $u$ WITH DE GIORGI

In this appendix we get the first lemma of De Giorgi, which is an  $L^\rho - L^\infty$  estimate for  $u$ . To this end, we define a relevant class of energy estimate which we call the De Giorgi class by analogy with quasi-linear elliptic and parabolic studies [DiB95, Lie96].

**Proposition 19** (The De Giorgi class). *Considering the system of cross-diffusion equations given by (1.1), when  $\alpha > 0$  the following energy estimate holds true for any constant  $\rho > 2$  and any  $k \in \mathbb{R}$ ,*

$$\begin{aligned} & \int_{\Omega} (u-k)_+^\rho(t, x) dx + \iint_{(s,t) \times \Omega} |\nabla (u-k)_+^{\frac{\rho}{2}}|^2(\tau, x) dx d\tau \\ & \lesssim \int_{\Omega} (u-k)_+^\rho(s, x) dx + \left( \iint_{(s,t) \times \Omega} (u-k)_+^\rho(\tau, x) dx d\tau \right)^{\frac{\rho-2}{\rho}} \\ & + \iint_{(s,t) \times \Omega} (u-k)_+^{\rho-1}(\tau, x) dx d\tau \end{aligned} \quad (\text{A.1})$$

where the constant only depends on the initial data,  $D$ ,  $T$ ,  $\Omega$ ,  $\rho$ .

*Proof of Proposition 19.* We multiply the equation for  $u$  by  $(u-k)_+^{\rho-1}$  and integrate on  $(s, t) \times \Omega$ ,

$$\begin{aligned} & \iint_{(s,t) \times \Omega} \frac{\partial_t (u-k)_+^\rho}{\rho} dx dt + \iint_{(s,t) \times \Omega} \beta d_\beta \frac{v}{\langle v \rangle^{2-\beta}} u (u-k)_+^{\frac{\rho}{2}-1} \nabla v \cdot \nabla (u-k)_+^{\frac{\rho}{2}} dx dt \\ & + \iint_{(s,t) \times \Omega} \left( (d_u + d_\alpha \left( 1 + \alpha \frac{u^2}{\langle u \rangle^2} \right) \langle u \rangle^\alpha + d_\beta \langle v \rangle^\beta) |\nabla (u-k)_+^{\frac{\rho}{2}}|^2 \right) dx dt \\ & = \iint_{(s,t) \times \Omega} (u-k)_+^{\rho-1} u (r_u - r_a \langle u \rangle^\alpha - r_b \langle v \rangle^b) dx dt, \end{aligned}$$

which using the fact that  $-\langle x \rangle^a \leq -x^a$  and  $x(r_u - r_a x^\alpha) \leq C$  for all  $x \geq 0$ , simplifies into

$$\begin{aligned} & \int_{\Omega} \frac{1}{\rho} (u-k)_+^\rho(t, x) dx dt + \iint_{(s,t) \times \Omega} \beta d_\beta \frac{v}{\langle v \rangle^{2-\beta}} u (u-k)_+^{\frac{\rho}{2}-1} \nabla v \cdot \nabla (u-k)_+^{\frac{\rho}{2}} dx dt \\ & + \iint_{(s,t) \times \Omega} d_u |\nabla (u-k)_+^{\frac{\rho}{2}}|^2 dx dt \\ & \leq \int_{\Omega} \frac{1}{\rho} (u-k)_+^\rho(s, x) dx dt + \iint_{(s,t) \times \Omega} (u-k)_+^{\rho-1} dx dt. \end{aligned} \quad (\text{A.2})$$

Applying Young's inequality and then a Hölder inequality to the second term of the left hand side yields

$$\begin{aligned} & - \iint_{(s,t) \times \Omega} \beta d_\beta \frac{v}{\langle v \rangle^{2-\beta}} u (u-k)_+^{\frac{\rho}{2}-1} \nabla v \cdot \nabla (u-k)_+^{\frac{\rho}{2}} dx dt \\ & \leq \iint_{(s,t) \times \Omega} \frac{(\beta d_\beta)^2}{2d_u} \frac{v^2}{\langle v \rangle^{4-2\beta}} u^2 |\nabla v|^2 (u-k)_+^{\rho-2} dx dt + \iint_{(s,t) \times \Omega} \frac{d_u}{2} |\nabla (u-k)_+^{\frac{\rho}{2}}|^2 dx dt \end{aligned} \quad (\text{A.3})$$



$$\leq \frac{(\beta d_\beta)^2}{2d_u} (1 + \max v^{2\beta}) \left( \iint_{\Omega_T} u^{2\rho} dx dt \right)^{\frac{1}{\rho}} \left( \iint_{\Omega_T} |\nabla v|^{2\rho} dx dt \right)^{\frac{1}{\rho}} \left( \iint_{(s,t) \times \Omega} (u-k)_+^\rho dx dt \right)^{\frac{\rho-2}{\rho}} + \iint_{(s,t) \times \Omega} \frac{d_u}{2} |\nabla (u-k)_+^{\frac{\rho}{2}}|^2 dx dt.$$

Combining (A.2) and (A.3) and using Lemmas 7 and 16 gives the result.  $\square$

**First Lemma of De Giorgi.** We get from the De Giorgi class the first lemma of De Giorgi, which is a  $L^\rho - L^\infty$  estimate to conclude that  $u$  is bounded.

**Lemma 20** (First Lemma of De Giorgi,  $L^\rho - L^\infty$  estimate). *Let  $0 \leq \tau_1 < \tau_2 < T$  and  $\rho > m+2$ . There exists a positive universal constant  $\delta$  only depending on the initial data,  $D, T, \Omega, m, \rho$  such that for any nonnegative function  $u : \Omega_T \rightarrow \mathbb{R}$  satisfying the energy estimate (A.1) the following implication holds true. If*

$$\iint_{(\tau_1, T) \times \Omega} u^\rho dx dt \leq \delta,$$

then we have

$$u \leq \frac{1}{2} \quad \text{in } (\tau_2, T) \times \Omega.$$

*Proof.* In this proof  $C > 0$  will denote a constant which will only depend on the initial data,  $D, T, \Omega, m, \rho$ . We define for any  $t \in (0, T)$ ,  $\Omega_{t,T} = (t, T) \times \Omega$  and

$$U_k = \iint_{\Omega_{t_k, T}} (u - c_k)_+^\rho dx dt,$$

where  $t_k = \tau_1 + (\tau_2 - \tau_1) \left(1 - \frac{1}{2^k}\right)$  and  $c_k = \frac{1}{2} (1 - 2^{-k})$ . We notice that  $t_k$  goes from  $\tau_1$  to  $\tau_2$  and  $c_k$  from 0 to  $\frac{1}{2}$ . We would like to prove that  $U_k$  satisfies the following induction formula

$$U_k \leq C^k U_{k-2}^\lambda,$$

where  $C > 0$  is a universal constant and  $\lambda > 1$  also. Defining  $V_k = U_{2k}$ , the sequence  $(V_k)$  satisfy

$$V_k \leq C^k V_{k-1}^\lambda,$$

and applying [Gue20, Lemma 3.12], we deduce that  $V_k = U_{2k}$  tends to 0 when  $V_0 = U_0$  is small enough. Moreover we have  $U_0 = \iint_{\Omega_{\tau_1, T}} u_+^\rho dx dt$  and  $U_\infty = \iint_{\Omega_{\tau_2, T}} \left(u - \frac{1}{2}\right)_+^2 dx dt = 0$  and we deduce the result.

Let us prove the induction formula. Let us define the Sobolev exponent

$$\sigma = \begin{cases} \frac{2m}{m-2} & \text{if } m > 2, \\ 5 & \text{if } m = 2, \\ +\infty & \text{if } m = 1, \end{cases}$$

in the following Sobolev inequality, for almost every  $t \in (t_k, t)$ ,

$$\|(u - c_k)_+^{\frac{\rho}{2}}(t, \cdot)\|_{L^\sigma(\Omega)} \leq C(m) \|(u - c_k)_+^{\frac{\rho}{2}}(t, \cdot)\|_{H^1(\Omega)}, \quad (\text{A.4})$$

where  $C(m)$  is a constant which only depends on the dimension  $m$ , see [AF03]. Using a Hölder inequality, we have

$$U_k = \iint_{\Omega_{t_k, T}} (u - c_k)_+^\rho dx dt \leq \int_{t_k}^T \left( \int_{\Omega} (u - c_k)_+^{\frac{\rho\sigma}{2}}(t, \cdot) dx \right)^{\frac{2}{\sigma}} |\{u(t, \cdot) \geq c_k\} \cap \Omega|^{1 - \frac{2}{\sigma}} dt. \quad (\text{A.5})$$

Since  $\{u(t, \cdot) \geq c_k\} = \{u(t, \cdot) \geq c_{k-1} + 2^{-k-1}\}$ , we deduce that for  $t \in (t_k, T)$

$$\begin{aligned} |\{u(t, \cdot) \geq c_k\} \cap \Omega|^{1 - \frac{2}{\sigma}} &\leq |\{u(t, \cdot) \geq c_{k-1} + 2^{-k-1}\} \cap \Omega|^{1 - \frac{2}{\sigma}} \\ &\leq \left( 2^{\rho(k+1)} \int_{\Omega} (u - c_{k-1})_+^\rho(t, \cdot) dx \right)^{1 - \frac{2}{\sigma}} \\ &\leq C^k \left( \sup_{t \in (t_k, T)} \int_{\Omega} (u - c_{k-1})_+^\rho(t, \cdot) dx \right)^{1 - \frac{2}{\sigma}}. \end{aligned} \quad (\text{A.6})$$

We can use the first part of the inequality defining the De Giorgi class in Lemma 19 with  $s$  integrated in  $(t_{k-1}, t_k)$  to bound the supremum and obtain in (A.6),

$$\begin{aligned} &|\{u(t, \cdot) \geq c_k\} \cap \Omega|^{1 - \frac{2}{\sigma}} \\ &\leq C^k \left( \iint_{\Omega_{t_{k-1}, T}} (u - c_{k-1})_+^\rho dx dt + \left( \iint_{\Omega_{t_{k-1}, T}} (u - c_{k-1})_+^\rho dx dt \right)^{\frac{\rho-2}{\rho}} + \iint_{\Omega_{t_{k-1}, T}} (u - c_{k-1})_+^{\rho-1} dx dt \right)^{1 - \frac{2}{\sigma}} \\ &\leq C^k \left( \iint_{\Omega_{t_{k-1}, T}} (u - c_{k-1})_+^\rho dx dt + \iint_{\Omega_{t_{k-1}, T}} 1_{\{u \geq c_{k-1}\}} dx dt \right)^{1 - \frac{2}{\sigma}} \\ &\quad + C^k \left( \iint_{\Omega_{t_{k-1}, T}} (u - c_{k-1})_+^\rho dx dt + \iint_{\Omega_{t_{k-1}, T}} 1_{\{u \geq c_{k-1}\}} dx dt \right)^{\frac{\rho-2}{\rho}(1 - \frac{2}{\sigma})} \end{aligned} \quad (\text{A.7})$$

where we used that  $(u - c_{k-1})_+^{\rho-1} \leq (u - c_{k-1})_+^\rho + 1_{\{u \geq c_{k-1}\}}$  to get the last bound. Since we have

$$\begin{aligned} \iint_{\Omega_{t_{k-1}, T}} 1_{\{u \geq c_{k-1}\}} dx dt &= |\{u \geq c_{k-1}\} \cap \Omega| \\ &\leq |\{u(t, \cdot) \geq c_{k-2} + 2^{-k}\} \cap \Omega| \\ &\leq 2^{\rho k} \int_{\Omega} (u - c_{k-2})_+^\rho \\ &\leq 2^{\rho k} U_{k-2}, \end{aligned}$$

we deduce using (A.7),

$$\begin{aligned} |\{u(t, \cdot) \geq c_k\} \cap \Omega|^{1 - \frac{2}{\sigma}} &\leq C^k \left( (U_{k-1} + U_{k-2})^{1 - \frac{2}{\sigma}} + (U_{k-1} + U_{k-2})^{\frac{\rho-2}{\rho}(1 - \frac{2}{\sigma})} \right) \\ &\leq C^k \left( U_{k-2}^{1 - \frac{2}{\sigma}} + U_{k-2}^{\frac{\rho-2}{\rho}(1 - \frac{2}{\sigma})} \right) \end{aligned} \quad (\text{A.8})$$

We notice that the last bound is independent of the variable  $t$  so it remains to bound  $\int_{t_k}^T \left( \int_{\Omega} (u - c_k)_+^{\frac{\rho\sigma}{2}} dx \right)^{\frac{2}{\sigma}} dt$  in (A.5). Using the Sobolev inequality (A.4) and the second part

of the inequality defining the De Giorgi class in Lemma 19 with  $s$  integrated in  $(t_{k-1}, t_k)$ , we deduce

$$\begin{aligned}
& \int_{t_k}^t \left( \int_{\Omega} (u - c_k)_+^{\frac{\rho\sigma}{2}} dx \right)^{\frac{2}{\sigma}} dt \\
& \leq C \left( \iint_{\Omega_{t_k, T}} (u - c_k)_+^{\rho} dx dt + \iint_{\Omega_{t_k, T}} |\nabla_x (u - c_k)_+^{\frac{\rho}{2}}|^2 dx dt \right) \\
& \leq C \left( \iint_{\Omega_{t_{k-1}, T}} (u - c_{k-1})_+^{\rho} dx dt + \left( \iint_{\Omega_{t_{k-1}, T}} (u - c_{k-1})_+^{\rho} dx dt \right)^{\frac{\rho-2}{\rho}} + \iint_{\Omega_{t_{k-1}, T}} (u - c_{k-1})_+^{\rho-1} dx dt \right) \\
& \leq C \left( \iint_{\Omega_{t_{k-1}, T}} (u - c_{k-1})_+^{\rho} dx dt + \iint_{\Omega_{t_{k-1}, T}} 1_{\{u \geq c_{k-1}\}} dx dt \right) \\
& \quad + C \left( \iint_{\Omega_{t_{k-1}, T}} (u - c_{k-1})_+^{\rho} dx dt + \iint_{\Omega_{t_{k-1}, T}} 1_{\{u \geq c_{k-1}\}} dx dt \right)^{\frac{\rho-2}{\rho}} \\
& \leq C^k \left( U_{k-2} + U_{k-2}^{\frac{\rho-2}{\rho}} \right). \tag{A.9}
\end{aligned}$$

By definition  $U_k$  is non-increasing so assuming that  $U_0 < 1$ , we have  $U_k < 1$  for every  $k \geq 0$ . Combining (A.8) and (A.7) and assuming  $U_0 < 1$ , we deduce that  $U_k$  satisfies the formula

$$U_k \leq C^k \left( U_{k-2} + U_{k-2}^{\frac{\rho-2}{\rho}} \right) \left( U_{k-2}^{1-\frac{2}{\sigma}} + U_{k-2}^{\frac{\rho-2}{\rho}(1-\frac{2}{\sigma})} \right) \leq C^k U_{k-2}^{\lambda},$$

with  $\lambda = \frac{\rho-2}{\rho} \left( 2 - \frac{2}{\sigma} \right) > 1$  for  $\rho > m + 2$  which ends the proof using [Gue20, Lemma 3.12] choosing  $\delta < \min \left( C^{-\frac{\lambda^2}{(\lambda-1)^2}}, 1 \right)$ .  $\square$

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## REFERENCES

- [AF03] Robert A Adams and John JF Fournier. *Sobolev spaces*, volume 140. Elsevier, 2003.
- [Ama89] Herbert Amann. Dynamic theory of quasilinear parabolic systems. III. Global existence. *Math. Z.*, 202(2):219–250, 1989.
- [Ama90] Herbert Amann. Dynamic theory of quasilinear parabolic equations. II. Reaction-diffusion systems. *Differential Integral Equations*, 3(1):13–75, 1990.
- [Ama93] Herbert Amann. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In *Function spaces, differential operators and nonlinear analysis (Friedrichroda, 1992)*, volume 133 of *Teubner-Texte Math.*, pages 9–126. Teubner, Stuttgart, 1993.
- [BL12] Jöran Bergh and Jörgen Löfström. *Interpolation spaces: an introduction*, volume 223. Springer Science & Business Media, 2012.
- [BM19] Haïm Brezis and Petru Mironescu. Where Sobolev interacts with Gagliardo-Nirenberg. *J. Funct. Anal.*, 277(8):2839–2864, 2019.

- [CLY04] Y. S. Choi, Roger Lui, and Yoshio Yamada. Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion. *Discrete Contin. Dyn. Syst.*, 10(3):719–730, 2004.
- [DiB93] Emmanuele DiBenedetto. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993.
- [DiB95] Emmanuele DiBenedetto. *Partial differential equations*. Birkhäuser Boston, Inc., Boston, MA, 1995.
- [DLMT15] L. Desvillettes, T. Lepoutre, A. Moussa, and A. Trescases. On the entropic structure of reaction-cross diffusion systems. *Communications in Partial Differential Equations*, 40(9):1705–1747, 2015.
- [DT15] L. Desvillettes and A. Trescases. New results for triangular reaction cross diffusion system. *J. Math. Anal. Appl.*, 430(1):32–59, 2015.
- [GA73] ME Gilpin and FJ Ayala. Global models of growth and competition. *Proc Natl Acad Sci U S A*, page 70(12):3590–3593, 1973.
- [GJ72] ME Gilpin and KE Justice. Animal dispersion in relation to social behavior. *Nature*, pages 236(8):273–303, 1972.
- [Gue20] Jessica Guerand. Quantitative regularity for parabolic De Giorgi classes. hal-02069086, January 2020.
- [HNP15] L. Hoang, T. Nguyen, and T. Phan. Gradient estimates and global existence of smooth solutions to a cross-diffusion system. *SIAM Journal on Mathematical Analysis*, 47(3):2122–2177, 2015.
- [Le21] Dung Le. On the global existence of a class of strongly coupled parabolic systems. arXiv:1910.07260, November 2021.
- [Lie96] Gary M. Lieberman. *Second order parabolic differential equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
- [LNN03] Dung Le, Linh Viet Nguyen, and Toan Trong Nguyen. Shigesada-Kawasaki-Teramoto model on higher dimensional domains. *Electron. J. Differential Equations*, pages No. 72, 12, 2003.
- [LNW98] Yuan Lou, Wei-Ming Ni, and Yaping Wu. On the global existence of a cross-diffusion system. *Discrete Contin. Dynam. Systems*, 4(2):193–203, 1998.
- [LSU68] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.
- [Nir59] L Nirenberg. On elliptic partial differential equations. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 13(2):115–162, 1959.
- [PT90] M. A. Pozio and A. Tesi. Global existence of solutions for a strongly coupled quasilinear parabolic system. *Nonlinear Anal.*, 14(8):657–689, 1990.
- [SKT79] Nanako Shigesada, Kohkichi Kawasaki, and Ei Teramoto. Spatial segregation of interacting species. *J. Theoret. Biol.*, 79(1):83–99, 1979.
- [Tre16] A. Trescases. On triangular reaction cross-diffusion systems with possible self-diffusion. *Bull. Sci. Math.*, 140(7):796–829, 2016.
- [TW11] Youshan Tao and Michael Winkler. Boundedness in a quasilinear parabolic-parabolic keller-segel system with subcritical sensitivity. *Journal of Differential Equations*, 252, 06 2011.
- [TW19] Youshan Tao and Michael Winkler. Boundedness and stabilization in a population model with cross-diffusion for one species. *Proc. London Math. Soc.*, 119(3):1598–1632, 2019.
- [VT08] Phan Van Tuoc. On global existence of solutions to a cross-diffusion system. *Journal of Mathematical Analysis and Applications*, 343(2):826–834, 2008.
- [Wan05] Yi Wang. The global existence of solutions for a cross-diffusion system. *Acta Math. Appl. Sin. Engl. Ser.*, 21(3):519–528, 2005.
- [Yag93] Atsushi Yagi. Global solution to some quasilinear parabolic system in population dynamics. *Nonlinear Anal.*, 21(8):603–630, 1993.
- [Yam95] Yoshio Yamada. Global solutions for quasilinear parabolic systems with cross-diffusion effects. *Nonlinear Anal.*, 24(9):1395–1412, 1995.

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