# Persistence of regularity of the solution to a hyperbolic boundary value problem in domain with corner

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Université du Littoral Côte d'Opale, LMPA
50 rue Ferdinand Buisson CS 80699 62228 Calais, France
Email: antoine.benoit@univ-littoral.fr
Phone: +333 21463651

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**Abstract**

This article is about the question of the persistence of regularity for the solution to hyperbolic boundary value in the quarter-space. More precisely we will both consider the pure boundary value problem and the initial boundary value problem and we propose a functional space, based upon the high order Sobolev space in which a control of the data of the problem leads to a control of the solution (in the same space). The space proposed here contains the tangential Sobolev space. The analysis borrows some ideas of the study of characteristic boundary value problems in the half-space for which the good derivative to consider is known to be the tangential derivative $x_1\partial_1$ instead of the normal derivative $\partial_1$. For quarter-space problems the good quantity to consider will be the radial derivative $x_1\partial_1 + x_2\partial_2$ and then we recover the control of tangential derivative $x_1\partial_1$ and $x_2\partial_2$ using explicit formulas in polar coordinates. The regularity of the solution is then established intrinsically by adapting the method introduced by the author to deal with half-space problems without using regularization methods.

**AMS subject classification**:
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# Contents

# 1 Introduction

We are interested in the well-posedness in high order Sobolev spaces of hyperbolic initial boundary value problems defined in a quadrant. For $N \in \mathbb{N}^*$, $d \geq 2$ we consider the following system of partial differential equations:

$$\begin{cases} L(\partial)u := \partial_t u + A_1\partial_1 u + A_2\partial_2 u + \sum_{j=3}^d A_j\partial_j u = f & \text{in } (t, x_1, x_2, x') \in [0, \infty[ \times \mathbb{R}^2_+ \times \mathbb{R}^{d-2}, \\ B_1 u_{|x_1=0} = g_1 & \text{for } (t, x_2, x') \in [0, \infty[ \times \mathbb{R}_+ \times \mathbb{R}^{d-2}, \\ B_2 u_{|x_2=0} = g_2 & \text{for } (t, x_1, x') \in [0, \infty[\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-2}, \\ u_{|t=0} = u_0 & \text{for } (x_1, x_2, x') \in \mathbb{R}^2_+ \times \mathbb{R}^{d-2}, \end{cases} \tag{1}$$

where the interior coefficients $A_j \in \mathbf{M}_{N\times N}(\mathbb{R})$ are given matrices and where the boundary matrices $B_1 \in \mathbf{M}_{p_1\times N}(\mathbb{R})$, $B_2 \in \mathbf{M}_{p_2\times N}(\mathbb{R})$ are also given and encode generic boundary conditions (the number of lines in the boundary conditions namely $p_1$ and $p_2$, that is the number of required boundary conditions on each side of the quarter space, namely the number of positive eigenvalues of $A_1$ and $A_2$ respectively. In order that we have true boundary value problems we will assume in the following that $0 < p_1, p_2 < N$.

In this article we will also consider the pure boundary value problem associated to (1) that is to say the same problem but defined on the whole time line $\mathbb{R}$.

$$\begin{cases} L(\partial)u = f & \text{in } (t, x_1, x_2, x') \in \mathbb{R} \times \mathbb{R}^2_+ \times \mathbb{R}^{d-2}, \\ B_1 u_{|x_1=0} = g_1 & \text{for } (t, x_2, x') \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-2}, \\ B_2 u_{|x_2=0} = g_2 & \text{for } (t, x_1, x') \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d-2}. \end{cases} \tag{2}$$

The aim of this article is to give some results about the persistence of the regularity for the solution to the problems (1) and (2). More precisely the question can crudely be reformulated as: "if the sources of the problem have some regularity then what about the solution ?".

Before to give an exposition of the main results of the article. Let us have a brief state of art about hyperbolic boundary value problems in domain with corners.

It is fair to say that such problems have a rather long history because they appear in the 60/70s in the literature in the articles of Sarason [Sarason, 1962], Osher [Osher, 1973]-[Osher, 1974] and also in the work of Sarason and Smoller [Sarason and Smoller, 7475] about geometric optics expansions. But such problems have then been in somewhat a little forgotten until the recent works of Huang and Temam [Huang and Temam, 2014], Huang [Huang, 2015], Halpern and Rauch [Halpern and Rauch, 2017], Métivier and Rauch [Métivier and Rauch, 2017] and the author [Benoit, 2015].

Before to describe a little more the above results let us insist on the fact that even if it is a rather old problem then very a little is known about the strong well-posedness of hyperbolic boundary value problems in domain with corners. And thus even in the case of non regular (meaning only $L^2$) solutions.

Indeed compared to the more classical geometry of the half-space for which a complete characterization (the so-called Kreiss-Lopatinskii condition) of the boundary matrices leading to strong well-posedness has been achieved by Kreiss [Kreiss, 1970], then in the corner geometry such a result is far to be well-understood and seems to be a rather long range question.

The most significant work for such a full characterization comes from the work of Osher [Osher, 1973] in which, by adapting the construction of a Kreiss type symmetrizor, the author achieves to show an *a priori* energy estimate but with a non explicit number of losses of derivatives so that it is far to be sufficient to establish a $L^2$-strong well-posedness theory. Moreover in this work the existence of a solution is not considered.

However on the one hand, some of the more recent articles are in a somewhat more optimistic. Let us briefly described the results of these works:

- In [Huang and Temam, 2014]-[Huang, 2015] the authors consider the problem in a square (so that there are four boundary conditions) and show by using Hile-Yossida theorem that for some suitable boundary conditions the problem is strongly well-posed in $L^2$. However in these contributions the author do not try to have a whole characterizations of the boundary conditions leading to $L^2$-strong well-posedness.

- In [Halpern and Rauch, 2017] the authors consider three dimensional corners and they show that for the particular class of maximal dissipative boundary conditions (see Definition ) then we have a good $L^2$ well-posedness theory. The method used is elliptic regularization which requires in particular that the number of conditions on each side of the boundary are equal (meaning $p_1 = p_2$). Assumption that we will not do in this contribution.

- In [Benoit, 2015], the author uses the strong=weak lemma of [Sarason, 1962] in order to show that under a structural assumption on the matrix $A_1^{-1}A_2$ (referred as Sarason matrix in the following) for two dimensional corners we have a good $L^2$-well-posedness theory at least when the boundary condition are dissipative.

On the other hand the recent work [Métivier and Rauch, 2017] is not so optimistic because it shows that if Sarason matrix admits a non trivial Jordan block then we can have non uniqueness of the solution although the problem is linear... The idea os such a counterexample to uniqueness is to give two solutions the first one having $L^2$ traces while the second one has not. Consequently in this article in order to avoid such pathological behaviour we will work with $L^2$ traces.

In this article we essentially propose two theorems establishing the persistence of the regularity of the solution to (1) and (2) separately. The results differ however a little.

Indeed the result for the pure boundary value problem (2) assumes that we have a good $L^2$-well-posedness theory and then gives a persistence of regularity result in some functional space that will be described below. In particular from the result of [Benoit, 2015] when the boundary condition are dissipative then we have such a $L^2$-well-posed condition but because we do not require in the proof the boundary conditions to be dissipative this method may apply to more generic boundary conditions. Let us stress that in such a framework the functional space $\mathscr{H}$ (even if this space is not so user-friendly) encoding the regularity of the solution is the same than the one encoding the regularity of the data. As a consequence we do not have any loss of regularity and we have a sharp persistence of regularity result.

While the result for the initial boundary value problem (1) do not require any $L^2$-well posedness assumption, in particular it shows that the problem is automatically $L^2$-well posed. But to operate the method

has to be restricted to dissipative boundary conditions. We propose here two results of persistence of regularity. The first one is a result where both the data and the solution are in $\mathscr{H}$, however because $\mathscr{H}$ is essentially a tangential Sobolev space it is not very well-adaptated for initial datum. This result thus needs some well-preparedness of the initial datum in order to compensate. The second result holds for an initial boundary value problem with datum in, the user-friendly, space $H^s$ but the solution will only be in some kind of conormal Sobolev space so that wa have somehow losses of regularity.

One of the principal interest of such persistence of regularity results is that, apart from the fact that it is a really satisfactory result, it is often a first step to treat non linear problems. Indeed to treat non linear problems then at some point an $L^\infty$ norm is required and such a bound is often obtain by Sobolev embeddings of the high order Sobolev space $H^s$.

Let us indicate that for elliptic partial differential equations, in domain with corners then the elliptic regularization phenomenon is known to be less good with corners than without. wWe refer to the full exposition of [Grisvard, 1985]. For hyperbolic problems for which, of course, no regularization, occurs one may think that the analogous phenomenon holds and thus gives rise to a loss of regularity which can be rather delicate to handle with for non linear problems. Our main results however show that such a loss does not appear.

In this article we will not obtain a persistence of regularity in the usual Sobolev space $H^s$ so that such an $L^\infty$ embedding is not immediate. But we have results in some weighted tangential Sobolev space (without giving a precise description see Paragraph 2.3.1, let us say that this space is generated by $x_1\partial_1$ and $x_2\partial_2$ instead of $\partial_1$ and $\partial_2$) for the pure boundary value problem (2) and in some conormal Sobolev space (we refer to Paragraph 2.3.2 for a precise definition) for the initial boundary value problem (1).

The establishement of such persistence of regularity result in $H^s$, using two tangential derivatives to control one normal derivatives like in [Guès, 1993] or [Secchi, 1996] will be given in a forthcoming publication.

Let us explain why these tangential derivatives $x_1\partial_1$ and $x_2\partial_2$ naturally come into play. Such operators of derivation naturally appear for characteristic, meaning that the boundary $A_1$ is singular, boundary value problems in the half-space and thus they also appear for corner problem because in some sense they are characteristic problems.

To be more precise on this claim we consider the half-space problem

$$\begin{cases} L(\partial)u = f & \text{in } \mathbb{R}_t \times \mathbb{R}_+ \times \mathbb{R}^{d-1}, \\ Bu_{|x_1=0} = g & \text{on } \mathbb{R}_t \times \mathbb{R}^{d-1}, \end{cases} \tag{3}$$

for which we assume that we have a $L^2$ good well-posedness theory. Then the classical method to derive the regularity of the solution to non characteristic problems if the sources $f$ and $g$ are regular (let us say $H^1$) is to mollify the equation with respect to the tangential variables $(t, x')$ and then to use the explicit equation in the interior

$$\partial_1 u = A_1^{-1}\Big(f - \partial_t u - \sum_{j=2}^{d} A_j \partial_j u\Big), \tag{4}$$

to recover the whole regularity $H^1$.

Of course for characteristic problems such a method fails because we do not have this explicit equation any more. To overcome this difficulty the classical way (we refer to [Rauch, 1985]-[Guès, 1993] or [Secchi, 1996]) is to consider the tangential derivative $x_1\partial_1$ instead of $\partial_1$. Indeed it satisfies

$$\begin{cases} L(\partial)x_1\partial_1 u = x_1\partial_1 f + A_1 u = x_1\partial_1 f + \left(f - \partial_t u - \sum_{j=2}^{d} A_j\partial_j u\right), \\ B(x_1\partial_1 u)_{|x_1=0} = 0, \end{cases}$$

where the source term in the interior equation only involves the source and tangential derivatives of $u$ so that the $L^2$ well posedness theory applies.

The common point of hyperbolic boundary value problem with corners and characteristic problems is the lack of the explicit formula (4). Indeed for corner problem one can not mollify with respect to the two

normal variables $x_1$ and $x_2$ without changing the traces. We can only mollify with respect to $(t, x')$ so that the equation in the interior only gives that $A_1\partial_1 u + A_2\partial_2 u \in L^2$ which is far to be sufficient to recover that $\partial_1 u$ and $\partial_2 u$ are in $L^2$.

Moreover a new difficulty proper to corner problem is that because there are "too many" normal directions the usual proofs for the half-space for establishing the existence of a regular solution which rely on a full mollification in the tangential variables fail. So one needs to obtain the existence of a solution in a more intrinsic way. To do so we will adapt the recent result of [Benoit, ] form the half-space to the quarter space. Before to describe a little more these method to obtain regular solution without regularization let us conclude with the link between characteristic and corner problems.

Once we have understood the similarities between characteristic problems and corner problems. Then the whole sketch of proof starts to be clear. If one considers the tangential derivative $x_1\partial_1 u$ then it solves

$$\begin{cases} L(\partial)x_1\partial_1 u = x_1\partial_1 f + \left(f - \partial_t u - \sum_{j=2}^d A_j\partial_j u - A_2\partial_2 u\right), \\ B_1(x_1\partial_1 u)_{|x_1=0} = 0, \\ B_2(x_1\partial_1 u)_{|x_2=0} = x_1\partial_1 g_2, \end{cases} \tag{5}$$

and the method fails because of the last term in the right hand side of the interior equation of (5). But if one considers instead the radial derivative $\mathbf{r}\partial_{\mathbf{r}} = x_1\partial_1 + x_2\partial_2$ then it solves

$$\begin{cases} L(\partial)\mathbf{r}\partial_{\mathbf{r}} u = \mathbf{r}\partial_{\mathbf{r}} f + \left(f - \partial_t u - \sum_{j=3}^d A_j\partial_j u\right), \\ B_1(\mathbf{r}\partial_{\mathbf{r}} u)_{|x_1=0} = x_2\partial_2 g_1, \\ B_2(\mathbf{r}\partial_{\mathbf{r}} u)_{|x_2=0} = x_1\partial_1 g_2, \end{cases} \tag{6}$$

so that the source in the interior equation of (6) is now $L^2$ and the well-posedness applies exactly like for characteristic problems to give that $\mathbf{r}\partial_{\mathbf{r}} u \in L^2$.

Then to recover from this fact that each $x_1\partial_1 u$ and $x_2\partial_2 u$ are in $L^2$ we get a full use of polar coordinates. Indeed in polar coordinates $(\mathbf{r}, \theta)$ the equation essentially reads

$$\mathscr{A}(\theta)\partial_\theta \widetilde{u} = \widetilde{f} + \mathscr{M}(\theta)\mathbf{r}\partial_{\mathbf{r}}\widetilde{u}, \tag{7}$$

where $\mathscr{A}(\theta)$ and $\mathscr{M}(\theta)$ are explicit matrices. Assuming that $\mathscr{A}(\theta)$ is invertible which is the case under a variation on Sarason hypothesis for well-posedness (see [Sarason, 1962]) then (7) gives an explicit formula for the angular derivative $\partial_\theta = -x_2\partial_1 + x_2\partial_2$ and plays the role of (4) for non characteristics problems.

Once we have the control of the angular derivative $\partial_\theta u$ and of the radial derivative $\mathbf{r}\partial_{\mathbf{r}}$ then one can recover a control of $x_1\partial_1 u$ and $x_2\partial_2$ separately and this essentially ends up the establishment of an *a priori* energy estimate for the pure boundary value problem in domain with a corner.

In order to conclude for the pure boundary value problem we have to construct a regular solution. As already mentioned we here need to use a new analysis compared to the half-space geometry because in the quarter space one can not regularize the solution with respect to $x_1$ or $x_2$. The construction of a regular solution without this regularization method is made by a rather straightforward modification of the duality method of Lax-Phillips [Lax and Phillips, 1960] from the space $L^2$ to some regular space containing the regularity of the radial derivative $(\mathbf{r}\partial_{\mathbf{r}})^\alpha u$. And then in a second time we show that the constructed solution in fact lies in the target space $\mathscr{H}$. The ideas of the duality method are rather classical and are recalled in Paragraph 4.3.

For the initial boundary value problem, following an idea of [[Benzoni-Gavage, 2007]-Chapter 3] already used in [Benoit, ], the construction is made directly by considering (1) like the pure evolution problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u = Au & \text{for } t > 0, \\ u_{|t=0} = u_0 & \text{on } \Gamma, \end{cases}$$

and use Lumer-Phillips Theorem. In order to do so we need to show that both $A$ and $A^*$ are (quasi)dissipative. This can be done if the boundary conditions in (1) (which are hidden in the domain of definition of $A$) are maximal dissipative and if moreover the base Hilbert space used in the definition of $D(A)$ is chosen in such a way that we have an essentially skew-adjoint operator operator meaning that $A^* = -A + \Phi'$ where $\Phi'$ stands for some linear continuous operator.

The paper is organized as follows, Section 2 contains some definitions and states the main results in Paragraph 2.3 namely Theorems 2.1, 2.2 and 2.3. Because Theorem 2.1 requires the fact that the corner problem is $L^2$ well-posed, this assumption is discussed at the end of Section 2 in Paragraph 2.4 using a result of [Benoit, 2015] whose (unpublished) proof is given for the sake of completeness.

The proof of Theorem 2.1 is divided in two parts. The first one deals with an *a priori* energy estimate which is demonstrated in Section 3 and then the second one deals with the regularity of the solution it is given in Section 4, more precisely in Paragraph 4.3.

The proof of Theorem 2.2 and 2.3 are more straightforward because they only relies on the construction to a regular solution, the energy estimate being a consequence of the construction and are given in Paragraph 4.2.

Both of the regularity results of the solution are shown assuming that we have some required duality formula, stated in Paragraph 4.1.2 and whose proofs are postponed to Paragraph 4.4.

Section 5 contains some examples of physical interest. At last Section 6 gives a conclusion and draws some prospects about boundary value problems in corner domains.

# 2 Assumptions, notations and main result

## 2.1 Notations and definitions

### 2.1.1 Notations

In this paragraph we introduce some notations, definitions and we recall some rather well-known facts for hyperbolic boundary value problems.

Firstly we introduce the following sets

$$\Omega := \mathbb{R}_t \times \mathbb{R}^2_{+,(x_1,x_2)} \times \mathbb{R}^{d-2}_{x'}, \partial\Omega_1 := \mathbb{R}_t \times \mathbb{R}_{+,x_2} \times \mathbb{R}^{d-2}_{x'} \simeq \mathbb{R}_t \times \mathbb{R}_{+,x_1} \times \mathbb{R}^{d-2}_{x'} := \partial\Omega_2 \text{ and } \Gamma := \mathbb{R}^2_{+,(x_1,x_2)} \times \mathbb{R}^{d-2}_{x'}.$$

We also define

$$\partial\Gamma_1 := \mathbb{R}_{+,x_2} \times \mathbb{R}^{d-2}_{x'} \simeq \mathbb{R}_{+,x_1} \times \mathbb{R}^{d-2}_{x'} := \partial\Gamma_2.$$

For $a$, $b \in \mathbb{R}$ the notations $[\![a, b]\!]$ stands for the set of integers between $a$ and $b$. That is to say $[\![a, b]\!] = [a, b] \cap \mathbb{Z}$.

Throughout the article the notations $C_A$ stands for a positive constant depending on the parameter(s) $A$. This constant may be modify from one line to the other without any change of notation.

For a functional space $X$, the dependency $X_\gamma$ implies that the integrations in the definition of $X_\gamma$ depend on the time variable, the notation $X$ without $\gamma$ means that these integrations do not depend on $t$.

In all the article for a multi-index $\delta$, the differentiation operator $\partial^\delta$ will stands for the tangential differentiation operator. This operator can change (but for simplicity of the notations will keep the same notation) depending on if we are considering a subspace of $\Gamma$ or a subspace of $\Omega$. More precisely we have the two alternatives

- If we are working on $\Gamma$ then $\delta := (\delta_3, ..., \delta_d) \in \mathbb{N}^{d-2}$ and $\partial^\delta := \partial_3^{\delta_3} \cdots \partial_d^{\delta_d}$.
- If we are working on $\Omega$ then $\delta := (\delta_0, \delta_3, ..., \delta_d) \in \mathbb{N}^{1+d-2}$ and $\partial^\delta := \partial_t^{\delta_0} \partial_3^{\delta_3} \cdots \partial_d^{\delta_d}$.

We hope that this common notation will not create any confusion.

For $p, q \in \mathbb{N}^*$ and a matrix $A \in \mathbf{M}_{p\times q}(\mathbb{R})$ the notation $A^T$ stands for the transpose of $A$.

### 2.1.2 Definitions

We introduce the following definitions for hyperbolic boundary value problems

**Definition 2.1 (Symmetric operator)** *We say that the operator $L(\partial)$ is symmetric if for all $j \in [\![1, d]\!]$ we have $A_j^T = A_j$.*

**Definition 2.2 (Non characteristic boundary)** *For $j = 1, 2$ we say that the boundary $\partial\Omega_j$ is non characteristic for the operator $L(\partial)$ if $\det A_j \neq 0$. Let $\mu_j$ be the number of positive eigenvalues of $A_j$ then $p_j$ the number of lines of $B_j$ satisfies $p_j = \mu_j$.*

In the following we will also frequently considered the following particular type of boundary conditions, already mentioned in the introduction, namely dissipative boundary conditions. In the half-space geometry these boundary conditions are known to not be the most generic ones for which the strong well-posedness of the boundary value problem occurs (these boundary conditions are characterized by the so-called uniform Kreiss-Lopatinskii condition [Kreiss, 1970]). But they have the advantage to be easily algebraically determinable conditions for which we have strong well-posedness (for such well-posedness result we refer for example to [[Benzoni-Gavage, 2007]-Chapter 3]). These particular boundary conditions are also commonly used when one deals with non linear problems see for instance [Guès, 1993] of [Secchi, 1996].

**Definition 2.3 (Maximal dissipative boundary condition)** *For $j = 1, 2$, we say that the boundary condition $B_j$ is maximal dissipative if the following properties are satisfied*

- *$\forall\, u \in \ker B_j$, we have $\langle A_j u, u\rangle \leq 0$.*
- *$\ker B$ is not a proper subspace of some linear subspace on which $A_j$ is non-positive.*

**Definition 2.4 (Strictly dissipative boundary condition)** *For $j = 1, 2$, we say that the boundary condition is strictly dissipative if we have*

- *$\forall\, u \in \ker B_j$, $u \neq 0$, $\langle A_j u, u\rangle < 0$.*
- *$\ker B$ is maximal for the previous property.*
- *$B$ is onto.*

*In particular it implies that there exists $\varepsilon_j$, $C_j > 0$ such that for all $u \in \mathbb{R}^N$ we have the inequality*

$$\varepsilon_j |u|^2 + \langle A_j u, u\rangle \leq C_j |B_j u|^2.$$

## 2.2 Assumptions

In this work we essentially use two assumptions, the first one ensures that we have a good $L^2$-well posedness theory for (2), the second one is a spectral condition on Sarason matrix made to ensure that we have enough explicit formulas in order to recover a good persistence of regularity result.

### 2.2.1 The $L^2$-well posedness Assumption

To deal with the pure boundary value problem (2) we will require the following assumption which ensures that the problem (2) comes with a good $L^2$-well posedness theory. We stress that this assumption is not required for in the initial boundary value problem (1).

**Assumption 2.1 ($L^2$-well posedness)** *We assume that the pure boundary value problem (2) is $L^2$-well posed in the following sense. For all $\gamma > 0$ for all $f \in L^2_\gamma(\Omega)$, $(g_1, g_2) \in L^2_\gamma(\partial\Omega_1) \times L^2_\gamma(\partial\Omega_2)$ there exists a unique solution $u$ to (2), $u \in L^2_\gamma(\Omega)$, its traces $(u_{|x_1=0}, u_{|x_2=0}) \in L^2_\gamma(\partial\Omega_1) \times L^2_\gamma(\partial\Omega_2)$ and $u$ satisfies the energy estimate*

$$\gamma \|u\|^2_{L^2_\gamma(\Omega)} + \|u_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|u_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)} \leq C\left(\frac{1}{\gamma}\|f\|^2_{L^2_\gamma(\Omega)} + \|g_1\|^2_{L^2_\gamma(\partial\Omega_1)} + \|g_2\|^2_{L^2(\partial\Omega_2)}\right), \quad (8)$$

*where the constant $C > 0$ does not depend on $\gamma$. In* (8) *the spaces with weight $L^2_\gamma$ are defined by: for $X \subset \Omega$*

$$L^2(X) := \{u \in \mathcal{D}'(X) \setminus e^{-\gamma t}u \in L^2(X)\},\ \textit{normed by}\ \|u\|^2_{L^2_\gamma(X)} := \int_X e^{-2\gamma t}|u(t,x)|^2\, dx\, dt.$$

This kind of concept of well posedness is a straightforward generalization of the one of [Kreiss, 1970] in the half-space geometry. Let us however point that for corner problems the requirement that the traces are $L^2$ in not harmless at all. Indeed it permits to avoid the pathological counterexample to uniqueness of [Métivier and Rauch, 2017] in which under a spectral property on the Sarason matrix the authors manage to construct a solution with traces in $L^2$ and an other one with traces which are not in $L^2$ giving consequently a non uniqueness result.

Let us stress that Assumption 2.1 is not harmless. We refer to Paragraph 2.4 for the proof that this assumption is satisfied for symmetric non characteristic problems with strictly dissipative boundary conditions.

#### 2.2.2 Assumption on Sarason matrix

As we will see in Paragraph 2.4 and as it is pointed in the work of Sarason [Sarason, 1962] and in the work of Métivier and Rauch [Métivier and Rauch, 2017], the Sarason matrix namely the matrix $A_1^{-1}A_2$ has a large influence on the behaviour of the solution to hyperbolic corner problems. To establish our persistence of regularity result we will also require that this matrix satisfies a spectral property. More precisely we assume the following

**Assumption 2.2** *Assume that the boundary $\partial\Omega_1$ is non-characteristic then we assume that the real eigenvalues of Sarason matrix $A_1^{-1}A_2$ (if they exist) are negative.*

Let us comment that compared to the assumption in [Sarason, 1962] (see Theorem 2.5) Assumption 2.2 is a little stronger because essentially the assumption of Sarason is the same except that it is required to hold on Jordan blocks only. As pointed in [Métivier and Rauch, 2017] if Sarason matrix has a non trivial Jordan block then it can lead to non uniqueness if we allow in the definition of strong well-posedness a weak enough control of the trace.

Assumption 2.2 has the advantage to be an easily algebraically checkable assumption and as we will see in Section 5 it is also satisfied by a lot of examples of physical interest.

Without enter into technical details, Assumption 2.2 is used in the proofs in order to recover from the control of the radial derivatives $(\mathbf{r}\partial_{\mathbf{r}})^\alpha u$ for $\alpha \in [\![0, s]\!]$ the whole control of the anisotropic derivatives $x_1^{\alpha_1+\alpha_2-k}x_2^k\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ where $\alpha_1+\alpha_2 = \alpha$, $k \in [\![0,\, \alpha]\!]$. So that if Assumption 2.2 fails the results of persistence of regularity still hold but with a rather weak control of the solution.

### 2.3 Main results

In this paragraph we state the main results of the article. Firstly we state the result of persistence of regularity for the pure boundary value problem (2) (see Theorem 2.1) and then we state the analogous results for the initial boundary value problem (see Theorems 2.2 and 2.3).

#### 2.3.1 Persistence of the regularity for the pure boundary value problem

In order to state precisely our main result we need to introduce some norms and functional spaces.

We define for $s \in \mathbb{N}^*$, and $0 \le p \le s$ the following modification of the classical Sobolev space $H^s(\Omega)$,

$$\mathbf{H}^{s,p}_\gamma(\Omega) := \Big\{u \in \mathcal{D}'(\Omega) \setminus \ \text{for}\ \alpha,\beta \in \mathbb{N}, \delta \in \mathbb{N}^{d-1}, \alpha+\beta+|\delta| \le s \ \text{and}\ \alpha+\beta \le p,\\ \forall k \in [\![0, \alpha+\beta]\!],\ x_1^k x_2^{\alpha+\beta-k}\partial_1^\alpha\partial_2^\beta\partial^\delta u \in L^2_\gamma(\Omega)\Big\}$$

for the whole space and let for $k \in [\![1,2]\!]$

$$\mathbf{H}^{s,p}_{\gamma}(\partial\Omega_k) := \Big\{ u \in \mathcal{D}'(\partial\Omega_k) \setminus \text{ for } \alpha \in \mathbb{N}, \delta \in \mathbb{N}^{d-1}, \alpha + |\delta| \leq s \text{ and } \alpha \leq p, \\ (x_{3-k}\partial_{3-k})^{\alpha}\partial^{\delta} u \in L^2_{\gamma}(\Omega) \Big\}$$

for the boundaries. Where for a multi-index $\delta = (\delta_0, \delta_3, ..., \delta_d) \in \mathbb{N}^{1+d-2}$ we defined $\partial^{\delta} := \partial_t^{\delta_0}\partial_3^{\delta_3}\cdots\partial_d^{\delta_d}$.

We introduce the following norms on $\mathbf{H}^{s,p}_{\gamma}(\Omega)$; for $u \in \mathbf{H}^{s,p}_{\gamma}(\Omega)$

$$\|u\|^2_{\mathbf{H}^{s,p}_{\gamma}(\Omega)} := \sum_{\alpha+\beta\leq p, \alpha+\beta+|\delta|\leq s} \sum_{k=0}^{\alpha+\beta} \|x_1^k x_2^{\alpha+\beta-k}\partial_1^{\alpha}\partial_2^{\beta}\partial^{\delta} u\|^2_{L^2_{\gamma}(\Omega)}$$

the norm on the boundaries being defined similarly.

In the previously defined Sobolev type spaces, the index $s$ stands for the maximal degree of regularity while $p$ stands for the maximal degree of regularity with respect to the normal derivatives $\partial_1$ and $\partial_2$. Note that we have weights with respect to the variables $x_1$ and $x_2$ and that the space includes the usual tangential Sobolev space $H^{s,p}_{\gamma,\tan}$ (see [Rauch, 1985]) defined by

$$H^{s,p}_{\gamma,\tan}(\Omega) := \left\{ u \in \mathcal{D}'(\Omega) \setminus (x_1\partial_1)^{\alpha}(x_2\partial_2)^{\beta}\partial^{\delta} u \in L^2_{\gamma}(\Omega), \forall\, \alpha+\beta \leq p,\ \alpha+\beta+|\delta| \leq s \right\}.$$

Indeed choosing $k = \alpha$ in the norm on $\mathbf{H}^{s,p}_{\gamma}(\Omega)$ and using the fact that $(x_k\partial_k)^{\alpha} = \sum_{p=0}^{\alpha} \lambda_p^{\alpha} x_k^p \partial_k^p$ for $(\lambda_p^{\alpha})_{\alpha,p} \subset \mathbb{R}$ shows that $H^{s,p}_{\gamma,\tan}(\Omega) \subset \mathbf{H}^{s,p}_{\gamma}(\Omega)$ which is consequently an extension of $H^{s,p}_{\gamma,\tan}(\Omega)$ with anisotropic weights in $x_1$ and $x_2$. This anisotropy comes from the fact that in the following proof we will have to consider the angular derivatives $\partial_{\theta} := -x_2\partial_1 + x_1\partial_2$, which mix the weights in front of the differentiation operators $\partial_1$ and $\partial_2$ and not only the radial derivative $\mathbf{r}\partial_{\mathbf{r}} = x_1\partial_1 + x_2\partial_2$.

In the following we will use the following space for $\mathbf{r} := \sqrt{x_1^2 + x_2^2}$

$$\mathscr{H}^s_{\gamma}(\Omega) := \left\{ u \in \mathcal{D}'(\Omega) \setminus \forall\, l \in [\![0,s]\!],\ \mathbf{r}^l u \in \mathbf{H}^{s,s-l}_{\gamma}(\Omega) \right\},$$

it is equipped with the obvious norm; for $u \in \mathscr{H}^s_{\gamma}(\Omega)$

$$\|u\|^2_{\mathscr{H}^s_{\gamma}(\Omega)} := \sum_{\ell=0}^{s} \|\mathbf{r}^l u\|^2_{\mathbf{H}^{s,s-\ell}_{\gamma}(\Omega)}.$$

We define the spaces $\mathscr{H}^s_{\gamma}(\partial\Omega_k)$, for $k \in \{1,2\}$ similarly.

Our first main result establishes a strong well-posedness result for the pure boundary value problem in high order weighted Sobolev spaces. The first part of the theorem gives some *a priori* energy estimate for regular solutions while the second statement gives the regularity of the solution in the high order Sobolev type spaces $\mathscr{H}^s_{\gamma}(\Omega)$.

**Theorem 2.1** *Let $s \geq 0$ under Assumptions 2.1 and 2.2 then there exist $\gamma_0 := \gamma_0(s) \geq 1$ and $C := C_s > 0$ such that if for $\gamma \geq \gamma_0$, $u \in \mathscr{H}^s_{\gamma}(\Omega)$ is such that $L(\partial)u \in \mathscr{H}^s_{\gamma}(\Omega)$, $B_1 u_{|x_1=0} \in \mathscr{H}^s_{\gamma}(\partial\Omega_1)$ and $B_2 u_{|x_2=0} \in \mathscr{H}^s_{\gamma}(\partial\Omega_2)$ then we have the a priori energy estimate: $\forall\, \gamma \geq \gamma_0$*

$$\|u\|^2_{\mathscr{H}^s_{\gamma}(\Omega)} + \|u_{|x_1=0}\|^2_{\mathscr{H}^s_{\gamma}(\partial\Omega_1)} + \|u_{|x_2=0}\|^2_{\mathscr{H}^s_{\gamma}(\partial\Omega_2)} \\ \leq C\left( \|L(\partial)u\|^2_{\mathscr{H}^s_{\gamma}(\Omega)} + \|B_1 u_{|x_1=0}\|^2_{\mathscr{H}^s_{\gamma}(\partial\Omega_1)} + \|B_2 u_{|x_2=0}\|^2_{\mathscr{H}^s_{\gamma}(\partial\Omega_2)} \right). \tag{9}$$

*Conversely under Assumptions 2.2 assume that for some $\gamma > 0$ $f \in \mathscr{H}^s_{\gamma}(\Omega)$, $(g_1, g_2) \in \mathscr{H}^s_{\gamma}(\partial\Omega_1) \times \mathscr{H}^s_{\gamma}(\partial\Omega_2)$ and consider*

$$\begin{cases} L(\partial)u = f & \text{in } \Omega, \\ B_1 u_{|x_1=0} = g_1 & \text{on } \partial\Omega_1, \\ B_2 u_{|x_2=0} = g_2 & \text{on } \partial\Omega_2. \end{cases} \tag{10}$$

*We assume that the dual problem to* (10) *comes with a good high order well-posedness theory see Assumption 4.1. Then the unique solution $u$ to* (10) *given by Assumption 2.1 lies in $\mathscr{H}^s_\gamma(\Omega)$. Moreover $u$ satisfies the energy estimate there exist $\underline{\gamma}$, $C>0$ such that $\forall\, \gamma \geq \underline{\gamma}$*

$$\gamma \|u\|^2_{\mathscr{H}^s_\gamma(\Omega)} \leq C\left(\frac{1}{\gamma}\|f\|^2_{\mathscr{H}^s_\gamma(\Omega)} + \|g_1\|^2_{\mathscr{H}^s_\gamma(\partial\Omega_1)} + \|g_2\|^2_{\mathscr{H}^s_\gamma(\partial\Omega_2)}\right), \tag{11}$$

*where $C$ depends on the coefficients of $L(\partial)$ and on $s$.*

The *a priori* energy estimate that is to say the first statement of Theorem 2.1 is demonstrated in Section 3, the regularity of the solution is then obtained by a slight modification of the duality method of [Lax and Phillips, 1960] see also [Benoit, ] is demonstrated in Paragraph 4.3.

From a persistence of regularity point of view Theorem 2.1 is really satisfactory because it shows that the solution $u$ has exactly the same regularity as the sources.

Moreover it is a quite positive, interesting and encouraging fact that the spaces $\mathscr{H}^s_\gamma(\Omega)$ contain all the tangential derivatives. Indeed in [Secchi, 1996] and [Guès, 1993] such control of the tangential derivatives $(x_1\partial_1)^\alpha u$ was used in order to recover the control of the normal derivative $\partial_1 u$ which implies the $L^\infty_x$ bound via Sobolev embedding.

For corner problems we can also recover the control of the normal derivatives by using two tangential derivatives to obtain such a $H^s$ regularity result of the solution. This claim is however behind of the scope of the present article and will follows in a forthcoming publication.

We conclude this paragraph by the following corollary of Theorem 2.1 showing that for symmetric non characteristic problems with strictly dissipative boundary conditions then we have a good persistence of regularity theory.

**Corollary 2.1** *We assume that the pure boundary value problem* (2) *is symmetric with non characteristic strictly dissipative boundary conditions and that we have Assumption 2.2. Let $s \in \mathbb{N}$ be given then there exist $\underline{\gamma} > 0$ such that for all $\gamma > \underline{\gamma}$ if the sources $f \in \mathscr{H}^s_\gamma(\Omega)$, $(g_1, g_2) \in \mathscr{H}^s_\gamma(\partial\Omega_1) \times \mathscr{H}^s_\gamma(\partial\Omega_2)$ then* (2) *admits a unique solution $u \in \mathscr{H}^s_\gamma(\Omega)$. This solution satisfies the energy estimate* (11).

**Proof :** The proof is just a direct consequence of Theorem 2.1. For symmetric operators with non characteristic strictly dissipative boundary conditions because Assumption 2.2 implies the assumptions of Theorem 2.5, it applies and gives the $L^2$ well-posedness result. As a consequence Assumption 2.1 holds.

From the first statement of Theorem 2.1 the energy estimate (9) holds for regular solutions.

It is a well-known fact that strictly dissipative boundary conditions are suitable for the dual problem so that the dual problem to (2) satisfies the *a priori* energy estimate (9) this implies in particular Assumption 4.1. So that the second statement of Theorem 2.1 applies and this completes the proof.

□

### 2.3.2 Persistence of the regularity for initial boundary value problems

Our second main result deals with the initial boundary value problem (1) for which by linearity we assume[1] to simplify that $f$, $g_1$, $g_2 \equiv 0$. An interesting point to be noticed is that contrary to Theorem 2.1 where the $L^2$-well posedness is assumed in order to derive the strong well-posedness in higher oder Sobolev type spaces then no $L^2$-well posedness is assumed for these results. This comes from the fact that these results are based on Lumer-Philips theorem which gives automatically the existence of a regular solution (which is in particular $L^2$).

Moreover let us insist on the fact that, in terms of regular well-posedness for the initial boundary value problem, we establish the so-called semigroup well-posedness of the system (1) that is to say that we obtain a control of the $L^\infty_t(\widetilde{H}^s(\Gamma))$ norm of the solution where $\widetilde{H}^s(\Gamma)$ is some suitable functional space based upon $H^s(\Gamma)$. In terms of well-posedness it is the best framework that one can hopes.

[1] Using a remark of [[Benzoni-Gavage, 2007] page 114] here we may also consider $f \not\equiv 0$ but the method really requires that $g_1 \equiv g_2 \equiv 0$ in order the boundary conditions to be incorporated in the domain of $A$

Unfortunately compared to Theorem 2.1 for which the data and the solution relies on the same functional space the counterpart in Theorem 2.3 is that the initial solution $u_0$ and the solution $u$ will not live in the same functional space. The reason of this is essentially that the tangential Sobolev generated by $x_1\partial_1$ and $x_2\partial_2$ can not be compared with the usual Sobolev space.

In the following we assume that the initial datum $u_0$ lies in some weighted Sobolev space $\mathscr{D}^s(\Gamma)$ and we will obtain that the solution $u$ to (1) lies in some kind of conormal Sobolev space. Without enter into the details of the definition of this space let us that say it essentially acts like the usual conormal space (that is that near the boundary $\{x_k = 0\}$ we have an estimate for $(x_k\partial_k)^s$ and of $\partial_k^s$ away from this boundary). Near the corner we only have an estimate for $(x_1\partial_1 + x_2\partial_2)^s$ with each a rather weak estimate, but which can be transformed into the same control than the one of Theorem 2.1.

To state our main results let us introduce some functional spaces and ideas.

Firstly the starting point to solve the initial boundary value problem is to write it as the pure evolution problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u = Au & \text{for } t \in [0,\infty[\,, \\ u_{|t=0} = u_0, \end{cases} \tag{12}$$

where

$$Au := -\sum_{j=1}^{d} A_j\partial_j u,$$

and where the boundary conditions in (1) are encoded in the domain $D(A)$ of $A$. This approach has first been introduced by Benzoni-Gavage and Serre in [[Benzoni-Gavage, 2007]-Chapter 3] for constant coefficients boundary value problems in the half-space and then have been extended by the author in [Benoit, ] to variable or characteristic coefficients.

To solve (12) we use Lumer-Phillips Theorem (see Theorem 4.1) which applies if we can justify that both $A$ and its dual operator $A^*$ are (quasi)dissipative (see Definition 4.2).

Then because we are now considering regular solution to an initial boundary value problem then some compatibility conditions are expected (and necessary) on the datum $u_0$. In the following discussion assume that the solution is regular enough for the traces to be well-defined. Then we should have

$$\left[B_1 u_{|x_1=0}\right]_{|t=0} = \left[B_2 u_{|x_2=0}\right]_{|t=0} = 0 \text{ so that } u_{0_{|x_1=0}} \in \ker B_1 \text{ and } u_{0_{|x_2=0}} \in \ker B_2,$$

condition which is referred as the compatibility condition of order zero.

More generically let us define for $k \geq 1$, $u^k := (\partial_t^k u)_{|t=0}$ and $u^0 = u_0$ then with $A = -\sum_{j=1}^{d} A_j\partial_j$ because on the one hand $u$ is solution to (1) we have $u^k = A^k u_0$ but on the other hand

$$\partial_t^k B_1 u_{|x_1=0} = \partial_t^k B_2 u_{|x_2=0} = 0 \text{ so that taking the trace } u^k_{|x_1=0} \in \ker B_1 \text{ and } u^k_{|x_2=0} \in \ker B_2. \tag{13}$$

This motivates the following definition

**Definition 2.5** *Let $s \in \mathbb{N}$ we say that $u_0$ satisfies the compatibility conditions up to the order $\sigma \leq s-1$ if (13) is satisfied for all $k \in [\![0, \sigma-1]\!]$. The compatibility condition of order $-1$ being the empty condition.*

To state the well-posedness result we introduce the following functional spaces For $s \in \mathbb{N}$, $0 \leq p \leq s$ we first define the Hilbert space for $X \subset \Gamma$:

$$H^{s,p}_{rad}(X) := \{u \in \mathcal{D}'(X) \setminus \forall\, 0 \leq \alpha \leq p,\, \forall\, \delta \in \mathbb{N}^{d-2} \text{ s.t. } \alpha + |\delta| \leq s \text{ we have } (\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial^\delta u \in L^2(X)\},$$

where for $\delta := (\delta_3, ..., \delta_d) \in \mathbb{N}^{d-2}$, $\partial^\delta := \partial_3^{\delta_3} \cdots \partial_d^{\delta_d}$ and where $\mathbf{r}\partial_{\mathbf{r}} := x_1\partial_1 + x_2\partial_2$. It is a Hilbert space according to the norm, for $u \in H^{s,p}_{rad}(X)$

$$\|u\|^2_{H^{s,p}_{rad}(X)} := \sum_{\alpha=0}^{p} \sum_{|\delta| \leq s-\alpha} \|(\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial^\delta u\|^2_{L^2(X)}.$$

We insist on the fact that in $H^{s,p}_{rad}(X)$ we do not require any control on the angular derivative $\partial_\theta u$ and/or on the anisotropic weighted derivatives $x_1^{\alpha+\beta-k}x_2^k\partial_1^\alpha\partial_2^\beta u$, $k\in[\![0,\alpha+\beta]\!]$.

Then we consider the following functional spaces $\mathcal{X}^s_{rad}(\Gamma)$ and $\mathcal{Y}^s(\Gamma)$ in which we have good duality formulas.

Firstly we define the space $\mathcal{X}^s_{rad}(\Gamma)$ as a subspace of $H^{s,0}_{rad}(\Gamma)$ as follows

$$\mathcal{X}^s_{rad}(\Gamma):=\left\{u\in\mathcal{D}'(\Gamma)\setminus\ \forall\, 0\le l\le s,\ \mathbf{r}^l u\in H^{s,s-l}_{rad}(\Gamma)\right\}.$$

In such a functional space the persistence of regularity result is the following

**Theorem 2.2** *Let $s\in\mathbb{N}$ and $u_0\in\mathcal{X}^s_{rad}(\Gamma)$ be given and satisfying the compatibility conditions up to the order $s-1$. We also assume that $u_0$ satisfies the well-preparedness assumptions*

$$\begin{cases} A^{k+1}u_0\in\mathcal{X}^{s-k}_{rad}(\Gamma) & \forall\, k\in[\![0,s-1]\!],\\ A^k u_{0_{|x_j=0}}\in\mathcal{X}^{s-k}_{rad}(\partial\Gamma_j) & \forall\, k\in[\![0,s]\!],\forall\, j\in[\![1,2]\!],\\ \|Au_0\|_{\mathcal{X}^s_{rad}(\Gamma)}\le C\|u_0\|_{\mathcal{X}^s_{rad}(\Gamma)} & \end{cases}\tag{14}$$

*We assume that the initial boundary value problem (1) is symmetric, non characteristic for each side of the boundary with strictly dissipative boundary conditions and that Assumption 2.2 holds.*

*Then (1) admits a unique solution $u\in\cap_{k=0}^s\mathscr{C}^k_t([0,\infty[;\mathscr{H}^{s-k}(\Gamma))$. Moreover there exists $\omega,C_s>0$ such that we have the energy estimate*

$$\forall\, t\in[0,\infty[\,,\ \forall k\in[\![0,s]\!],\ \|\partial_t^k u(t)\|_{\mathscr{H}^{s-k}(\Gamma)}\le C_s e^{\omega t}\|u_0\|_{\mathcal{X}^s_{rad}(\Gamma)},\tag{15}$$

*where the space $\mathscr{H}^p(\Gamma)$ are defined exactly like the space $\mathscr{H}^p_\gamma(\Omega)$ excepted that the integrations are made on the space $\Gamma$ instead of the time/space $\Omega$.*

Secondly to state our second main result we define the space Hilbert space $\mathcal{Y}^s(\Gamma)$ by

$$\mathcal{Y}^s(\Gamma):=\left\{u\in\mathcal{D}'(\Gamma)\setminus\forall\, l\in[\![0,s]\!],\ \mathbf{r}^l u\in H^{s,s-l}_{rad}(\Gamma)\text{ and }\|u\|_{\mathcal{Y}^s(\Gamma)}<\infty\right\},$$

where the norm $\|\cdot\|_{\mathcal{Y}^s(\Gamma)}$ comes from the scalar product

$$\begin{aligned}\langle u,v\rangle_{\mathcal{Y}^s(\Gamma)}:=\sum_{l=0}^s\sum_{\alpha=0}^{s-l}\sum_{|\delta|\le s-\alpha}&\int_{\mathscr{C}}\left\langle(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l u,(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l v\right\rangle\mathrm{d}x\\ &+\int_{\mathscr{B}_1}\sum_{p=0}^\alpha\langle\binom{\alpha}{p}\underline{\partial}_{1,p}(x_2\partial_2)^{\alpha-p}\partial^\delta\mathbf{r}^l u,\binom{\alpha}{p}\underline{\partial}_{1,p}(x_2\partial_2)^{\alpha-p}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x\\ &+\int_{\mathscr{B}_2}\sum_{p=0}^\alpha\langle\binom{\alpha}{p}(x_1\partial_1)^p\underline{\partial}_{2,\alpha-p}\partial^\delta\mathbf{r}^l u,\binom{\alpha}{p}(x_1\partial_1)^p\underline{\partial}_{2,\alpha-p}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x\\ &+\int_{\mathscr{I}}\sum_{p=0}^\alpha\langle\binom{\alpha}{p}\underline{\partial}_{1,p}\underline{\partial}_{2,\alpha-p}\partial^\delta\mathbf{r}^l u,\binom{\alpha}{p}\underline{\partial}_{1,p}\underline{\partial}_{2,\alpha-p}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x,\end{aligned}\tag{16}$$

where we defined

$$\begin{aligned}&\mathscr{C}:=\{x\in\Gamma\setminus 0\le x_1,\ x_2<1\},\quad \mathscr{I}:=\{x\in\Gamma\setminus x_1,\ x_2\ge 1\},\\ &\mathscr{B}_1:=\{x\in\Gamma\setminus x_1>1,\ 0<x_2\le 1\}\text{ and }\mathscr{B}_2:=\{x\in\Gamma\setminus 0<x_1<1,\ x_2\ge 1\}\end{aligned}$$

and where the differentiation operators $\underline{\partial}_{1,\cdot}$ and $\underline{\partial}_{2,\cdot}$ are defined by

$$\forall\, p\in[\![0,\alpha]\!],\ \underline{\partial}_{1,p}:=\sum_{q=1}^p\lambda_q^p\partial_1^q\text{ and }\underline{\partial}_{2,\alpha-p}:=\sum_{q=1}^{\alpha-p}\lambda_q^{\alpha-p}\partial_2^q.\tag{17}$$

The scalars $\lambda_i^j$ being fixed and defined in (90). We decompose

$$\langle u,v\rangle_{\mathcal{Y}^s(\Gamma)} := \langle u,v\rangle_{\mathcal{Y}^s(\mathscr{C})} + \langle u,v\rangle_{\mathcal{Y}^s(\mathscr{B}_1)} + \langle u,v\rangle_{\mathcal{Y}^s(\mathscr{B}_2)} + \langle u,v\rangle_{\mathcal{Y}^s(\mathscr{I})},$$

with obvious definitions.

The space in which lies the solution to the initial boundary value problem is then defined by

$$\mathscr{Y}^s(\Gamma) := \left\{ u \in \mathcal{D}'(\Gamma) \setminus \forall\, 0 \le l \le s,\ \mathbf{r}^l u \in H^{s,s-l}_{rad}(\Gamma) \text{ and } \|u\|_{\mathscr{Y}^s(\Gamma)} < \infty \right\},$$

where the norm on $\mathscr{Y}^s(\Gamma)$ comes from the scalar product

$$\langle u,v\rangle_{\mathscr{Y}^s(\Gamma)} := \langle u,v\rangle_{\mathscr{H}^s(\mathscr{C})} + \langle u,v\rangle_{\mathcal{Y}^s(\mathscr{B}_1)} + \langle u,v\rangle_{\mathcal{Y}^s(\mathscr{B}_2)} + \langle u,v\rangle_{H^s(\mathscr{I})},$$

where $\mathscr{H}^s(\mathscr{C})$ has the same definition than $\mathscr{H}^s_\gamma(\mathscr{C})$ except that we do not integrate with respect to the time variable (so that we take the norm $\|\cdot\|_{L^2_x(\mathscr{C})}$ instead of $\|\cdot\|_{L^2_\gamma(\mathbb{R}\times\mathscr{C})}$). $H^s(\mathscr{I})$ being the usual Sobolev space of order $s$.

Finally for the initial condition we define the space

$$\mathscr{D}^s(\Gamma) := \left\{ u \in H^s(\Gamma) \setminus \forall\, l \in [\![0,s]\!],\ \mathbf{r}^l u \in H^{s-l}(\Gamma) \right\},$$

it is equipped with the norm $\|\cdot\|_{\mathscr{D}(\Gamma)}$ defined by

$$\forall\, u \in \mathscr{D}^s(\Gamma),\ \|u\|^2_{\mathscr{D}(\Gamma)} := \sum_{l=0}^{s} \|\mathbf{r}^l u\|^2_{H^s(\Gamma)}.$$

One can see that for $u \in \mathcal{Y}^s(\Gamma)$ (or $\mathscr{Y}^s(\Gamma)$) we have $\|u\|_{\mathcal{Y}^s(\Gamma)} \le C_s \|u\|_{\mathscr{D}^s(\Gamma)}$ so that $\mathscr{D}^s(\Gamma) \subset \mathcal{Y}^s(\Gamma)$.

With such definitions $\mathcal{Y}^s(\Gamma)$ acts like a conormal Sobolev space except at the corner where we only have a control of the radial derivatives, while $\mathscr{Y}^s(\Gamma)$ is truly a conormal Sobolev space because it contains the tangential derivatives $(x_1\partial_1)^{\alpha_1}(x_2\partial_2)^{\alpha_2}$ near the corner.

The well-posedness result in such a space is the following:

**Theorem 2.3** *Let $s \in \mathbb{N}$ and $u_0 \in \mathscr{D}^s(\Gamma)$ satisfying the compatibility conditions* (13) *up to the order $s-1$. We assume that the initial boundary value problem* (1) *is symmetric, with non characteristic maximal dissipative boundary condition as defined in Paragraph 2.1.2. Let us also assume that Sarason matrix satisfies Assumption 2.2. Then there exists a unique solution $u \in \cap_{k=0}^s \mathscr{C}^k_t([0,\infty[\,;\mathscr{Y}^{s-k}(\Gamma))$ to* (1)*. Moreover we have the energy estimates that there exist $\omega$, $C_s > 0$ such that*

$$\forall\, t \in [0,\infty[\,,\ \forall k \in [\![0,s]\!],\ \|\partial_t^k u(t)\|_{\mathscr{Y}^s(\Gamma)} \le C_s e^{\omega t} \|u_0\|_{\mathscr{D}^s(\Gamma)}. \tag{18}$$

We end up this presentation paragraph by some comments on Theorems 2.2 and 2.3

- Let us first remark that for $s = 0$ the conditions imposed on the datum $u_0$ are void so that $u_0$ can be chosen arbitrary in $L^2(\Gamma)$. In particular this gives for free the $L^2$ well-posedness result.
- Theorem 2.2 gives from (only) a control on the radial derivatives of the initial datum $u_0$ a complete tangential control of the solution.
- It seems rather delicate to bypass the well-preparedness assumption (14). Indeed the issue here is that if $u_0 \in \mathcal{X}^{s+1}(\Gamma)$ then it is clear for all tangential $j$, $\partial_j u_0 \in \mathcal{X}^s(\Gamma)$ but because $\mathcal{X}^s_{rad}(\Gamma)$ is defined with the derivatives $x_1\partial_1$ and $x_2\partial_2$ so that it is not obvious that $A_1\partial_1 u_0 \in \mathcal{X}^s_{rad}(\Gamma)$ and $A_2\partial_2 u_0 \in \mathcal{X}^s_{rad}(\Gamma)$. This well-preparedness assumption is however removed in Theorem 2.3 but we pay some price on the regularity on the solution.

- Concerning the persistence of regularity point of view Theorem 2.2 is sharp but it is rather fair to say that compared to the data $u_0$ we have some losses of regularity for the solution $u$ near the boundaries in 2.3. There are two kinds of losses.

  1. The first one comes from the fact that in $\mathscr{C}$ we only have a control of the tangential derivatives while $u_0$ is controlled by the normal derivatives.
  2. The second point is more delicate and comes from the definition of $\mathcal{Y}^s(\Gamma)$. Indeed near the boundaries (to fix the ideas let us consider $\mathscr{B}_1$) we only recover a control of the unmixed derivatives $\sum_{p=0}^{\alpha} \underline{\partial}_{1,p}(x_2\partial_2)^{\alpha-p}$ and not a whole control of the decoupled derivatives $\partial_1^{\alpha}(x_2\partial_2)^{\beta}$ for $\alpha+\beta \leq s$. This comes from the fact that near the boundaries the operators of differentiation are not defined by $x_1\partial_1$ and $x_2\partial_2$ but one of then is defined by $\partial_1$, the other one being defined by $x_2\partial_2$ so that we can not use the radial and angular derivatives in order to recover the control of the uncoupled derivatives. However if we have a control of one of the normal derivative $\partial_1$ or $\partial_2$ near the corner some partition of unity arguments enable us to show that the control near the boundary can in fact be transformed into the usual Sobolev one.

## 2.4 Comments about the $L^2$-well-posedness Assumption: the $L^2$ theory

The following paragraph exposes a framework for which Assumption 2.1 is known to hold. The results exposed here are generalizations of the ones of [Benoit, 2015] to variable coefficients. We stress in particular that it is the only part of the article where the coefficients are authorized to vary.

Assumption 2.1 is of course far to be harmless. Indeed the full-characterization of the boundary matrices $B_1$ and $B_2$ leading to the $L^2$-well posedness of the quarter space problem boundary value problem is far to be clearly understood instead of the one of the analogous problem in the half-space (see for example [Kreiss, 1970]-[Chazarain and Piriou, 1981]-[Benzoni-Gavage, 2007]).

However, there is a framework, the one of the so-called strictly dissipative boundary conditions for which Assumption 2.1 can be verified. Let us introduce the following definitions with are straightforward generalizations to the ones of Paragraph 2.1.2 to variable coefficients

**Definition 2.6 (Strictly dissipative boundary conditions)** *The boundary condition $B_1$ (resp. $B_2$) on the side $\partial\Omega_1$ (resp. $\partial\Omega_2$) is said to be strictly dissipative if*

$$\forall\,(t,x_2,x')\in\partial\Omega_1,\ \forall\,u\in\ker B_1(t,x_2,x'),u\neq 0,\ \text{we have}\ \langle A_1(t,0,x_2,x')u,u\rangle<0.$$

$$(\text{resp. }\forall\,(t,x_1,x')\in\partial\Omega_2,\ \forall\,u\in\ker B_2(t,x_1,x'),u\neq 0,\ \text{we have}\ \langle A_2(t,x_1,0,x')u,u\rangle<0)\,.$$

*It implies (we refer to [[Benzoni-Gavage, 2007]-Chapter 3] for a proof) that if $i=1,2$ there exist $\varepsilon_i,C_i>0$ such that for all $u\in\mathbb{R}^N$ we have the inequality*

$$\varepsilon_i|u|^2+\left\langle\widetilde{A_i}(t,x_{3-i},x')u,u\right\rangle\leq C_i|B_i(t,x',x_{3-i})u|^2. \tag{19}$$

*where $\widetilde{A_1}(t,x_2,x')=A_1(t,0,x_2,x')$ and $\widetilde{A_2}(t,x_1,x')=A_2(t,x_1,0,x')$. We also require that $B_i$ is onto and that $\ker B_i$ is maximal for the previous property.*

We will also require the coefficients of the corner problem to be symetric in the following sense.

**Definition 2.7 (Symmetric coefficients)** *The hyperbolic operator $L(\partial)$ is said to be symmetric if*

$$\forall\,(t,x)\in\Omega,\ A_j^T(t,x)=A_j(t,x).$$

Finally let us assume that the two sides of the boundary are non characteristic in the sense that

**Definition 2.8 (Non characteristic boundary)** *The side of the boundary $\partial\Omega_1$ (resp. $\partial\Omega_2$) is said to be non characteristic if*

$$\forall\,(t,x)\in\Omega,\ \det A_1(t,x)\neq 0\,(\textit{resp. }\ \det A_2(t,x)\neq 0)\,.$$

The proof exposed bellow strongly relies on the "weak=strong result" of Sarason [Sarason, 1962], the other arguments being classical. Indeed the proof is a three steps proof

1. Establish an *a priori* energy estimate is made in Paragraph 2.4.1 (which comes from free thanks to the strict dissipativity).

2. Construct a weak solution. In order to do so we will use a slight modification of the so-called duality method of [Lax and Phillips, 1960]. We refer to Paragraph 2.4.2

3. Show that the weak solution is a strong one to conclude. This result comes from the "weak=strong result" of Sarason [Sarason, 1962]. It is the technical part of the proof. For a sake of completeness the result of Sarason is recalled in Paragraph 2.4.3

The result is the following:

**Theorem 2.4** *We consider the corner problem*

$$\begin{cases} L(\partial)u = f & \text{in } \Omega, \\ B_1 u_{|x_1=0} = g_1 & \text{on } \partial\Omega_1, \\ B_2 u_{|x_2=0} = g_2 & \text{on } \partial\Omega_2, \end{cases} \tag{20}$$

*for which we assume that the coefficients $A_j$, satisfy*

$$A_1 = A_1(x_1, x_2) \in \mathscr{C}^1_b(\mathbb{R}^2_+),\ A_2 = A_2(x_1, x_2) \in \mathscr{C}^1_b(\mathbb{R}^2_+),\ \text{and } \forall\, j \geq 3\ A_j = A_j(t, x) \in W^{1,\infty}(\Omega),$$

*$\mathscr{C}^1_b$ being the et of $\mathscr{C}^1$ functions with bounded derivatives and where the boundary matrix $B_j = B_j(x_{3-j})$ lies in $\mathscr{C}^0(\mathbb{R}_+)$. Assume that this problem admits non characteristic strictly dissipative boundaries in the sense of Definitions 2.6 and 2.8 and that it is symetric in the sense of Definition 2.7. Assume moreover that in a neighborhood $\mathscr{V}$ of $(0,0)$ the eigenvalues $\lambda$ of the matrix $A_1^{-1}A_2$ satisfies the following*

1. *The multiplicity of $\lambda$ remains constant on $\mathscr{V}$.*

2. *The real eigenvalues and the imaginary parts of the complex eigenvalues remains either positive either negative away from zero on $\mathscr{V}$.*

3. *In its Jordan decomposition $(A_1^{-1}A_2)(0,0)$ does not admit a non trivial Jordan block associated to a real positive eigenvalue.*

*There exists*[2] *$\gamma_0 > 0$ such that for all $\gamma \geq \gamma_0$ if $f \in L^2_\gamma(\Omega)$, $(g_1, g_2) \in L^2_\gamma(\partial\Omega_1) \times L^2_\gamma(\partial\Omega_2)$ then the problem (20) admits a unique solution $u \in L^2_\gamma(\Omega)$ with traces $(u_{|x_1=0}, u_{|x_2=0}) \in L^2_\gamma(\partial\Omega_1) \times L^2_\gamma(\partial\Omega_2)$. Moreover the solution $u$ satisfies the energy estimate: there exists $C > 0$ such that for all $\gamma \geq \gamma_0$*

$$\begin{aligned} \gamma\|u\|^2_{L^2_\gamma(\Omega)} + \|u_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|u_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)} \\ \leq C\left(\frac{1}{\gamma}\|L(\partial)u\|^2_{L^2_\gamma(\Omega)} + \|B_1 u_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|B_2 u_{|x_2=0}\|^2_{L^2(\partial\Omega_2)}\right). \end{aligned} \tag{21}$$

The proof of this result occupies the three following paragraphs.

### 2.4.1 *A priori* energy estimate

Because we are in the particular framework of strictly dissipative boundary conditions, with symmetric matrices $A_j$, the *a priori* energy estimate which is the technical point in the analysis of Kreiss [Kreiss, 1970], which is far to be well established for corner problems, comes for free. Indeed we have the following proposition

**Proposition 2.1** *Assume that the problem*

$$\begin{cases} L(\partial)u + Du = f & \text{in } \Omega, \\ B_1(t, x_2, x')u_{|x_1=0} = g_1 & \text{on } \partial\Omega_1, \\ B_2(t, x_1, x')u_{|x_2=0} = g_2 & \text{on } \partial\Omega_2, \end{cases}$$

[2] The coefficient $\gamma_0$ only depends on the dimension $d$ and the coefficients $A_j$

*where $D \in L^\infty(\Omega, \mathbf{M}_{N\times N}(\mathbb{R}))$, is symmetric with strictly dissipative boundary conditions. Also assume that the coefficients $A_j \in W^{1,\infty}(\Omega)$ then there exists $\gamma_0 > 0$ such that if $u$ is a regular enough solution to (2) then $u$ satisfies the energy estimate: there exists $C > 0$ such that for all $\gamma \geq \gamma_0$ then*

$$\gamma\|u\|^2_{L^2_\gamma(\Omega)} + \|u_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|u_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)} \leq C\left(\frac{1}{\gamma}\|L(\partial)u\|^2_{L^2_\gamma(\Omega)} + \|B_1 u_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|B_2 u_{|x_2=0}\|^2_{L^2(\partial\Omega_2)}\right),$$

*where we stress that the constant $C$ depends on the coefficients but not on $\gamma$.*

**Proof :** We introduce $v = e^{-\gamma t}u$ where $u$ stands for a regular solution to (20). We have $L(\partial)v + \gamma v + Dv = e^{-\gamma t}L(\partial)u$. Multiplying by $v$ and integrating over $\Omega$ gives from the symmetry assumption

$$2\gamma\|v\|^2_{L^2(\Omega)} - \int_{\partial\Omega_1}\left\langle A_{1_{|x_1=0}}v_{|x_1=0}, v_{|x_1=0}\right\rangle \mathrm{d}t\,\mathrm{d}x'\,\mathrm{d}x_2 - \int_{\partial\Omega_2}\left\langle A_{2_{|x_2=0}}v_{|x_2=0}, v_{|x_2=0}\right\rangle \mathrm{d}t\,\mathrm{d}x'\,\mathrm{d}x_1 + \int_\Omega \langle Dv, v\rangle\ \mathrm{d}t\,\mathrm{d}x = 2\int_\Omega e^{-\gamma t}L(\partial)u\cdot v\,\mathrm{d}t\,\mathrm{d}x + \sum_{j=1}^d \int_\Omega \langle(\partial_j A_j)v, v\rangle\ \mathrm{d}t\,\mathrm{d}x.$$

Let $M := \max_{j\in[\![1,d]\!]}\|\partial_j A_j\|_{L^\infty(\Omega)}$ and choose for example $\underline{\gamma} := 2Md$. Then for all $\gamma \geq \underline{\gamma}$ we have from Young inequality combined with the strict dissipativity property (19)

$$(\gamma - \|D\|_{L^\infty(\Omega)})\|v\|^2_{L^2(\Omega)} - \|v_{|x_1=0}\|^2_{L^2(\partial\Omega_1)} + \|v_{|x_2=0}\|^2_{L^2(\partial\Omega_1)} \leq C\left(\frac{1}{\gamma}\|e^{-\gamma\cdot}L(\partial)u\|^2_{L^2(\Omega)} + \|B_1 v_{|x_1=0}\|^2_{L^2(\partial\Omega_1)} + \|B_2 v_{|x_2=0}\|^2_{L^2(\partial\Omega_1)}\right)$$

so that if we choose for example $\gamma_0 := \max(\underline{\gamma}, 2\|D\|_{L^\infty(\Omega)})$ we obtain the desired estimate for $u$.

□

### 2.4.2 Existence of a weak solution

The existence of a weak solution to (2) is shown by duality and follows the same ideas that the ones exposed in [Lax and Phillips, 1960] see also [[Benzoni-Gavage, 2007]-Section 9.2].

Such method will also be a little refined in Paragraph 4.2 in order to establish the regularity of the solution to the pure boundary value problem (2) so that recalling this method here is a good preliminary work.

However the definitions of weak solutions are the ones used in [Sarason, 1962] and differ a little from the one used in [Lax and Phillips, 1960]. This little modification is made to apply the "weak=strong" result of [Sarason, 1962]. Let us first introduce the so-called dual problem of (2) and the functional spaces that will be used in the following.

**Definition 2.9 (Dual problem of (2))** *The pure boundary value problem*

$$\begin{cases} L^*(\partial)v = \widetilde{f} & \text{in } \Omega, \\ C_1(t, x_2, x')v_{|x_1=0} = \widetilde{g}_1 & \text{on } \partial\Omega_1, \\ C_2(t, x_1, x')v_{|x_2=0} = \widetilde{g}_2 & \text{on } \partial\Omega_2, \end{cases} \tag{22}$$

*is a dual problem of (2) if for all regular functions $u$ and $v$ we have the duality formula:*

$$\langle L(\partial)u, v\rangle_{L^2(\Omega)} - \langle u, L^*(\partial)v\rangle_{L^2(\Omega)} = -\sum_{j=1}^2 \left\langle N_j u_{|x_j=0}, C_j v_{|x_j=0}\right\rangle_{L^2(\partial\Omega_j)} + \left\langle B_j u_{|x_j=0}, M_j v_{|x_j=0}\right\rangle_{L^2(\partial\Omega_j)},$$

*for some matrices $N_j, C_j \in \mathbf{M}_{(N-p_j)\times N}(\mathbb{R})$, $M_j \in \mathbf{M}_{p_j\times N}(\mathbb{R})$ satisfying the decomposition*

$$\forall\,(t, x_{3-j}, x') \in \partial\Omega_j,\ \widetilde{A}_j = C_j^T N_j + M_j^T B_j. \tag{23}$$

The following proposition is a standard result whose proof can be find for example in [[Benzoni-Gavage, 2007]-Lemma 9.4].

**Proposition 2.2** *Define*[3]

$$L^*(\partial) := -\partial_t - \sum_{j=1}^{d} A_j^T \partial_j - \sum_{j=1}^{d} \partial_j A_j^T,$$

*then for $j = 1, 2$ there exist matrices $N_j, C_j \in (\mathscr{C}^0 \cap L^\infty)(\partial\Omega_j, \mathbf{M}_{(N-p_j)\times N}(\mathbb{R}))$ and a matrix $M_j \in (\mathscr{C}^0 \cap L^\infty)(\partial\Omega_j, \mathbf{M}_{p_j\times N}(\mathbb{R}))$ such that the pure boundary value problem (22) is a dual problem of (2).*

*The matrices $C_j$ are characterized by*

$$\forall\, (t, x_{3-j}, x') \in \partial\Omega_j,\ \ker C_j = (A_j \ker B_j)^\perp, \tag{24}$$

*the matrices $N_j$ are onto and satisfy*

$$\ker B_j \cap \ker N_j = \{0\}.$$

*Finally for all matrices characterized by (24) if the boundary conditions of (2) are strictly dissipative then the boundary condition of the dual problem (22) are strictly dissipative.*

*In particular, for later purposes let us remark that if the operator $L(\partial)$ is symmetric it implies that we have the following sign property: for $j \in \{1, 2\}$*

$$\forall\, v \in \ker C_j,\ v \neq 0,\ \langle \widetilde{A}_j v, v\rangle > 0. \tag{25}$$

We introduce the following functional space in order to define the weak solutions of (2). For $\chi \in \mathbb{R}$ we introduce the Hilbert product space $\mathscr{S}_\chi := L^2_\chi(\Omega) \times L^2_\chi(\partial\Omega_1) \times L^2(\partial\Omega_2)$ equipped with the product norm

$$\text{for } (u, \overline{u}^1, \overline{u}^2) \in \mathscr{S}_\chi\ ,\ \|(u, \overline{u}^1, \overline{u}^2)\|_{\mathscr{S}_\chi} := \sqrt{\|u\|^2_{L^2_\chi(\Omega)} + \|\overline{u}^1\|^2_{L^2_\chi(\partial\Omega_1)} + \|\overline{u}^2\|^2_{L^2_\chi(\partial\Omega_2)}}.$$

Following [Sarason, 1962] we define the following notions of weak solutions

**Definition 2.10 (Weak solutions to $L(\partial)u = f$)** *Let $\gamma > 0$ and $f \in L^2_\gamma(\Omega)$. We then say that the triplet $U = (u, \overline{u}^1, \overline{u}^2) \in \mathscr{S}_\gamma$ is a weak solution to $L(\partial)u = f$ if for all $v \in H^1_{-\gamma}(\Omega)$ there holds*

$$\langle f, v\rangle_{L^2(\Omega)} - \langle u, L^*(\partial)v\rangle_{L^2(\Omega)} = -\left\langle A_1\overline{u}^1, v_{|x_1=0}\right\rangle_{L^2(\partial\Omega_1)} - \left\langle A_2\overline{u}^2, v_{|x_2=0}\right\rangle_{L^2(\partial\Omega_2)}.$$

**Definition 2.11 (Weak solutions to (2))** *Let $\gamma > 0$ and $(f, g_1, g_2) \in \mathscr{S}_\gamma$. We then say that the triplet $U = (u, \overline{u}^1, \overline{u}^2) \in \mathscr{S}_\gamma$ is a weak solution to (2) if $U$ is a weak solution of $L(\partial)u = f$ in the sense of Definition 2.10 and if moreover we have*

$$B_1\overline{u}^1 = g_1 \text{ and } B_2\overline{u}^2 = g_2.$$

Note that with such a definition we do not require $\overline{u}^1$ and $\overline{u}^2$ to be the traces of $u_{|x_1=0}$ and $u_{|x_2=0}$ so that we are free to modify a little the traces values obtained via the usual use of Riesz representation theorem. The main result of this paragraph is the following proposition:

**Proposition 2.3 (Existence of a weak solution of (2))** *Let $\gamma > 0$ and $(f, g_1, g_2) \in \mathscr{S}_\gamma$. Assume that the pure boundary value problem (2) is symmetric, non characteristic with strictly dissipative boundary conditions then it has a weak solution $U \in \mathscr{S}_\gamma$ in the sense of Definition 2.11*

**Proof :** Let $\gamma > 0$ we introduce the space

$$X_{-\gamma} := \left\{\mathscr{L}^* v := (L^*(\partial)v, C_1 v_{|x_1=0}, C_2 v_{|x_2=0}) \text{ where } v \in H^1_{-\gamma}(\Omega)\right\}.$$

From Proposition 2.2 there is no loss of generality by assuming that the matrices $C_1$, $C_2$ are strictly dissipative for the dual problem (22) so that from Proposition 2.1 we have the dual energy estimate (8) that is

$$\|v\|_{\mathscr{S}_{-\gamma}} \leq C_\gamma \|\mathscr{L}^* v\|_{\mathscr{S}_{-\gamma}}. \tag{26}$$

[3]Note that this part of the proof does not require the symmetry of the coefficients.

Let $(f, g_1, g_2) \in \mathscr{H}_\gamma$ we introduce the form $\ell : X_{-\gamma} \to \mathbb{R}$ defined by

$$\ell(\mathscr{L}^* v) := \langle f, v \rangle_{L^2(\Omega;\mathbb{R}^N)} + \left\langle g_1, M_1 v_{|x_1=0} \right\rangle_{L^2(\partial\Omega_1;\mathbb{R}^{p_1})} + \left\langle g_2, M_2 v_{|x_2=0} \right\rangle_{L^2(\partial\Omega_2;\mathbb{R}^{p_2})}.$$

We have from Cauchy-Schwarz and Young inequalities

$$|\ell(\mathscr{L}^* v)| \leq C \left[ \|v\|^2_{L^2_{-\gamma}(\Omega)} + \|v_{|x_1=0}\|^2_{L^2_{-\gamma}(\partial\Omega_1)} + \|v_{|x_2=0}\|^2_{L^2_{-\gamma}(\partial\Omega_2)} \right]$$

where $C$ depends in particular on $f, g_1, g_2$ and on $\|M_1\|_{L^\infty(\partial\Omega_1)}$, $\|M_2\|_{L^\infty(\partial\Omega_2)}$. Using the *a priori* energy estimate (26) we obtain $|\ell(\mathscr{L}^* v)| \leq C_\gamma \|\mathscr{L}^* v\|_{\mathscr{S}_{-\gamma}}$. Consequently from Hahn-Banach theorem the form $\ell$ can be extended by continuity up to $\mathscr{S}_{-\gamma}$ and from Riesz representation theorem there exists $\underline{U} := (\underline{u}, \underline{u}^1, \underline{u}^2) \in \mathscr{S}_{-\gamma}$ such that

$$\ell(\mathscr{L}^* v) = \langle u, L^*(\partial) v \rangle_{L^2(\Omega)} + \left\langle \underline{u}^1, C_1 v_{|x_1=0} \right\rangle_{L^2(\partial\Omega_1)} + \left\langle \underline{u}^2, C_2 v_{|x_2=0} \right\rangle_{L^2(\partial\Omega_2)},$$

so that from the definition of $\ell$ the triplet $\underline{U}$ satisfies

$$\begin{aligned} \langle f, v \rangle_{L^2(\Omega)} - \langle u, L^*(\partial) v \rangle_{L^2(\Omega)} = &- \left\langle g_1, M_1 v_{|x_1=0} \right\rangle_{L^2(\partial\Omega_1)} + \left\langle \underline{u}^1, C_1 v_{|x_1=0} \right\rangle_{L^2(\partial\Omega_1)} \\ &- \left\langle g_2, M_2 v_{|x_2=0} \right\rangle_{L^2(\partial\Omega_2)} + \left\langle \underline{u}^1, C_2 v_{|x_1=0} \right\rangle_{L^2(\partial\Omega_2)}. \end{aligned} \tag{27}$$

In the rest of the proof we modify the boundary couple $(\underline{u}^1, \underline{u}^2)$ in some $(\overline{u}^1, \overline{u}^2)$ defined in such a way that the right hand side of (27) becomes $-(\langle A_1 \overline{u}^1, v_{|x_1=0} \rangle_{L^2(\partial\Omega_1)} + \langle A_2 \overline{u}^2, v_{|x_2=0} \rangle)_{L^2(\partial\Omega_2)}$ showing that $\overline{U} := (u, \overline{u}^1, \overline{u}^2) \in \mathscr{H}_\gamma$ is a weak solution to $L(\partial)u = f$. To conclude we verify that such a $\overline{U}$ in fact gives a weak solution to (2).

Because for all $(t, x_{3-j}, x') \in \partial\Omega_j$, the matrices $N_j$ and $B_j$ are onto we can write $\underline{u}^j = N_j \widetilde{u}^j$ and $g_j = B_j h_j$ with $\widetilde{u}^j, h_j \in L^2_\gamma(\Omega;\mathbb{R}^N)$. So that

$$- \left\langle g_j, M_j v_{|x_j=0} \right\rangle_{L^2(\partial\Omega_j)} + \left\langle \underline{u}^j, C_j v_{|x_j=0} \right\rangle_{L^2(\partial\Omega_j)} = - \left\langle A_j \overline{u}^j, v_{|x_j=0} \right\rangle_{L^2(\partial\Omega_j)},$$

with

$$\overline{u}^j := A_j^{-1} \left( M_j^T B_j h_j - C_j^T N_j \widetilde{u}^j \right).$$

Consequently the triplet $\overline{U} := (u, \overline{u}^1, \overline{u}^2) \in \mathscr{H}_\gamma$ is a weak solution to $L(\partial)u = f$.

To conclude we verify that for $j = 1, 2$ we have $B_j \overline{u}^j = g_j$. From the definition of $\overline{u}^j$ we can directly compute

$$B_j \overline{u}^j = -B_j A_j^{-1} C_j^T N_j \widetilde{u}^j + B_j A_j^{-1} M_j^T B_j h_j,$$

and the desired equality comes from the relations $B_j A_j^{-1} C_j^T N_j = 0$ and $B_j A_j^{-1} M_j^T B_j = B_j$ which are direct consequences of decomposition (23) and the definition of the dual problem.

Indeed because $\mathbb{R}^N = \ker N_j \oplus \ker B_j$ it is sufficient to compute $B_j A_j^{-1} C_j^T N_j u$ for $u \in \ker B_j$ and $B_j A_j^{-1} M_j^T B_j v$ for $v \in \ker N_j$. But using (23) we can write

$$\begin{aligned} B_j A_j^{-1} C_j^T N_j u &= B_j A_j^{-1} \left( A_j - M_j^T B_j \right) u = 0, \\ B_j A_j^{-1} M_j^T B_j v &= B_j A_j^{-1} \left( A_j - C_j^T N_j \right) v = B_j v. \end{aligned}$$

□

### 2.4.3 Uniqueness of the weak solution

To show the uniqueness of the weak solution and that the conclusions of Assumption 2.1 are satisfied for symmetric non-characteristics operators with strictly dissipative boundary conditions we use the "weak=strong" lemma of [Sarason, 1962]. In order to do so we introduce the following definition for strong solutions.

**Definition 2.12 (Strong solution(s) to $L(\partial)u = f$)** *Let $\gamma > 0$ and $f \in L^2_\gamma(\Omega)$ then $U = (u, \overline{u}^1, \overline{u}^2) \in \mathscr{S}_\gamma$ is said to be a strong solution to $L(\partial)u = f$ if there exists a sequence $(u^n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ such that*

$$\lim_{n\to\infty} \|U - (u^n, u^n_{|x_1=0}, u^n_{|x_2=0})\|^2_{\mathscr{S}_\gamma} + \|L(\partial)u^n - f\|^2_{L^2_\gamma(\Omega)} = 0.$$

In order to show that weak solutions are in fact strong ones we will apply the result of [Sarason, 1962]. Let us recall precisely the statement of this result[4].

**Theorem 2.5 ([Sarason, 1962]-Theorem 3.1)** *Consider $L := \partial_1 + A(x_1, x_2)\partial_2 + B(x_1, x_2)$ be an operator acting on $\mathbb{R}^2_+$. We assume that*

1. *The matrices $A \in \mathscr{C}^1(\mathbb{R}^2_+)$ and $B \in \mathscr{C}^0(\mathbb{R}^2_+)$.*
2. *Let $\lambda \in \mathbb{R}$ be a real eigenvalue of $A(0,0)$ corresponding to a non-simple eigenspace of $A(0,0)$, then the line $y = \lambda x$ lies outside of $\Omega$ (that is $\lambda < 0$).*
3. *In a neighborhood of $(0,0)$ the matrix $A$ does not change type (meaning that its eigenvalues remain either purely real either complex).*

*Let $U = (u, u^1, u^2) \in \mathscr{S}_\gamma$ be a weak solution to $Lu = f$ then $U$ is a strong solution in the sense of Definition 2.12.*

Note that the assumptions of Theorem 2.4 on $A_1^{-1}A_2$ combined with the fact that the coefficients $A_1$ and $A_2$ do not depend on the tangential variables $(t, x')$ are made in order to apply the previous theorem to the matrix $A := A_1^{-1}A_2$.

With this result in hand then we can easily conclude the proof of Theorem 2.4. Indeed let a triplet $U = (u, \overline{u}^1, \overline{u}^2) \in \mathscr{S}_\gamma$ be a weak solution of (20) given by Proposition 2.3 in particular it is a weak solution to $L(\partial)u = f$ in the sense of Definition 2.10. Then by Theorem 2.5, $U$ is a strong solution to $L(\partial)u = f$ that is there exists a sequence $(u^n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ such that for all $\gamma > 0$

$$\lim_{n\to\infty} \|U - (u^n, u^n_{|x_1=0}, u^n_{|x_2=0})\|^2_{\mathscr{S}_\gamma} + \|L(\partial)u^n - f\|^2_{L^2_\gamma(\Omega)} = 0.$$

However $(u^n)_{n\in\mathbb{N}}$ is regular so that it satisfies the *a priori* energy estimate of Proposition 2.1 that is to say that for all $\gamma \geq \gamma_0$ we have

$$\begin{aligned}\gamma\|u^n\|^2_{L^2_\gamma(\Omega)} + \|u^n_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|u^n_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)} \\ \leq C\left(\frac{1}{\gamma}\|L(\partial)u^n\|^2_{L^2_\gamma(\Omega)} + \|B_1 u^n_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|B_2 u^n_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)}\right).\end{aligned}$$

From the triangle inequality we thus have

$$\begin{aligned}\gamma\|u\|^2_{L^2_\gamma(\Omega)} + \|\overline{u}^1\|^2_{L^2_\gamma(\partial\Omega_1)} + \|\overline{u}^2\|^2_{L^2_\gamma(\partial\Omega_2)} \\ \leq C_\gamma\|U - (u^n, u^n_{|x_1=0}, u^n_{|x_2=0})\|^2_{\mathscr{S}_\gamma} \\ + C\left(\frac{1}{\gamma}\|L(\partial)u^n\|^2_{L^2_\gamma(\Omega)} + \|B_1 u^n_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|B_2 u^n_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)}\right).\end{aligned}$$

So that passing to the limit $n$ goes to $\infty$ we obtain that for all $\gamma \geq \gamma_0$:

$$\gamma\|u\|^2_{L^2_\gamma(\Omega)} + \|\overline{u}^1\|^2_{L^2_\gamma(\partial\Omega_1)} + \|\overline{u}^2\|^2_{L^2_\gamma(\partial\Omega_2)} \leq C\left(\frac{1}{\gamma}\|f\|^2_{L^2_\gamma(\Omega)} + \|B_1\overline{u}^1\|^2_{L^2_\gamma(\partial\Omega_1)} + \|B_2\overline{u}^2_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)}\right).$$

However recall that $U$ is a weak solution to (20) so that we have $B_1\overline{u}^1 = g_1$ and $B_2\overline{u}^2 = g_2$ thus the weak solution $U$ satisfies the energy estimate (21). The problem (20) being linear it automatically implies the uniqueness of the weak solution. This ends up the proof of Theorem 2.4, showing consequently that Assumption 2.1 applies to a non empty set of boundary value problems.

[4]Note that we simplify a little the statement of the result of [Sarason, 1962]. Indeed the result of [Sarason, 1962] holds for generic corner domains and note only for our quarter space $\mathbb{R}^2_+$.

# 3 The *a priori* energy estimate

In this section we want to show that any (regular) solution to the pure boundary value problem

$$\begin{cases} L(\partial)u = f & \text{in } \Omega, \\ B_1 u_{|x_1=0} = g_1 & \text{on } \partial\Omega_1, \\ B_2 u_{|x_2=0} = g_2 & \text{on } \partial\Omega_2, \end{cases} \tag{28}$$

satisfies the energy estimate (9).

The proof of such high order energy estimate is classically made by iteration on $s$. In Paragraph 3.1 we give the initialization for $s = 1$. Note that we could also use Theorem 2.4 in order to initialize at $s = 0$ but because the step $s = 1$ contains all the main ingredients of the proof it is a better initialization step to illustrate our purpose. Then the iteration process is described in Paragraph 3.2. The three main ides of the proof are the followings:

1. We estimate the radial derivatives $(\mathbf{r}\partial_{\mathbf{r}})^k u$, for all $k \in [\![0, s]\!]$. This is made using Assumption 2.1.
2. We use the equation in polar coordinates to obtain explicit formulas for the $(\mathbf{r}\partial_{\mathbf{r}})^{k-l}\partial_\theta^l u$ for $k \in [\![0, s]\!]$ and $l \in [\![0, k]\!]$ essentially in terms of the radial derivatives. This uses Assumption 2.2.
3. To conclude we then use explicit formulas to recover the control of the anisotropic weighted derivatives $x_1^{\alpha+\beta-p} x_2^p \partial_1^\alpha \partial_2^\beta$ for $\alpha + \beta \in [\![0, s]\!]$ and $p \in [\![0, \alpha + \beta]\!]$ in terms of the mixed derivatives $(\mathbf{r}\partial_{\mathbf{r}})^{k-l}\partial_\theta^l u$.

## 3.1 The case $s = 1$

In this paragraph we give the proof of Theorem for $s = 1$ in order to initialize the iterative process. We thus consider $u \in \mathscr{H}^1_\gamma(\Omega)$ and we shall estimate

$$\|u\|^2_{\mathbf{H}^{1,1}(\Omega)} + \|\mathbf{r}u\|^2_{\mathbf{H}^{1,0}(\Omega)} + \sum_{p=1}^{2} \|u_{|x_p=0}\|^2_{\mathbf{H}^{1,1}(\partial\Omega_p)} + \|x_{3-p}u_{|x_p=0}\|^2_{\mathbf{H}^{1,0}(\partial\Omega_p)},$$

which by definition of the Sobolev spaces $\mathbf{H}^{1,p}(X)$ amounts to estimate

$$\begin{aligned} \sum_{\alpha+\beta=1} \sum_{k=0}^{1} \|x_1^k x_2^{1-k} \partial_1^\alpha \partial_2^\beta u\|^2_{L^2_\gamma(\Omega)} + \sum_{|\delta|\le 1} \|\partial^\delta u\|^2_{L^2_\gamma(\Omega)} + \sum_{|\delta|\le 1} \|\partial^\delta \mathbf{r}u\|^2_{L^2_\gamma(\Omega)} \\ + \sum_{p=1}^{2} \|x_{3-p}\partial_{3-p}u\|^2_{L^2_\gamma(\partial\Omega_p)} + \sum_{|\delta|\le 1} \|\partial^\delta u\|^2_{L^2_\gamma(\partial\Omega_p)} + \sum_{|\delta|\le 1} \|\partial^\delta x_{3-p}u\|^2_{L^2_\gamma(\partial\Omega_p)}, \end{aligned} \tag{29}$$

where we recall that for $\delta = (\delta_0, \delta_3, ..., \delta_d) \in \mathbb{N}^{1+d-2}$, $\partial^\delta := \partial_t^{\delta_0}\partial_3^{\delta_3} \cdots \partial_d^{\delta_d}$.

Firstly let $j \in \{t\} \cup [\![3, d]\!] := \mathscr{T}$, $\mathscr{T}$ being the set of tangential indices. In order to simplify the notations let $A_t := I_{d\times d}$. Then clearly $\partial_j u$ solves the boundary value problem[5]

$$\begin{cases} L(\partial)\partial_j u = \partial_j L(\partial)u & \text{in } \Omega, \\ B_1\partial_j u_{|x_1=0} = \partial_j B_1 u_{|x_1=0} & \text{on } \partial\Omega_1, \\ B_2\partial_j u_{|x_2=0} = \partial_j B_2 u_{|x_2=0} & \text{on } \partial\Omega_2, \end{cases}$$

so that from Assumption 2.1 we obtain the following energy estimate for tangential derivatives: for all $j \in \mathscr{T}$

$$\begin{aligned} \gamma\|\partial_j u\|^2_{L^2_\gamma(\Omega)} + \|\partial_j u_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|\partial_j u_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)} \\ \le C\left(\frac{1}{\gamma}\|\partial_j L(\partial)u\|^2_{L^2_\gamma(\Omega)} + \|\partial_j B_1 u_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|\partial_j B_2 u_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)}\right), \end{aligned} \tag{30}$$

[5]Note that we use in a non trivial way the fact that $A_1$ and $A_2$ do not depend on $(t, x')$

This equation gives the estimations for the second and the fifth terms in (29). So it remains to estimate the terms $x_1\partial_1 u$, $x_2\partial_1 u$, $x_1\partial_2 u$ and $x_2\partial_2 u$ appearing in the first and the third terms and also the $\mathbf{r}\partial_j u$ and their traces in order to conclude.

The first estimates are the most interesting ones. They are done in two steps, firstly we use the equation in cartesian coordinates to obtain an estimate on the radial derivative $x_1\partial_1 u + x_2\partial_2 u = \mathbf{r}\partial_{\mathbf{r}} u$ and then we use the formulation of the equation in polar coordinates to estimate the polar derivative $\partial_\theta u = -x_2\partial_1 u + x_1\partial_2 u$.

Finally some simple and explicit computations then permit to express each term $x_1\partial_1 u$, $x_2\partial_1 u$, $x_1\partial_2 u$ and $x_2\partial_2 u$ in terms of $\mathbf{r}\partial_{\mathbf{r}} u$ and $\partial_\theta u$ in order to conclude.

We apply $\mathbf{r}\partial_{\mathbf{r}}$ to the evolution equation of (28) and we use the commutator relations

$$\mathbf{r}\partial_{\mathbf{r}}\partial_1 = \partial_1\mathbf{r}\partial_{\mathbf{r}} - \partial_1 \text{ and } \mathbf{r}\partial_{\mathbf{r}}\partial_2 = \partial_2\mathbf{r}\partial_{\mathbf{r}} - \partial_2,$$

to obtain that $\mathbf{r}\partial_{\mathbf{r}} u$ satisfies the problem

$$\begin{cases} L(\partial)\mathbf{r}\partial_{\mathbf{r}} u = \mathbf{r}\partial_{\mathbf{r}} L(\partial)u + \underbrace{A_1\partial_1 u + A_2\partial_2 u}_{=L(\partial)u - \sum_{j\in\mathscr{T}} A_j\partial_j u} & \text{in } \Omega, \\ B_1(\mathbf{r}\partial_{\mathbf{r}} u)_{|x_1=0} = B_1 x_2\partial_2 u_{|x_1=0} = x_2\partial_2 B_1 u_{|x_1=0} & \text{on } \partial\Omega_1, \\ B_2(\mathbf{r}\partial_{\mathbf{r}} u)_{|x_2=0} = B_1 x_1\partial_1 u_{|x_2=0} = x_1\partial_1 B_2 u_{|x_2=0} & \text{on } \partial\Omega_2, \end{cases}$$

where we used the classical trick consisting in expressing the normal derivative in terms of the equation $L(\partial)u = A_1\partial_1 u + A_2\partial_2 u + \sum_{j\in\mathscr{T}} A_j\partial_j u$ already used in characteristics boundary value problems in the half-space (see [Rauch, 1985])

So that from Assumption 2.1 combined with (30) we obtain the estimate

$$\begin{aligned} \gamma\|\mathbf{r}\partial_{\mathbf{r}} u\|^2_{L^2_\gamma(\Omega)} + \|x_2\partial_2 u_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|x_1\partial_1 u_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)} & \qquad (31) \\ \leq C\left(\frac{1}{\gamma}\|\mathbf{r}\partial_{\mathbf{r}} L(\partial)u\|^2_{L^2_\gamma(\Omega)} + \frac{1}{\gamma}\|L(\partial)u\|^2_{L^2_\gamma(\Omega)} + \|x_2\partial_2 B_1 u_{|x_1=0}\|_{L^2_\gamma(\partial\Omega_1)} + \|x_1\partial_1 B_1 u_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)}\right) \\ + \frac{C}{\gamma}\sum_{j\in\mathscr{T}}\|\partial_j u\|^2_{L^2_\gamma(\Omega)} \\ \leq C\Big(\frac{1}{\gamma}\|r\partial_r L(\partial)u\|^2_{L^2_\gamma(\Omega)} + \frac{1}{\gamma}\|L(\partial)u\|^2_{L^2_\gamma(\Omega)} + \frac{1}{\gamma^3}\sum_{j\in\mathscr{T}}\|\partial_j L(\partial)u\|^2_{L^2_\gamma(\Omega)} \\ + \sum_{k=1}^{2}\|x_{3-k}\partial_{3-k} B_k u_{|x_k=0}\|_{L^2_\gamma(\partial\Omega_k)} + \frac{1}{\gamma^2}\sum_{j\in\mathscr{T}}\|\partial_j B_k u_{|x_k=0}\|^2_{L^2_\gamma(\partial\Omega_k)}\Big). \end{aligned}$$

We complete the estimates for the terms in cartesian coordinates by estimating $\mathbf{r}\partial_j u$ for $j \in \mathscr{T}$. We apply $\partial_j$ to (28) and multiply by $\mathbf{r}$ to obtain that $\mathbf{r}\partial_j u$ satisfies

$$\begin{cases} L(\partial)\mathbf{r}\partial_j u = \mathbf{r}\partial_j L(\partial)u + (\partial_1\mathbf{r})A_1\partial_j u + (\partial_2\mathbf{r})A_2\partial_j u & \text{in } \Omega, \\ B_1(\mathbf{r}\partial_j u)_{|x_1=0} = x_2\partial_j B_1 u_{|x_1=0} & \text{on } \partial\Omega_1, \\ B_2(\mathbf{r}\partial_j u)_{|x_2=0} = x_1\partial_j B_2 u_{|x_2=0} & \text{on } \partial\Omega_2, \end{cases}$$

using the fact that $\|\partial_1\mathbf{r}\|_{L^\infty(\Omega)}$, $\|\partial_2\mathbf{r}\|_{L^\infty(\Omega)} \leq 1$ we can apply Assumption 2.1 to derive the estimate

$$\begin{aligned} \gamma\|\mathbf{r}\partial_j u\|^2_{L^2_\gamma(\Omega)} + \|x_2\partial_j u_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|x_1\partial_j u_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)} & \qquad (32) \\ \leq C\left(\frac{1}{\gamma}\|\mathbf{r}\partial_j L(\partial)u\|^2_{L^2_\gamma(\Omega)} + \|x_2\partial_j B_1 u_{|x_1=0}\|_{L^2_\gamma(\partial\Omega_1)} + \|x_1\partial_j B_2 u_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)}\right) + \frac{C}{\gamma}\|\partial_j u\|^2_{L^2_\gamma(\Omega)} \\ \leq C\Big(\frac{1}{\gamma}\|\mathbf{r}\partial_j L(\partial)u\|^2_{L^2_\gamma(\Omega)} + \frac{1}{\gamma^3}\|\partial_j L(\partial)u\|^2_{L^2_\gamma(\Omega)} \\ + \sum_{k=1}^{2}\|x_{3-k}\partial_j B_k u_{|x_k=0}\|_{L^2_\gamma(\partial\Omega_k)} + \frac{1}{\gamma^2}\|\partial_j B_k u_{|x_k=0}\|^2_{L^2_\gamma(\partial\Omega_k)}\Big). \end{aligned}$$

This gives the estimates for the third and the sixth terms in (29).

We now turn to the estimate on $\partial_\theta u$. In polar coordinates let $\widetilde{u}(\mathbf{r},\theta) := u(\mathbf{r}\cos\theta, \mathbf{r}\sin\theta)$. Then $\widetilde{u}$ solves the boundary value problem (28)

$$\begin{cases} \mathscr{A}(\theta)\partial_\theta\widetilde{u} + \mathscr{M}(\theta)\mathbf{r}\partial_{\mathbf{r}}\widetilde{u} + \mathbf{r}\sum_{j\in\mathscr{T}} A_j\partial_j\widetilde{u} = \mathbf{r}\widetilde{L(\partial)u} & \text{for } (t,\mathbf{r},\theta,x') \in \mathbb{R}\times\mathbb{R}_+\times\left[0,\frac{\pi}{2}\right]\times\mathbb{R}^{d-2}, \\ B_1\widetilde{u}_{|\theta=\frac{\pi}{2}} = B_1 u_{|x_1=0}, & \text{for } (t,\mathbf{r},x') \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^{d-2}, \\ B_2\widetilde{u}_{|\theta=0} = B_2 u_{|x_2=0}, & \text{for } (t,\mathbf{r},x') \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^{d-2}, \end{cases} \tag{33}$$

where the matrices $\mathscr{A}(\theta) := \cos\theta A_2 - \sin\theta A_1$ and $\mathscr{M}(\theta) := \cos\theta A_1 + \sin\theta A_2$. We now use the Assumption on the spectrum of $A_2^{-1}A_1$ in order to invert the matrix $\mathscr{A}(\theta)$ in the evolution equation of (33).

**Lemma 3.1** *Under Assumption 2.2 the matrix $\mathscr{A}(\theta)$ is uniformly invertible. That is for all $\theta \in \left[0,\frac{\pi}{2}\right]$, $\mathscr{A}(\theta)$ is invertible that is there exists $C>0$ such that*

$$\|\mathscr{A}^{-1}\|_{L^\infty(\left[0,\frac{\pi}{2}\right])} \leq C,$$

*where $C$ does not depend on $\theta$.*

*As a consequence because $\mathscr{A} \in (\mathscr{C}^\infty \cap L^\infty)(\left[0,\frac{\pi}{2}\right]; \mathbf{M}_{N\times N}(\mathbb{R}))$ we also have that for all $n \geq 0$*

$$\|\partial_\theta^n \mathscr{A}^{-1}\|_{L^\infty(\left[0,\frac{\pi}{2}\right])} \leq C_n.$$

**Proof :** The matrix $\mathscr{A}$ is clearly continuous with respect to $\theta$ and we can remark that at the boundaries $\mathscr{A}(\pi/2) = -A_1$ and $\mathscr{A}(0) = A_2$ are invertible. For $\theta \in \left]0,\frac{\pi}{2}\right[$ then we can write

$$\mathscr{A}(\theta) = \cos(\theta)A_1\left(A_1^{-1}A_2 - \tan(\theta)\right),$$

so that $\det\mathscr{A}(\theta) = 0$ is equivalent to say that $\tan(\theta)$ is an eigenvalue of $A_1^{-1}A_2$ which is impossible because of Assumption 2.2 on the spectrum of $A_1^{-1}A_2$.

The second statement of the lemma is then shown inductively using the explicit formula for the derivative $\partial_\theta^n\mathscr{A}^{-1}$ in terms of the $\partial_\theta^m\mathscr{A}^{\pm1}$ with $m<n$.

□

So that from the triangle inequality we obtain the estimate:

$$\begin{aligned}\|\partial_\theta u\|_{L^2_\gamma(\Omega)} = \|\partial_\theta\widetilde{u}\|_{L^2_\gamma(\widetilde{\Omega},\mathbf{r}\,\mathrm{d}\mathbf{r})} \leq C\|\mathscr{A}^{-1}\|_{L^\infty(\left[0,\frac{\pi}{2}\right])}\Big(&\|\mathscr{M}\|_{L^\infty(\left[0,\frac{\pi}{2}\right])}\|\mathbf{r}\partial_{\mathbf{r}}\widetilde{u}\|_{L^2_\gamma(\widetilde{\Omega},\mathbf{r}\,\mathrm{d}\mathbf{r})} \\ &+ \|\mathbf{r}\widetilde{L(\partial)u}\|_{L^2_\gamma(\widetilde{\Omega},\mathbf{r}\,\mathrm{d}\mathbf{r})} + \sum_{j\in\mathscr{T}}\|\mathbf{r}\partial_j\widetilde{u}\|_{L^2_\gamma(\widetilde{\Omega},\mathbf{r}\,\mathrm{d}\mathbf{r})}\Big),\end{aligned}$$

where $\widetilde{\Omega}$ stands for the quarter space in polar coordinates. So that

$$\|\partial_\theta u\|_{L^2_\gamma(\Omega)} \leq C\Big(\|\mathbf{r}\partial_{\mathbf{r}}u\|_{L^2_\gamma(\Omega)} + \|\mathbf{r}L(\partial)u\|_{L^2_\gamma(\Omega)} + \sum_{j\in\mathscr{T}}\|\mathbf{r}\partial_j u\|_{L^2_\gamma(\Omega)}\Big).$$

Consequently $\|\partial_\theta u\|_{L^2_\gamma(\Omega)}$ can be estimated by the right hand-side of (31) and (32).

To conclude we observe that

$$\begin{cases} \cos\theta\partial_1 u + \sin\theta\partial_2 u = \partial_{\mathbf{r}}\widetilde{u} \\ -\mathbf{r}\sin\theta\partial_1 u + \mathbf{r}\cos\theta\partial_2 u = \partial_\theta\widetilde{u} \end{cases} \Leftrightarrow \begin{cases} \mathbf{r}\partial_2 u = \mathbf{r}\sin\theta\partial_{\mathbf{r}}\widetilde{u} + \cos\theta\partial_\theta\widetilde{u}, \\ \mathbf{r}\partial_1 u = \mathbf{r}\cos\theta\partial_{\mathbf{r}}\widetilde{u} - \sin\theta\partial_\theta\widetilde{u}. \end{cases} \tag{34}$$

Multiplying the second line of (34) by $\cos\theta$ and $\sin\theta$ gives

$$x_1\partial_1 u = \cos^2\theta\mathbf{r}\partial_{\mathbf{r}}\widetilde{u} - \cos\theta\sin\theta\partial_\theta\widetilde{u} \text{ and } x_2\partial_1 u = \sin\theta\cos\theta\mathbf{r}\partial_{\mathbf{r}}\widetilde{u} - \sin^2\theta\partial_\theta\widetilde{u},$$

so that for $k \in [\![1,2]\!]$

$$\|x_k\partial_1 u\|^2_{L^2_\gamma(\Omega)} \le \|\mathbf{r}\partial_{\mathbf{r}} u\|^2_{L^2_\gamma(\Omega)} + \|\partial_\theta u\|^2_{L^2_\gamma(\Omega)} \tag{35}$$

$$\le C\Bigg( \|\mathbf{r}L(\partial)u\|^2_{L^2_\gamma(\Omega)} + \frac{1}{\gamma^2}\sum_{j\in\mathscr{T}} \|\partial_j \mathbf{r}L(\partial)u\|^2_{L^2_\gamma(\Omega)}$$

$$+ \frac{1}{\gamma^2}\Big[\|L(\partial)u\|^2_{L^2_\gamma(\Omega)} + \|r\partial\mathbf{r}L(\partial)u\|^2_{L^2_\gamma(\Omega)} + \frac{1}{\gamma^2}\sum_{j\in\mathscr{T}} \|\partial_j L(\partial)u\|^2_{L^2_\gamma(\Omega)}\Big]$$

$$+ \frac{1}{\gamma}\sum_{k=1}^{2} \|x_{3-k}\partial_{3-k}B_k u_{|x_k=0}\|^2_{L^2_\gamma(\partial\Omega_k)} + \sum_{j\in\mathscr{T}} \|x_{3-k}\partial_j B_k u_{|x_k=0}\|^2_{L^2_\gamma(\partial\Omega_k)} + \frac{1}{\gamma^2}\|\partial_j B_k u_{|x_k=0}\|^2_{L^2_\gamma(\partial\Omega_k)}\Bigg).$$

Proceeding similarly for the first line of (34) gives the desired bounds on $x_1\partial_2 u$ and $x_2\partial_2 u$. From now on let us fix $\gamma \ge 1$ in order to simplify the expression of the right hand side of (35). Combining (30), (32) and (35) we obtain the desired energy estimate:

$$\|u\|^2_{\mathscr{H}^1_\gamma(\Omega)} + \|u_{|x_1=0}\|^2_{\mathscr{H}^1_\gamma(\partial\Omega_1)} + \|u_{|x_2=0}\|^2_{\mathscr{H}^1_\gamma(\partial\Omega_2)}$$
$$\le C\left(\|L(\partial)u\|^2_{\mathscr{H}^1_\gamma(\Omega)} + \|B_1 u_{|x_1=0}\|^2_{\mathscr{H}^1_\gamma(\partial\Omega_1)} + \|B_2 u_{|x_2=0}\|^2_{\mathscr{H}^1_\gamma(\partial\Omega_2)}\right),$$

as desired. This ends up the initialization.

## 3.2 The case $s > 1$

In order to show the result for $s > 1$ we need to introduce the following Sobolev type space: for $X \subset \Omega$

$$\widetilde{H}^{s,p}_\gamma(X) := \Big\{ u \in \mathcal{D}'(X) \setminus (\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial_\theta^\beta \partial^\gamma u \in L^2_\gamma(\Omega) \text{ for } \alpha,\beta \in \mathbb{N}, \delta \in \mathbb{N}^{d-1},\ \alpha+\beta+|\delta| \le s \text{ and } \alpha+\beta \le p \Big\}$$

and

$$\widetilde{H}^{s,p,\natural}_\gamma(X) := \left\{ u \in \mathcal{D}'(X) \setminus (\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial^\delta u \in L^2_\gamma(\Omega) \text{ for } \alpha \in \mathbb{N}, \delta \in \mathbb{N}^{d-1},\ \alpha \le p \text{ and } \alpha+|\delta| \le s \right\}.$$

The space $\widetilde{H}^{s,p}_\gamma$ (resp. $\widetilde{H}^{s,p,\natural}_\gamma$) has full regularity $s$ and contains at most $p$ derivatives with respect to $\mathbf{r}\partial_{\mathbf{r}}$ and $\partial_\theta$ (resp. $\mathbf{r}\partial_{\mathbf{r}}$ only).

We introduce the following norm on $\widetilde{H}^{s,p}_\gamma(X)$; for $u \in \widetilde{H}^{s,p}_\gamma(X)$

$$\|u\|^2_{\widetilde{H}^{s,p}_\gamma(X)} := \sum_{\alpha+\beta\le p, \alpha+\beta+|\delta|\le s} \|(\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial_\theta^\beta \partial^\delta u\|^2_{L^2_\gamma(X)},$$

and we define similarly a norm on $\widetilde{H}^{s,p,\natural}_\gamma(X)$ for $u \in \widetilde{H}^{s,p,\natural}_\gamma(X)$

$$\|u\|^2_{\widetilde{H}^{s,p,\natural}_\gamma(X)} := \sum_{\alpha\le p, \alpha+|\delta|\le s} \|(\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial^\delta u\|^2_{L^2_\gamma(X)}.$$

The proof essentially follows the one made for the initialization and is made in four steps:

1. We estimate the radial derivatives $(\mathbf{r}\partial_{\mathbf{r}})^\alpha$, $0 \le \alpha \le s$ and the boundary terms in the initialization of the proof of Lemma 3.2.
2. Then we use iteratively such estimates to control the mixed derivatives $(\mathbf{r}\partial_{\mathbf{r}})^{\alpha-k}\partial_\theta^k$, for $0 \le k \le \alpha$. This gives the auxiliary estimate of Proposition 3.1.
3. We show that the right hand side of (36) can be bounded by the $\mathscr{H}^s_\gamma$-norm of the sources (see (51)).
4. Finally we show that the interior term in the left hand side of (36) (namely the term involving the mixed derivatives $(\mathbf{r}\partial_{\mathbf{r}})^{\alpha-k}\partial_\theta^k$ can be bounded by above by the $\mathscr{H}^s_\gamma$-norm of the solution (see equation (56)).

We first show the following estimate in the spaces $\widetilde{H}^{s,p}_\gamma(X)$ and $\widetilde{H}^{s,\natural}_\gamma(X)$.

**Proposition 3.1** *For all $s\in\mathbb{N}$ there exists $C:=C_s>0$ such that the following estimate holds*

$$\begin{aligned}\sum_{\ell=0}^{s}\|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)}+\sum_{\ell=0}^{s}\|x_2^\ell u_{|x_1=0}\|^2_{\widetilde{H}^{s,s-\ell,\natural}_\gamma(\partial\Omega_1)}+\sum_{\ell=0}^{s}\|x_1^\ell u_{|x_2=0}\|^2_{\widetilde{H}^{s,s-\ell,\natural}_\gamma(\partial\Omega_2)}\qquad(36)\\ \leq C\bigg(\|L(\partial)u\|_{\widetilde{H}^{s,s,\natural}_\gamma(\Omega)}+\sum_{\ell=1}^{s}\|\mathbf{r}^\ell L(\partial)u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)}\\ +\sum_{\ell=0}^{s}\|x_2^\ell B_1u_{|x_1=0}\|^2_{\widetilde{H}^{s,s-\ell,\natural}_\gamma(\partial\Omega_1)}+\sum_{\ell=0}^{s}\|x_1^\ell B_2u_{|x_2=0}\|^2_{\widetilde{H}^{s,s-\ell,\natural}_\gamma(\partial\Omega_2)}\bigg).\end{aligned}$$

**Proof of Proposition 3.1** We proceed by iteration. Because of the definitions of $\widetilde{H}^{1,0}$ and $\widetilde{H}^{1,0,\natural}$. The case $s=1$ has been considered in Paragraph 3.1.

Let $s\in\mathbb{N}$ be such that we have the estimate

$$\begin{aligned}\sum_{\ell=0}^{s}\|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)}+\sum_{\ell=0}^{s}\|x_2^\ell u_{|x_1=0}\|^2_{\widetilde{H}^{s,s-\ell,\natural}_\gamma(\partial\Omega_1)}+\sum_{\ell=0}^{s}\|x_1^\ell u_{|x_2=0}\|^2_{\widetilde{H}^{s,s-\ell,\natural}_\gamma(\partial\Omega_2)}\qquad(37)\\ \leq C\bigg(\|L(\partial)u\|_{\widetilde{H}^{s,s,\natural}_\gamma(\Omega)}+\sum_{\ell=1}^{s}\|\mathbf{r}^\ell L(\partial)u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)}\\ +\sum_{\ell=0}^{s}\|x_2^\ell B_1u_{|x_1=0}\|^2_{\widetilde{H}^{s,s-\ell,\natural}_\gamma(\partial\Omega_1)}+\sum_{\ell=0}^{s}\|x_1^\ell B_2u_{|x_2=0}\|^2_{\widetilde{H}^{s,s-\ell,\natural}_\gamma(\partial\Omega_2)}\bigg),\end{aligned}$$

we want to show the analogous result for $s=s+1$.

We have

$$\sum_{\ell=0}^{s+1}\|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s+1,s+1-\ell}_\gamma(\Omega)}=\sum_{\ell=0}^{s}\|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s+1,s-\ell}_\gamma(\Omega)}+\sum_{\ell=0}^{s+1}\sum_{|\delta|\leq\ell}\sum_{k=0}^{s-\ell+1}\|(\mathbf{r}\partial_{\mathbf{r}})^{s+1-k-\ell}\partial_\theta^k\partial^\delta\mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)}.\qquad(38)$$

Note that the second sum in the right hand side of (38) can also be labelled by $k\in[\![0,s+1]\!]$ and $\ell\in[\![0,s+1-k]\!]$.

To deal with the first term we write

$$\begin{aligned}\sum_{\ell=0}^{s}\|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s+1,s-\ell}_\gamma(\Omega)}&=\sum_{\ell=0}^{s}\sum_{\alpha+\beta\leq s-\ell}\sum_{|\delta|=s+1-\ell-(\alpha+\beta)}\|(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial_\theta^\beta\partial^\delta\mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)}+\sum_{\ell=0}^{s}\|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)}\\ &\leq\Big(\sum_{\ell=0}^{s}\sum_{j\in\mathscr{T}}\sum_{\alpha+\beta\leq s-\ell}\sum_{|\delta'|=s-\ell-(\alpha+\beta)}\|(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial_\theta^\beta\partial^{\delta'}\partial_j\mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)}+\sum_{\ell=0}^{s}\|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)}\Big)\\ &\leq C\Big(\sum_{\ell=0}^{s}\sum_{j\in\mathscr{T}}\|\mathbf{r}^\ell\partial_j u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)}+\|L(\partial)u\|^2_{\widetilde{H}^{s,s,\natural}_\gamma(\Omega)}+\sum_{\ell=1}^{s}\|\mathbf{r}^\ell L(\partial)u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)}\\ &+\sum_{\ell=0}^{s}\|x_2^\ell B_1u_{|x_1=0}\|^2_{\widetilde{H}^{s,s-\ell,\natural}_\gamma(\partial\Omega_1)}+\sum_{\ell=0}^{s}\|x_1^\ell B_2u_{|x_2=0}\|^2_{\widetilde{H}^{s,s-\ell,\natural}_\gamma(\partial\Omega_2)}\Big),\end{aligned}$$

from the estimate (37). To conclude we then use (37) with $u = \partial_j u$ so that

$$\begin{aligned}
&\sum_{\ell=0}^{s}\sum_{j\in\mathscr{T}} \|\mathbf{r}^\ell \partial_j u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)} \\
&\qquad \le C\Big(\|L(\partial)\partial_j u\|^2_{\widetilde{H}^{s,s,\natural}_\gamma(\Omega)} + \sum_{\ell=1}^{s}\|\mathbf{r}^\ell L(\partial)\partial_j u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)} \\
&\qquad\qquad + \sum_{\ell=0}^{s}\|x_2^\ell B_1 \partial_j u_{|x_1=0}\|^2_{\widetilde{H}^{s,s-\ell,\natural}_\gamma(\partial\Omega_1)} + \sum_{\ell=0}^{s}\|x_1^\ell B_2 \partial_j u_{|x_2=0}\|^2_{\widetilde{H}^{s,s-\ell,\natural}_\gamma(\partial\Omega_2)}\Big) \\
&\qquad \le C\Big(\|L(\partial)u\|^2_{\widetilde{H}^{s+1,s,\natural}_\gamma(\Omega)} + \sum_{\ell=1}^{s}\|\mathbf{r}^\ell L(\partial)u\|^2_{\widetilde{H}^{s+1,s-\ell}_\gamma(\Omega)} \\
&\qquad\qquad + \sum_{\ell=0}^{s}\|x_2^\ell B_1 u_{|x_1=0}\|^2_{\widetilde{H}^{s+1,s-\ell,\natural}_\gamma(\partial\Omega_1)} + \sum_{\ell=0}^{s}\|x_1^\ell B_2 u_{|x_2=0}\|^2_{\widetilde{H}^{s+1,s-\ell,\natural}_\gamma(\partial\Omega_2)}\Big)
\end{aligned}$$

where we used the fact that $\big[\mathbf{r}^\ell L(\partial), \partial_j\big] = 0$ for $j \in \mathscr{T}$. So that we have

$$\begin{aligned}
\sum_{\ell=0}^{s}\|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s+1,s-\ell}_\gamma(\Omega)} \le C\Big(&\|L(\partial)u\|^2_{\widetilde{H}^{s+1,s,\natural}_\gamma(\Omega)} + \sum_{\ell=1}^{s}\|\mathbf{r}^\ell L(\partial)u\|^2_{\widetilde{H}^{s+1,s-\ell}_\gamma(\Omega)} \\
&+ \sum_{\ell=0}^{s}\|x_2^\ell B_1 u_{|x_1=0}\|^2_{\widetilde{H}^{s+1,s-\ell,\natural}_\gamma(\partial\Omega_1)} + \sum_{\ell=0}^{s}\|x_1^\ell B_2 u_{|x_2=0}\|^2_{\widetilde{H}^{s+1,s-\ell,\natural}_\gamma(\partial\Omega_2)}\Big).
\end{aligned} \tag{39}$$

To deal with the second term of the right hand side of (38) we will use the following lemma

**Lemma 3.2** *Let $s \in \mathbb{N}$ then for all $k \in [\![0, s+1]\!]$, for all $\ell \in [\![0, s+1-k]\!]$ we have*

$$\begin{aligned}
&\|(\mathbf{r}\partial_{\mathbf{r}})^{s+1-k-\ell}\partial_\theta^k \partial^\delta \mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)} \\
&\qquad \le C\Big(\|L(\partial)u\|^2_{\widetilde{H}^{s+1,s+1,\natural}_\gamma(\Omega)} + \sum_{p=1}^{s+1}\|\mathbf{r}^p L(\partial)u\|_{\widetilde{H}^{s+1,s+1-p}(\Omega)} \\
&\qquad\qquad + \sum_{p=0}^{s+1}\|x_2^p B_1 u_{|x_1=0}\|_{\widetilde{H}^{s+1,s+1-p,\natural}(\partial\Omega_1)} + \sum_{p=0}^{s+1}\|x_1^p B_2 u_{|x_2=0}\|_{\widetilde{H}^{s+1,s+1-p,\natural}(\partial\Omega_2)}\Big).
\end{aligned} \tag{40}$$

*For all $\ell \in [\![0, s+1]\!]$ we also have the boundary estimate*

$$\begin{aligned}
&\|(x_2\partial_2)^{s+1-\ell}\partial^\delta x_2^\ell u_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|(x_1\partial_1)^{s+1-\ell}\partial^\delta x_1^\ell u_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)} \\
&\qquad \le C\Big(\|L(\partial)u\|^2_{\widetilde{H}^{s+1,s+1,\natural}_\gamma(\Omega)} + \sum_{p=1}^{s+1}\|\mathbf{r}^p L(\partial)u\|_{\widetilde{H}^{s+1,s+1-p}(\Omega)} \\
&\qquad\qquad + \sum_{p=0}^{s+1}\|x_2^p B_1 u_{|x_1=0}\|_{\widetilde{H}^{s+1,s+1-p,\natural}(\partial\Omega_1)} + \sum_{p=0}^{s+1}\|x_1^p B_2 u_{|x_2=0}\|_{\widetilde{H}^{s+1,s+1-p,\natural}(\partial\Omega_2)}\Big)
\end{aligned} \tag{41}$$

With Lemma 3.2 in hand (36) at the step $s+1$ follows from (38). So that it concludes the proof of Proposition 3.1.

□

We now turn to the proof of Lemma 3.2.

**Proof of Lemma 3.2** To show (40) we proceed by finite iteration on $k$. The proof of (41) comes from the initialization.

For $k=0$ we have to show that for all $\ell \in [\![0, s+1]\!]$ and $\forall\, |\delta| \leq \ell$ we have

$$\begin{aligned}
&\|(\mathbf{r}\partial_{\mathbf{r}})^{s+1-\ell}\partial^{\delta}\mathbf{r}^{\ell}u\|^2_{L^2_\gamma(\Omega)} \qquad (42)\\
&\quad\leq C\Big(\|L(\partial)u\|^2_{\widetilde{H}^{s+1,s+1,\natural}_\gamma(\Omega)} + \sum_{p=1}^{s+1}\|\mathbf{r}^p L(\partial)u\|^2_{\widetilde{H}^{s+1,s+1-p}(\Omega)}\\
&\qquad + \sum_{p=0}^{s+1}\|x_2^p B_1 u_{|x_1=0}\|^2_{\widetilde{H}^{s+1,s+1-p,\natural}(\partial\Omega_1)} + \sum_{p=0}^{s+1}\|x_1^p B_2 u_{|x_2=0}\|^2_{\widetilde{H}^{s+1,s+1-p,\natural}(\partial\Omega_2)}\Big).
\end{aligned}$$

For $s, \ell, \delta$ fixed we have

$$\begin{aligned}
L(\partial)(\mathbf{r}\partial_{\mathbf{r}})^{s+1-\ell}\partial^{\delta}\mathbf{r}^{\ell}u &= \sum_{p=0}^{s+1-\ell}\binom{s+1-\ell}{p}(\mathbf{r}\partial_{\mathbf{r}})^p L(\partial)\partial^{\delta}\mathbf{r}^{\ell}u\\
&\quad - \sum_{p=0}^{s-\ell}\binom{s+1-\ell}{p}(\mathbf{r}\partial_{\mathbf{r}})^p \sum_{j\in\mathscr{T}} A_j\partial_j\partial^{\delta}u\\
&:= \mathscr{F}_{1,s,\ell} + \mathscr{F}_{2,s,\ell}
\end{aligned}$$

Moreover from the binomial formula we have the boundary values

$$\big[(\mathbf{r}\partial_{\mathbf{r}})^{s+1-\ell}\mathbf{r}^{\ell}u\big]_{|x_1=0} = \left[\sum_{p=0}^{s+1-\ell}\binom{s+1-\ell}{p}(x_1\partial_1)^p(x_2\partial_2)^{s+1-\ell-p}\mathbf{r}^{\ell}u\right]_{|x_1=0} = (x_2\partial_2)^{s+1-\ell}x_2^{\ell}u_{|x_1=0},$$

and

$$\big[(\mathbf{r}\partial_{\mathbf{r}})^{s+1-\ell}\mathbf{r}^{\ell}u\big]_{|x_2=0} = (x_1\partial_1)^{s+1-\ell}x_1^{\ell}u_{|x_2=0},$$

so that $(\mathbf{r}\partial_{\mathbf{r}})^{s+1-\ell}\partial^{\delta}\mathbf{r}^{\ell}u$ satisfies the boundary value problem

$$\begin{cases}
L(\partial)(\mathbf{r}\partial_{\mathbf{r}})^{s+1-\ell}\partial^{\delta}\mathbf{r}^{\ell}u = \mathscr{F}_{1,s,\ell} + \mathscr{F}_{2,s,\ell} & \text{in } \Omega,\\
B_1((\mathbf{r}\partial_{\mathbf{r}})^{s+1-\ell}\partial^{\delta}\mathbf{r}^{\ell}u)_{|x_1=0} = (x_2\partial_2)^{s+1-\ell}x_2^{\ell}\partial^{\delta}B_1 u_{|x_1=0} & \text{on } \partial\Omega_1,\\
B_2((\mathbf{r}\partial_{\mathbf{r}})^{s+1-\ell}\partial^{\delta}\mathbf{r}^{\ell}u)_{|x_2=0} = (x_1\partial_1)^{s+1-\ell}x_1^{\ell}\partial^{\delta}B_2 u_{|x_2=0} & \text{on } \partial\Omega_2,
\end{cases} \qquad (43)$$

so that from Assumption 2.1 we have

$$\begin{aligned}
&\gamma\|(\mathbf{r}\partial_{\mathbf{r}})^{s+1-\ell}\partial^{\delta}\mathbf{r}^{\ell}u\|^2_{L^2_\gamma(\Omega)} + \|(x_2\partial_2)^{s+1-\ell}\partial^{\delta}x_2^{\ell}u_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|(x_1\partial_1)^{s+1-\ell}\partial^{\delta}x_1^{\ell}u_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)} \qquad (44)\\
&\quad\leq C\Big(\frac{1}{\gamma}\|\mathscr{F}_{1,s,\ell}\|^2_{L^2_\gamma(\Omega)} + \frac{1}{\gamma}\|\mathscr{F}_{2,s,\ell}\|^2_{L^2_\gamma(\Omega)}\\
&\qquad + \|(x_2\partial_2)^{s+1-\ell}x_2^{\ell}\partial^{\delta}B_1 u_{|x_1=0}\|^2_{L^2_\gamma(\partial\Omega_1)} + \|(x_1\partial_1)^{s+1-\ell}x_1^{\ell}\partial^{\delta}B_2 u_{|x_2=0}\|^2_{L^2_\gamma(\partial\Omega_2)}\Big).
\end{aligned}$$

To conclude we then use the following bounds

$$\begin{aligned}
\|\mathscr{F}_{2,s,\ell}\|^2_{L^2_\gamma(\Omega)} &\leq \|C\mathbf{r}^{\ell}u\|^2_{\widetilde{H}^{s+1,s-\ell,\natural}_\gamma(\Omega)} \qquad (45)\\
&\leq C\Big(\|L(\partial)u\|^2_{\widetilde{H}^{s+1,s,\natural}_\gamma(\Omega)} + \sum_{\ell=1}^{s}\|\mathbf{r}^{\ell}L(\partial)u\|^2_{\widetilde{H}^{s+1,s-\ell}_\gamma(\Omega)}\\
&\quad + \sum_{\ell=0}^{s}\|x_2^{\ell}B_1 u_{|x_1=0}\|^2_{\widetilde{H}^{s+1,s-\ell,\natural}_\gamma(\partial\Omega_1)} + \sum_{\ell=0}^{s}\|x_1^{\ell}B_2 u_{|x_2=0}\|^2_{\widetilde{H}^{s+1,s-\ell,\natural}_\gamma(\partial\Omega_2)}\Big)
\end{aligned}$$

which holds from (39).

For dealing with the term $\mathscr{F}_{1,s,\ell}$ we first remark that for $\ell = 0$ we have

$$\|\mathscr{F}_{1,s,0}\|^2_{L^2_\gamma(\Omega)} \leq C\sum_{p=0}^{s+1} \|(\mathbf{r}\partial_{\mathbf{r}})^p \partial^\delta L(\partial)u\|^2_{L^2_\gamma(\Omega)} \leq C\|L(\partial)u\|^2_{H^{s+1,s+1,\natural}_\gamma(\Omega)}.$$

For $\ell \geq 1$ we decompose

$$\begin{aligned} L(\partial)\mathbf{r}^\ell u &= (\partial_1 \mathbf{r}^\ell)A_1 u + (\partial_2 \mathbf{r}^\ell)A_2 u + \mathbf{r}^\ell L(\partial)u \\ &= \ell\left[(\partial_1 r)A_1 + (\partial_2 r)A_2\right]\mathbf{r}^{\ell-1}u + \mathbf{r}^\ell L(\partial)u. \end{aligned}$$

So that

$$\begin{aligned} \|\mathscr{F}_{1,s,\ell}\|^2_{L^2_\gamma(\Omega)} \leq & \sum_{p=0}^{s+1-\ell} \|(\mathbf{r}\partial_{\mathbf{r}})^p \mathbf{r}^{\ell-1}\partial^\delta u\|^2_{L^2_\gamma(\Omega)} + \sum_{p=0}^{s+1-\ell} \|(\mathbf{r}\partial_{\mathbf{r}})^p \mathbf{r}^\ell \partial^\delta L(\partial)u\|^2_{L^2_\gamma(\Omega)} \\ \leq & \|\mathbf{r}^{\ell-1}u\|^2_{\widetilde{H}^{s+1,s-(\ell-1),\natural}_\gamma(\Omega)} + \|\mathbf{r}^\ell L(\partial)u\|^2_{\widetilde{H}^{s+1,s+1-\ell,\natural}_\gamma(\Omega)} \\ \leq & C\Big(\|L(\partial)u\|^2_{\widetilde{H}^{s+1,s+1,\natural}_\gamma(\Omega)} + \sum_{\ell=1}^{s+1} \|\mathbf{r}^\ell L(\partial)u\|^2_{\widetilde{H}^{s+1,s+1-\ell}_\gamma(\Omega)} \\ & + \sum_{\ell=0}^{s} \|x_2^\ell B_1 u_{|x_1=0}\|^2_{\widetilde{H}^{s+1,s-\ell,\natural}_\gamma(\partial\Omega_1)} + \sum_{\ell=0}^{s} \|x_1^\ell B_2 u_{|x_2=0}\|^2_{\widetilde{H}^{s+1,s-\ell,\natural}_\gamma(\partial\Omega_2)}\Big) \end{aligned}$$

where we used (39) for $\ell' = \ell - 1$. Consequently for all $\ell \in [\![0, s+1]\!]$ we have

$$\begin{aligned} \|\mathscr{F}_{1,s,\ell}\|^2_{L^2_\gamma(\Omega)} \leq C\Big(&\|L(\partial)u\|^2_{\widetilde{H}^{s+1,s+1,\natural}_\gamma(\Omega)} + \sum_{\ell=1}^{s+1} \|\mathbf{r}^\ell L(\partial)u\|^2_{\widetilde{H}^{s+1,s+1-\ell}_\gamma(\Omega)} \\ &+ \sum_{\ell=0}^{s} \|x_2^\ell B_1 u_{|x_1=0}\|^2_{\widetilde{H}^{s+1,s-\ell,\natural}_\gamma(\partial\Omega_1)} + \sum_{\ell=0}^{s} \|x_1^\ell B_2 u_{|x_2=0}\|^2_{\widetilde{H}^{s+1,s-\ell,\natural}_\gamma(\partial\Omega_2)}\Big). \end{aligned} \tag{46}$$

As a consequence, (42) (the initalization of (40) at $k = 0$) follows from (44)-(45) and (46). This also gives (41). This concludes the initialization part of Lemma 3.2.

We assume that for fixed $k$, the estimate (40) holds for all $\ell \in [\![0, s+1-k]\!]$. We want to estimate the $\|(\mathbf{r}\partial_{\mathbf{r}})^{s-k-\ell}\partial_\theta^{k+1}\partial^\delta \mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)}$ for $\ell \in [\![0, s]\!]$. From the equation in polar coordinates (see (33)) we express

$$\partial_\theta \widetilde{u} = \mathscr{A}^{-1}(\theta)\left[\widetilde{\mathbf{r}L(\partial)u} - \mathscr{M}(\theta)\mathbf{r}\partial_{\mathbf{r}}\widetilde{u} - \sum_{j\in\mathscr{T}} A_j \partial_j\partial_j\partial_j \widetilde{u}\right],$$

so that

$$\begin{aligned} \|(\mathbf{r}\partial_{\mathbf{r}})^{s-k-\ell}\partial_\theta^{k+1}\partial^\delta \mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)} \leq & \underbrace{\|(\mathbf{r}\partial_{\mathbf{r}})^{s-k-\ell}\partial^\delta\partial_\theta^k \mathscr{A}^{-1}(\theta)\mathbf{r}\widetilde{L(\partial)\mathbf{r}^\ell u}\|^2_{L^2_\gamma(\Omega)}}_{:=\mathscr{I}_1} \\ & + \underbrace{\|(\mathbf{r}\partial_{\mathbf{r}})^{s+1-k-\ell}\partial^\delta\partial_\theta^k \mathscr{A}^{-1}(\theta)\mathscr{M}(\theta)\mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)}}_{:=\mathscr{I}_2} \\ & + \underbrace{\sum_{j\in\mathscr{T}} \|(\mathbf{r}\partial_{\mathbf{r}})^{s+1-k-\ell}\partial^\delta\partial_\theta^k \mathscr{A}^{-1}(\theta)A_j\mathbf{r}^\ell\partial_j u\|^2_{L^2_\gamma(\Omega)}}_{:=\mathscr{I}_3}, \end{aligned} \tag{47}$$

and we estimate each of the $\mathscr{I}_j$ separately.

For $\mathscr{I}_1$ we first decompose

$$rL(\partial)\mathbf{r}^\ell u = [(\partial_1 r)A_1 + (\partial_1 r)A_2]\,\mathbf{r}^\ell u + \mathbf{r}^{\ell+1}L(\partial)u,$$

and then we apply Leibniz formula on the products $\mathscr{A}^{-1}\times u$ and $\mathscr{A}^{-1}\times L(\partial)u$ so that

$$\partial_\theta^k \mathscr{A}^{-1}(\theta) rL(\partial)\mathbf{r}^\ell u = \sum_{p=0}^{k}\binom{k}{p}\mathscr{A}^{-(k-p)}(\theta)\,[(\partial_1 r)A_1 + (\partial_1 r)A_2]\,\mathbf{r}^\ell \partial_\theta^p u + \sum_{p=0}^{k}\binom{k}{p}\mathscr{A}^{-(k-p)}(\theta)\partial_\theta^p \mathbf{r}^{\ell+1}L(\partial)u.$$

It follows using the uniform bound on $\mathscr{A}^{-(p)}$ (see Lemma 3.1) that

$$\begin{aligned}\mathscr{I}_1 \leq & \Big(\sum_{p=0}^{k}\|(\mathbf{r}\partial_{\mathbf{r}})^{s-k-\ell}\partial_\theta^p\partial^\delta \mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)} + \sum_{p=0}^{k}\|(\mathbf{r}\partial_{\mathbf{r}})^{s+1-k-(\ell+1)}\partial_\theta^p\partial^\delta \mathbf{r}^{\ell+1}L(\partial)u\|^2\Big) \\ \leq & C\left(\|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)} + \|\mathbf{r}^\ell L(\partial)u\|^2_{\widetilde{H}^{s+1,s+1-\ell}_\gamma(\Omega)}\right)\end{aligned} \tag{48}$$

and we use (37) to estimate the first term in the right hand side.

For $\mathscr{I}_2$ we use Leibniz formula on the product $(\mathscr{A}^{-1}\mathscr{M}(\theta))\times u$ and we bound each of the derivatives of $\mathscr{A}^{-1}\mathscr{M}$ to obtain

$$\begin{aligned}\mathscr{I}_2 \leq & C\sum_{p=0}^{k}\|(\mathbf{r}\partial_{\mathbf{r}})^{s+1-k-\ell}\partial^\delta\partial_\theta^p\mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)} \\ \leq & C\Big(\sum_{p=0}^{k-1}\|(\mathbf{r}\partial_{\mathbf{r}})^{s+1-k-\ell}\partial^\delta\partial_\theta^p\mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)} + \|(\mathbf{r}\partial_{\mathbf{r}})^{s+1-k-\ell}\partial^\delta\partial_\theta^k\mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)}\Big), \\ \leq & C\Big(\|\mathbf{r}^\ell u\|_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)} + \|L(\partial)u\|^2_{\widetilde{H}^{s+1,s+1,\natural}_\gamma(\Omega)} + \sum_{p=1}^{s+1}\|\mathbf{r}^p L(\partial)u\|^2_{\widetilde{H}^{s+1,s+1-p}(\Omega)} \\ & + \sum_{p=0}^{s+1}\|x_2^p B_1 u_{|x_1=0}\|^2_{\widetilde{H}^{s+1,s+1-p,\natural}(\partial\Omega_1)} + \sum_{p=0}^{s+1}\|x_1^p B_2 u_{|x_2=0}\|^2_{\widetilde{H}^{s+1,s+1-p,\natural}(\partial\Omega_2)}\Big)\end{aligned} \tag{49}$$

because (40) holds at the order $k$. And once again we conclude by estimating the first term from (37).

Finally to deal with the term $\mathscr{I}_3$ we use once again Leibniz formula and we majorate the derivatives of $\mathscr{A}^{-1}$ to recover that

$$\mathscr{I}_3 \leq C\sum_{p=0}^{k}\|(r\partial)^{s-k+\ell}\partial^\delta\partial_j\partial_\theta^p\mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)} \leq C\|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s+1,s-\ell}_\gamma(\Omega)}. \tag{50}$$

This time we used (39) to conclude.

We use (48), (49) and (50) in (47). This shows that the estimate (40) holds at the order $k+1$. This concludes the iterative process in the proof of Lemma 3.2.

□

This concludes the proof of the first two steps in the proof of the *a priori* energy estimate. It remains to show the first one it is made in Proposition 3.2 and the fourth one in Proposition 3.3

**Proposition 3.2** *For all $s\in\mathbb{N}^*$, there exists $C:=C_s>0$ such that*

$$\|L(\partial)u\|_{\widetilde{H}^{s,s,\natural}_\gamma(\Omega)} + \sum_{l=1}^{s}\|\mathbf{r}^l L(\partial)u\|^2_{\widetilde{H}^{s,s-l}_\gamma(\Omega)} \leq C\|L(\partial)u\|_{\mathscr{H}^s_\gamma(\Omega)}. \tag{51}$$

**Proof of Proposition 3.2** We evaluate each term in the left hand side of (51).

- For the first term, by definition we have

$$\|L(\partial)u\|^2_{\widetilde{H}^{s,s,\natural}_\gamma(\Omega)} = \sum_{m=0}^{s} \sum_{|\delta|\leq s-m} \|(\mathbf{r}\partial_{\mathbf{r}})^m \partial^\delta L(\partial)u\|^2_{L^2_\gamma(\Omega)}.$$

The operator $(\mathbf{r}\partial_{\mathbf{r}})^m$ can be made precise in terms of $\partial_1$ and $\partial_2$ More precisely we have the following formula; for a function $f$ depending on $x_1$ and $x_2$

$$(\mathbf{r}\partial_{\mathbf{r}})^m f = \sum_{p=0}^{m} \sum_{\alpha=1}^{p} \sum_{\beta=1}^{m-p} \binom{m}{p} \lambda^p_\alpha \lambda^{m-p}_\beta x_1^\alpha x_2^\beta \partial_1^\alpha \partial_2^\beta f, \tag{52}$$

where the value of the scalars $\lambda^{\cdot}_{\cdot}$ is of little interest for our discussion (we refer to (90) for a precise definition). So that

$$\begin{aligned}
\|L(\partial)u\|^2_{\widetilde{H}^{s,s,\natural}_\gamma(\Omega)} \leq & C_s \sum_{m=0}^{s} \sum_{|\delta|\leq s-m} \sum_{\alpha=1}^{p} \sum_{\beta=1}^{m-p} \|x_1^\alpha x_2^\beta \partial_1^\alpha \partial_2^\beta \partial^\delta L(\partial)u\|^2_{L^2_\gamma(\Omega)}. \\
\leq & C_s \sum_{m=0}^{s} \sum_{|\delta|\leq s-m} \sum_{\alpha+\beta\leq m} \sum_{k=0}^{\alpha+\beta} \|x_1^k x_2^{\alpha+\beta-k} \partial_1^\alpha \partial_2^\beta \partial^\delta L(\partial)u\|^2_{L^2_\gamma(\Omega)} \\
\leq & C_s \|L(\partial)u\|_{\mathscr{H}^s_\gamma(\Omega)}
\end{aligned}$$

- For fixed $l \in [\![1, s]\!]$ we have

$$\|\mathbf{r}^l L(\partial)u\|^2_{\widetilde{H}^{s,s-l}_\gamma(\Omega)} = \sum_{n+m\leq s-l} \sum_{|\delta|\leq s-(n+m)} \|(\mathbf{r}\partial_{\mathbf{r}})^m \partial_\theta^s \partial^\delta \mathbf{r}^l L(\partial)u\|^2_{L^2_\gamma(\Omega)}. \tag{53}$$

It is not so clear if we have a nice formula for $\partial^n_\theta f$ but we have the following lemma

**Lemma 3.3** *For all $n \in \mathbb{N}$ there exist $(P^n_{\alpha,\beta})_{0\leq\alpha,\beta\leq n}$ which is sum of monomial of order $\alpha+\beta$ such that we have the equality*

$$\partial^n_\theta f = \sum_{\alpha=0}^{n+1} \sum_{\beta=0}^{n-\alpha} P^n_{\alpha,\beta} \partial_1^\alpha \partial_2^\beta f. \tag{54}$$

**Proof of Lemma 3.3** We show this lemma by iteration. Clearly (54) holds for $n = 0$. Assume that (54) holds at some order $n$. Then

$$\begin{aligned}
\partial^{n+1}_\theta f = & \sum_{\alpha=0}^{n} \sum_{\beta=0}^{n-\alpha} \left(-x_2\partial_1 P_{\alpha,\beta} + x_1 \partial_2 P^n_{\alpha,\beta}\right) \partial_1^\alpha \partial_2^\beta f \\
& - \sum_{\alpha=1}^{n+1} \sum_{\beta=0}^{n+1-\alpha} x_2 P^n_{\alpha-1,\beta} \partial_1^\alpha \partial_2^\beta f + \sum_{\alpha=0}^{n} \sum_{\beta=1}^{n+1-\alpha} x_1 P^n_{\alpha,\beta-1} \partial_1^\alpha \partial_2^\beta f.
\end{aligned}$$

Define

$$Q^n_{\alpha,\beta} := \begin{cases} -x_2\partial_1 P^n_{\alpha,\beta} + x_1 \partial_2 P^n_{\alpha,\beta} & \text{if } 0 \leq \alpha \leq n\,,\ 0 \leq \beta \leq n-\alpha, \\ 0 & \text{if } \alpha = n+1\,,\ \beta = 0, \\ 0 & \text{if } \alpha = 0\,,\ \beta = n+1, \end{cases}$$

$$R^n_{\alpha,\beta} := \begin{cases} -x_2 P^n_{\alpha-1,\beta} & \text{if } 1 \leq \alpha \leq n+1\,,\ 0 \leq \beta \leq n+1-\alpha, \\ 0 & \text{if } \alpha = 0\,,\ 0 \leq \beta \leq n+1, \end{cases}$$

and

$$S^n_{\alpha,\beta} := \begin{cases} x_1 P^n_{\alpha,\beta-1} & \text{if } 0 \le \alpha \le n\,,\ 1 \le \beta \le n+1-\alpha, \\ 0 & \text{if } 0 \le \alpha \le n+1\,,\ \beta = 0, \end{cases}$$

which are all sums of monomial so that we have

$$\partial^{n+1} f = \sum_{\alpha=0}^{n} \sum_{\beta=0}^{n+1-\alpha} (Q^n_{\alpha,\beta} + R^n_{\alpha,\beta} + S^n_{\alpha,\beta}) \partial_1^\alpha \partial_2^\beta.$$

□

We now complete the bound for the right hand side of (53) by making the right hand side more precise. We have from Leibniz formula

$$\begin{aligned} &\|(\mathbf{r}\partial_{\mathbf{r}})^m \partial_\theta^n \partial^\delta \mathbf{r}^l L(\partial) u\|^2_{L^2_\gamma(\Omega)} \\ &\quad = \| \sum_{p=0}^{m} \sum_{\alpha'=0}^{p} \sum_{\beta'=0}^{m-p} \binom{m}{p} \lambda^p_{\alpha'} \lambda^{m-p}_{\beta'} x_1^{\alpha'} x_2^{\beta'} \partial_1^{\alpha'} \partial_2^{\beta'} \sum_{\alpha=0}^{n} \sum_{\beta=0}^{n-\alpha} P^n_{\alpha,\beta} \partial_1^\alpha \partial_2^\beta \partial^\delta \mathbf{r}^l L(\partial) u\|^2_{L^2_\gamma(\Omega)} \\ &\quad \le C_m \sum_{p=0}^{m} \sum_{\alpha'=0}^{p} \sum_{\beta'=0}^{m-p} \sum_{\alpha=0}^{n} \sum_{\beta=0}^{n-\alpha} \sum_{\gamma'=0}^{\alpha'} \sum_{\eta'=0}^{\beta'} \|x_1^{\alpha'} x_2^{\beta'} (\partial_1^{\alpha'-\gamma'} \partial_2^{\beta'-\eta'} P^n_{\alpha,\beta}) \partial_1^{\alpha+\gamma'} \partial_2^{\beta+\eta'} \partial^\delta \mathbf{r}^l L(\partial) u\|^2_{L^2_\gamma(\Omega)}, \end{aligned} \tag{55}$$

and to conclude we shall justify that in the right hand side of (55) the order of derivation with respect to $\partial_1$ and $\partial_2$ is bounded by $m+n$ and that the polynomial in front of this factor is sum of monomial of order $\alpha+\gamma'+\beta+\eta'$. Firstly because of the set of indices in the sums we have

$$\alpha + \gamma' \beta + \mu' \le \alpha + \alpha' + n - \alpha + \beta' \le n + m,$$

so that the order of derivatives with respect to $\partial_1$ and $\partial_2$ are bounded by $n+m$ like in the definition of $\mathscr{H}^s_\gamma$.

Secondly we shall justify that the polynomial in factor of the derivation operator has the good degree with respect to $x_1$, $x_2$. If $(\partial_1^{\alpha'-\gamma'} \partial_2^{\beta'-\eta'} P^n_{\alpha,\beta})$ vanishes then the associated norm vanishes and it can be bounded by any term appearing in the norm of $\mathscr{H}^s_\gamma(\Omega)$ to fix the ideas we use the trivial bound

$$0 = \|x_1^{\alpha'} x_2^{\beta'} (\partial_1^{\alpha'-\gamma'} \partial_2^{\beta'-\mu'} P^n_{\alpha,\beta}) \partial_1^{\alpha+\gamma'} \partial_2^{\beta+\eta'} \partial^\delta \mathbf{r}^l L(\partial) u\|^2_{L^2_\gamma(\Omega)} \le \|x_1^{\alpha+\gamma'} x_2^{\beta+\eta'} \partial_1^{\alpha+\gamma'} \partial_2^{\beta+\eta'} \partial^\delta \mathbf{r}^l L(\partial) u\|^2_{L^2_\gamma(\Omega)}.$$

If $(\partial_1^{\alpha'-\gamma'} \partial_2^{\beta'-\eta'} P^n_{\alpha,\beta}) \neq 0$ it is thus a sum of monomials of degree $\alpha+\beta-\alpha'-\beta'+\gamma'+\eta'$ so that the term $(\partial_1^{\alpha'-\gamma'} \partial_2^{\beta'-\mu'} P^n_{\alpha,\beta})$ is of degree $\alpha+\gamma'+\beta+\eta'$ like the derivative. Using the crude of majoration of the cardinal to avoid the extra sums we thus have justified that

$$\|\mathbf{r}^l L(\partial) u\|_{\widetilde{H}^{s,s-l}_\gamma(\Omega)} \le C_s \|L(\partial) u\|_{\mathscr{H}^s_\gamma(\Omega)},$$

this completes the proof of Proposition 3.2.

□

**Proposition 3.3** *For all $s \in \mathbb{N}^*$, there exists $C := C_s > 0$ such that*

$$I_s := \sum_{\ell=0}^{s} \sum_{\alpha+\beta \le s-\ell, \alpha+\beta+|\delta| \le s} \sum_{k=0}^{\alpha+\beta} \|x_1^k x_2^{\alpha+\beta-k} \partial_1^\alpha \partial_2^\beta \partial^\delta \mathbf{r}^\ell u\| \le \sum_{\ell=0}^{s} \|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)}. \tag{56}$$

**Proof of Proposition 3.3** We proceed by iteration on $s$. The initialisation (56) for $s=1$ has been established in Paragraph 3.1 (see equation (35)).

We then assume that $I_s \leq \sum_{\ell=0}^{s} \|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)}$ for some $s \geq 1$ and we show that the same holds for $I_{s+1}$. We decompose

$$\begin{aligned} I_{s+1} =& \underbrace{\sum_{\ell=0}^{s} \sum_{|\delta|\leq \ell} \sum_{\alpha+\beta=s+1-\ell} \sum_{k=0}^{\alpha+\beta} \|x_1^k x_2^{\alpha+\beta-k} \partial_1^\alpha \partial_2^\beta \partial^\delta \mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)}}_{:=J_{s+1}} \\ &+ \underbrace{\sum_{\ell=0}^{s} \sum_{\alpha+\beta\leq s-\ell, \alpha+\beta+|\delta|=s+1-\ell} \sum_{k=0}^{\alpha+\beta} \|x_1^k x_2^{\alpha+\beta-k} \partial_1^\alpha \partial_2^\beta \partial^\delta \mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)}}_{:=K_{s+1}} \\ &+ I_s + \sum_{|\delta|\leq s+1} \|\mathbf{r}^{s+1} \partial^\delta u\|^2_{L^2_\gamma(\Omega)}. \end{aligned}$$

So that

$$I_{s+1} \leq J_{s+1} + K_{s+1} + \sum_{\ell=0}^{s} \|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)} + \|\mathbf{r}^{s+1} u\|^2_{\widetilde{H}^{s+1,0}_\gamma(\Omega)}. \tag{57}$$

Relabelling the sums in $J_{s+1}$ and $K_{s+1}$ we can thus write

$$\begin{aligned} J_{s+1} =& \sum_{\ell=0}^{s} \sum_{|\delta|\leq \ell} \sum_{p=0}^{s+1-\ell} \sum_{k=0}^{s+1-\ell} \|x_1^k x_2^{s+1-\ell-k} \partial_1^{s+1-\ell-p} \partial_2^p \partial^\delta \mathbf{r}^\ell u\|^2_{L^2_\gamma(\Omega)} \\ K_{s+1} =& \sum_{\ell=0}^{s} \sum_{\alpha+\beta\leq s-\ell, \alpha+\beta+|\delta|=s-\ell} \sum_{j\in\mathscr{T}} \sum_{k=0}^{\alpha+\beta} \|x_1^k x_2^{\alpha+\beta-k} \partial_1^\alpha \partial_2^\beta \partial^\delta \mathbf{r}^\ell \partial_j u\|^2_{L^2_\gamma(\Omega)} \end{aligned}$$

By assumption we thus have

$$K_{s+1} \leq \sum_{\ell=0}^{s} \sum_{j\in\mathscr{T}} \|\mathbf{r}^\ell \partial_j u\|^2_{\widetilde{H}^{s,s-\ell}_\gamma(\Omega)} \leq \sum_{\ell=0}^{s} \|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s+1,s-\ell}_\gamma(\Omega)} \tag{58}$$

Let $\delta$ and $\ell$ be fixed we define $v = v_{\ell,\delta} := \mathbf{r}^\ell \partial^\delta u$ the aim of the following is to express the derivatives $\partial_1^{s+1-\ell-p}\partial_2^p v$ appearing in $J_{s+1}$ as sums of the derivatives $(\mathbf{r}\partial_{\mathbf{r}})^{s+1-\ell-k}\partial_\theta^k v$ for $k \in [\![0, s+1-\ell]\!]$.

Let $N \geq 1$ be fixed, $k \in [\![0, N-1]\!]$ and let $v = v(x_1, x_2)$ be a given regular function, we want to isolate the terms of higher order of differentiation in terms of $\partial_1$ and $\partial_2$ in $(\mathbf{r}\partial_{\mathbf{r}})^{N-1-k}\partial_\theta^k v$. We thus decompose

$$(\mathbf{r}\partial_{\mathbf{r}})^{N-1-k}\partial_\theta^k v = \sum_{p=0}^{N-1} \alpha_p^{N,k} \partial_1^{N-1-p}\partial_2^p + p_k^N := L_k^N + p_k^N, \tag{59}$$

where for all $k \in [\![0, N-1]\!]$, $p \in [\![0, N-1]\!]$ $\alpha_p^{N,k}$ is a sum of monomials of degree $N$ and where for all $k \in [\![0, N-1]\!]$, $p_k^N$ reads as a sum of monomial of degree $\alpha+\beta$ times $\partial_1^\alpha \partial_2^\beta v$ with $\alpha+\beta < N-1$ that is

$$p_k^N := \sum_{\alpha+\beta\leq N-2} \sum_{k=0}^{\alpha+\beta} \psi_{\alpha,\beta,k} x_1^k x_2^{\alpha+\beta-k} \partial_1^\alpha \partial_2^\beta v, \tag{60}$$

where $\psi_{\alpha,\beta,k} \in \mathbb{R}$.

Indeed let $\rho$ be a monomial of degree $N$ with respect to $x_1$ and $x_2$ we thus have

$$(\mathbf{r}\partial_{\mathbf{r}})(\rho\partial_1^{N-p}\partial_2^p v) = \underbrace{x_1\rho}_{d^\circ = N+1} \partial_1^{N+1-p}\partial_2^p v + \underbrace{x_2\rho}_{d^\circ=N+1} \partial_1^{N-p}\partial_2^{p+1} v + \underbrace{[x_1(\partial_1\rho) + x_2(\partial_2\rho)]}_{d^\circ = N} \partial_1^{N-p}\partial_2^p v$$

and the same holds for the operator $\partial_\theta$.

We consider the system of equations

$$\mathscr{S}^N := \begin{cases} L_0^N = (\mathbf{r}\partial_{\mathbf{r}})^{N-1}v - p_0^N := \ell_1^N, \\ L_1^N = (\mathbf{r}\partial_{\mathbf{r}})^{N-1}v - p_1^N := \ell_2^N, \\ \vdots \\ L_{N-2}^N = \mathbf{r}\partial_{\mathbf{r}}\partial_\theta^{N-2}v - p_{N-2}^N := \ell_{N-2}^N, \\ L_{N-1}^N = \partial_\theta^{N-1}v - p_{N-1}^N := \ell_{N-1}^N, \end{cases} \Leftrightarrow A^N U^N = L^N \tag{61}$$

with $A_{i,j}^N = \alpha_j^{N,i}$, $U^N := \left[\partial_1^{N-1}v, \partial_1^{N-2}\partial_2 v, ..., \partial_2^{N-1}v\right]^T \in \mathbb{R}^N$ and $L^N = \left[\ell_0^N, \ell_1^N, ..., \ell_{N-1}^N\right]^T \in \mathbb{R}^N$.

The following lemma describes the solution to (61)

**Lemma 3.4** *For all $N \geq 2$ and $k \in [\![0, N-1]\!]$ there exist $\psi_0^{N,k}, ..., \psi_{N-1}^{N,k}$ functions, $\mathscr{C}^\infty$ with respect to $\theta$, such that the solution $U^N$ of $A^N U^N = L^N$ satisfies*

$$\forall k \in [\![0, N-1]\!],\ \mathbf{r}^{N-1}U_k^N = \sum_{p=0}^{N-1} \psi_p^{N,k}(\theta)\ell_p^N,$$

*where we stress that the $\psi_p^{N,k}$ do not depend on $r$ and are bounded with respect to $\theta$.*

**Proof of Lemma 3.4** We proceed by iteration on $N$. The initialisation has been done in Paragraph 3.1 indeed (34) gives

$$\mathbf{r}U_0^2 = \cos\theta\ell_0^2 - \sin\theta\ell_1^2 \text{ and } \mathbf{r}U_1^2 = \sin\theta\ell_0^2 + \cos\theta\ell_1^2,$$

so that the $\psi_\cdot^{0,\cdot}$ are readable.

We now assume that the solution of $A^N U^N = L^N$ satisfies $\mathbf{r}^{N-1}U_k^N = \sum_{p=0}^{N-1} \psi_p^{N,k}(\theta)\ell_p^N$ for some $\lambda_p^{N,k}$ and we want to show that the same holds for the solution of $\mathscr{S}^{N+1}$.

The aim of the following is to show that we can easily solve the system $\mathscr{S}^{N+1}$ in terms of the solution of $\mathscr{S}^N$. Indeed we decompose the terms $(\mathbf{r}\partial_{\mathbf{r}})^{N-k}\partial_\theta^k$ like in (59). Because $(\mathbf{r}\partial_{\mathbf{r}})^N = (x_1\partial_1 + x_2\partial_2)(\mathbf{r}\partial_{\mathbf{r}})^{N-1}$ and $\partial_\theta^N = (-x_2\partial_1 + x_1\partial_2)\partial_\theta^{N-1}$ we have

$$\alpha_p^{N+1,0} := \begin{cases} x_1\alpha_0^{N,0} & p = 0 \\ x_1\alpha_p^{N,0} + x_2\alpha_{p-1}^{N,0} & p \in [\![1, N-1]\!] \\ x_2\alpha_N^{N,0} & p = N \end{cases} \text{ and } \alpha_p^{N+1,N} := \begin{cases} -x_2\alpha_0^{N,N-1} & p = 0 \\ -x_2\alpha_p^{N,N-1} + x_1\alpha_{p-1}^{N,N-1} & p \in [\![1, N-1]\!] \\ x_1\alpha_s^{N,N-1} & p = N \end{cases}$$

and for $k \in [\![1, N-1]\!]$ we have

$$\alpha_p^{N+1,k} := \begin{cases} x_1\alpha_0^{N,k} & p = 0 \\ x_1\alpha_p^{N,k} + x_2\alpha_{p-1}^{N,k} & p \in [\![1, N-1]\!] \\ x_2\alpha_N^{N,k} & p = N \end{cases} \text{ or } \alpha_p^{N+1,k} := \begin{cases} -x_2\alpha_0^{N,k-1} & p = 0 \\ -x_2\alpha_p^{N,k-1} + x_1\alpha_{p-1}^{N,k-1} & p \in [\![1, N-1]\!] \\ x_1\alpha_N^{N,k-1} & p = N \end{cases}$$

depending on that we write $(\mathbf{r}\partial_{\mathbf{r}})^{N-k}\partial_\theta^k = (\mathbf{r}\partial_{\mathbf{r}})\left((\mathbf{r}\partial_{\mathbf{r}})^{N-1-k}\partial_\theta^k\right)$ that corresponds to $L_k^N$ or that we write $(\mathbf{r}\partial_{\mathbf{r}})^{N-k}\partial_\theta^k = \partial_\theta\left((\mathbf{r}\partial_{\mathbf{r}})^{N-1-(k-1)}\partial_\theta^{k-1}\right)$ corresponding to $L_{k-1}^N$. Consequently for $k \in [\![0, N-1]\!]$ we thus have

$$\begin{aligned} x_1 L_{k+1}^{N+1} + x_2 L_k^{N+1} =& -x_1x_2\alpha_0^{N,k}\partial_1^N v + \sum_{p=1}^{N-1}(-x_1x_2\alpha_p^{N,k} + x_1^2\alpha_{p-1}^{N,k})\partial_1^{N-p}\partial_2^p v + x_1^2\alpha_N^{N,k}\partial_2^N v \\ &+ x_1x_2\alpha_0^{N,k}\partial_1^N v + \sum_{p=1}^{N-1}(x_1x_2\alpha_p^{N,k} + x_2^2\alpha_{p-1}^{N,k})\partial_1^{N-p}\partial_2^p v + x_2^2\alpha_N^{N,k}\partial_2^N v \\ =& \mathbf{r}^2\sum_{p=0}^{N}\alpha_p^{N,k}\partial_1^{N-p}\partial_2^p\partial_2 v. \end{aligned}$$

So that by substituting $L_k^{N+1}$ by $x_1 L_{k+1}^{N+1} + x_2 L_k^{N+1}$ we obtain

$$A^{N+1}U^{N+1} = L^{N+1} \Leftrightarrow \begin{cases} L_0^{N+1} = \ell_0^{N+1}, \\ A^N \overline{U}^N = \overline{L}^N, \end{cases}$$

where we defined $\overline{U}^N = \mathbf{r}^2 \left[\partial_1^{N-1}\partial_2 v, \partial_1^{N-1}\partial_2^2 v, ..., \partial_2^N v\right]^T = \mathbf{r}^2 \left[U_1^{N+1}, ..., U_N^{N+1}\right]^T$ and $\overline{L}_k^N := \left[\overline{\ell}_0^N, ..., \overline{\ell}_{N-1}^N\right]^T$ where $\overline{\ell}_k^N := x_1 \ell_{k+1}^{N+1} + x_2 \ell_k^{N+1}$.

So that by assumption we have that for all $k \in [\![1, N]\!]$

$$\mathbf{r}^{N+1} U_k^{N+1} = \sum_{p=0}^{N-1} \psi_p^{N,k}(\theta) \overline{\ell}_p = \sum_{p=0}^{N-1} \psi_p^{N,k}(\theta)(x_1 \ell_{p+1}^{N+1} + x_2 \ell_p^{N+1}),$$

so that writing $x_1 = \mathbf{r}\cos\theta$ and $x_2 = \mathbf{r}\sin\theta$ and simplifying by $\mathbf{r}$ gives

$$\mathbf{r}^N U_k^{N+1} = \sum_{p=0}^{N} \psi_p^{N+1,k}(\theta) \ell_p^{N+1},$$

where we defined

$$\psi_p^{N+1,k} := \begin{cases} \sin\theta \psi_0^{N,k} & p = 0, \\ \sin\theta \psi_p^{N,k} + \cos\theta \psi_{p-1}^{N,k} & p \in [\![1, N-1]\!], \\ \cos\theta \psi_{N-1}^{N,k} & p = N. \end{cases}$$

To recover the missing coordinate $U_0^{N+1}$ we substitute for $k \in [\![0, N-1]\!]$, $L_k^{N+1}$ by $x_1 L_k^{N+1} - x_2 L_k^{N+1}$ and we obtain the equivalence

$$\mathscr{S}^{N+1} \Leftrightarrow \begin{cases} A^N \underline{U}^N = \underline{L}^N, \\ L_N^{N+1} = \ell_N^{N+1}, \end{cases},$$

where $\underline{U}^N := \mathbf{r}^2 \left[U_0^N, U_1^N, ..., U_{N-1}^N\right]$ and $\underline{L} = \left[x_1 \ell_0^N - x_2 \ell_1^N, ..., x_1 \ell_{N-1}^N - x_2 \ell_N^N\right]^T$ and we conclude exactly like in the previous case.

□

We apply the result of Lemma 3.4 with $N = s + 2 - \ell \geq 2$ and to the function $v := v_{\delta,\ell} = \mathbf{r}^\ell \partial^\delta u$. We thus can write $A^N U^N = L^N$ with $U^{s+2-\ell} = \left[\partial_1^{s+1-\ell} v, ..., \partial_2^{s+1-\ell} v\right]^T \in \mathbb{R}^{s+2-\ell}$ and where the datum $L^{s+2-\ell} = \left[\ell_0^{s+2-\ell}, ..., \ell_{s+1-\ell}^{s+2-\ell}\right]^T \in \mathbb{R}^{s+2+\ell}$ where we recall that by definition $\ell_q^N := (\mathbf{r}\partial_{\mathbf{r}})^{N-1-q} \partial_\theta^q v - p_q^N$ where .

Consequently we have that for all $p \in [\![0, s+1-\ell]\!]$

$$\mathbf{r}^{s+1-\ell} \partial_1^{s+1-\ell-p} \partial_2^p \partial^\delta \mathbf{r}^\ell u = \sum_{q=0}^{s+1-\ell} \psi_q^{s+1-\ell,p}(\theta) \ell_q^{s+2-\ell},$$

from which we deduce that for all $k \in [\![0, s+1-\ell]\!]$

$$x_1^k x_2^{s+1-k} \partial_1^{s+1-\ell-p} \partial_2^p \partial^\delta \mathbf{r}^\ell u = \sum_{q=0}^{s+1-\ell} \widetilde{\psi}_q^{s+1-\ell,p}(\theta) \ell_q^{s+2-\ell} \tag{62}$$

where $\widetilde{\psi}_q^{s+1-\ell,p}(\theta) := \cos^k\theta \sin^{s+1-k}\theta \psi_q^{s+1-\ell,p}(\theta)$ is bounded with respect to $\theta$.

It follows, using in particular (60) that

$$\begin{aligned} &\|x_1^k x_2^{s+1-k} \partial_1^{s+1-\ell-p} \partial_2^p \partial^\delta \mathbf{r}^\ell u\|_{L^2_\gamma(\Omega)}^2 \\ &\qquad \leq \sum_{q=0}^{s+1-\ell} \|(\mathbf{r}\partial_{\mathbf{r}})^{s+1-\ell-q} \partial_\theta^q \partial^\delta \mathbf{r}^\ell u\|_{L^2_\gamma(\Omega)}^2 + \sum_{q=0}^{s+1-\ell} \|p_q^{s+2-\ell}\|_{L^2_\gamma(\Omega)}^2 \\ &\qquad \leq \|\partial^\delta \mathbf{r}^\ell u\|_{\widetilde{H}^{s+1-\ell,s+1-\ell}_\gamma(\Omega)}^2 + C_{s,\ell} \sum_{\alpha+\beta \leq s-\ell} \sum_{k'=0}^{\alpha+\beta} \|x_1^{k'} x_2^{\alpha+\beta-k'} \partial_1^\alpha \partial_2^\beta \mathbf{r}^\ell \partial^\delta u\|_{L^2_\gamma(\Omega)}^2. \end{aligned}$$

To obtain our estimate on $J_{s+1}$ we sum the above inequality with respect to $k$, $p$, $\ell$ and $\delta$. We thus have

$$\begin{aligned}
J_{s+1} \leq & C_s\Big(\sum_{\ell=0}^{s}\sum_{|\delta|\leq\ell}\|\partial^\delta \mathbf{r}^\ell u\|^2_{\widetilde{H}^{s+1-\ell,s+1-\ell}_\gamma(\Omega)} + \sum_{\ell=0}^{s}\sum_{|\delta|\leq\ell}\sum_{\alpha+\beta\leq s-\ell}\sum_{k'=0}^{\alpha+\beta}\|x_1^{k'}x_2^{\alpha+\beta-k'}\partial_1^\alpha\partial_2^\beta\mathbf{r}^\ell\partial^\delta u\|^2_{L^2_\gamma(\Omega)}\Big)\\
\leq & C_s\Big(\sum_{\ell=0}^{s}\|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s+1,s+1-\ell}_\gamma(\Omega)} + I_s\Big)\\
\leq & C_s\sum_{\ell=0}^{s}\|\mathbf{r}^\ell u\|^2_{\widetilde{H}^{s+1,s+1-\ell}_\gamma(\Omega)}.
\end{aligned} \tag{63}$$

To conclude we combined (57), (58) and (63). This shows that (51) holds for $s+1$ instead of $s$ and ends up the proof of Proposition 3.3.

□

To end up the proof of the a priori energy estimate at the order $s+1$ we combine Propositions 3.1, 3.2 and 3.3.

□

# 4 Regularity of the solution

In this section we establish the regularity of the solution to initial boundary value problem (1) and to the pure boundary value problem (2). This section thus contains the proof of the second statement of Theorem 2.1 and the ones of Theorems 2.2 and 2.3. For convenience we recall here the two problems; we will consider

$$\begin{cases} L(\partial)u = f & \text{for } (t,x)\in\Omega\\ B_1 u_{|x_1=0} = g_1 & \text{on } (t,x_2,x')\in\partial\Omega_1,\\ B_2 u_{|x_2=0} = g_2 & \text{on } (t,x_1,x')\in\partial\Omega_2, \end{cases} \tag{64}$$

that is to say the **pure** boundary value problem, and

$$\begin{cases} L(\partial)u = 0 & \text{for } (t,x)\in\Omega,\\ B_1 u_{|x_1=0} = 0 & \text{on } (t,x_2,x')\in\partial\Omega_1,\\ B_2 u_{|x_2=0} = 0 & \text{on } (t,x_1,x')\in\partial\Omega_2,\\ u_{|t=0} = u_0 & \text{on } x\in\Gamma, \end{cases} \tag{65}$$

the **initial** boundary value problem.

The existence of a regular solution to (64) comes from a slight adaptation of the duality method introduced in Lax-Phillips [Lax and Phillips, 1960] while the existence for (65) comes from the application of Lumer-Phillips theorem [Lumer and Phillips, 1961]. We refer to [Benoit, ] for examples of applications of these methods in the simpler geometry of the half-space.

In order to apply the method based upon Lumer-Phillips theorem we should express (65) as a pure evolution problem so that we write it under the following form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u = Au & \text{for } t>0,\\ u_{|t=0} = u_0 & \text{on } t=0, \end{cases} \tag{66}$$

where $A$ is the operator defined by $Au := -\sum_{j=1}^{d} A_j\partial_j u$ on some domain which in particular encodes in its definition the boundary conditions of (65) (we refer to Proposition 4.1 for a precise definition of $D(A)$).

In the following we first recall for completeness the version of Lumer-Phillips that we will use in order to show the existence of the solution to (65) that is to say the one involving the dissipativity of the operator $A$ combined with the one for its dual operator $A^*$.

Similarly the main assumption for the duality method of Lax-Phillips is to assume that the dual problem to (64) satisfies an *a priori* energy estimate and then to recover from this injectivity property on the adjoint operator the surjectivity of the primal operator giving the existence of a solution.

So that both methods of construction of a solution to (64) and (65) rely in fact on a dual operator defined on some suitable base Hilbert spaces.

The required duality formulas are stated in Paragraph 4.1.2 but because they are a little heavy to demonstrate their proof are postponed to Paragraph 4.1.2. The main feature to keep in mind is that $A$ is essentially skew-adjoint, meaning that in some suitable Hilbert space we can write the dual operator $A^*$ can be written under the form

$$A^* = -A + \Phi \tag{67}$$

where $\Phi$ is some linear continuous operator.

Paragraph 4.1.1 contains the version of Lumer-Phillips theorem that we will use in the following and gives some elements of functional analysis.

Then assuming that the decomposition (67) holds, Paragraph 4.2 describes the Lumer-Phillips method in two different base Hilbert space and thus gives the proof of Theorems 2.2 and 2.3 while Paragraph 4.3 deals with Lax-Phillips method and thus gives the proof of the second statement of Theorem 2.1.

## 4.1 Preliminaries

### 4.1.1 Functional analysis framework

In this paragraph we recall for a sake of completeness and for convenience the following version of Lumer-Phillips theorem that will be used to show the existence of a regular solution to (66).

In the following $(X, \|\cdot\|_X)$ stands for a Banach space and $A$ denotes an operator defined on some domain $D(A)$. We recall the following definition for dissipative/quasidissipative operators.

**Definition 4.1 (Dissipative operator)** *Consider $A : D(A) \to X$ an operator. We say that $A$ is dissipative if*

$$\forall\, \lambda > 0,\ \forall\, u \in D(A),\ \|(\lambda I - A)u\|_X \geq \lambda \|u\|_X.$$

**Definition 4.2 (Quasidissipative operator)** *Let $A : D(A) \to X$ an operator. We say that $A$ is $\omega$-quasidissipative if there exists $\omega \geq 0$ such that $A - \omega I$ is dissipative.*

For such operators we have the following corollary of Hille-Yosida theorem characterizing the strongly continuous semigroups of (quasi)contractions

**Theorem 4.1 ([Lumer and Phillips, 1961])** *Let $A : D(A) \to X$ be a closed densely defined operator. We assume that $A$ is $\omega$-quasidissipative and that there exists $\lambda_0 > \omega$ such that $\lambda_0 I - A$ is onto then $A$ generates a strongly continuous semigroup of quasicontractions $T$ that is to say that*

$$\exists\, \omega > 0,\ s.t.\ \forall\, t \in [0, \infty[\,,\ \|T(t)\|_X \leq e^{\omega t}.$$

As pointed in the beginning of this section we will not show that $\lambda_0 I - A$ is onto but we change this requirement by some property on the dual operator $A^*$. This is why we rather consider the following corollary of Theorem 4.1 in which the second statement just describes the definition of a strongly continuous semigroup of quasicontractions on the solution $u$ to the initial boundary value problem (66).

**Theorem 4.2** *Let $A$ be a closed densely defined operator $D(A) \to X$ if $A$ and $A^*$ are both $\omega$-quasidissipative then $A$ generates a strongly continuous semigroup of quasicontractions on $X$.*

*In particular the differential equation*

$$\begin{cases} \frac{d}{dt}u = Au & t \in [0,\infty[\,, \\ u_{|t=0} = u_0, \end{cases}$$

*has a unique solution* $u \in \mathscr{C}^0([0,\infty[\,, D(A)) \cap \mathscr{C}^1([0,\infty[\,, X)$ *which satisfies the energy estimate*

$$\exists\, \omega > 0,\ \forall\, t \in [0,\infty[\,,\ \|u(t)\|_X \leq e^{\omega t}\|u_0\|_X.$$

### 4.1.2 Duality formulas

The only thing that is needed in order to show the regularity of the solution to problems (64) and (65) is the following duality formulas for $A$ on $\mathcal{X}^s_{rad}(\Gamma)$ or $\mathcal{Y}^s(\Gamma)$ (we recall that these spaces are introduced in Paragraph 2.3.2).

Because the establishment of such duality formulas is rather heavy and that it is not computation free we postpone the proofs to Paragraph 4.4.

**Proposition 4.1** *Let* $s \geq 0$ *and let* $\mathcal{Z} := \mathcal{X}^s_{rad}(\Gamma)$ *or* $\mathcal{Z} := \mathcal{Y}^s(\Gamma)$ *then we have the following characterization of* $A$, $D(A)$, $A^*$ *and* $D(A^*)$*:*

*Let* $A : D(A) \to \mathcal{Z}$ *be defined by* $Au := -\sum_{j=1}^d A_j \partial_j u$ *on*

$$D(A) := D_s(A) = \{u \in \mathcal{Z} \setminus Au \in \mathcal{Z}\,,\ \forall j \in \{1,2\}\,,\ u_{|x_j=0} \in \mathcal{Z} \text{ satisfies } u_{|x_j=0} \in \ker B_j\}.$$

*Then the dual operator of* $A^* : D(A^*) \to \mathcal{Z}$ *is defined on*

$$D(A^*) := D_s(A^*) = \{v \in \mathcal{Z} \setminus -Av \in \mathcal{Z}\,,\ \forall j \in \{1,2\}\,,\ v_{|x_j=0} \in \mathcal{Z} \text{ satisfies } v_{|x_j=0} \in \ker C_j\},$$

*where the boundary matrices* $C_j$ *are such that we have the decomposition (see equation* (23)*)*

$$A_j := M_j^T B_j + C_j^T N_j. \tag{68}$$

*Moreover there exists a linear continuous operator* $\Phi : \mathcal{Z} \to \mathcal{Z}$ *such that we have the characterization*

$$A^* = -A + \Phi.$$

To conclude let us remark that $A$ and thus $A^*$ are closed operators and that they are densely defined because both $D(A)$ and $D(A^*)$ contains $\mathcal{D}(\Gamma)$.

## 4.2 The initial boundary value problem

In the following assuming that we have the duality formulas exposed above we expose the proofs of Theorems 2.2 and 2.3. The proof of Theorem 2.3 is given in Paragraph **??** and the one of Theorem 2.2 is given in Paragraph 4.2.2

### 4.2.1 Existence of a solution to (65) in $\mathcal{Y}^s(\Gamma)$; proof of Theorem 2.3

In this paragraph we use Lumer-Phillips theorem in the form of Theorem 4.2 with $X := \mathcal{Y}^s(\Gamma)$ in order to show the existence of a regular solution to (66).

The sketch of the proof is the following:

1. Classically when one deals with compatible initial datum we will first show the result when the initial datum $u_0$ as one extra regularity and one extra compatibility condition. We then obtain the desired result by regularization of the initial datum.
2. We first apply Theorem 4.2 to (66). This is made possible because thanks to the maximal dissipativity of the boundary conditions $A$ is dissipative. We also use Proposition 4.1 in order to state the same result for $A^*$. This gives the existence of a solution $u \in \mathscr{C}^0([0,\infty[\,, D(A)) \cap \mathscr{C}^1([0,\infty[\,, \mathcal{Y}^s(\Gamma))$ to (66).

3. Then we iteratively apply Theorem 4.2 to $\partial_t^k(66)$ for $k \in \{1, ..., s\}$ which will give the desired regularity of the solution in terms of the space $\mathcal{Y}^s(\Gamma)$.

4. To conclude we have to justify that near the corner we can recover from the control of the radial derivative $(\mathbf{r}\partial_{\mathbf{r}})^\alpha \mathbf{r}^l u$ the control of all the anisotrops derivatives $x_1^{\alpha_1+\alpha_2-k} x_2^k \partial_1^{\alpha_1}\partial_2^{\alpha_2} u$ this is made in two steps

   (a) Firstly we use Assumption 2.2 which permits to bound explicitly the radial/angular derivatives $(\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial_\theta^{s-l-\alpha} u$ in terms of the radial derivatives $(\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial^\delta u$.

   (b) Then from the explicit estimation (56) which controls the desired $x_1^{\alpha_1+\alpha_2-k} x_2^k \partial_1^{\alpha_1}\partial_2^{\alpha_2} u$ in terms of the $(\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial_\theta^{s-l-\alpha} u$.

   We conclude by showing that away from the boundary we can recover the $H^s$ control of the solution from the control of the $\sum_{p=0}^{\alpha} \underline{\partial}_{1,p}\underline{\partial}_{2,\alpha-p}$.

In the following we will intensively use the following lemma.

**Lemma 4.1** *Let $k \in \mathbb{N}$ and $X := X_k = \mathcal{Y}^k(\Gamma)$ then if the boundary conditions of (65) are maximal dissipative then the operators $A : D_k(A) \to X$ and $A^* : D_k(A^*) \to X$ are quasidissipative in the sense of Definition 4.2.*

**Proof :** We first establish the dissipativity of $A$. Let $\lambda > 0$, $\omega > 0$ to be fixed below and $u \in D(A)$, we define $f := ((\lambda+\omega)I - A)u$. We have from Cauchy-Schwarz inequality

$$(\lambda+\omega)\|u\|_X^2 - \langle Au\,,\, u\rangle_X \leq \|f\|_X \cdot \|u\|_X.$$

The remaining of the proof consist in give a bound for the term $-\langle Au\,,\, u\rangle_X$. In order to do so, we borrow the following formula, namely equation (97), obtained during the determination of the adjoint operator. More precisely we can write that for all $u \in D(A)$

$$-\langle Au, u\rangle_X = -\frac{1}{2}\langle u, \Phi u\rangle_X - \frac{1}{2}\mathcal{B}_1 - \frac{1}{2}\mathcal{B}_2, \tag{69}$$

where $\Phi$ is the operator given by Proposition 4.1 and where the boundary integrals $\mathcal{B}_1$ and $\mathcal{B}_2$ are explicitly given by

$$\mathcal{B}_1 = \sum_{l=0}^{k}\sum_{\alpha=0}^{k-l}\sum_{|\delta|\leq k-\alpha}\int_{\partial\Gamma_1}\langle A_1(x_2\partial_2)^\alpha\partial^\delta x_2^l u_{|x_1=0}, (x_2\partial_2)^\alpha\partial^\delta x_2^l u_{|x_1=0}\rangle\,\mathrm{d}x'\,\mathrm{d}x_2,$$

$$\mathcal{B}_2 = \sum_{l=0}^{k}\sum_{\alpha=0}^{k-l}\sum_{|\delta|\leq k-\alpha}\int_{\partial\Gamma_2}\langle A_2(x_1\partial_1)^\alpha\partial^\delta x_1^l u_{|x_2=0}, (x_1\partial_1)^\alpha\partial^\delta x_1^l u_{|x_2=0}\rangle\,\mathrm{d}x'\,\mathrm{d}x_1.$$

We remark that from the definition of $D(A)$ the term $(x_2\partial_2)^\alpha\partial^\delta x_2^l u_{|x_1=0}$ (resp. $(x_1\partial_1)^\alpha\partial^\delta x_1^l u_{|x_2=0}$) appearing in the scalar product defining $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) is in $\ker B_1$ (resp. $\ker B_2$). Indeed these terms only involve tangential derivatives of $u_{|x_1=0} \in \ker B_1$ or $u_{|x_2=0} \in \ker B_2$.

So that from the definition of maximal dissipative boundary conditions (see Definition 4.1) we have $\mathcal{B}_1, \mathcal{B}_2 \leq 0$ and thus these terms are signed and can be neglected.

Consequently we have

$$(\lambda+\omega)\|u\|_X^2 - \frac{1}{2}|\langle u, \Phi u\rangle_X| \leq \|f\|_X \cdot \|u\|_X,$$

so that it suffices to choose $\omega \geq \frac{\|\Phi\|}{2}$ to obtain the desired dissipativity property.

The dissipativity of the adjoint $A^*$ follows essentially the same lines. So we will only give the main ingredients. Let $\lambda > 0$ and $\omega > 0$ to be chosen large enough we define for $v \in D(A^*)$, $f := (\lambda+\omega)v - A^*v$. So using Proposition 4.1 and equation (69) we have

$$(\lambda+\omega)\|v\|_X^2 + \mathcal{B}_1 + \mathcal{B}_2 - \frac{1}{2}\langle v\,,\, \Phi v\rangle_X \leq \|f\| \cdot \|v\|.$$

Because now $v \in D(A^*)$ we have in particular $v_{|x_1=0} \in \ker C_1$ and $v_{|x_2=0} \in \ker C_2$ so that we have $\mathcal{B}_1, \mathcal{B}_2 \geq 0$ from the remark[6] (25). The desired dissipativity property follows by choosing $\omega \geq \frac{\|\Phi\|}{2}$.

□

We first show Theorem 2.3 when $u_0 \in \mathscr{D}^{s+1}(\Gamma)$ satisfies the compatibility conditions up to the order $s$ we will then in a second time establish the result only for $u_0 \in \mathscr{D}^s(\Gamma)$ satisfying the compatibility conditions up to the order $s-1$.

We want to use Lemma 4.1 for $k = s$ to apply Theorem 4.2 to (66). Assume for a while that $u_0 \in D_s(A)$ then from Theorem 4.2 there exists a unique solution $u \in \mathscr{C}^0([0,\infty[\,, D_s(A)) \cap \mathscr{C}^1([0,\infty[\,, \mathcal{Y}^s(\Gamma))$ which satisfies the energy estimate

$$\forall\, t \in [0,\infty[\,,\ \|u(t)\|_{\mathcal{Y}^s(\Gamma)} \leq e^{\omega t}\|u_0\|_{\mathcal{Y}^s(\Gamma)} \leq \|u_0\|_{\mathscr{D}^s(\Gamma)}. \tag{70}$$

We now justify that $u_0 \in D_s(A)$. We have $u_0 \in \mathscr{D}^{s+1}(\Gamma) \subset \mathscr{D}^s(\Gamma) \subset \mathcal{Y}^s(\Gamma)$ and $Au_0 \in \mathscr{D}^s(\Gamma) \subset \mathcal{Y}^s(\Gamma)$. Then for $j \in \{1,2\}$; $u_{|x_j=0} \in \mathscr{D}^{s+1/2}(\partial\Gamma_j) \subset \mathcal{Y}^s(\partial\Gamma_j)$ and because of the compatibility condition of order 0

$$\begin{cases} B_1 u_{0_{|x_1=0}} = 0, \\ B_2 u_{0_{|x_2=0}} = 0. \end{cases}$$

We now want to apply Theorem 4.1 to $\partial_t u$. Such function satisfies

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\partial_t u = A\partial_t u, \\ (\partial_t u)_{|t=0} = Au_0 := u^1, \end{cases}$$

so that we should justify that $u^1 \in D_{s-1}(A)$. We have $u^1 = Au_0 \in \mathcal{D}^s(\Gamma) \subset \mathcal{Y}^{s-1}(\Gamma)$ and $Au^1 = A^2u_0 \in \mathcal{D}^{s-1}(\Gamma) \subset \mathcal{Y}^{s-1}(\Gamma)$. Moreover for $j \in \{1,2\}$; $u_{|x_j=0} \in \mathcal{D}^{s-1/2}(\partial\Gamma_j) \subset \mathcal{D}^{s-1}(\partial\Gamma_j) \subset \mathcal{Y}^{s-1}(\Gamma)$ satisfies

$$\begin{cases} B_1 u^1_{|x_1=0} = 0, \\ B_2 u^1_{|x_2=0} = 0, \end{cases}$$

from the compatibility condition of order one.

So that $\partial_t u \in \mathscr{C}^0([0,\infty[\,, D_{s-1}(A)) \cap \mathscr{C}^1([0,\infty[\,, \mathcal{Y}^{s-1}_{rad}(\Gamma))$ satisfies the energy estimate

$$\forall t \in [0,\infty[\,,\ \|\partial_t u(t)\|_{\mathcal{Y}^{s-1}(\Gamma)} \leq e^{\omega t}\|Au_0\|_{\mathcal{Y}^{s-1}_{rad}(\Gamma)} \leq Ce^{\omega t}\|u_0\|_{\mathscr{D}^s(\Gamma)}, \tag{71}$$

where $C$ depends on the coefficients $A_j$.

Proceeding iteratively we thus obtain that $u \in \cap_{k=0}^s \mathscr{C}^k_t([0,\infty[\,; \mathcal{Y}^{s-k}(\Gamma))$ satisfies the energy estimates

$$\forall t \in [0,\infty[\,,\ \forall\, k \in [\![0,s]\!],\ \|\partial_t^k u(t)\|_{\mathcal{Y}^{s-k}(\Gamma)} \leq C_s e^{\omega t}\|u_0\|_{\mathscr{D}^s(\Gamma)}. \tag{72}$$

We now have to justify that near the corner $\partial_t^k u(t)$ is not only in $\mathcal{Y}^{s-k}(\mathscr{C})$ but that it is in $\mathscr{H}^{s-k}(\mathscr{C})$. We also have to justify that the similar result holds in the interior that is to say that we can recover the fact that $\partial_t^k u(t) \in H^{s-k}(\mathscr{I})$ from the fact that it is in $\mathcal{Y}^{s-k}(\mathscr{I})$.

- Extra regularity near the corner: to fix the ideas let us consider the case $k = 0$ (the others cases being similar). In order to show that $u(t) \in \mathscr{H}^s(\mathscr{C})$ from the explicit bound (56) (in which we do not integrate with respect to $t$) which gives a bound for the anisotropic weighted derivatives in $\mathscr{H}^s(\mathscr{C})$ in terms of the radial and angular derivatives, it is sufficient to justify that the $\|(\mathbf{r}\partial_{\mathbf{r}})^{s-k-l}\partial_\theta^k\partial^\delta \mathbf{r}^l u\|_{L^2(\mathscr{C})}$ are finite for all $0 \leq k \leq s-l$.

  Note that because $u \in \mathcal{Y}^s(\Gamma)$ the previous result holds for $k = 0$. We then shall recover the control of $\|(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial_\theta^{s-l-\alpha}\partial^\delta \mathbf{r}^l u\|_{L^2(\mathscr{C})}$ iteratively like it as been done in the proof of the *a priori* energy estimate

[6] This point of the proof uses in a non trivial way the fact that we can express the dual operator as $-A$ plus some operator in order to recover the good sign property.

given in Paragraph 3.2. From (47) (in which we only remove the integration with respect to $t$) we have the relation

$$\begin{aligned}
&\|(\mathbf{r}\partial_{\mathbf{r}})^{s-1-k-l}\partial_\theta^{k+1}\partial^\delta \mathbf{r}^l\widetilde{u}\|^2_{L^2(\widetilde{\mathscr{C}},\mathbf{r}\partial_{\mathbf{r}})} && (73)\\
&\qquad\leq \|(\mathbf{r}\partial_{\mathbf{r}})^{s-k-l}\partial^\delta\partial_\theta^k\mathscr{A}^{-1}(\theta)\mathscr{M}(\theta)\mathbf{r}^l\widetilde{u}\|^2_{L^2(\widetilde{\mathscr{C}},\mathbf{r}\partial_{\mathbf{r}})} && (74)\\
&\qquad\quad+\sum_{j\in\mathscr{T}}\|(\mathbf{r}\partial_{\mathbf{r}})^{s-1-k-l}\partial^\delta\partial_\theta^k\mathscr{A}^{-1}(\theta)A_j\mathbf{r}^l\partial_j\widetilde{u}\|^2_{L^2(\widetilde{\mathscr{C}},\mathbf{r}\partial_{\mathbf{r}})} && (75)\\
&\qquad\leq C_s\left(\|(\mathbf{r}\partial_{\mathbf{r}})^{s-k-\ell}\partial^\delta\partial_\theta^k\mathbf{r}^l\widetilde{u}\|^2_{L^2(\widetilde{\mathscr{C}},\mathbf{r}\partial_{\mathbf{r}})}+\|(\mathbf{r}\partial_{\mathbf{r}})^{s-1-k-l}\partial^{\delta+1}\partial_\theta^k\mathbf{r}^l\widetilde{u}\|^2_{L^2(\widetilde{\mathscr{C}},\mathbf{r}\partial_{\mathbf{r}})}\right),
\end{aligned}$$

where $\widetilde{\mathscr{C}}$ stands for the image of $\mathscr{C}$ in polar coordinates and where we use Assumption 2.2, Leibniz formula and the slight abuse of notation, for $j\in\mathscr{T}$, $\partial_j\partial^\delta=\partial^{\delta+1}$ in the last line. So that in cartesian coordinates we have

$$\|(\mathbf{r}\partial_{\mathbf{r}})^{s-1-k-l}\partial_\theta^{k+1}\partial^\delta\mathbf{r}^l u\|^2_{L^2(\mathscr{C})}\leq C_s\left(\|(\mathbf{r}\partial_{\mathbf{r}})^{s-k-\ell}\partial^\delta\partial_\theta^k\mathbf{r}^l u\|^2_{L^2(\mathscr{C})}+\|(\mathbf{r}\partial_{\mathbf{r}})^{s-1-k-l}\partial^{\delta+1}\partial_\theta^k\mathbf{r}^l u\|^2_{L^2(\mathscr{C})}\right),$$

equation which gives the desired result iteratively.

- Extra regularity in the interior : Because of the definition of $\mathcal{Y}^{s-k}(\mathscr{I})$ the result holds for $k=s$. We here proof of the fact that we can recover the result that $\partial_t^{s-1}u\in H^1(\mathscr{I})$ from the fact that it is in $\mathcal{Y}^1(\mathscr{I})$. Because of the inversibility of $A_1$ we can write

$$\|\partial_1 u\|_{L^2(\mathscr{I})}\leq C\Big(\|\partial_t^s u\|_{L^2(\mathscr{I})}+\|\partial_2\partial_t^{s-1}u\|_{L^2(\mathscr{I})}+\sum_{j=3}^d\|\partial_j\partial_t^{s-1}\partial_j u\|_{L^2(\mathscr{I})}\Big),$$

so that from the triangle inequality

$$\|\partial_1 u+\partial_2 u\|_{L^2(\mathscr{I})}\geq\left|(1-C)\|\partial_1\partial_t^{s-1}u\|_{L^2(\mathscr{I})}-C\|\partial_t^s u\|_{L^2(\mathscr{I})}-C\sum_{j=3}^d\|\partial_j\partial_t^{s-1}\partial_j u\|_{L^2(\mathscr{I})}\right|. \tag{76}$$

The left hand side of (76) being finite because $\partial_t^{s-1}u\in\mathcal{Y}^1(\Gamma)$. We separate several cases

  - If $C>1$ we thus obtain $(C-1)\|\partial_1\partial_t^{s-1}u\|_{L^2(\mathscr{I})}\leq\|\partial_1 u+\partial_2 u\|_{L^2(\mathscr{I})}$. So that we obtain that $\partial_1\partial_t^{s-1}u\in L^2(\mathscr{I})$.
  - For $C<1$
    - if $(1-C)\|\partial_1\partial_t^{s-1}u\|_{L^2(\mathscr{I})}\leq C\|\partial_t^s u\|_{L^2(\mathscr{I})}+C\sum_{j=3}^d\|\partial_j\partial_t^{s-1}\partial_j u\|_{L^2(\mathscr{I})}$. Then the result is immediate.
    - if $(1-C)\|\partial_1\partial_t^{s-1}u\|_{L^2(\mathscr{I})}\geq C\|\partial_t^s u\|_{L^2(\mathscr{I})}+C\sum_{j=3}^d\|\partial_j\partial_t^{s-1}\partial_j u\|_{L^2(\mathscr{I})}$ then we have from (76)

$$(1-C)\|\partial_1\partial_t^{s-1}u\|_{L^2(\mathscr{I})}\leq\|\partial_1 u+\partial_2 u\|_{L^2(\mathscr{I})}+C\|\partial_t^s u\|_{L^2(\mathscr{I})}+C\sum_{j=3}^d\|\partial_j\partial_t^{s-1}\partial_j u\|_{L^2(\mathscr{I})},$$

      so that $\partial_1\partial_t^{s-1}u\in L^2(\mathscr{I})$.

  To obtain the fact that $\partial_2\partial_t^{s-1}u\in L^2(\mathscr{I})$ we use the inversibility of $A_2$. We thus have justified that $\partial_t^{s-1}u\in H^1(\mathscr{I})$. The proof for $k\geq 2$ follows exactly the same lines and is omitted here.

We thus have justified that when $u_0\in\mathscr{D}^{s+1}(\Gamma)$ satisfies the compatibility conditions (13) up to the order $s$ then (65) admits a unique solution $u\in\cap_{k=0}^s\mathscr{C}_t^k([0,\infty[\,;\mathscr{Y}^{s-k}(\Gamma))$ and that we have the energy estimate

$$\forall\, t\in[0,\infty[\,,\ \forall\, k\in[\![0,s]\!],\ \|\partial_t^k u(t)\|_{\mathscr{Y}^{s-k}(\Gamma)}\leq C_s e^{\omega t}\|u_0\|_{\mathscr{D}^s(\Gamma)}. \tag{77}$$

To conclude the proof of Theorem 2.3 it is sufficient to show that the same result holds for $u_0 \in \mathscr{D}^s(\Gamma)$ satisfying the compatibility conditions (13) up to the order $s-1$. In order to do so we use the following regularization result:

**Lemma 4.2** *Let $s \in \mathbb{N}$ be fixed and $u_0 \in \mathscr{D}^s(\Gamma)$ satisfying the compatibility conditions* (13) *up to the order $s-1$ then there exists a sequence $(u_0^\nu)_{\nu\in\mathbb{N}} \subset \mathscr{D}^{s+1}(\Gamma)$ satisfying the compatibility conditions* (13) *up to the order $s$ such that $\lim_{\nu\to\infty} u_0^\nu = u_0$ in $\mathscr{D}^s(\Gamma)$.*

**Proof of Lemma 4.2** This lemma is rather immediate because we are working with homogeneous compatibility conditions so that it is sufficient to approximate $u_0$ in $\mathscr{D}^s(\Gamma)$ by a sequence of regular functions vanishing near the boundaries $\{x_1 = 0\}$ and $\{x_2 = 0\}$ such an approximate sequence in fact satisfies the compatibility conditions at any order.

□

We thus introduce the initial boundary value problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u^\nu = A u^\nu & \text{for } t > 0, \\ u^\nu_{|t=0} = u_0^\nu, \end{cases} \tag{78}$$

because $u_0^\nu$ is regular enough the previous discussion applies and thus for fixed $\nu$, (78) admits a unique solution $u^\nu \in \cap_{k=0}^s \mathscr{C}_t^k([0,\infty[\,;\mathscr{Y}^{s-k}(\Gamma))$ with the energy estimates (77). By linearity and from the energy estimates (77) we can show that $(u^\nu(t))_{\nu\in\mathbb{N}}$ is a Cauchy sequence so that it converges pointwise to some $u \in \cap_{k=0}^s \mathscr{C}_t^k([0,\infty[\,;\mathscr{Y}^{s-k}(\Gamma))$, passing to the limit in (78) and (77) shows that $u$ is a solution to (78) satisfying the energy estimates (77) in particular it is unique. This ends up the proof of Theorem 2.3.

□

### 4.2.2 Existence of a solution to (65) in $\mathcal{X}^s_{rad}(\Gamma)$; proof of Theorem 2.2

The proof of Theorem 2.2 follows essentially the same lines as the one given in the previous paragraph for Theorem 2.3.

In a first time we assume that $u_0 \in \mathcal{X}^s_{rad}(\Gamma)$ satisfies the compatibility conditions up to the order $s$.

Because it only relies on the fact that the dual operator is essentially skew-adjoint which is also true in $\mathcal{X}^k_{rad}(\Gamma)$ we have the analogous of Lemma 4.1

**Lemma 4.3** *Let $k \in \mathbb{N}$ and $X := X_k = \mathcal{X}^k_{rad}(\Gamma)$ then if the boundary conditions of* (65) *are maximal dissipative then the operators $A : D_k(A) \to X$ and $A^* : D_k(A^*) \to X$ are quasidissipative in the sense of Definition 4.2.*

We are then in position to apply Theorem 4.1. It is made possible because of the well-preparedness assumption (14). For instance for $k = s$ we have $u_0 \in D_s(A)$. Indeed we have $u_0 \in \mathcal{X}^s_{rad}(\Gamma)$, then from the well-preparedness assumption (14) we have $Au_0 \in \mathcal{X}^s_{rad}(\Gamma)$ and the trace $u_{0|x_j=0} \in \mathcal{X}^s_{rad}(\partial\Gamma_j)$ and finally $u_{0_{|x_j=0}} \in \ker B_j$ because of the compatibility condition of order zero. So that Theorem 4.1 gives the existence of a unique $u \in \mathscr{C}^0([0,\infty[\,, D_s(A)) \cap \mathscr{C}^1([0,\infty[\,, \mathcal{X}^s_{rad}(\Gamma))$ which satisfies the energy estimate

$$\forall\, t \in [0,\infty[\,,\ \|u(t)\|_{\mathcal{X}^s(\Gamma)} \leq e^{\omega t}\|u_0\|_{\mathcal{X}^s(\Gamma)}.$$

We then apply inductively Theorem 4.1 to $\partial_t u$ like in Paragraph 4.2.1. This is made possible because of the well-preparedness assumption 14. This shows that $u \in \cap_{k=0}^s \mathscr{C}_t^k([0,\infty[\,;\mathcal{X}^{s-k}_{rad}(\Gamma))$ satisfies the energy estimates

$$\forall t \in [0,\infty[\,,\ \forall\, k \in [\![0,s]\!],\ \|\partial_t^k u(t)\|_{\mathcal{X}^{s-k}(\Gamma)} \leq C_s e^{\omega t}\|u_0\|_{\mathcal{X}^s(\Gamma)},$$

where at each iteration we used the well-preparedness assumption $\|Au_0\|_{\mathcal{X}^s(\Gamma)} \leq C\|u_0\|_{\mathcal{X}^s(\Gamma)}$.

To conclude we reproduce the arguments of Paragraph 4.2.1 to justify that we can recover from the regularity $u \in \cap_{k=0}^s \mathscr{C}_t^k(\mathbb{R}_+, \mathcal{X}^{s-k}_{rad}(\Gamma))$ we can recover the full regularity of the anisotropic weighted derivatives

that is to say that $u \in \cap_{k=0}^{s} \mathscr{C}_t^k(\mathbb{R}_+, \mathscr{H}_{rad}^{s-k}(\Gamma))$ (we use (73) integrated over the full space instead of $\widetilde{\mathscr{C}}$). This gives the proof of Theorem 2.2 when $u_0$ satisfies one extra compatibility condition.

Finally we recover the result for $u_0 \in \mathcal{X}_{rad}^s(\Gamma)$ satisfying the compatibility conditions up to the order $s-1$ using an approximation procedure by regular vanishing functions near the boundaries exactly like in Lemma 4.2. This concludes the proof of Theorem 2.2.

## 4.3 The pure boundary value problem

This paragraph is devoted to the proof of the regularity of the solution to (64). This proof relies on a rather straightforward adaptation of the so-called duality method introduced in [Lax and Phillips, 1960]. The main ideas of the modified method are the following:

1. We first introduce a dual problem to the primal problem (64). This dual problem has already been introduced in Paragraph 4.1.2.
2. We assume that the dual problem to (64) satisfies an *a priori* energy estimate in high order based Sobolev space (more precisely in $\mathcal{X}_{rad,\gamma}^s(\Omega)$). So that in particular it gives us uniqueness of the solution to the dual problem.
3. Using this uniqueness property for the dual problem we recover the existence of a solution for the primal problem in $\mathcal{X}_{rad,\gamma}^s(\Omega)$.
4. Like in Paragraph 4.2 to recover the full regularity of the solution in $\mathscr{H}_\gamma^s(\Omega)$ we use the explicit formulas (47) and (56) giving the regularity of the angular derivatives $\partial_\theta^k$ in terms of the radial ones.

The most substantial modification that we make compared to the method of [Lax and Phillips, 1960] exposed above is that we will apply this method to high order Sobolev space (more precisely a space containing the radial derivatives $(\mathbf{r}\partial_{\mathbf{r}})^\alpha$) so that if we have an *a priori* energy estimate in high order Sobolev norms for the dual problem then the solution to the primal problem will inherit this regularity.

We consider the Hilbert space $\mathcal{X}_{rad}^s(\Gamma)$ encoding the control of the radial derivatives $(\mathbf{r}\partial_{\mathbf{r}})^k$, for $k \in [\![0, s-l]\!]$ of the weighted functions $\mathbf{r}^l u$. Because we have now to deal with the time variable we introduce the following functional spaces; for $\chi \in \mathbb{R}$ and $X \subset \Omega$ we define

$$\mathcal{X}_{rad,\chi}^s(X) := \{u \in \mathcal{D}'(X) \setminus \forall\, l \in \{0,...,s\}, \forall\, \alpha \in \{0,...,s-l\}, \forall\, |\delta| \leq s-\alpha,\, e^{-\chi t}(\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial^\delta \mathbf{r}^l u \in L^2(X)\}, \quad (79)$$

where the multi-index $\delta = (\delta_0, \delta_3, ..., \delta_d) \in \mathbb{N}^{1+d-2}$ and where the operator the differentiation $\partial^\delta$ now stands for $\partial := \partial_t^{\delta_0}\partial_3^{\delta_3}...\partial_d^{\delta_d}$. This space comes with the norm ; for $u \in \mathcal{X}_{rad,\chi}^s(X)$

$$\|u\|^2_{\mathcal{X}_{rad,\chi}^s(X)} := \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_X e^{-2\chi t}|(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta \mathbf{r}^l u(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x.$$

Proposition 4.1 applies without any change because we only add in $\mathcal{X}_{rad,\chi}^s(X)$ the tangential derivative $\partial_t$ and because the $A_j$ do not depend on $t$.

Consequently we have the duality formula: for all $u \in \mathcal{X}_{rad,\chi}^s(\Omega)$ satisfying that $u_{|x_1=0} \in \mathcal{X}_{rad,\chi}^s(\partial\Omega_1)$, $u_{|x_2=0} \in \mathcal{X}_{rad,\chi}^s(\partial\Omega_2)$ and for all $v \in \mathcal{X}_{rad,-\chi}^s(\Omega)$ such that $v_{|x_1=0} \in \mathcal{X}_{rad,-\chi}^s(\partial\Omega_1)$, $v_{|x_2=0} \in \mathcal{X}_{rad,-\chi}^s(\partial\Omega_2)$ we have the equality

$$\begin{aligned}\langle L(\partial)u, v\rangle_{\mathcal{X}_{rad,\chi}^s(\Omega);\mathcal{X}_{rad,-\chi}^s(\Omega)} - \langle u, L^*(\partial)v\rangle_{\mathcal{X}_{rad,\chi}^s(\Omega);\mathcal{X}_{rad,-\chi}^s(\Omega)} - \langle u, \Phi v\rangle_{\mathcal{X}_{rad,\chi}^s(\Omega);\mathcal{X}_{rad,-\chi}^s(\Omega)} \\ + \sum_{j=1}^{2}\langle B_j u_{|x_j=0}, M_j v_{|x_j=0}\rangle + \langle N_j u_{|x_j=0}, C_j v_{|x_j=0}\rangle = 0,\end{aligned} \quad (80)$$

where we defined $L^*(\partial) := -L(\partial)$ and where the duality products on the boundary are duality product $\langle\cdot,\cdot\rangle_{\mathcal{X}_{rad,\chi}^s(\partial\Omega_j);\mathcal{X}_{rad,-\chi}^s(\partial\Omega_j)}$.

Indeed we use equality (97) in which we do not have any cancellation of the boundary terms $\mathcal{B}_1$ and $\mathcal{B}_2$ so that we just express these terms using the decomposition (68).

For later purpose we define the "complete" (in the sense that it is the one for which we have the duality formula) adjoint operator $\underline{L}^*(\partial) := L^*(\partial) - \Phi$. In the following we consider as a dual problem to (64) the pure boundary value problem

$$\begin{cases} L^*(\partial)v = \widetilde{f} & \text{in } \Omega, \\ C_1 v_{|x_1=0} = \widetilde{g}_1 & \text{on } \partial\Omega_1, \\ C_2 v_{|x_2=0} = \widetilde{g}_2 & \text{on } \partial\Omega_2, \end{cases} \tag{81}$$

where $C_1$ and $C_2$ are defined in the decomposition (68).

Our main assumption of the current paragraph is the following. It states that the dual problem (81) admits an *a priori* energy estimate in the high order radial Sobolev space. More precisely

**Assumption 4.1** *Assume that for some $s \in \mathbb{N}$ and $\underline{\gamma} > 0$ for $\gamma \geq \underline{\gamma}$, if the solution $v \in \mathcal{X}^s_{rad,-\gamma}(\Omega)$ is such that $\widetilde{f} := L^*(\partial)v \in \mathcal{X}^s_{rad,-\gamma}(\Omega)$ and $\widetilde{g}_1 := C_1 v_{|x_1=0} \in \mathcal{X}^s_{rad,-\gamma}(\partial\Omega_1)$, $\widetilde{g}_2 := C_2 v_{|x_2=0} \in \mathcal{X}^s_{rad,-\gamma}(\partial\Omega_2)$ then there exists $C > 0$ such that for all $\gamma \geq \underline{\gamma}$ we have the energy estimate*

$$\begin{aligned} \gamma \|v\|^2_{\mathcal{X}^s_{rad}(\Omega)} + \|v_{|x_1=0}\|^2_{\mathcal{X}^s_{rad}(\partial\Omega_1)} + \|v_{|x_2=0}\|^2_{\mathcal{X}^s_{rad}(\partial\Omega_2)} \\ \leq C \left( \frac{1}{\gamma} \|\widetilde{f}\|^2_{\mathcal{X}^s_{rad}(\Omega)} + \|\widetilde{g}_1\|^2_{\mathcal{X}^s_{rad}(\partial\Omega_1)} + \|\widetilde{g}_2\|^2_{\mathcal{X}^s_{rad}(\partial\Omega_2)} \right). \end{aligned}$$

Let us indicate that from the results of Section 3 this assumption is not so restrictive or heavy. Indeed it is in fact sufficient to assume that the dual problem (81) is $L^2$-well posed. In particular it is well-known that symmetric problems with strictly dissipative boundary conditions have a good behaviour for the dual problem so that if the primal problem satisfies such assumptions then so do its dual problem (we refer for example to [Lax and Phillips, 1960] for a proof of this fact).

We now show the existence of a solution $u$ to the primal problem (64).

**Proposition 4.2** *Let $s \in \mathbb{N}$, under Assumption 4.1 there exists $\underline{\gamma}_0 > 0$ such that for all $\gamma \geq \underline{\gamma}_0$ if the sources $f \in \mathcal{X}^s_{rad,\gamma}(\Omega)$, $(g_1,\ g_2) \in \mathcal{X}^s_{rad,\gamma}(\partial\Omega_1) \times \mathcal{X}^s_{rad,\gamma}(\partial\Omega_2)$ then the primal pure boundary value problem (64) admits a unique solution $\underline{u} \in \mathcal{X}^s_{rad,\gamma}(\Omega)$. Moreover $\underline{u}$ satisfies the energy estimate, there exists $C > 0$ such that for all $\gamma \geq \underline{\gamma}_0$*

$$\gamma \|\underline{u}\|^2_{\mathcal{X}^s_{rad,\gamma}(\Omega)} \leq C \left( \frac{1}{\gamma} \|f\|^2_{\mathcal{X}^s_{rad,\gamma}(\Omega)} + \|g_1\|^2_{\mathcal{X}^s_{rad,\gamma}(\partial\Omega_1)} + \|g_2\|^2_{\mathcal{X}^s_{rad,\gamma}(\partial\Omega_2)} \right). \tag{82}$$

*If in addition $f \in \mathscr{H}^s_\gamma(\Omega)$ and $(g_1,\ g_2) \in \mathscr{H}^s_\gamma(\partial\Omega_1) \times \mathscr{H}^s_\gamma(\partial\Omega_2)$ then $\underline{u} \in \mathscr{H}^s_\gamma(\Omega)$. Finally if the primal problem (64) satisfies Assumption 2.1 then $\underline{u}$ satisfies the a priori energy estimate (11).*

**Proof :** The proof exposed here follows essentially the one used in [Benoit, ] see also [[Benzoni-Gavage, 2007]-Paragraph 4.5.3].

We introduce the following subset of $\mathcal{X}^s_{rad,-\gamma}(\Omega)$, $\mathbb{X}$ defined by

$$\mathbb{X} := \{ \underline{L}^*(\partial)v \text{ where } v \in \mathcal{X}^s_{rad,-\gamma}(\Omega) \text{ satisfies } C_1 v_{|x_1=0} = C_2 v_{|x_2=0} = 0 \},$$

and for given $f \in \mathcal{X}^s_{rad,\gamma}(\Omega)$, $(g_1, g_2) \in \mathcal{X}^s_{rad,\gamma}(\partial\Omega_1) \times \mathcal{X}^s_{rad,\gamma}(\partial\Omega_2)$ the following linear form $\ell : \mathbb{X} \to \mathbb{R}$ defined by

$$\begin{aligned} \ell(\underline{L}^*(\partial)v) :=& \langle f, v \rangle_{\mathcal{X}^s_{rad,\gamma}(\Omega), \mathcal{X}^s_{rad,-\gamma}(\Omega)} \\ &+ \langle g_1, N_1 v_{|x_1=0} \rangle_{\mathcal{X}^s_{rad,\gamma}(\partial\Omega_1), \mathcal{X}^s_{rad,-\gamma}(\partial\Omega_1)} + \langle g_2, N_2 v_{|x_2=0} \rangle_{\mathcal{X}^s_{rad,\gamma}(\partial\Omega_2), \mathcal{X}^s_{rad,-\gamma}(\partial\Omega_2)}. \end{aligned} \tag{83}$$

The main point of the following is to show that because of Assumption 4.1, $\ell$ is continuous on $\mathbb{X}$. Indeed Cauchy-Schwarz inequality combined with the energy estimate for (81) gives for $\gamma \geq \underline{\gamma}$

$$|\ell(\underline{L}^*(\partial)v)| \leq C \left( \frac{1}{\gamma} \|f\|^2_{\mathcal{X}^s_{rad,\gamma}(\Omega)} + \frac{1}{\sqrt{\gamma}} \left( \|g_1\|^2_{\mathcal{X}^s_{rad,\gamma}(\partial\Omega_1)} + \|g_2\|^2_{\mathcal{X}^s_{rad,\gamma}(\partial\Omega_1)} \right) \right) \cdot \|L^*(\partial)v\|^2_{\mathcal{X}^s_{rad,-\gamma}(\Omega)}, \tag{84}$$

and in order to show that $\ell$ is continuous we shall only replace $L^*(\partial)v$ by $\underline{L}^*(\partial)v$ in the right hand side of (84).

The triangle inequality and the definition of $\underline{L}^*(\partial)$ give

$$\|\underline{L}^*(\partial)v\|_{\mathcal{X}^s_{rad,-\gamma}(\Omega)} \geq \left| \|L^*(\partial)v\|_{\mathcal{X}^s_{rad,-\gamma}(\Omega)} - \|\Phi v\|_{\mathcal{X}^s_{rad,-\gamma}(\Omega)} \right|. \tag{85}$$

However using the fact that $\Phi$ is continuous and the energy estimate for the dual problem we have the estimate

$$\|\Phi v\|_{\mathcal{X}^s_{rad,-\gamma}(\Omega)} \leq C\|v\|_{\mathcal{X}^s_{rad,-\gamma}(\Omega)} \leq \frac{C}{\gamma}\|L^*(\partial)v\|_{\mathcal{X}^s_{rad,-\gamma}(\Omega)},$$

so that we can choose $\gamma$ large enough such that the right hand of (85) is positive and reads like a positive multiple of $\|\underline{L}^*(\partial)v\|_{\mathcal{X}^s_{rad,-\gamma}(\Omega)}$. This shows the continuity of $\ell$ on $\mathbb{X}$.

From Hahn-Banach theorem we extend $\ell$ to the whole space $\mathcal{X}^s_{rad,-\gamma}$ and then from Riesz representation theorem we deduce the existence of a unique $\underline{u} \in \mathcal{X}^s_{rad,\gamma}(\Omega)$ such that

$$\ell(\underline{L}^*(\partial)v) = \langle \underline{u}, \underline{L}^*(\partial)v\rangle_{\mathcal{X}^s_{rad,\gamma}(\Omega),\mathcal{X}^s_{rad,-\gamma}(\Omega)}. \tag{86}$$

Moreover from the characterization of $\|\underline{u}\|_{\mathcal{X}^s_{rad,\gamma}(\Omega)}$ given by Riesz representation theorem we have for $\gamma$ large enough the energy estimate

$$\gamma\|\underline{u}\|^2_{\mathcal{X}^s_{rad,\gamma}(\Omega)} \leq C\left(\frac{1}{\gamma}\|f\|^2_{\mathcal{X}^s_{rad,\gamma}(\Omega)} + \|g_1\|^2_{\mathcal{X}^s_{rad,\gamma}(\partial\Omega_1)} + \|g_2\|^2_{\mathcal{X}^s_{rad,\gamma}(\partial\Omega_1)}\right), \tag{87}$$

this gives estimate (82).

We shall now verify that such $\underline{u}$ is a solution to (64). We choose $v \in \mathcal{D}(\Omega)$ and combine (83) and (86) with the duality formula (80) to obtain that for all $v \in \mathcal{D}(\Omega)$,

$$\langle f, v\rangle_{\mathcal{X}^s_{rad,\gamma}(\Omega),\mathcal{X}^s_{rad,-\gamma}} = \langle \underline{u}, \underline{L}^*(\partial)v\rangle_{\mathcal{X}^s_{rad,\gamma}(\Omega),\mathcal{X}^s_{rad,-\gamma}} = \langle L(\partial)\underline{u}, v\rangle_{\mathcal{X}^s_{rad,\gamma}(\Omega),\mathcal{X}^s_{rad,-\gamma}}$$

so that $L(\partial)\underline{u} = f$. Then choosing $v \in \mathcal{D}(\Omega)$ satisfying $C_j v_{|x_j=0} = 0$ gives, using the fact that $L(\partial)\underline{u} = f$, the equality $\langle B\underline{u}_{|x_j=0} - g_j, M_j v_{|x_j=0}\rangle$. The matrix $M_j$ being onto the right hand side can be replaced by any test function so that we end up with the boundary condition $B_j\underline{u}_{|x_j=0} = g_j$. This shows that $\underline{u}$ is a solution to (64).

To conclude we have to justify that if $f \in \mathscr{H}^s_\gamma(\Omega)$ and $(g_1,\ g_2) \in \mathscr{H}^s_\gamma(\partial\Omega_1) \times \mathscr{H}^s_\gamma(\partial\Omega_2)$ then $\underline{u} \in \mathscr{H}^s_\gamma(\Omega)$. It is essentially made like in Paragraph 4.2.

It is once again a direct consequence of (51) and (56) which permits to control respectively the norm of the solution in $\mathcal{X}^s_{rad,\gamma}$ by the norm of $f \in \mathscr{H}^s_\gamma(\Omega)$ and $(g_1,\ g_2) \in \mathscr{H}^s_\gamma(\partial\Omega_1) \times \mathscr{H}^s_\gamma(\partial\Omega_2)$. The details are omitted here.

□

## 4.4 Proof of the duality formulas

In this paragraph we show that in both $\mathcal{X}^s_{rad}(\Gamma)$ and $\mathcal{Y}^s(\Gamma)$, the adjoint $A^*$ of $A$ can be written under the form $A^* = -A + \phi$, with $\Phi$ linear continuous giving thus the proof of Proposition 4.1.

As we will see in the core of the proofs it is crucial for the analysis that the base Hilbert space $\mathcal{X}^s_{rad}(\Gamma)$ or $\mathcal{Y}^s(\Gamma)$ is defined with respect to the radial derivative and not the tangential ones.

Let us point that from now to the end of the article the multi-index $\delta := (\delta_3, ..., \delta_d) \in \mathbb{N}^{d-2}$ so that the operator $\partial^\delta := \partial_3^{\delta_3} \cdots \partial_d^{\delta_d}$.

### 4.4.1 Duality formula in $\mathcal{X}^s_{rad}(\Gamma)$

Let $u, v \in \mathcal{X}^s_{rad}(\Gamma)$ we have

$$\begin{aligned}\langle Au, v\rangle_{\mathcal{X}^s_{rad}(\Gamma)} =& -\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\le s-\alpha}\int_\Gamma \sum_{j=3}^{d}\big\langle (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta \mathbf{r}^l A_j\partial_j u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l v\big\rangle\,\mathrm{d}x\\ &-\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\le s-\alpha}\int_\Gamma \sum_{j=1}^{2}\big\langle (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta \mathbf{r}^l A_j\partial_j u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l v\big\rangle\,\mathrm{d}x,\\ :=& -\mathcal{I}^{tan}-\mathcal{I}^{nor}.\end{aligned}$$

We first establish the duality formula for the integral term $\mathcal{I}^{tan}$ which is simpler because it involves less commutators than $\mathcal{I}^{nor}$.

Because the coefficients $A_j$ are constant we have

$$(\mathbf{r}\partial_{\mathbf{r}})^\alpha A_j\mathbf{r}^l\partial^\delta u = A_j(\mathbf{r}\partial_{\mathbf{r}})^\alpha\mathbf{r}^l\partial^\delta u. \tag{88}$$

So that we have

$$\mathcal{I}^{tan} := \mathcal{J}^{tan},$$

where

$$\mathcal{J}^{tan} := \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\le s-\alpha}\int_\Gamma\sum_{j=3}^{d}\langle A_j\partial_j(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial_1^q\partial_2^r\mathbf{r}^l\partial^\delta u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x$$

We conclude the derivation of the duality formula involving $\mathcal{I}^{tan}$ by making the integration by parts in $\underline{\mathcal{J}}^{tan}$ because the operators $\partial_j$ are tangential we have, using the symmetry of the coefficients,

$$\mathcal{J}^{tan} = -\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\le s-\alpha}\int_\Gamma\sum_{j=3}^{d}\langle(\mathbf{r}\partial_{\mathbf{r}})^\alpha\mathbf{r}^l\partial^\delta u, A_j\partial_j(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l v)\rangle\,\mathrm{d}x.$$

So that there holds that

$$-\mathcal{I}^{tan} = \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\le s-\alpha}\int_\Gamma\sum_{j=3}^{d}\langle(\mathbf{r}\partial_{\mathbf{r}})^\alpha\mathbf{r}^l\partial^\delta u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l A_j\partial_j v\rangle\,\mathrm{d}x = \overline{\mathcal{I}}^{tan}, \tag{89}$$

where for a bilinear form $\mathcal{I} := i(u, v)$ the notation $\overline{\mathcal{I}}$ stands for $\overline{\mathcal{I}} := i(v, u)$.

We now repeat essentially the same computations to deal with the term containing the normal derivatives that is to say $\mathcal{I}^{nor}$. The method is however modify firstly because $\mathbf{r}^l$ depends on $x_1$ and $x_2$ so we have one commutator to deal with. Then, we also have to take care of the fact that $\partial_j$ and $(\mathbf{r}\partial_{\mathbf{r}})^\alpha$ do not commute any more so that there is an other commutator to handle. Finally note that we also have to deal with boundary terms during the integration by parts.

We have

$$\begin{aligned}\mathcal{I}^{nor} =& \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\le s-\alpha}\int_\Gamma\sum_{j=1}^{2}\big\langle A_j(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\partial_j\mathbf{r}^l u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l v\big\rangle\,\mathrm{d}x\\ &-\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\le s-\alpha}\int_\Gamma\sum_{j=1}^{2}\big\langle lA_j(\partial_j r)(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^{l-1}u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l v\big\rangle\,\mathrm{d}x\\ :=& \widetilde{\mathcal{J}}^{nor}-\mathcal{L}^{nor},\end{aligned}$$

where $\mathcal{L}^{nor} := \ell(u, v)$ is bilinear with respect to $(u, v)$.

We now deal with the commutator $\partial_j$ and $(\mathbf{r}\partial_{\mathbf{r}})^\alpha$ in $\widetilde{\mathcal{J}}^{nor}$. In order to do so for a function $f$ depending on $(t,x)$ we first write from the binomial formula

$$(\mathbf{r}\partial_{\mathbf{r}})^\alpha f := \sum_{p=0}^{\alpha} \binom{\alpha}{p} (x_1\partial_1)^p (x_2\partial_2)^{\alpha-p} f.$$

Then we decompose the compositions of the tangential derivatives operators under the form

$$\forall\, n,m\in\mathbb{N},\ (x_1\partial_1)^n f := \sum_{q=1}^{n} \lambda_q^n x_1^q \partial_1^q f \text{ and } (x_2\partial_2)^m f := \sum_{r=1}^{m} \lambda_r^m x_2^r \partial_2^r f,$$

where the scalars $(\lambda_p^n)_{n,p\in\mathbb{N}}$ are characterized by the relations

$$\begin{cases} \lambda_p^N = \lambda_{p-1}^N + p\lambda_p^N & \forall N \geq p-1, \\ \lambda_p^1 = \cdots = \lambda_p^{p-2} = 0; \lambda_p^{p-1} = 1. \end{cases} \tag{90}$$

With these notations in hand we have

$$\begin{aligned} \widetilde{\mathcal{J}}^{nor} =& \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_\Gamma \sum_{j=1}^{2} \langle A_j\partial_j(\mathbf{r}\partial_{\mathbf{r}})^\alpha \mathbf{r}^l\partial^\delta u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta \mathbf{r}^l v\rangle\,\mathrm{d}x \\ &- \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_\Gamma \langle \sum_{p=0}^{\alpha}\sum_{q=1}^{p}\sum_{r=1}^{\alpha-p} \binom{\alpha}{p} q\lambda_q^p\lambda_r^{\alpha-p} x_1^{q-1}x_2^r \partial_1^q\partial_2^r \mathbf{r}^l A_1\partial^\delta u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta \mathbf{r}^l v\rangle\,\mathrm{d}x \\ &- \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_\Gamma \langle \sum_{p=0}^{\alpha}\sum_{q=1}^{p}\sum_{r=1}^{\alpha-p} \binom{\alpha}{p} r\lambda_q^p\lambda_r^{\alpha-p} x_1^{q}x_2^{r-1} \partial_1^q\partial_2^r \mathbf{r}^l A_2\partial^\delta u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta \mathbf{r}^l v\rangle\,\mathrm{d}x \\ &:= \breve{\mathcal{J}}^{nor} - \mathcal{M}_1^{nor} - \mathcal{M}_2^{nor}. \end{aligned} \tag{91}$$

The terms $\mathcal{M}_1^{nor}$ and $\mathcal{M}_2^{nor}$ being bilinear with respect to $u$ and $v$ so that we write $\mathcal{M}_k^{nor} := m_k(u,v)$. For later purpose let use remark that by definition we have

$$\mathcal{M}_1^{nor} = \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_\Gamma \langle A_1[\partial_1, (\mathbf{r}\partial_{\mathbf{r}})^\alpha]\partial^\delta \mathbf{r}^l u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta \mathbf{r}^l v\rangle\,\mathrm{d}x, \tag{92}$$

$$\mathcal{M}_2^{nor} = \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_\Gamma \langle A_2[\partial_2, (\mathbf{r}\partial_{\mathbf{r}})^\alpha]\partial^\delta \mathbf{r}^l u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta \mathbf{r}^l v\rangle\,\mathrm{d}x. \tag{93}$$

We end up with

$$\breve{\mathcal{J}}^{nor} = \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_\Gamma \sum_{j=1}^{2} \langle A_j\partial_j(\mathbf{r}\partial_{\mathbf{r}})^\alpha \mathbf{r}^l\partial^\delta u), (\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta \mathbf{r}^l v\rangle\,\mathrm{d}x. \tag{94}$$

We now perform the integration by parts in the right hand side of (94) we obtain using the symmetry of

the coefficients

$$\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_\Gamma\sum_{j=1}^{2}\langle\partial_j\partial_j((\mathbf{r}\partial_\mathbf{r})^\alpha\mathbf{r}^l\partial^\delta u),(\mathbf{r}\partial_\mathbf{r})^\alpha\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x =$$

$$-\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_\Gamma\sum_{j=1}^{2}\langle(\mathbf{r}\partial_\mathbf{r})^\alpha\mathbf{r}^l\partial^\delta u),A_j\partial_j[(\mathbf{r}\partial_\mathbf{r})^\alpha\partial^\delta\mathbf{r}^l v\rangle]\,\mathrm{d}x$$

$$-\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_{\partial\Gamma_1}\langle A_1[(\mathbf{r}\partial_\mathbf{r})^\alpha\mathbf{r}^l\partial^\delta u]_{|x_1=0},[(\mathbf{r}\partial_\mathbf{r})^\alpha\partial^\delta\mathbf{r}^l v]_{|x_1=0}\rangle\,\mathrm{d}x'\,\mathrm{d}x_2$$

$$-\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_{\partial\Gamma_2}\langle A_2[(\mathbf{r}\partial_\mathbf{r})^\alpha\mathbf{r}^l\partial^\delta u]_{|x_2=0},[(\mathbf{r}\partial_\mathbf{r})^\alpha\partial^\delta\mathbf{r}^l v]_{|x_2=0}\rangle\,\mathrm{d}x'\,\mathrm{d}x_1,$$

$$:=-\underline{\widetilde{\mathcal{J}}}^{nor}-\mathcal{B}_1-\mathcal{B}_2.$$

Reiterating exactly the same computations as the ones performed so far in order to express $\underline{\widetilde{\mathcal{J}}}^{nor}$ in terms of $(\mathbf{r}\partial_\mathbf{r})^\alpha\mathbf{r}^l\partial^\delta A_j\partial_j v$ instead of $A_j\partial_j(\mathbf{r}\partial_\mathbf{r})^\alpha\mathbf{r}^l\partial^\delta v$ gives at the end of the day:

$$\begin{aligned}-\mathcal{I}^{nor}=&\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_\Gamma\sum_{j=1}^{2}\langle(\mathbf{r}\partial_\mathbf{r})^\alpha\mathbf{r}^l\partial^\delta u,(\mathbf{r}\partial_\mathbf{r})^\alpha\partial^\delta\mathbf{r}^l A_j\partial_j v\rangle]\,\mathrm{d}x\\&+\mathcal{L}^{nor}+\overline{\mathcal{L}}^{nor}+\mathcal{M}_1^{nor}+\overline{\mathcal{M}}_1^{nor}+\mathcal{M}_2^{nor}+\overline{\mathcal{M}}_2^{nor}+\mathcal{B}_1+\mathcal{B}_2,\end{aligned}\tag{95}$$

where we used the notations for $\mathcal{I}\in\{\mathcal{L}^{nor},\mathcal{M}_1^{nor}\}$ associated to the bilinear form $i$, $\overline{\mathcal{I}}:=i(v,u)$.

Summing (89) and (95) then leads to the formula

$$\begin{aligned}\langle Au,v\rangle_{\mathcal{X}^s_{rad}(\Gamma)}=&-\langle u,Av\rangle_{\mathcal{X}^s_{rad}(\Gamma)}+\mathcal{B}_1+\mathcal{B}_2\\&+\mathcal{L}^{nor}+\overline{\mathcal{L}}^{nor}+\mathcal{M}_1^{nor}+\overline{\mathcal{M}}_1^{nor}+\mathcal{M}_2^{nor}+\overline{\mathcal{M}}_2^{nor}.\end{aligned}\tag{96}$$

To end up the proof of Proposition 4.1 we should first cancel the boundary terms in (96). This step is made by imposing some boundary conditions on the domains of $A$ and $A^*$. Then we show that all the bilinear forms appearing in the right hand side, namely $\mathcal{L}^{nor}$, $\mathcal{M}_1^{nor}$ and $\mathcal{M}_2^{nor}$ (and the analogous terms with the notation $\bar{\cdot}$), of (96) are continuous on $\mathcal{X}^s_{rad}(\Gamma)\times\mathcal{X}^s_{rad}(\Gamma)$. So that for each bilinear form $i$, Riesz representation theorem combined with Hahn-Banach theorem permit us to write

$$\mathcal{I}=i(u,v)=\langle u,\varphi_i v\rangle$$

where $\varphi_i:\mathcal{X}^s_{rad}(\Gamma)\to\mathcal{X}^s_{rad}(\Gamma)$ is linear continuous and satisfies $\|\varphi_i\|_{\mathcal{L}(\mathcal{X}^s_{rad}(\Gamma))}=\|i\|_{\mathcal{B}(\mathcal{X}^s_{rad}(\Gamma)\times\mathcal{X}^s_{rad}(\Gamma))}$.

Summing all the $\varphi_i$ gives the linear continuous operator $\Phi$ of Proposition 4.1 and completes the proof.

Consequently we conclude the proof of Proposition 4.1 by showing the three following lemmas. The first one asserts that the boundary terms vanish as expected. The two others show the boundedness of the various operators appearing in the duality formula.

More precisely in Lemma 4.5 we show the boundeness of $\mathcal{L}^{nor}$ essentially by the application of Cauchy-Scharwz inequality.

Lemma 4.6 establishes the result for the operator $\mathcal{M}_1^{nor}$ and $\mathcal{M}_2^{nor}$. As we will see in the proof this result is a little more tricky and uses in a non trivial way that our base Hilbert space is defined via tangential derivatives.

**Lemma 4.4** *With $A:D(A)\to\mathcal{X}^s_{rad}(\Gamma)$ and $A^*:D(A^*)\to\mathcal{X}^s_{rad}(\Gamma)$ defined in Proposition 4.1 we have*

$$\mathcal{B}_1=\mathcal{B}_2=0.$$

**Lemma 4.5** *Let $\mathcal{L}^{nor}$ be previously defined. Then for all $s \geq 0$ there exist a continuous linear operator $\varphi_\ell : \mathcal{X}^s_{rad}(\Gamma) \to \mathcal{X}^s_{rad}(\Gamma)$ such that the following equality holds for all $u$, $v \in \mathcal{X}^s_{rad}(\Gamma)$*

$$\mathcal{L}^{nor} = \langle u, \varphi_\ell v \rangle_{\mathcal{X}^s_{rad}(\Gamma)}.$$

*The analogous result also holds for $\overline{\mathcal{L}}^{nor}$.*

**Lemma 4.6** *Let $\mathcal{M}_1^{nor}$ and $\mathcal{M}_2^{nor}$ be defined as above. Then there exists a continuous linear operator $\varphi_m : \mathcal{X}^s_{rad}(\Gamma) \to \mathcal{X}^s_{rad}(\Gamma)$ such that the following equality holds for all $u$, $v \in \mathcal{X}^s_{rad}(\Gamma)$*

$$\mathcal{M}_1^{nor} + \mathcal{M}_2^{nor} = \langle u, \varphi_m v \rangle_{\mathcal{X}^s_{rad}(\Gamma)}.$$

*The analogous result holds for the operator $\overline{\mathcal{M}}_1^{nor} + \overline{\mathcal{M}}_2^{nor}$.*

So that with Lemmas 4.5 and 4.6 in hand we can show that for all $u$, $v \in \mathcal{X}^s_{rad}(\Gamma)$ we have the identity

$$\langle Au, v \rangle_{\mathcal{X}^s_{rad}(\Gamma)} = -\langle u, Av \rangle_{\mathcal{X}^s_{rad}(\Gamma)} + \mathcal{B}_1 + \mathcal{B}_2 + \langle u, \Phi v \rangle_{\mathcal{X}^s_{rad}(\Gamma)}, \tag{97}$$

while with both Lemmas 4.4, 4.5 and 4.6 we end up with the duality formula that for all $u \in D(A)$ and for all $v \in D(A^*)$

$$\langle Au, v \rangle_{\mathcal{X}^s_{rad}(\Gamma)} = -\langle u, Av \rangle_{\mathcal{X}^s_{rad}(\Gamma)} + \langle u, \Phi v \rangle_{\mathcal{X}^s_{rad}(\Gamma)}. \tag{98}$$

We start by the proof of Lemma 4.4

**Proof of Lemma 4.4** This lemma is a direct consequence of the definition of the dual boundary conditions.

We only show that for $u \in D(A)$ and $v \in D(A^*)$, $\mathcal{B}_1 = 0$. In order to do so, we recall the decomposition of $A_1$ in terms of the boundary matrix $B_1$ and the dual matrix $C_1$:

$$A_1 := M_1^T B_1 + C_1^T N_1.$$

So that using this decomposition in the expression of $\mathcal{B}_1$ gives

$$\begin{aligned}\mathcal{B}_1 =& \sum_{l=0}^{s} \sum_{\alpha=0}^{s-l} \sum_{|\delta| \leq s-\alpha} \int_{\partial\Gamma_1} \langle B_1[(\mathbf{r}\partial_{\mathbf{r}})^\alpha \mathbf{r}^l \partial^\delta u]_{|x_1=0}, M_1[(\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial^\delta \mathbf{r}^l v]_{|x_1=0} \rangle \, \mathrm{d}x' \, \mathrm{d}x_2 \\ &+ \sum_{l=0}^{s} \sum_{\alpha=0}^{s-l} \sum_{|\delta| \leq s-\alpha} \int_{\partial\Gamma_1} \langle N_1[(\mathbf{r}\partial_{\mathbf{r}})^\alpha \mathbf{r}^l \partial^\delta u]_{|x_1=0}, C_1[(\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial^\delta \mathbf{r}^l v]_{|x_1=0} \rangle \, \mathrm{d}x' \, \mathrm{d}x_2.\end{aligned} \tag{99}$$

However for all functions $f$ depending on $x_1$ and $x_2$ we have

$$[(\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial^\delta \mathbf{r}^l f]_{|x_1=0} = (x_2\partial_2)^\alpha \partial^\delta x_2^l f_{|x_1=0},$$

so that the first term in the right hand side of (99) vanishes because by definition of $D(A)$ we have that $u_{|x_1=0} \in \ker B_1$. Similarly the second term in the right hand side of (99) vanishes because by definition of $v \in D(A^*)$.

□

We now proceed to the proof of Lemma 4.5.

**Proof of Lemma 4.5** Clearly it is sufficient to establish the result for the terms without the notation $\bar{\cdot}$.

As already mentioned it is sufficient to show that $\mathcal{L}^{nor}$ is bilinear continuous on $\mathcal{X}^s_{rad}(\Gamma) \times \mathcal{X}^s_{rad}(\Gamma)$, the existence of the operator defined in Lemma 4.5 follows from Riesz representation theorem.

We use Cauchy-Schwarz inequality twice to estimate

$$\begin{aligned}|\mathcal{L}^{nor}| \leq& \left( \sum_{l=1}^{s} \sum_{\alpha=0}^{s-l} \sum_{|\delta| \leq s-\alpha} \int_\Gamma \sum_{j=1}^{2} \left| l(\mathbf{r}\partial_{\mathbf{r}})^\alpha \mathbf{r}^{l-1} \partial^\delta A_j u \right|^2 \mathrm{d}x \right)^{1/2} \times \|v\|_{\mathcal{X}^s_{rad}(\Gamma)}, \\ \leq& C_{A_{1,2}} \left( \sum_{l=0}^{s-1} \sum_{\alpha=0}^{s-l-1} \sum_{|\delta| \leq s-\alpha} \int_\Gamma \left| (l+1)(\mathbf{r}\partial_{\mathbf{r}})^\alpha \mathbf{r}^l \partial^\delta u \right|^2 \mathrm{d}x \right)^{1/2} \times \|v\|_{\mathcal{X}^s_{rad}(\Gamma)}, \\ \leq& C_{A_{1,2}} \|u\|_{\mathcal{X}^s_{rad}(\Gamma)} \times \|v\|_{\mathcal{X}^s_{rad}(\Gamma)}.\end{aligned}$$

□

We conclude by the proof of Lemma 4.6.

**Proof of Lemma 4.6** We now turn to the boundedness of the operator $\mathcal{M}_1^{nor} + \mathcal{M}_2^{nor}$. In order to do so we use equation (92) combined with the following lemma

**Lemma 4.7** *Let $\alpha \in \mathbb{N}$, there exists an operator $T^\alpha$ such that we have the factorisations*

$$[\partial_1, (\mathbf{r}\partial_{\mathbf{r}})^\alpha] = T^\alpha \partial_1 \text{ and } [\partial_2, (\mathbf{r}\partial_{\mathbf{r}})^\alpha] = T^\alpha \partial_2. \tag{100}$$

*Moreover from equation* (91), *$T^\alpha$ is explicitly given by*

$$T^\alpha := \sum_{p=0}^{\alpha} \sum_{q=0}^{p-1} \sum_{r=1}^{\alpha-p} \binom{\alpha}{p} (q+1) \lambda_{q+1}^p \lambda_r^{\alpha-p} x_1^q x_2^r \partial_1^q \partial_2^r. \tag{101}$$

Before to give the proof of this lemma let us indicate that it is this result which make us working with the radial derivative $(\mathbf{r}\partial_{\mathbf{r}})^\alpha$ in the whole duality method and not directly with the derivatives $(x_1\partial_1)^{\alpha_1}(x_2\partial_2)^{\alpha_2}$ for which, at the end of the day, the solution will be regular with respect to. Indeed the previous lemma is true only for the radial derivative and fails for example if one considers the derivative $x_1\partial_1 x_2\partial_2$.

**Proof of Lemma 4.7** We proceed by iteration. For $\alpha = 0$ then the result is satisfied with $T^0 = 0$. Assume that for a given $\alpha \in \mathbb{N}$ we have $[\partial_1, (\mathbf{r}\partial_{\mathbf{r}})^\alpha] = T^\alpha \partial_1$ and $[\partial_2, (\mathbf{r}\partial_{\mathbf{r}})^\alpha] = T^\alpha \partial_2$. We then consider, using a standard property on commutators

$$\begin{aligned}
\left[\partial_1, (\mathbf{r}\partial_{\mathbf{r}})^{\alpha+1}\right] =& (\mathbf{r}\partial_{\mathbf{r}})^\alpha [\partial_1, \mathbf{r}\partial_{\mathbf{r}}] + [\partial_1, (\mathbf{r}\partial_{\mathbf{r}})^\alpha] \mathbf{r}\partial_{\mathbf{r}} \\
=& (\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial_1 + T^\alpha \partial_1 \mathbf{r}\partial_{\mathbf{r}} \\
=& (\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial_1 + T^\alpha [\partial_1, \mathbf{r}\partial_{\mathbf{r}}] + T^\alpha \mathbf{r}\partial_{\mathbf{r}}\partial_1.
\end{aligned}$$

So that we can write

$$\left[\partial_1, (\mathbf{r}\partial_{\mathbf{r}})^{\alpha+1}\right] = T^{\alpha+1}\partial_1 \text{ with } T^{\alpha+1} := (\mathbf{r}\partial_{\mathbf{r}})^\alpha + T^\alpha T^1 + T^\alpha \mathbf{r}\partial_{\mathbf{r}},$$

the definition of $T^{\alpha+1}$ being independent on $\partial_1$ we can then easily show by reiterating the same computations that we also have $\left[\partial_2, (\mathbf{r}\partial_{\mathbf{r}})^{\alpha+1}\right] = T^{\alpha+1}\partial_2$.

□

Using this result we have

$$\begin{aligned}
\mathcal{M}_1^{nor} + \mathcal{M}_2^{nor} =& \sum_{l=0}^{s} \sum_{\alpha=0}^{s-l} \sum_{|\delta| \le s-\alpha} \int_\Gamma \langle T^\alpha A_1 \partial_1 \partial^\delta \mathbf{r}^l u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial^\delta \mathbf{r}^l v\rangle + \langle T^\alpha A_2 \partial_2 \partial^\delta \mathbf{r}^l u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial^\delta \mathbf{r}^l v\rangle \, \mathrm{d}x \\
=& -\sum_{l=0}^{s} \sum_{\alpha=1}^{s-l} \sum_{|\delta| \le s-\alpha} \int_\Gamma \langle \sum_{p=1}^{\alpha-1} \sum_{q=0}^{p-1} \sum_{r=1}^{\alpha-p} \binom{\alpha}{p} (q+1) \lambda_{q+1}^p \lambda_r^{\alpha-p} x_1^q x_2^r \partial_1^q \partial_2^r A \partial^\delta \mathbf{r}^l u, (\mathbf{r}\partial_{\mathbf{r}})^\alpha \partial^\delta \mathbf{r}^l v\rangle \, \mathrm{d}x,
\end{aligned}$$

where we used the fact that from the definition of $A$, $A_1\partial_1 = -A - A_2\partial_2$ combined with the expression of $T^\alpha$ given in (101). Note that we also use the fact that $T^0 = 0$ in order to change the lower bound in the sum on $\alpha$ and we do not write the vanishing terms corresponding to $p = 0$ and $p = \alpha$ in the interior sum.

Using Cauchy-Schwarz inequality twice thus gives

$$\begin{aligned}
&|\mathcal{M}_1^{nor} + \mathcal{M}_2^{nor}| \\
&\le \left( \sum_{l=0}^{s} \sum_{\alpha=1}^{s-l} \sum_{|\delta| \le s-\alpha} \int_\Gamma \left| \sum_{p=0}^{\alpha-2} \sum_{q=1}^{p} \sum_{r=1}^{\alpha-1-p} \binom{\alpha}{p+1} (q+1) \lambda_q^{p+1} \lambda_r^{\alpha-1-p} x_1^q x_2^r \partial_1^q \partial_2^r A \partial^\delta \mathbf{r}^l u \right|^2 \mathrm{d}x \right)^{1/2} \times \|v\|_{\mathcal{X}^s_{rad}(\Gamma)} \\
&\le \left( \sum_{l=0}^{s} \sum_{\alpha=1}^{s-l} \sum_{|\delta| \le s-\alpha} C_\alpha \int_\Gamma \left| (\mathbf{r}\partial_{\mathbf{r}})^{\alpha-1} A \partial^\delta \mathbf{r}^l u \right|^2 \mathrm{d}x \right)^{1/2} \times \|v\|_{\mathcal{X}^s_{rad}(\Gamma)}.
\end{aligned}$$

To show that this right hand side is bounded by $C_s\|u\|_{\mathcal{X}^s_{rad}(\Gamma)}\times\|v\|_{\mathcal{X}^s_{rad}(\Gamma)}$ we use the same method than for characteristic boundary values problems in the half-space exposed in [Benoit, ].

We introduce, $\mathcal{Z}$, the following subset of $\mathcal{D}(\Gamma)$ defined by

$$\mathcal{Z}:=\left\{u\in\mathcal{D}(\Gamma)\setminus(A_1\partial_1+A_2\partial_2)\mathcal{X}^s_{rad}(\Gamma)u=0\right\},$$

then $\mathcal{Z}$ is non empty and for all $u\in\mathcal{Z}$ we have the identity

$$A\partial^\delta\mathbf{r}^l u=\sum_{j\in\mathcal{T}}A_j\partial_j\partial^\delta\mathbf{r}^l u+l(\partial_1\mathbf{r})\mathbf{r}^{l-1}A_1\partial^\delta u+l(\partial_2\mathbf{r})\mathbf{r}^{l-1}A_2\partial^\delta u,$$

so that on $\mathcal{Z}$ the operator $A$ only add a tangential derivative plus an operator of order zero.

Consequently we have for $u\in\mathcal{Z}$, with a slight abuse of notations

$$\begin{aligned}
&|\mathcal{M}_1^{nor}+\mathcal{M}_2^{nor}|\\
\leq&C_s\left(\sum_{l=1}^{s}\sum_{\alpha=1}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_\Gamma\left|(\mathbf{r}\partial_{\mathbf{r}})^{\alpha-1}\partial^\delta\mathbf{r}^{l-1}u\right|^2\,\mathrm{d}x+\sum_{l=0}^{s}\sum_{\alpha=1}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_\Gamma\left|(\mathbf{r}\partial_{\mathbf{r}})^{\alpha-1}\partial^{\delta+1}\mathbf{r}^l u\right|^2\,\mathrm{d}x\right)^{1/2}\times\|v\|_{\mathcal{X}^s_{rad}(\Gamma)}\\
\leq&C_s\left(\|u\|_{\mathcal{X}^{s-1}_{rad}(\Gamma)}+\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l-1}\sum_{0<|\delta|\leq s-\alpha}\int_\Gamma\left|(\mathbf{r}\partial_{\mathbf{r}})^{\alpha}\partial^\delta\mathbf{r}^l u\right|^2\,\mathrm{d}x\right)^{1/2}\times\|v\|_{\mathcal{X}^s_{rad}(\Gamma)}\\
\leq&C_s\|u\|_{\mathcal{X}^s_{rad}(\Gamma)}\times\|v\|_{\mathcal{X}^s_{rad}(\Gamma)}.
\end{aligned}$$

We can thus apply Hahn-Banach theorem which ensures that $\mathcal{M}_1^{nor}+\mathcal{M}_2^{nor}$ can be extended to the whole space $\mathcal{X}^s_{rad}(\Gamma)^2$ such that the extension satisfies $|\mathcal{M}_1^{nor}+\mathcal{M}_2^{nor}|\leq C\|u\|_{\mathcal{X}^s_{rad}(\Gamma)}\times\|v\|_{\mathcal{X}^s_{rad}(\Gamma)}$ on $\mathcal{X}^s_{rad}(\Gamma)^2$. Then Riesz representation theorem gives the existence of some linear continuous operator $\varphi_m:\mathcal{X}^s_{rad}(\Gamma)\to\mathcal{X}^s_{rad}(\Gamma)$ such that we have the equality

$$\forall\, u,\, v\in\mathcal{X}^s_{rad}(\Gamma),\ \mathcal{M}_1^{nor}+\mathcal{M}_2^{nor}=\langle u,\varphi_m(v)\rangle_{\mathcal{X}^s_{rad}(\Gamma)}.$$

□

### 4.4.2 Duality formula in $\mathcal{Y}^s(\Gamma)$

In this paragraph we use the computations exposed in the previous paragraph in order to derive the needed duality formula in the functional space $\mathcal{Y}^s(\Gamma)$.

Before to establish the duality formula we introduce some notations. The set of index of the partition of $\Gamma$ is denoted by $\Gamma^\flat:=\{\mathscr{C},\mathscr{B}_1,\mathscr{B}_2,\mathscr{I}\}$. We then define the following differentiation operators (all depending on $\alpha$, we omit this dependency in order to simplify the notations)

$$\partial_{\mathscr{C}}:=(\mathbf{r}\partial_{\mathbf{r}})^\alpha,\partial_{\mathscr{B}_1}:=\sum_{p=0}^{\alpha}\binom{\alpha}{p}\underline{\partial}_{1,p}(x_2\partial_2)^{\alpha-p},\partial_{\mathscr{B}_2}:=\sum_{p=0}^{\alpha}\binom{\alpha}{p}(x_1\partial_1)^p\underline{\partial}_{2,\alpha-p}\text{ and }\partial_{\mathscr{I}}:=\sum_{p=0}^{\alpha}\binom{\alpha}{p}\underline{\partial}_{1,p}\underline{\partial}_{2,\alpha-p}.$$

Let $u,\, v\in\mathcal{Y}^s(\Gamma)$ be given, we shall compute

$$\langle Au,v\rangle_{\mathcal{Y}^s}:=-\mathcal{I}^{tan}-\mathcal{I}^{nor}$$

where we defined

$$\begin{aligned}
\mathcal{I}^{tan}:=\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}&\int_{\mathscr{C}}\sum_{j=3}^{d}\langle A_j\partial_j(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l u,(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta v\rangle\,\mathrm{d}x+\int_{\mathscr{B}_1}\sum_{j=3}^{d}\langle\partial_{\mathscr{B}_1}\partial^\delta A_j\partial_j u,\partial_{\mathscr{B}_1}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x\\
&+\int_{\mathscr{B}_2}\sum_{j=3}^{d}\langle\partial_{\mathscr{B}_2}A_j\partial_j\mathbf{r}^l u,\partial_{\mathscr{B}_2}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x+\int_{\mathscr{I}}\sum_{j=3}^{d}\langle\partial_{\mathscr{I}}\partial^\delta A_j\partial_j\mathbf{r}^l u,\partial_{\mathscr{I}}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x
\end{aligned}$$

and

$$\mathcal{I}^{nor} := \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_{\mathscr{C}}\sum_{j=1}^{2}\langle A_j\partial_j(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l u,(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x + \int_{\mathscr{B}_1}\sum_{j=1}^{2}\langle\partial_{\mathscr{B}_1}\partial^\delta A_j\partial_j\mathbf{r}^l u,\partial_{\mathscr{B}_1}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x$$
$$+\int_{\mathscr{B}_2}\sum_{j=1}^{2}\langle\partial_{\mathscr{B}_2}A_j\partial_j\mathbf{r}^l u,\partial_{\mathscr{B}_2}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x + \int_{\mathscr{I}}\sum_{j=1}^{2}\langle\partial_{\mathscr{I}}\partial^\delta A_j\partial_j\mathbf{r}^l u,\partial_{\mathscr{I}}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x$$

Like for the duality formula in $\mathcal{X}^s_{rad}(\Gamma)$ we treat the terms $\mathcal{I}^{tan}$ and $\mathcal{I}^{nor}$ separately. For $\mathcal{I}^{tan}$ we can reiterate the same computations as the ones performed in Paragraph 4.4.1, we do not have any commutator to deal with because the $A_j$ are constant. Indeed the operators $\partial_j$ is tangential so that the integration by parts $\langle\partial_j A_j\partial^\delta u,\partial^\delta v\rangle$ becomes $\langle\partial^\delta u,A_j\partial_j\partial^\delta v\rangle$ does not see the decomposition of $\Gamma$ and does not involve any boundary term.

We thus can write

$$-\mathcal{I}^{tan} = \overline{\mathcal{I}}^{tan}.$$

We then proceed to the term involving the normal derivatives namely $\mathcal{I}^{nor}$. The computations follows essentially the ones performed for $\mathcal{I}^{tan}$ see also the ones of Paragraph 4.4.1. However we have to pay attention to two points:

1. The first one is that because of the partition of $\Gamma$ boundary terms along $\{x_1=1\}$ or $\{x_2=1\}$ will appear when we make the integration by parts with respect to $\partial_j$. However as we will see this boundary terms vanish because of the definition of the operators $\underline{\partial}_{1,p}$ and $\underline{\partial}_{2,\alpha-p}$.

2. The second one is that in the proof of the boundedness of the operator $\mathcal{M}^{nor}_1+\mathcal{M}^{nor}_2$ which relies on the commutators of $\partial_1$ and $\partial_2$ with $(\mathbf{r}\partial_{\mathbf{r}})^\alpha$ we used in a non trivial way the fact that our base Hilbert space was defined with respect to the radial derivative to avoid the bad terms $A_1\partial_1$ and $A_2\partial_2$ coming from the definition of $A$ (see Lemma 4.7). This is not true any more because the radial derivative only acts on $\mathscr{C}$, not on the whole space.

   What saves the day is that, to fix the ideas we consider the area $\mathscr{B}_1$, we can use the equation in order to express the bad derivative $A_2\partial_2 = -A - A_1\partial_1$. We then treat the term involving $A$ like in Lemma 4.6. Concerning the term involving $A_1\partial_1$ this term can be controlled in $\mathscr{B}_1$ because the operator $\underline{\partial}_{1,p}$ naturally gives the control of the normal derivative $\partial_1$.

Reiterating the same kind of computations than the ones performed for $\mathcal{I}^{tan}$ (but in which we deal with the commutators $[\mathbf{r}^l,\partial_j]$ and $[\partial_{\mathcal{A}},\partial_j]$) we obtain that

$$\mathcal{I}^{nor} = -\mathcal{M}^{nor}_1-\mathcal{M}^{nor}_2-\sum_{\mathcal{A}\in\Gamma^\flat}\mathcal{L}^{nor}_{\mathcal{A}}+\underbrace{\sum_{\mathcal{A}\in\Gamma^\flat}\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_{\mathcal{A}}\sum_{j=1}^{2}\langle\partial_j(A_j\partial_{\mathcal{A}}\partial^\delta\mathbf{r}^l u),\partial_{\mathcal{A}}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x}_{:=\underline{\mathcal{I}}^{nor}}, \tag{102}$$

Where for $\mathcal{A}\in\Gamma^\flat$ the operator $\mathcal{L}^{nor}_{\mathcal{A}}$ is defined by

$$\mathcal{L}^{nor}_{\mathcal{A}} := \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_{\mathcal{A}}\sum_{j=1}^{2}\langle l(\partial_j r)\partial_{\mathcal{A}}\partial^\delta\mathbf{r}^{l-1}A_j u,\partial_{\mathcal{A}}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x,$$

and where we used the fact that for all $p\in[\![0,\alpha]\!]$ we have the identity $[\underline{\partial}_{1,p},\partial_1]=[\underline{\partial}_{2,\alpha-p},\partial_2]=0$ in order to obtain

$$\mathcal{M}^{nor}_1 := \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_{\mathscr{C}}\langle A_1[\partial_1,(\mathbf{r}\partial_{\mathbf{r}})^\alpha]\partial^\delta\mathbf{r}^l u,(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x \tag{103}$$
$$+\int_{\mathscr{B}_2}\langle\sum_{p=0}^{\alpha}\underline{\partial}_{2,\alpha-p}A_1[\partial_1,(x_1\partial_1)^p]\partial^\delta\mathbf{r}^l u,\partial_{\mathcal{B}_2}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x,$$

and

$$\mathcal{M}_2^{nor} := \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\le s-\alpha}\int_{\mathscr{C}}\langle A_2[\partial_2,(\mathbf{r}\partial_{\mathbf{r}})^\alpha]\partial^\delta\mathbf{r}^l u,(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x \tag{104}$$
$$+\int_{\mathscr{B}_1}\langle\sum_{p=0}^{\alpha}\underline{\partial}_{1,p}A_2[\partial_2,(x_2\partial_2)^{\alpha-p}]\partial^\delta\mathbf{r}^l u,\partial_{\mathcal{B}_1}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x.$$

The commutators in (103) and (104) can be explicited like in (91).

The end of the determination of the adjoint operator is rather clear. We make the integrations by parts in $\underline{\mathcal{J}}^{nor}$ and then we reiterate exactly the same computations as the ones exposed from the beginning of this paragraph on the term $A_j\partial_j(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta v$. Let us stress that from the definition of the operators $\underline{\partial}_{1,p}$ and $\underline{\partial}_{2,\alpha-p}$ the boundary terms along $\{x_1=1\}$ or $\{x_2=1\}$ appearing during the integration by parts vanish. Indeed let us for example consider the term involving $A_1\partial_1$ in $\underline{\mathcal{I}}^{nor}$. We fix $l\,,\alpha$ and $\delta$ and let $f:=\partial^\delta\mathbf{r}^l u$ and $g:=\partial^\delta\mathbf{r}^l v$ then we have

$$\begin{aligned}
&\sum_{\mathcal{A}\in\Gamma^\flat}\int_{\mathcal{A}}\langle A_1\partial_1\partial_{\mathcal{A}}f,\partial_{\mathcal{A}}g\rangle\,\mathrm{d}x\\
&\quad=-\sum_{\mathcal{A}\in\Gamma^\flat}\int_{\mathcal{A}}\langle\partial_{\mathcal{A}}f,A_1\partial_1\partial_{\mathcal{A}}g\rangle\,\mathrm{d}x+\widetilde{\mathcal{B}}_1\\
&\qquad+\int_0^1\int_{\mathbb{R}^{d-2}}\langle A_1\left[(\mathbf{r}\partial_{\mathbf{r}})^\alpha f\right]_{|x_1=1},\left[(\mathbf{r}\partial_{\mathbf{r}})^\alpha g\right]_{|x_1=1}\rangle\,\mathrm{d}x'\,\mathrm{d}x_2\\
&\qquad+\int_1^\infty\int_{\mathbb{R}^{d-2}}\langle\sum_{p=0}^{\alpha}A_1\left[\binom{\alpha}{p}\underline{\partial}_{2,\alpha-p}(x_1\partial_1)^p f\right]_{|x_1=1},\sum_{p=0}^{\alpha}\left[\binom{\alpha}{p}\underline{\partial}_{2,\alpha-p}(x_1\partial_1)^p g\right]_{|x_1=1}\rangle\,\mathrm{d}x'\,\mathrm{d}x_2\\
&\qquad-\int_0^1\int_{\mathbb{R}^{d-2}}\langle\sum_{p=0}^{\alpha}A_1\left[\binom{\alpha}{p}\underline{\partial}_{1,p}(x_2\partial_2)^{\alpha-p} f\right]_{|x_1=1},\sum_{p=0}^{\alpha}\left[\binom{\alpha}{p}\underline{\partial}_{1,p}(x_2\partial_2)^{\alpha-p} g\right]_{|x_1=1}\rangle\,\mathrm{d}x'\,\mathrm{d}x_2\\
&\qquad-\int_1^\infty\int_{\mathbb{R}^{d-2}}\sum_{p=0}^{\alpha}\langle A_1\left[\binom{\alpha}{p}\underline{\partial}_{1,p}\underline{\partial}_{2,\alpha-p} f\right]_{|x_1=1},\sum_{p=0}^{\alpha}\left[\binom{\alpha}{p}\underline{\partial}_{1,p}\underline{\partial}_{2,\alpha-p} g\right]_{|x_1=1}\rangle\,\mathrm{d}x'\,\mathrm{d}x_2,
\end{aligned}\tag{105}$$

where $\widetilde{\mathcal{B}}_1$ is defined bellow.

Then recall that we write $(\mathbf{r}\partial_{\mathbf{r}})^\alpha f=\sum_{p=0}^{\alpha}\sum_{q=1}^{p}\sum_{r=1}^{\alpha-p}\binom{\alpha}{p}\lambda_q^p\lambda_r^{\alpha-p}x_1^q x_2^r\partial_1^q\partial_2^r f$ so that

$$\begin{aligned}
\left[(\mathbf{r}\partial_{\mathbf{r}})^\alpha f\right]_{|x_1=1}&=\sum_{p=0}^{\alpha}\sum_{q=1}^{p}\sum_{r=1}^{\alpha-p}\binom{\alpha}{p}\lambda_q^p\lambda_r^{\alpha-p}x_2^r(\partial_1^q\partial_2^r f)_{|x_1=0}\\
&=\sum_{p=0}^{\alpha}\binom{\alpha}{p}(x_2\partial_2)^{\alpha-p}(\underline{\partial}_{1,p}f)_{|x_1=0},
\end{aligned}$$

and consequently the first boundary term in the right hand side of (105) is compensated by the third one and similarly the second term in the right hand side of (105) is compensated by the fourth one.

Consequently we end up with the duality formula

$$\langle Au,v\rangle_{\mathcal{Y}}=-\langle u,Av\rangle_{\mathcal{Y}}+\widetilde{\mathcal{B}}_1+\widetilde{\mathcal{B}}_2+\mathcal{M}_1^{nor}+\overline{\mathcal{M}}_1^{nor}+\mathcal{M}_2^{nor}+\overline{\mathcal{M}}_2^{nor}+\sum_{\mathcal{A}\in\Gamma^\flat}\mathcal{L}_{\mathcal{A}}^{nor} \tag{106}$$

where $\widetilde{\mathcal{B}}_1$ and $\widetilde{\mathcal{B}}_2$ stand for the boundary terms along $\{x_1=0\}$ and $\{x_2=0\}$appearing in the integration by

parts. They are given by for $k \in \{1,2\}$:

$$\widetilde{\mathcal{B}}_k := -\sum_{\alpha=0}^{s}\sum_{|\delta|\leq s-\alpha}\int_0^1\int_{\mathbb{R}^{d-2}}\langle\left[(x_{3-k}\partial_{3-k})^\alpha\partial^\delta u\right]_{|x_k=0},\left[(x_{3-k}\partial_{3-k})^\alpha\partial^\delta v\right]_{|x_k=0}\rangle\,\mathrm{d}x'\,\mathrm{d}x_{3-k}$$
$$-\sum_{\alpha=0}^{s}\sum_{|\delta|\leq s-\alpha}\int_1^\infty\int_{\mathbb{R}^{d-2}}\langle\left[\underline{\partial}_{3-k,\alpha}\partial^\delta u\right]_{|x_k=0},\left[\underline{\partial}_{3-k,\alpha}\partial^\delta v\right]_{|x_k=0}\rangle\,\mathrm{d}x'\,\mathrm{d}x_{3-k}.$$

In order to conclude we have to show the analogous of Lemmas 4.4 and 4.6 for the new Hilbert space $\mathcal{Y}^s(\Gamma)$. The proofs of Lemma 4.4 in the space $\mathcal{Y}^s(\Gamma)$ follow exactly the same as the one in the space $\mathcal{X}^s_{rad}(\Gamma)$ so that they are omitted. Similarly the proof of the fact that the $\mathcal{L}^{nor}_{\mathcal{A}}$ give rise to bilinear continuous forms follow the same lines than Lemma 4.5.

Because on the other hand Lemma 4.6 relies on the fact that we work with the radial derivative $(\mathbf{r}\partial_{\mathbf{r}})^\alpha$ which is not true any more on the whole space then we give in the following its proof.

**Lemma 4.8** *Let $\mathcal{M}^{nor}_1$ and $\mathcal{M}^{nor}_2$ be defined in* (103) *and* (104) *then there exists a linear continuous operator $\varphi'_m : \mathcal{Y}^s(\Gamma)\to\mathcal{Y}^s(\Gamma)$ such that we have the equality*

$$\forall\, u,v\in\mathcal{Y}^s(\Gamma),\ \mathcal{M}^{nor}_1+\mathcal{M}^{nor}_2=\langle u,\varphi'_m v\rangle_{\mathcal{Y}^s(\Gamma)}.$$

*The analogous result holds for $\overline{\mathcal{M}}^{nor}_1+\overline{\mathcal{M}}^{nor}_2$.*

**Proof of Lemma 4.8** We have

$$\mathcal{M}^{nor}_1+\mathcal{M}^{nor}_2=\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_{\mathscr{C}}\langle(A_1[\partial_1,(\mathbf{r}\partial_{\mathbf{r}})^\alpha]+A_2\left[\partial_2,(\mathbf{r}\partial_{\mathbf{r}})^\alpha\right]\partial^\delta\mathbf{r}^l u,(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x \tag{107}$$
$$+\int_{\mathscr{B}_2}\sum_{p=0}^{\alpha}\langle\underline{\partial}_{2,\alpha-p}A_1\left[\partial_1,(x_1\partial_1)^p\right]\partial^\delta\mathbf{r}^l u,\partial_{\mathscr{B}_2}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x$$
$$+\int_{\mathscr{B}_1}\sum_{p=0}^{\alpha}\langle\underline{\partial}_{1,p}A_2\left[\partial_2,(x_2\partial_2)^{\alpha-p}\right]\partial^\delta\mathbf{r}^l u,\partial_{\mathscr{B}_1}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x.$$

For the first term in the right hand side of (107) we have radial derivatives so that we can proceed exactly like in Paragraph 4.4.1 (see Lemma 4.6) we thus obtain the bound

$$\left|\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_{\mathscr{C}}\langle(A_1[\partial_1,(\mathbf{r}\partial_{\mathbf{r}})^\alpha]+A_2\left[\partial_2,(\mathbf{r}\partial_{\mathbf{r}})^\alpha\right]\partial^\delta\mathbf{r}^l u,(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x\right|\leq C_s\|u\|_{\mathcal{Y}^s(\Gamma)}\times\|v\|_{\mathcal{Y}^s(\Gamma)}. \tag{108}$$

To conclude we shall then establish such bounds for the others terms in the right hand side of (107). In the following we give the bound for the term defined on $\mathscr{B}_2$, the analysis is the same for the one defined on $\mathscr{B}_1$. We explicit $[\partial_1,(x_1\partial_1)^p]=\sum_{q=1}^p\lambda^p_q x_1^{q-1}\partial_1^q$ so that using the definition of $A$ we have

$$\sum_{p=0}^{\alpha}\underline{\partial}_{2,\alpha-p}A_1\left[\partial_1,(x_1\partial_1)^p\right]\partial^\delta\mathbf{r}^l u=\underbrace{-\sum_{p=1}^{\alpha}\sum_{q=0}^{p-1}(q+1)\underline{\partial}_{2,\alpha-p}\lambda^p_{q+1}x_1^q\partial_1^q A\partial^\delta\,\mathbf{r}^l u}_{:=\mathcal{N}^\alpha_A}$$
$$\underbrace{-\sum_{p=1}^{\alpha}\sum_{q=0}^{p-1}(q+1)\underline{\partial}_{2,\alpha-p}\lambda^p_{q+1}x_1^q\partial_1^q A_2\partial_2\partial^\delta\,\mathbf{r}^l u}_{:=\mathcal{N}^\alpha_{A_2}},$$

so that we are leading to consider the terms $\mathcal{N}_A$ and $\mathcal{N}_{A_2}$ separately. By Cauchy-Schwarz inequality and the triangle inequality we have

$$\begin{aligned}\Big|\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_{\mathscr{B}_2}\langle\sum_{p=0}^{\alpha}\underline{\partial}_{2,\alpha-p}\left[\partial_1,(x_1\partial_1)^p\right]A_1\partial^\delta\mathbf{r}^l u,\partial_{\mathscr{B}_2}\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x\Big|\\ \leq & C\left(\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_{\mathscr{B}_2}\left|\mathcal{N}_A^\alpha\mathbf{r}^l u\right|^2\,\mathrm{d}x\right)^{1/2}\times\|v\|_{\mathcal{Y}^s(\Gamma)}\\ & + C\left(\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_{\mathscr{B}_2}\left|\mathcal{N}_{A_2}^\alpha\mathbf{r}^l u\right|^2\,\mathrm{d}x\right)^{1/2}\times\|v\|_{\mathcal{Y}^s(\Gamma)},\end{aligned}$$

and to conclude we just have to study the numbers of derivatives in $\mathcal{N}_{A_2}^\alpha$ and in $\mathcal{N}_A^\alpha$. For convenience we recall that by definition of the base space $\mathcal{Y}^s(\Gamma)$ in the area $\mathscr{B}_2$ we can take normal derivatives $\partial_2$ and tangential derivatives $(x_1\partial_1)$ of $\partial^\delta u$. Using the definition of $\underline{\partial}_{2,\alpha-p}$ we have

$$\begin{aligned}|\mathcal{N}_{A_2}^\alpha\mathbf{r}^l u|\leq & C_{s,A}\left(\sum_{p=1}^{\alpha}\left|\sum_{q=1}^{p-1}\sum_{r=1}^{\alpha-p}x_1^q\partial_1^q\partial_2^{r+1}\partial^\delta\mathbf{r}^l u\right|+\sum_{p=1}^{\alpha}\sum_{r=1}^{\alpha-p}|\partial_2^{r+1}\partial^\delta\mathbf{r}^l u|\right)\\ \leq & C_s\left(\sum_{p=1}^{\alpha}(x_1\partial_1)^{p-1}\left|\sum_{r=1}^{\alpha-p}\partial_2^{r+1}\partial^\delta\mathbf{r}^l u\right|+\sum_{p=0}^{\alpha-1}\left|\underline{\partial}_{2,\alpha-p}\partial^\delta\mathbf{r}^l u\right|\right)\\ \leq & C_s\left(\sum_{p=0}^{\alpha-1}|(x_1\partial_1)^p\partial_{2,\alpha-p}\partial^\delta\mathbf{r}^l u|+\sum_{p=0}^{\alpha-1}\left|\underline{\partial}_{2,\alpha-p}\partial^\delta\mathbf{r}^l u\right|\right),\end{aligned}$$

so that we obtain the desired bound (the second term in the right hand side being bounded inductively)

$$\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_{\mathscr{B}_2}\left|\mathcal{N}_{A_2}^\alpha\mathbf{r}^l u\right|^2\,\mathrm{d}x\leq C_s\|u\|^2_{\mathcal{Y}^s(\Gamma)}. \tag{109}$$

To conclude we reiterate the same computations for the term $\mathcal{N}_A^\alpha$ (without the normal derivative $\partial_2$) we obtain

$$|\mathcal{N}_A^\alpha\mathbf{r}^l u|\leq C_s\left(\sum_{p=0}^{\alpha-1}|(x_1\partial_1)^p\partial_{2,\alpha-1-p}A\partial^\delta\mathbf{r}^l u|+\sum_{p=1}^{\alpha}\left|\underline{\partial}_{2,\alpha-p}A\partial^\delta\mathbf{r}^l u\right|\right), \tag{110}$$

in which we remark that each operator of differentiation in the right hand side in now of order $\alpha-1$. We conclude by introducing like in Paragraph 4.4.1 the subspace $\mathcal{Z}$ of $\mathcal{D}(\Gamma)$ by(

$$\mathcal{Z}:=\left\{u\in\mathcal{D}(\Gamma)\setminus A_1\partial_1+A_2\partial_2)u=0\right\}.$$

For $u\in\mathcal{Z}$, $A$ adds one derivative with respect to $\partial^\delta$ and on zero order term. Because the operators in the right hand side of (110) are of order $\alpha-1$ we have the desired estimate that for all $u\in\mathcal{Z}$

$$\sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int_{\mathscr{B}_2}\left|\mathcal{N}_A^\alpha\mathbf{r}^l u\right|^2\,\mathrm{d}x\leq\|u\|^2_{\mathcal{Y}^s(\Gamma)}, \tag{111}$$

estimate that can be extended to the whole space $\mathcal{Y}^s(\Gamma)$ thanks to Hahn-Banach Theorem. So that from (107)-(109) and (111) the form $\mathcal{M}_1^{nor}+\mathcal{M}_2^{nor}$ is bilinear continuous on $\mathcal{Y}^s(\Gamma)\times\mathcal{Y}^s(\Gamma)$. Riesz representation theorem then concludes the proof.

□

# 5 Applications

In this section we give some examples of physical interest for which Assumption 2.2 about the spectrum of the corner matrix $A_1^{-1}A_2$ is satisfied. We borrow some of the examples of [Huang and Temam, 2014]. The list exposed here being probably far to be exhaustive.

## 5.1 The wave example

In the following we consider the $2d$-wave equation

$$\partial_t^2 u - (\partial_1^2 + \partial_2^2)u = f \text{ for } (t,x) \in \mathbb{R} \times \mathbb{R}_+^2.$$

It is well-known that this scalar second order partial differential equation can be written in the following first order system of partial differential equation

$$\begin{cases} \partial_t U + A_1\partial_1 U + \partial_2 U = \widetilde{f} & \text{in } \Omega, \\ B_1 U_{|x_1=0} = \widetilde{g}_1 & \text{on } \partial\Omega_1, \\ B_2 U_{|x_2=0} = \widetilde{g}_2 & \text{on } \partial\Omega_2, \\ U_{|t=0} = \widetilde{u}_0 & \text{on } \Gamma, \end{cases} \tag{112}$$

for suitable boundary matrices $B_1$, $B_2 \in \mathbf{M}_{1\times 2}(\mathbb{R})$ so that it is a simple exercise to determine the boundary matrices giving rise to strictly dissipative boundary conditions.

In (112) the coefficients $A_1$, $A_2$ are given by

$$A_1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We thus have

$$A_1^{-1}A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

which clearly has $\pm i$ as eigenvalues so that Assumption 2.2 is satisfied.

## 5.2 Inviscid shallow water equation

The inviscid shallow water equation (SWE) in two dimensions reads after linearisation around the constant state $(u_0, v_0, \phi_0) \in \mathbb{R}^2 \times \mathbb{R}_+$, $u_0, v_0 \neq 0$ in the form

$$\partial_t U + A_1\partial_1 U + A_2\partial_2 U + CU = 0, \tag{113}$$

where $U := (u, v, \phi)^T$ is the vector encoding the velocity $(u, v) \in \mathbb{R}^2$ and the height of the fluid $\phi \in \mathbb{R}_+$ and where the coefficients $A_1$ $A_2$ and $C$ are explicitly given by

$$A_1 := \begin{bmatrix} u_0 & 0 & g \\ 0 & u_0 & 0 \\ \phi_0 & 0 & u_0 \end{bmatrix}, \; A_2 := \begin{bmatrix} v_0 & 0 & 0 \\ 0 & v_0 & g \\ 0 & \phi_0 & v_0 \end{bmatrix} \text{ and } C := \begin{bmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $f$ stands for the Coriolis parameter and where $g > 0$ is the gravitational acceleration. In the following we assume that $u_0^2 \neq g\phi_0$, $v_0^2 \neq g\phi_0$ and $u_0^2 + v_0^2 \neq g\phi_0$ so that in particular the matrices $A_1$ and $A_2$ are invertible.

Moreover it is easy to see that their eigenvalues are respectively given by $\lambda_{1,0} = u_0$, $\lambda_{1,\pm} = u_0 \pm \sqrt{u_0^2 + g\phi_0}$ and $\lambda_{2,0} = v_0$, $\lambda_{2,\pm} = v_0 \pm \sqrt{v_0^2 + g\phi_0}$, so that independently on the sign of $u_0$ and $v_0$, $A_1$ and $A_2$ have at least one positive eigenvalue so that there is non trivial boundary conditions to be imposed in addition to the partial differential equation (113).

We now discuss Assumption 2.2. We consider $\lambda \in \mathbb{C}$ an eigenvalue of $A_1^{-1}A_2$ it is characterized by

$$\begin{aligned} \det(A_2 - \lambda A_1) = 0 &\Leftrightarrow (v_0 - \lambda u_0) \cdot \left[(u_0^2 - \phi_0 g)\lambda^2 - 2u_0 v_0 \lambda + v_0^2 - \phi_0 g\right] = 0, \\ &\Leftrightarrow (v_0 - \lambda u_0) P(\lambda) = 0 \end{aligned}$$

showing that $\frac{v_0}{u_0}$ is a (real) eigenvalue of $A_1^{-1}A_2$. So that in order that Assumption 2.2 holds we assume that $u_0$ and $v_0$ have opposite sign. As a consequence the fluid of linearisation can be incoming/outgoing or outgoing/incoming, in order that Assumption 2.2 holds.

To conclude the study of Assumption 2.2 we investigate the roots of $P$. Its discriminant $\Delta = 4\phi_0 g(u_0^2 + v_0^2 - g\phi_0)$ so that if $|(u_0, v_0)|^2 < g\phi_0$ the roots of $P$ have non trivial imaginary part and thus Assumption 2.2 applies.

If $|(u_0, v_0)|^2 \geq g\phi_0$, the roots of $P$ are real and they are given by

$$\lambda_- := \frac{u_0 v_0 - \sqrt{\phi_0 g(u_0^2 + v_0^2 - \phi_0 g)}}{u_0^2 + \phi_0 g} < 0 \text{ and } \lambda_+ := \frac{u_0 v_0 + \sqrt{\phi_0 g(u_0^2 + v_0^2 - \phi_0 g)}}{u_0^2 + \phi_0 g},$$

equation in which we see that $\lambda_+ < 0$ if and only if $u_0^2 v_0^2 > \phi_0 g(u_0^2 + v_0^2 - \phi_0 g)$. So that Assumption 2.2 can be satisfied even if $|(u_0, v_0)|^2 \geq g\phi_0$.

So that to apply Corollary 2.1 or Theorem 2.3 then assuming that the boundary conditions are strictly dissipative only the symmetry assumption of the coefficients is missing. Equation (113) is of course not symmetric but it is Friedrichs symmetrizable. Indeed if we introduce the diagonal matrix

$$S := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{g}{\phi_0} \end{bmatrix} > 0,$$

and the change of unknown $V := S^{1/2}U$ then we recover the equation

$$\partial_t V + \widetilde{A}_1 \partial_1 V + \widetilde{A}_2 \partial_2 V + \widetilde{C} V = 0, \tag{114}$$

where

$$\widetilde{A}_1 := S^{1/2} A_1 S^{-1/2} = \begin{bmatrix} u_0 & 0 & \sqrt{g\phi_0} \\ 0 & u_0 & 0 \\ \sqrt{g\phi_0} & 0 & u_0 \end{bmatrix} \text{ and } \widetilde{A}_2 := S^{1/2} A_2 S^{-1/2} = \begin{bmatrix} v_0 & 0 & 0 \\ 0 & v_0 & \sqrt{g\phi_0} \\ 0 & \sqrt{g\phi_0} & v_0 \end{bmatrix} \tag{115}$$

are now symmetric. Of course because of its special form Assumption 2.2 on $A_1^{-1}A_2$ is equivalent to the one on $\widetilde{A}_1^{-1}\widetilde{A}_2$ so that Corollary 2.1 and Theorem 2.3 apply to (114).

## 5.3 Euler equation

In this paragraph we consider the $2d$-Euler equation

$$\text{for } (t,x) \in \mathbb{R} \times \mathbb{R}_+^2, \quad \begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \rho^{-1} \nabla p = 0, \\ \partial_t e + \mathbf{u} \cdot \nabla e + \rho^{-1} p \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{116}$$

where $\rho \in \mathbb{R}_+$ stands for the density of the fluid, $\mathbf{u} := (u, v) \in \mathbb{R}^2$ denotes the velocity and where $e \in \mathbb{R}_+$ stands for the internal energy. In (116) we close the system by imposing the pressure law $p = p(\rho, e)$. After linearisation around the constant state $(u_0, v_0, \rho_0, e_0)$, we use the notation $\mathbf{u}_0 := (u_0, v_0)$ with $u_0, v_0 \neq 0$, Euler equation reads under the form

$$\partial_t U + A_1 \partial_1 U + \partial_2 U = 0, \text{ for } (t,x) \in \mathbb{R} \times \mathbb{R}_+^2$$

where $U := (u, v, \rho, e)^T$ and where the coefficients are given by

$$A_1 := \begin{bmatrix} u_0 & 0 & \frac{1}{\rho_0}\partial_\rho p_0 & \frac{1}{\rho_0}\partial_e p_0 \\ 0 & u_0 & 0 & 0 \\ \rho_0 & 0 & u_0 & 0 \\ \frac{1}{\rho_0} p_0 & 0 & 0 & u_0 \end{bmatrix} \text{ and } A_2 := \begin{bmatrix} v_0 & 0 & 0 & 0 \\ 0 & v_0 & \frac{1}{\rho_0}\partial_\rho p_0 & \frac{1}{\rho_0}\partial_e p_0 \\ 0 & \rho_0 & v_0 & 0 \\ 0 & \frac{1}{\rho_0} p_0 & 0 & v_0 \end{bmatrix},$$

where we introduce the notations $p_0 := p(\rho_0, e_0)$, $\partial_\rho p_0 := (\partial_\rho p)(p_0, e_0)$ and $\partial_e p_0 := (\partial_e p)(p_0, e_0)$. For later purpose, assuming that the system (116) is hyperbolic so that we have $\partial_\rho p_0 + \frac{1}{\rho_0^2} p_0 \partial_e p_0 \geq 0$, we also introduce the local speed of sound $c := \sqrt{\partial_\rho p_0 + \frac{1}{\rho_0^2} p_0 \partial_e p_0}$.

The eigenvalues $\lambda_1$ of $A_1$ and $\lambda_2$ of $A_2$ are given by

$$\lambda_{1,0} = u_0,\ \lambda_{2,0} := v_0 \text{ and } \lambda_{1,\pm} = u_0 \pm c,\ \lambda_{2,\pm} = v_0 \pm c.$$

So that to ensure that both $A_1$ and $A_2$ have at least one positive eigenvalue and not four we impose that if $u_0 < 0$ (resp. $v_0 < 0$) then $-u_0 < c$ (resp. $-v_0 < c$) and if $u_0 > 0$ (resp. $v_0 > 0$) then $u_0 < c$ and $v_0 < c$.

We now turn to a discussion about Assumption 2.2. We have the following characterization of the eigenvalues of $A_1^{-1}A_2$

$$\begin{aligned}\det(A_1^{-1}A_2 - \lambda I) = 0 \Leftrightarrow & (v_0 - \lambda u_0)^2 \cdot \left[(u_0^2 - c^2)\lambda^2 - 2u_0 v_0 \lambda + v_0^2 - c^2\right] = 0 \\ \Leftrightarrow & (v_0 - \lambda u_0)^2 P(\lambda) = 0.\end{aligned}$$

Consequently $\lambda := \frac{v_0}{u_0}$ is a real eigenvalue of multiplicity two so that in order that Assumption 2.2 holds we assume that $u_0$ and $v_0$ have opposite sign. The fluid of linearisation is thus outgoing/incoming or incoming/outgoing.

To fix the ideas we assume that $u_0 < 0$ while $v_0 > 0$. We conclude the discussion by considering the roots of $P$. Its discriminant is given by $\Delta := 4c^2(u_0^2 + v_0^2 - c^2)$. So that if $|\mathbf{u}_0| < c$ meaning that the reference state is subsonic then the roots of $P$ have non trivial imaginary part and Assumption 2.2 holds.

Conversely if the fluid is supersonic meaning that $|\mathbf{u}_0| > c$ then the roots of $P$ are given by

$$\lambda_- := \frac{u_0 v_0 - c\sqrt{|\mathbf{u}_0|^2 - c^2}}{u_0^2 - c^2} \text{ and } \lambda_+ := \frac{u_0 v_0 + c\sqrt{|\mathbf{u}_0|^2 - c^2}}{u_0^2 - c^2},$$

and we remark that because we imposed $-u_0 < c$ then $\lambda_- > 0$ independently on the value of $\mathbf{u}_0$ so that Assumption 2.2 fails.

We can not apply directly the results of the article to (116) because of the lack of symmetry of the coefficients. However it is well-known that Euler equations are Friedrichs symmetrizable. Indeed following for example [[Benzoni-Gavage, 2007]-Paragraph 13.2.3] in the independent set of variables $(p, \mathbf{u}, s)$ where $s$ stands for an entropy we can write (116) under the form

$$S(\partial_t + \widetilde{A}_1 \partial_1 + \widetilde{A}_2 \partial_2) \begin{bmatrix} p \\ \mathbf{u} \\ s \end{bmatrix} = 0,$$

where the matrices $S$ is definite positive and where the matrices $S\widetilde{A}_1$ are symmetric. Our analysis can be extended *mutatis mutandis* to systems with a definite positive matrix $S$ in front of the $\partial_t$. Once again because Assumption 2.2 for $A_1^{-1}A_2$ is equivalent to the one on $\widetilde{A}_1^{-1}\widetilde{A}_2$ Theorems 2.1 or 2.3 can be apply to Euler equations if the boundary conditions are strictly dissipative for instance. We refer to [[Benzoni-Gavage, 2007]-Paragraph 14.2] for a discussion about the dissipative boundary conditions for Euler equation.

# 6 Comments, conclusion and prospects

In this article we give several results (namely Theorems 2.1 and 2.2 and 2.3) establishing a persistence of regularity result for hyperbolic boundary value problems defined in the quarter space. The first one, Theorem 2.1, deals with the pure boundary value problem (2) and gives a sharp result in terms of persistence of regularity because the space in which we control the solution is the same as the one of the data of the problem.

The second one, Theorem 2.3 applies to the initial boundary value problem (1). On the one hand, it is really satisfactory because it applies to the natural space $H^s(\Gamma)$ but one the other hand because away

from the corner we can not totally decouple the derivatives using the trick consisting in expressing these derivatives in terms of the radial and angular derivatives then Theorem 2.3 comes with a somewhat loss of regularity which will be studied in a future contribution. More precisely we will show that up to pay some extra tangential derivatives (in the spirit of [Guès, 1993]-[Secchi, 1996]) we can control the normal derivatives $\partial_1 u$ and $\partial_2 u$ in the whole space.

In order to simplify as much as possible the exposition we choose to work in this article with constant coefficients establishing the analogous of Theorems 2.1 or 2.3 for variable coefficients is a required step if one wants to deal with quasilinear problems. However let us point that in some case the analysis of this article can be easily extend to variable coefficients.

Indeed let us assume that the coefficients $A_j$ only depend on the tangential space variable $x'$. Then the previous analysis applies if we take care to the two following points

- The first one is the establishment of the *a priori* energy estimate of Section 3. But a careful reading of the proof show that imposing such variable coefficients will only influence the proof when one uses the energy estimates to recover the control of the tangential derivatives $\partial_j u$, $j \in \mathscr{T}$ and of the radial derivatives $(\mathbf{r}\partial_{\mathbf{r}})^\alpha$. In both cases $x'$-variable coefficients only add a commutator in these estimates which is a zero order term so that it can be easily controlled, assuming that $\gamma$ is large enough.

- The second one is in the establishement of the duality formulas of Paragraph 4.4. The proofs being unchanged except that we have to deal with some extra commutators. For example when we deal with the commutator $\left[\partial^\delta, A_j\partial_j\right]$ then the bilinear form

$$\mathcal{J}(u,v) := \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}\int\sum_{j=3}^{d}\langle(\mathbf{r}\partial_{\mathbf{r}})^\alpha\mathbf{r}^l\sum_{\beta\leq\delta}\binom{\delta}{\beta}\partial_j(\partial^{\delta-\beta}A_j\partial^\beta u),(\mathbf{r}\partial_{\mathbf{r}})^\alpha\partial^\delta\mathbf{r}^l v\rangle\,\mathrm{d}x,$$

  will appear in the duality formula. However it is not difficult to show that like $\mathcal{L}^{nor}$ this bilinear form is continuous so that it will only contribute in the duality formula to the operator $\Phi$ and the remaining of the analysis is unchanged.

The same modifications can also be performed to deal with variable coefficients with respect to time except that at present time we can only deal with the pure boundary value problem (2). Indeed the establishment of the duality formula in $\mathcal{X}^s_{rad,\gamma}(\Omega)$ can be performed even if we have a dependency with respect to $t$. In the author's knowledge it is the first example of well-posedness for time depending coefficients boundary value problems in corner domains.

Similarly the proof of the duality formula $\mathcal{Y}^s(\Gamma)$ does not see such a dependence. However the point explaining that we can not deal with time variable coefficient for the initial boundary value problem (1) is that the domain of the operators $D(A)$ now depend on $t$ so that a precise analysis shall be performed. We however have good reasons to believe that some existing results in the litterature (like for example the well-known [Kato, 1956]) can be used to deal with such time depending coefficients.

Establish the analogous of Theorems 2.1 and 2.3 with coefficients depending on the normal variables $x_1$ or $x_2$ seems to be however more delicate than the dependency with respect to $(t, x')$. Indeed,

- The derivation of the *a priori* energy estimate and more precisely the decoupling of the derivatives using the radial and the angular derivatives is a little more delicate because in the expression of the angular derivatives the matrices $\mathscr{M}$ and $\mathscr{A}$ now depends on $x_1$ and $x_2$ giving rising to a lot of extra terms.

- The second point is in the establishment of the duality formulas. When the $A_j$ depend on $x_1$ and $x_2$

then when we have to deal with the commutator $[(\mathbf{r}\partial_{\mathbf{r}})^{\alpha}, A_j]$ then the following bilinear forms appear:

$$\mathcal{J}_2^{tan} := \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}$$
$$\int_{\Gamma}\sum_{j=1}^{d}\langle\sum_{p=0}^{\alpha}\sum_{q=1}^{p}\sum_{r=1}^{\alpha-p}\sum_{t=0}^{q-1}\sum_{k=0}^{r-1}\binom{q}{t}\binom{r}{k}\binom{\alpha}{p}\lambda_q^p\lambda_r^{\alpha-p}x_1^q x_2^r\partial_j\left[(\partial_1^{q-t}\partial_2^{r-k}A_j)\partial_1^t\partial_2^k\mathbf{r}^l\partial^{\delta}u\right],(\mathbf{r}\partial_{\mathbf{r}})^{\alpha}\partial^{\delta}\mathbf{r}^l v\rangle\,\mathrm{d}x$$

$$\mathcal{J}_3^{tan} := \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}$$
$$\int_{\Gamma}\sum_{j=1}^{d}\langle\sum_{p=0}^{\alpha}\sum_{q=1}^{p}\sum_{r=1}^{\alpha-p}\sum_{t=0}^{q-1}\binom{q}{t}\binom{\alpha}{p}\lambda_q^p\lambda_r^{\alpha-p}x_1^q x_2^r\partial_j\left[(\partial_1^{q-t}A_j)\partial_1^t\partial_2^r\mathbf{r}^l\partial^{\delta}u\right],(\mathbf{r}\partial_{\mathbf{r}})^{\alpha}\partial^{\delta}\mathbf{r}^l v\rangle\,\mathrm{d}x$$

$$\mathcal{J}_4^{tan} := \sum_{l=0}^{s}\sum_{\alpha=0}^{s-l}\sum_{|\delta|\leq s-\alpha}$$
$$\int_{\Gamma}\sum_{j=1}^{d}\langle\sum_{p=0}^{\alpha}\sum_{q=1}^{p}\sum_{r=1}^{\alpha-p}\sum_{k=0}^{r-1}\binom{r}{k}\binom{\alpha}{p}\lambda_q^p\lambda_r^{\alpha-p}x_1^q x_2^r\partial_j\left[(\partial_2^{r-k}A_j)\partial_1^q\partial_2^k\mathbf{r}^l\partial^{\delta}u\right],(\mathbf{r}\partial_{\mathbf{r}})^{\alpha}\partial^{\delta}\mathbf{r}^l v\rangle\,\mathrm{d}x.$$

Contrary to the other commutators like $\mathcal{L}^{nor}$ it seems delicate to show that these terms give rise to continuous bilinear forms. One assumption that may help is to assume that the coefficients are constants outside some compact set of $\mathbb{R}^2_+\times\mathbb{R}^{d-2}$ so that one can recover the same index on $x_1^q$ and on $\partial_1^t$ in the previous formulas. However the main difficulty is then that one have to control the uncorrelated derivatives $(x_1\partial_1)^q(x_2\partial_2)^r u$ in terms of $(\mathbf{r}\partial_{\mathbf{r}})^{\alpha}$ which only gives a control of the sum of the correlated derivatives $(x_1\partial_1)^p(x_2\partial_2)^{\alpha-p}u$ for $p\in[\![0,\alpha]\!]$.

Consequently showing the analogous of Theorems 2.1 and 2.3 for normal variable coefficients os left for future studies.

Finally we would like to discuss some prospects about the use of Sobolev embedding in order to recover from our persistence of regularity results the $L^\infty$ norm of the solution $u$. As already mentioned before the persistence of regularity result Theorem 2.1 because it gives a decorrelation of the derivatives in the full space and not only near the corner like Theorem 2.3 shall be more convenient for this purpose.

Because we have in particular a full control of the tangential derivatives $(x_1\partial_1)^{\alpha}(x_2\partial_2)^{\beta}$, $\alpha+\beta\leq s$ then we are not really far to an $H^s$ estimate. Indeed following the result of [Guès, 1993] or [Secchi, 1996] for example to recover the control of $\partial_1 u$ then it is sufficient to control $(x_1\partial_1)\partial_1 u$ that is to say a control of the mixed tangential/normal derivative.

Because as already pointed our problem has a large similarity with characteristic boundary velur problems, we have reasons to believe that the analysis of [Guès, 1993] and [Secchi, 1996] can be used. However one key point in these analysis is that to initialize the proof they use that some part of the normal derivative $\partial_1 u^I$ can be estimated for free (more precisely $u^I$ stands for the part of the solution lying in the space on whoch $A_1$ is invertible). The control of the full normal derivative $\partial_1 u$ is then recover via the control of the tangential ones.

In our problem we can of course invert $A_1$ on the whole space but we only have then that $\partial_1 u$ depends on $A_1^{-1}A_2\partial_2 u$ which is not known to be $L^2$. So that if one wants to use the analysis of characteristic boundary value problems then one shall find a subspace on which we can infer that both $\partial_1 u^I$ and $\partial_2 u^I$ are in $L^2$. This point is also left for future studies.

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