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Interior estimates in the sup-norm for a class of generalized functions with integral representations

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Abstract

In this paper we construct apriori estimates for the first order derivatives in the sup-norm for first order meta-monogenic functions, generalized monogenic functions satisfying a differential equation with an anti-monogenic right hand side and generalized meta-monogenic functions satisfying a differential equation with an anti-meta-monogenic right hand side. We obtain such estimates through integral representations of these classes of functions and give an explicit expression for the corresponding constants appearing in the estimates. Then we show how initial value problems can be solved in case an interior estimate is true in the function spaces under consideration. All related functions are in a Clifford type algebra.

Keywords: Interior estimates of solutions of elliptic equations, Initial value problems of Cauchy-Kovaleskaya type, Meta-monogenic functions, Clifford type algebras.

Resumen

En este artículo nosotros construimos estimados a priori en la norma del supremo, para las derivadas de primer orden de funciones meta-monogénicas de primer orden,

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de funciones monogénicas generalizadas satisfaciendo una ecuación diferencial con un lado derecho anti-monogénico y de funciones meta-monogénicas generalizadas satisfaciendo una ecuación diferencial con un lado derecho anti-meta-monogénico. Tales estimados los obtenemos a través de representaciones integrales de esas clases de funciones y damos una expresión explícita para las correspondientes constantes que aparecen en los estimados. Finalmente, nosotros mostramos cómo problemas de valores iniciales pueden ser resueltos en caso que tengamos un estimado interior en el espacio de funciones considerado. Todas las funciones relacionadas con los resultados de este trabajo tienen valores en un álgebra tipo Clifford.

Palabras claves: Estimados interiores de soluciones de ecuaciones elípticas, Problemas de valores iniciales de tipo Kovaleskaya, Funciones meta-monogénicas, álgebras tipo Clifford.

1 Motivation of the paper

Let Ω be a bounded domain in \mathbb{R}^{n+1} whose points are denoted by $x = (x_0, \dots, x_n)$. Let D_λ be the first order meta-monogenic operator of \mathbb{R}^{n+1} (see [1, 2])

$$D_\lambda = D - \lambda = \left(\frac{\partial}{\partial x_0} + \sum_{j=1}^n e_j \frac{\partial}{\partial x_j} \right) - \lambda, \quad (1)$$

where $\lambda \in \mathbb{R}$. The Clifford-algebra-valued solutions u of the equation $D_\lambda u = 0$ are calling first order meta-monogenic functions. From now on, whenever we say meta-monogenic we mean first order meta-monogenic.

A Clifford-algebra-valued function $u = u(x) = \sum_A u_A$ satisfying a differential equation of the form

$$D_\lambda u = F(x, u, \partial_0 u, \partial_1 u, \dots, \partial_n u), \quad \partial_i = \frac{\partial}{\partial x_i}, \quad i = 0, 1, \dots, n, \quad (2)$$

is called a generalized meta-monogenic function. In this article we consider the case when the right-hand side of equation (2) has the form

$$F(x, u) = \sum_A F_A(x) u_A,$$

where the coefficients $F_A(x)$ are Clifford-algebra-valued functions having continuous first order derivatives (see [3, 4]). If $\overline{D}_\lambda = \left(e_0 \partial_0 - \sum_{i=0}^n e_i \partial_i \right) - \lambda$ is

the conjugate meta-monogenic operator, then we will say that the equation

$$D_\lambda u = \sum_A F_A(x) u_A \quad (3)$$

is a differential equation with anti-meta-monogenic right-hand side if for each solution u the expression

$$\overline{D}_\lambda \left(\sum_A F_A(x) u_A \right) \quad (4)$$

is a linear combination of the u_A alone (not depending on the partial derivatives of the u_A) (see [5, 6]).

If $\lambda = 0$ in equation (2), then the functions u are called generalized monogenic and if $\lambda = 0$ in equations (3) and (4), then we have a differential equation with anti-monogenic right-hand side (see [6]).

Initial value problem of type

$$\partial_t u = \mathcal{F}(x, u, \partial_i u), \quad (5)$$

$$u(0, x) = \varphi(x), \quad (6)$$

can be solved by the contraction principle in case the initial function φ belongs to an associated space whose elements satisfy an interior estimate.

Definition 1. [7, 8] Let \mathcal{F} be a first order differential operator depending on t, x, u and $\partial_i u$ for $i = 0, 1, \dots, n$, while \mathcal{G} is a differential operator with respect to the spacelike variables x_i with coefficients not depending on time t . \mathcal{F} is said to be associated with \mathcal{G} if \mathcal{F} maps all solutions for the differential equation $\mathcal{G}u = 0$ into solutions of the same equation for a fixedly chosen t , i.e.,

$$\mathcal{G}u = 0 \Rightarrow \mathcal{G}(\mathcal{F}u) = 0.$$

The function space \mathcal{X} containing all the solutions for the differential equation $\mathcal{G}u = 0$ is called an **associated space** of \mathcal{F} .

This concept generalizes the fact that the space of holomorphic functions is associated to the complex differentiation $\frac{d}{dz}$ because the complex derivative of a holomorphic function is again holomorphic.

On the other hand, interior estimates describe the behaviour of the derivatives of a function near the boundary of a bounded domain:

Definition 2. *If u is given function in a domain Ω , and Ω' is a subdomain having positive distance $\text{dist}(\Omega', \partial\Omega)$ from the boundary $\partial\Omega$ of Ω , then an estimate as*

$$\left\| \frac{\partial u}{\partial x_j} \right\|_{\Omega'} \leq \frac{K}{\text{dist}(\Omega', \partial\Omega)} \|u\|_{\Omega}$$

is called an interior estimate, where K is a constant neither depending on u nor on Ω' .

Although initial value problems of the type (5), (6) can be solved using associated spaces (see [3, 9–14]), the famous Lewy example (see [15]) shows that there are differential equations as (5) not having any solution, in particular no initial value problem is solvable for these equations. It means that no associated space with an interior estimate for its elements, can exist.

The desired solution of the initial value problem (5),(6) can be found as a fixed point of a related operator as the following:

$$U(x, t) = \varphi(x) + \int_0^t F(\tau, x, u, \partial_j u) d\tau. \quad (7)$$

In order to construct fixed points of this operator in suitable function spaces, we have to estimate the integro-differential operator on the right-hand side of (7). This is a little bit difficult because the integrand contains derivatives with respect to the spacelike variables x_j . But the estimation is possible in case one uses the method of associated spaces. The existence of an associated differential operator where its elements satisfy an interior estimate, implies not only the solubility of the differential equation but also the possibility to choose arbitrary solutions of the associated differential equation as initial functions (see [4, 16–21]).

The classical Cauchy-Kovalevskaya problem was solved by W. Walter in [22] using fixed-point methods and a weighted sup-norm. Taking the initial functions from the classes of functions considered in this article and the weighted sup-norm, we can follow the idea of W. Walter to solve those initial value problems generated by the generalized functions discussed in this paper.

Interior estimates can be found for functions having boundary or domain integral representations and are possible for different norms in the case of solutions of elliptic equations (see [23–26]). For example, in order to obtain such an interior estimate for the complex derivative of holomorphic functions

it is used the Cauchy Integral Formula:

$$\left\| \frac{\partial u}{\partial x_j} \right\|_{\Omega'} \leq \frac{1}{\text{dist}(\Omega', \partial\Omega)} \|u\|_{\Omega}$$

where u is a complex valued function in a bounded domain Ω , and Ω' a subdomain having positive distance $\text{dist}(\Omega', \partial\Omega)$ from the boundary $\partial\Omega$ of Ω . Also, in higher dimensions, since monogenic functions are solutions of the Laplace equation, interior estimates in the supremum norm can be obtained from the Poisson Integral Formula for harmonic functions. Using the L_p -norm, interior estimates have been calculated using other integral representations for functions (see [26, 27]).

The goal of the present paper is to construct interior estimates in the supremum norm (sup-norm) associated to more general classes of functions called meta-monogenics, generalized monogenic functions satisfying a differential equation with an anti-monogenic right hand side and generalized meta-monogenic functions satisfying a differential equation with an anti-meta-monogenic right hand side, when these functions take its value in Clifford type algebras. We obtain such estimates through integral representations of these classes of functions (see [28–30]) and give an explicit expression for the corresponding constants appearing in the estimates. Then we show how initial value problems can be solved in case an interior estimate is true in the function space under consideration.

Finally, as the parameter-depending Clifford algebras that we consider are more general Clifford algebras, we cover more general partial differential equations. Hence our results on interior estimates in the sup-norm are generalizations of those that can be found in the framework of Classic Clifford algebras.

2 Preliminaries

2.1 Constant parameters depending Clifford algebras

Clifford algebras can be defined as equivalence classes in the ring $\mathcal{R}[X_1, \dots, X_n]$ of polynomials in n variables X_1, \dots, X_n with real coefficients, *where two polynomials are to be distinguished in case they differ at least in the order of the factors X_j in one of their terms*. Then two polynomials are said to be equivalent if their difference is a polynomial for which each term contains at least

one of the factors

$$X_j^2 + 1 \quad \text{or} \quad X_i X_j + X_j X_i . \quad (8)$$

Denoting X_j by e_j , $j = 1, \dots, n$, the structure polynomials (8) imply the well-known rules of the usual Clifford algebra \mathcal{A}_n :

$$\begin{aligned} e_j^2 &= -1, \quad j = 1, \dots, n, \quad \text{and} \\ e_i e_j &= -e_j e_i \quad \text{for } i \neq j. \end{aligned}$$

It is well known that \mathcal{A}_n extends the Euclidean space \mathbb{R}^{n+1} , which has the basis $e_0 = 1, e_1, \dots, e_n$ (see [31, 32]).

Similarly, we can obtain Clifford-type algebras if the structure polynomials (8) are replaced by

$$X_j^2 + \alpha_j \quad \text{and} \quad X_i X_j + X_j X_i - 2\gamma_{ij}, \quad (9)$$

where $i, j = 1, \dots, n$, $i \neq j$, and α_j and $\gamma_{ij} = \gamma_{ji}$, for $i \neq j$ are real constants (see [32, 33]). We call them Clifford algebras depending on parameters α_j and γ_{ij} or Clifford type algebra. If $n \geq 2$, the Clifford algebra generated by the structure polynomials (9) is denoted by $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ and for $n = 1$ we write $\mathcal{A}_1(2, \alpha)$. As in the case of the usual Clifford algebra $\mathcal{A}_n (= \mathcal{A}_n(2, 1, 0))$, we have from structure polynomials (9) the rules

$$\begin{aligned} e_j^2 &= -\alpha_j, \quad j = 1, \dots, n, \\ e_i e_j + e_j e_i &= 2\gamma_{ij}, \quad i \neq j, \end{aligned} \quad (10)$$

and it can be proven that its dimension is 2^n . For other more general Clifford type algebras see [29, 32].

We will consider functions u defined on a bounded domain Ω in \mathbb{R}^{n+1} having values in $\mathcal{A}_n(2, \alpha_i, \gamma_{ij})$, i.e., $u(x) = \sum_{A \in \Gamma_n} u_A(x) e_A$, where

$$u_A : \Omega \rightarrow \mathbb{R} \quad \Gamma_n = \{0, 1, 2, \dots, 12, 13, \dots, 123 \dots n\}.$$

The generalized Cauchy-Riemann operator and its conjugate in \mathbb{R}^{n+1} are defined by

$$D = \sum_{j=0}^n e_j \partial_j \quad \text{and} \quad \bar{D} = \partial_0 - \sum_{j=1}^n e_j \partial_j,$$

respectively, where $\partial_j = \frac{\partial}{\partial x_j}$, for $j = 1, \dots, n$.

Definition 3. A function $u \in C^1(\Omega, \mathcal{A}_n(2, \alpha_i, \gamma_{ij}))$ is called *left monogenic* if it satisfies the equation $Du = 0$ and *right monogenic* if it satisfies $uD = 0$. A left monogenic function will be called *monogenic*.

A straightforward calculation shows that for each twice continuously differentiable monogenic function, we have

$$\bar{D}Du = \partial_0^2 u + \sum_{j=1}^n \alpha_j \partial_j^2 u - 2 \sum_{i < j} \gamma_{ij} \partial_i \partial_j u = 0. \quad (11)$$

Example 1. In $\mathcal{A}_n(2, \alpha_j, 0)$, the equation $\bar{D}Du = 0$ can be hyperbolic or parabolic in case some of the α_j are negative or null respectively and can be elliptic if all of the α_j are positive.

Remark 1. Observe that a monogenic function in \mathcal{A}_n always satisfies the Laplace equation. In contrast, the Example 1 shows that monogenic functions in more general Clifford algebras satisfy different equations from the Laplace equation.

Using the rules (10), we get the following result:

Example 2. In $\mathcal{A}_2(2, \alpha, \beta, \gamma)$ the Cauchy-Riemann system $Du = 0$ for a Clifford-algebra-valued function $u = u_0 + u_1 e_1 + u_2 e_2 + u_{12} e_{12}$ reads

$$\begin{aligned} \partial_0 u_0 - \alpha \partial_1 u_1 + 2\gamma \partial_2 u_1 - \beta \partial_2 u_2 &= 0, \\ \partial_0 u_1 + \partial_1 u_0 + \beta \partial_2 u_{12} &= 0, \\ \partial_0 u_2 - \alpha \partial_1 u_{12} + \partial_2 u_0 + 2\gamma \partial_2 u_{12} &= 0, \\ \partial_0 u_{12} + \partial_1 u_2 - \partial_2 u_1 &= 0. \end{aligned}$$

Remark 2. After Example 2, we see that also the first order system $Du = 0$ is more general in a Clifford algebra $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ than the Cauchy-Riemann system for monogenic functions in \mathcal{A}_n .

2.2 Fundamental solutions for D_λ and $D_{-\lambda}$

Let Ω be a bounded domain in \mathbb{R}^{n+1} with sufficiently smooth boundary. Consider a $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ -valued function $u = \sum_{A \in \Gamma_n} u_A e_A$ defined in Ω and suppose

it is monogenic, then we get the homogeneous second order differential equation given by (11). We denote the operator $\overline{D}D$ by $\tilde{\Delta}$ and so we have

$$\tilde{\Delta}u = \partial_0^2 u + \sum_{j=1}^n \alpha_j \partial_j^2 u - 2 \sum_{i < j} \gamma_{ij} \partial_i \partial_j u = 0. \quad (12)$$

From now on we will suppose that the α_j are positive for $j = 0, \dots, n$ and the absolute values of the γ_{ij} are not too large (i.e. $|\gamma_{ij}| \leq \text{const}$ with a suitable constant). Then the equation (12) is elliptic and its coefficient matrix is

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1 & -\gamma_{12} & \cdots & -\gamma_{1n} \\ 0 & -\gamma_{21} & \alpha_2 & \cdots & -\gamma_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\gamma_{n1} & -\gamma_{n2} & \cdots & \alpha_n \end{pmatrix}. \quad (13)$$

As consequence of the ellipticity, (13) has an inverse matrix having the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & A_{11} & \cdots & A_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & A_{n1} & \cdots & A_{nn} \end{pmatrix}, \quad (14)$$

where $A_{ij} = A_{ji}$ (because of $\gamma_{ij} = \gamma_{ji}$). Using these coefficients, we can define for two points $\xi = (\xi_0, \xi_1, \dots, \xi_n)$ and $x = (x_0, x_1, \dots, x_n)$ of \mathbb{R}^{n+1} a non-Euclidean distance ρ by

$$\rho^2 = (x_0 - \xi_0)^2 + \sum_{i,j=1}^n A_{ij} (x_i - \xi_i)(x_j - \xi_j). \quad (15)$$

On the other hand, for a point $x \neq \xi$ we consider the usual Euclidean distance $r = |x - \xi|$. Then we can write $x - \xi = ry$, where $|y| = 1$. For such point the non Euclidean distance ρ_0 between the point y and $(0, \dots, 0)$ has a positive infimum c , i.e., $\rho_0 \geq c$. On the other hand, the definition (15) shows that $\rho = r\rho_0$. Hence we obtain

$$\rho \geq cr. \quad (16)$$

Using the distance ρ given by (15), the function $E_{-\lambda}(x, \xi)$ is defined by

$$E_{-\lambda}(x, \xi) = E(x, \xi) \cdot e^{-\lambda(x_0 - \xi_0)},$$

where $x, \xi \in \Omega$, $\lambda \in \mathbb{R}$ and

$$E(x, \xi) = \frac{1}{\omega_{n+1}} \frac{1}{\varrho^{n+1}} \left((x_0 - \xi_0) - \sum_{i,j=1}^n A_{ij} e_i(x_j - \xi_j) \right).$$

The function $E_{-\lambda}(x, \xi)$ is a fundamental solution for the operator $D_{-\lambda} = D + \lambda$ in $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ (see [28] and [2]).

Denoting

$$x' - \xi' = (x_0 - \xi_0) + \sum_{i,j=1}^n e_i A_{ij} (x_j - \xi_j)$$

and after some calculation, we can prove that $(x' - \xi') \cdot \overline{(x' - \xi')} = \varrho^2$ (see Lemma 1 in [34]). Taking $\varrho = \|x' - \xi'\|_{\varrho}$, the kernel $E_{-\lambda}(x, \xi)$ can be represented as

$$E_{-\lambda}(x, \xi) = \frac{1}{\omega_{n+1}} \frac{\overline{x' - \xi'}}{\|x' - \xi'\|_{\varrho}^{n+1}} \cdot \exp(-\lambda(x_0 - \xi_0)). \quad (17)$$

2.3 A Cauchy formula for meta-monogenic functions

Theorem 1. [2, 28] *Let $E_{-\lambda}(x, \xi)$ be a fundamental solution for $D_{-\lambda}$. An arbitrary twice continuously differentiable function u with values in the algebra $\mathcal{A}_n(2, \alpha_i, \gamma_{ij})$ can be represented in the form*

$$h(\alpha_j, \gamma_{ij}) \cdot u(\xi) = \int_{\partial\Omega} E_{-\lambda}(x, \xi) \cdot d\sigma \cdot u - \int_{\Omega} E_{-\lambda}(x, \xi) \cdot D_{\lambda} u \, dx, \quad (18)$$

for interior points ξ of Ω and where

$$h(\alpha_j, \gamma_{ij}) = - \int_{|x-\xi|=\varepsilon} E_{-\lambda}(x, \xi) \cdot d\sigma. \quad (19)$$

On the other hand, if we take u such that $D_{\lambda} u = 0$ in (18), then we can obtain a Cauchy Integral type Formula for the meta-monogenic function u :

$$h(\alpha_j, \gamma_{ij}) \cdot u(\xi) = \int_{\partial\Omega} E_{-\lambda}(x, \xi) \cdot d\sigma \cdot u. \quad (20)$$

2.4 Some important estimates

Suppose Ω is an arbitrary domain in \mathbb{R}^{n+1} with finite measure $m\Omega$. Then the **Schmidt Inequality** states

$$\int_{\Omega} \frac{1}{|x - \xi|^\alpha} d\xi \leq \frac{\omega_{n+1}}{n+1-\alpha} \left(\frac{m\Omega}{\tau_{n+1}} \right)^{1-\frac{\alpha}{n+1}}$$

provided $0 \leq \alpha < n+1$, where τ_{n+1} means the volume of the unit ball in \mathbb{R}^{n+1} and ω_{n+1} is the surface measure of its boundary (see [33]).

Let $u(x) = \sum_{A \in \Gamma_n} u_A e_A$, $v(x) = \sum_{B \in \Gamma_n} v_B e_B$ be a pair of $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ -valued functions. Consider the integral

$$\int_{\partial\Omega} v \cdot d\sigma \cdot u = \sum_{A, B \in \Gamma_n, j=0}^n \int_{\partial\Omega} v_B e_B \cdot e_j N_j d\mu \cdot u_A e_A,$$

where $d\sigma = \sum_{j=0}^n e_j N_j d\mu$, $N = (N_0, N_1, \dots, N_n)$ is the outer unit normal vector on $\partial\Omega$ and $d\mu$ is the scalar measure element of $\partial\Omega$. Since

$$\left| \int_{\partial\Omega} v_B e_B e_j N_j d\mu u_A e_A \right| = \left| e_B e_j e_A \int_{\partial\Omega} v_B N_j u_A d\mu \right| \leq b_{B,j,A} \int_{\partial\Omega} |v_B| |u_A| d\mu,$$

where $b_{B,j,A}$ is an upper bound for $|e_B e_j e_A|$, then

$$\left| \int_{\partial\Omega} v \cdot d\sigma \cdot u \right| \leq M^* \int_{\partial\Omega} |u| |v| d\mu, \quad (21)$$

where $M^* = (n+1)4^n \max_{B,j,A} b_{B,j,A}$.

Similar arguments leads to

$$\left| \int_{\partial\Omega} u \cdot d\sigma \right| \leq M_* \int_{\partial\Omega} |u| d\mu, \quad (22)$$

where $M_* = (n+1)2^n \max_{A,j} |e_A e_j|$.

3 Interior estimates

In this section we calculate interior estimates for meta-monogenic functions, generalized monogenic functions satisfying a differential equation with an anti-monogenic right-hand side and generalized meta-monogenic functions satisfying a differential equation with an anti-meta-monogenic right-hand side. For all cases we use integral representations given by fundamental solutions.

3.1 Meta-monogenic functions

Consider an $\mathcal{A}_n(2, \alpha_i, \gamma_{ij})$ -valued function $u(x) = \sum_{A \in \Gamma_n} u_A(x) e_A$ on a set \mathcal{K} .

We define the sup-norm for u as the maximum of the sup-norms of its 2^n real-valued components, i.e.,

$$\|u\| = \sup_{\mathcal{K}} |u(x)| := \max_{A \in \Gamma_n} (\sup_{\mathcal{K}} |u_A(x)|).$$

In order to obtain a interior estimate using the equation (20) for meta-monogenic functions we must compute the derivative of the fundamental solution $E_{-\lambda}$.

A straightforward calculation leads to

$$\begin{aligned} \partial_{x_s} E_{-\lambda}(x, \xi) &= -\frac{(n+1)}{\omega_{n+1}} \frac{(x' - \xi')_s \overline{(x' - \xi')}}{\|x' - \xi'\|_\varrho^{(n+3)}} \exp(-\lambda(x_0 - \xi_0)) \\ &+ \frac{1}{\omega_{n+1} \|x' - \xi'\|_\varrho^{(n+1)}} \left[\sum_{j=0}^n \hat{A}_{js} e_j - \lambda \hat{A}_{0s} \overline{(x' - \xi')} \right] \exp(-\lambda(x_0 - \xi_0)), \end{aligned}$$

$s = 0, 1, \dots, n$, where

$$\hat{A}_{0s} = \hat{A}_{j0} = 0, \quad s, j = 1, \dots, n, \quad \hat{A}_{00} = 1, \quad \hat{A}_{js} = -A_{js}, \quad s, j = 1, \dots, n$$

and

$$(x' - \xi')_s = -\sum_{j=0}^n \hat{A}_{js} (x_j - \xi_j), \quad s = 1, \dots, n, \quad (x' - \xi')_0 = (x_0 - \xi_0).$$

Therefore, denoting $\|x' - \xi'\|_\varrho$ by just ϱ and using (20) and (21) we have

$$\begin{aligned}
|h(\alpha_j, \gamma_{ij}) \cdot \partial_{\xi_s} u(\xi)| &= \\
&\left| \int_{\partial\Omega} \partial_{\xi_s} E_{-\lambda}(x, \xi) \cdot d\sigma \cdot u(x) \right| \leq M^* \int_{\partial\Omega} |\partial_{\xi_s} E_{-\lambda}(x, \xi)| |u(x)| d\mu \\
&\leq \frac{M^*}{\omega_{n+1}} \sup_{\partial\Omega} |u(x)| \left(\int_{\partial\Omega} \frac{n+1}{\varrho^{n+3}} |(x' - \xi')_s \overline{(x' - \xi')}| \exp(-\lambda(x_0 - \xi_0)) \right) d\mu \\
&\quad + \frac{M^*}{\omega_{n+1}} \sup_{\partial\Omega} |u(x)| \left(\int_{\partial\Omega} \frac{1}{\varrho^{n+1}} \left| \left(\sum_{j=0}^n \hat{A}_{js} e_j \right) \exp(-\lambda(x_0 - \xi_0)) \right| d\mu \right. \\
&\quad \left. + |\lambda| |\hat{A}_{0s}| \int_{\partial\Omega} \frac{1}{\varrho^{n+1}} |\overline{(x' - \xi')}| \exp(-\lambda(x_0 - \xi_0)) \right) d\mu.
\end{aligned}$$

Since Ω is a bounded domain, the exponential function satisfies

$$|\exp(-\lambda(x_0 - \xi_0))| \leq L,$$

for some constant L . Taking into account that

$$\begin{aligned}
\int_{|x-\xi|=r} d\mu &= r^n \omega_{n+1}, \\
|(x' - \xi')_s| &\leq \sum_{j=0}^n |\hat{A}_{js}| |x_j - \xi_j| \leq r \sum_{j=0}^n |\hat{A}_{js}|, \quad s = 0, 1, \dots, n, \\
|\overline{(x' - \xi')}| &\leq r(1 + \sum_{i,j=1}^n |A_{ij}|) \quad \text{and} \quad |u(x)v(x)| \leq a(\alpha_i, \gamma_{ij}) |u(x)| |v(x)|,
\end{aligned}$$

where $u, v \in \mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ and $a(\alpha_i, \gamma_{ij})$ is a positive constant depending only on α_i and γ_{ij} , using (16) and (22) with $E_{-\lambda}(x, \xi)$ instead of u , and integrating over $|x - \xi| = r$ instead of $\partial\Omega$, we obtain

$$|\partial_{\xi_s} u(\xi)| \leq \frac{K_n^*(\alpha_j, \gamma_{ij})}{r} \sup_{\overline{\Omega}} |u(x)|,$$

where

$$K_n^*(\alpha_j, \gamma_{ij}) = \frac{L \cdot M^*}{M_*} \left[\frac{a_1(\alpha_j, \gamma_{ij})}{c^{n+3}} \sum_{j=0}^n |\hat{A}_{js}| \left(1 + \sum_{i,j=1}^n |A_{ij}| \right) + \frac{a_2(\alpha_j, \gamma_{ij})}{c^{n+1}} \sum_{j=0}^n |\hat{A}_{js}| + \frac{a_3(\alpha_j, \gamma_{ij}) c_0 |\lambda| |\hat{A}_{0s}|}{c^{n+1}} \left(1 + \sum_{i,j=1}^n |A_{ij}| \right) \right]. \quad (23)$$

Here we are assuming that $h(\alpha_j, \gamma_{ij})$ possesses an inverse and c_0 is constant satisfying $\varrho \leq c_0$. Hence the limiting process $r \rightarrow d(\xi) = \inf_{x \in \partial\Omega} |x - \xi|$ leads to the interior estimate for the derivative of a meta-monogenic function and so we have proved the following theorem:

Theorem 2. *Suppose u is a meta-monogenic function in Ω with values in the parameter Clifford algebra $\mathcal{A}_n(2, \alpha, \gamma_{ij})$. If the Clifford constant $h(\alpha_j, \gamma_{ij})$ defined by (19) has an inverse and c_0 is constant satisfying $\varrho \leq c_0$, then the estimate*

$$|\partial_{\xi_s} u(\xi)| \leq \frac{K_n^*(\alpha_j, \gamma_{ij})}{d(\xi)} \|u\| \quad (24)$$

holds, where $K_n^*(\alpha_j, \gamma_{ij})$ is given by (23).

3.2 Generalized monogenic functions satisfying a differential equation with an anti-monogenic right-hand side

In this subsection, we will show the interior estimate for solutions of a partial differential equation whose right-hand side is anti-monogenic (see [6]).

Consider a Clifford type algebra-valued generalized monogenic function u satisfying a differential equation with anti-monogenic right-hand side of the form

$$Du = F(x, u) = \sum_{A \in \Gamma_n} F_A(x) u_A, \quad (25)$$

where the coefficients $F_A(x)$ are parameter-depending on a Clifford algebra-valued functions possessing continuous first-order derivatives. As the equation (25) has an anti-monogenic right-hand side, the expression

$$\bar{D} \left(\sum_{A \in \Gamma_n} F_A(x) u_A \right)$$

is a linear combination only of the components u_A . We denote this linear combination by

$$\sum_{A \in \Gamma_n} G_A(x) u_A.$$

Thus, each solution u of (25) satisfies the differential equation

$$\tilde{\Delta}u = \sum_{A \in \Gamma_n} G_A(x) u_A. \quad (26)$$

In [5], a sufficient condition for the right-hand side of the equation (25) to be anti-monogenic is given. The given criterion allows the construction of anti-monogenic functions in Clifford type algebras.

A fundamental solution of the elliptic equation $\tilde{\Delta}u = 0$ is the Clifford-parameter Newton potential

$$\tilde{K}(x, \xi) = \frac{-1}{(n-1)\omega_{n+1}} \cdot \frac{1}{\rho^{n-1}}$$

(see [29,30,32]). Therefore, we have that each generalized monogenic function satisfying the equation (26), can be represented in the form

$$u(x) = \int_{\Omega} \tilde{K}(x, \xi) \cdot q \cdot \sum_{A \in \Gamma_n} G_A(\xi) u_A(\xi) d\xi, \quad (27)$$

where q is a Clifford constant depending on the parameters α_i and γ_{ij} , $i, j = 1, \dots, n$. Consequently, an interior estimate of a generalized monogenic function solution of (26) can be obtained from (27).

Consider $G_A = \sum_{B \in \Gamma_n} g_{AB} e_B$, where g_{AB} are the real-valued components of the coefficients G_A . If C_G is a bound for the absolute values of g_{AB} , then

$$|G_A|^2 = \sum_{B \in \Gamma_n} |g_{AB}|^2 \leq 2^n C_G^2.$$

Since u has 2^n components u_A , we have

$$\sum_{A \in \Gamma_n} |G_A| |u_A| \leq 2^{n/2} \cdot 2^n C_G \|u\| = 2^{\frac{3}{2}n} C_G \|u\|. \quad (28)$$

The estimate (16) yields

$$|\partial_{x_j} \tilde{K}(x, \xi)| \leq \frac{|x_j - \xi_j|}{\omega_{n+1} c^{n+1} r^{n+1}} \leq \frac{1}{\omega_{n+1} c^{n+1}} \frac{1}{|x - \xi|^n}. \quad (29)$$

It means that the differentiated kernel $\partial_{x_j} \tilde{K}(x, \xi)$ is integrable.

From a straightforward calculation follows the next Lemma:

Lemma 1. *Consider two function $u = \sum_A u_A e_A$ and $v = \sum_A v_B e_B$ in $\mathcal{A}_n(2, \alpha, \gamma_{ij})$. Then*

i) $|uv| \leq 2^{n/2} M |u| |v|$, where $M = M(\alpha_i, \gamma_{ij})$ is a constant depending on the parameters α_i and γ_{ij} .

ii) $|\int_{\Omega} u(x) dx| \leq 2^{n/2} \int_{\Omega} |u(x)| dx$.

iii) $|u_1 \cdot u_2 \cdots u_m| \leq (2^{n/2} M)^{m-1} |u_1| |u_2| \cdots |u_m|$.

iv) $|\int_{\Omega} (u_1(x) \cdots u_m(x)) dx| \leq 2^{\frac{nm}{2}} M^{m-1} \int_{\Omega} |u_1(x)| \cdots |u_m(x)| dx$.

In order to obtain an interior estimate for the first order derivatives of u given by (27), we use the Lemma 1, the estimates (28), (29) and Schmidt's Inequality, which leads to

$$\left| \partial_{x_i} \left(\int_{\Omega} \tilde{K}(x, \xi) \cdot q \cdot \sum_{A \in \Gamma_n} G_A(\xi) u_A(\xi) d\xi \right) \right| \leq \frac{2^{\frac{5}{2}n} \cdot M^2 \cdot |q|}{c^{n+1}} \cdot \left(\frac{m\Omega}{\tau_{n+1}} \right)^{\frac{1}{n+1}} C_G \cdot \|u\|,$$

where $M = M(\alpha_i, \gamma_{ij})$ is a positive constant depending only on α_i and γ_{ij} .

Denoting the diameter of Ω by d_0 and the positive distance from $x \in \Omega$ to the boundary $\partial\Omega$ of Ω by $dist(x, \partial\Omega)$, we have

$$1 \leq \frac{d_0}{dist(x, \partial\Omega)}$$

everywhere in Ω . This gives us the following interior estimate:

$$\left| \frac{\partial u}{\partial x_i}(x) \right| \leq \frac{C^*}{dist(x, \partial\Omega)} \|u\|,$$

where

$$C^* = \frac{d_0 \cdot 2^{\frac{5}{2}n} \cdot M^2 \cdot |q|}{c^{n+1}} \cdot \left(\frac{m\Omega}{\tau_{n+1}} \right)^{\frac{1}{n+1}} \cdot C_G. \quad (30)$$

Therefore we have been proved the following theorem

Theorem 3. *Suppose u is a generalized monogenic $\mathcal{A}_n(2, \alpha, \gamma_{ij})$ -valued function in Ω . Then we have the interior estimate*

$$\left| \frac{\partial u}{\partial x_i}(x) \right| \leq \frac{C^*}{\text{dist}(x, \partial\Omega)} \|u\|, \quad (31)$$

where C^* is given by (30).

Remark 3. *If u is a monogenic function, we have $C_G = 0$ in (30) implying $C^* = 0$ and thus we do not get interior estimates for the function u by this way. Therefore to obtain an interior estimate for the function u , we can use a representation of the function u by boundary integral given by the Theorem 5.6 in [29].*

3.3 Generalized meta-monogenic functions satisfying a differential equation with an anti-meta-monogenic right-hand side

Now we will show an interior estimate for solutions of a partial differential equation whose right-hand side is anti-meta-monogenic. To do that, we will use the results of the previous subsection.

We consider the following equation

$$D_\lambda u = F(x, u) = \sum_{A \in \Gamma_n} \tilde{F}_A(x) u_A, \quad (32)$$

where the components $\tilde{F}_A(x)$ are continuously differentiable $\mathcal{A}_n(2, \alpha_j, \gamma_{ij})$ -valued functions. Suppose that the equation (32) is a differential equation with an anti-meta-monogenic right-hand side and denote the expression $\overline{D}_\lambda(\sum_{A \in \Gamma_n} \tilde{F}_A(x) u_A)$ by $\sum_{A \in \Gamma_n} \tilde{G}_A(x) u_A$. Then each solution of (32) satisfies the differential equation

$$\tilde{\Delta} u - 2\lambda \partial_0 u + \lambda^2 u = \sum_{A \in \Gamma_n} \tilde{G}_A(x) u_A.$$

Replacing the function u by

$$v = u \exp(-\lambda x_0)$$

we obtain

$$\tilde{\Delta}v = \sum_{A \in \Gamma_n} \tilde{G}_A(x)v_A.$$

Observe that the function v is also a generalized meta-monogenic function. Thus the interior estimate for $\frac{\partial v}{\partial x_i}$ can be obtained in the sup-norm in a similar form as in Subsection 3.2. Since Ω is a bounded domain, the definition of function v leads to an interior estimate for the generalized meta-monogenic function u . Therefore we have the following result:

Theorem 4. *Let u be a meta-monogenic function in Ω with values in the Clifford type algebra $\mathcal{A}_n(2, \alpha, \gamma_{ij})$. Suppose that \tilde{C}_G is a bound for the absolute values of the components of \tilde{G}_A and ν a bound for $\exp(\lambda x_0)$ in Ω . Denoting the diameter of Ω by d_0 and taking into account that*

$$1 \leq \frac{d_0}{\text{dist}(x, \partial\Omega)}$$

everywhere in Ω , then we have the following interior estimate

$$\left| \frac{\partial u}{\partial x_i}(x) \right| \leq \frac{\tilde{C}^*}{\text{dist}(x, \partial\Omega)} \|u\|, \quad (33)$$

where

$$\tilde{C}^* = \frac{d_0 \cdot 2^{\frac{5}{2}n} \cdot M^2 \cdot \nu \cdot |q|}{c^{n+1}} \cdot \left(\frac{m\Omega}{\tau_{n+1}} \right)^{\frac{1}{n+1}} \tilde{C}_G.$$

4 Applications of the interior estimates for the solvability of initial value problems

We consider the initial value problem

$$\partial_t u(t, x) = \mathcal{F}(t, x, u, \partial_j u) \quad (34)$$

$$u(0, x) = \varphi(x), \quad (35)$$

where $t \in [0, T]$ is the variable time, $x = (x_0, \dots, x_n)$ is a spacelike variable on \mathbb{R}^{n+1} , ∂_j is the differentiation operator with respect to x_j and $u = u(t, x)$ is an $\mathcal{A}_n(2, \alpha, \gamma_{ij})$ -valued function. The right-hand side \mathcal{F} is taken as linear in

the real valued components u_A and their first order derivatives with respect to the variables x_j . For example we can take \mathcal{F} as

$$\mathcal{F}(t, x, u, \partial_j u) = \sum_{i=0}^n A^{(i)}(t, x) \partial_i u(t, x) + B(t, x)u(t, x) + C(t, x), \quad (36)$$

where $A^{(i)}$ are twice continuously differentiable $\mathcal{A}_n(2, \alpha, \gamma_{ij})$ -valued functions and B and C are continuously differentiable $\mathcal{A}_n(2, \alpha, \gamma_{ij})$ -valued functions, for each $t \in [0, T]$ (see [10, 16]).

Furthermore, we assume that the initial function defined in a bounded domain in \mathbb{R}^{n+1} satisfies a certain partial differential equation

$$\mathcal{G}(u) = 0. \quad (37)$$

We will show that this initial value problem is solvable if the space of solutions of the equation (37) is associated to the operator \mathcal{F} and its elements satisfy an interior estimate.

In what follows, we will suppose that equation (37) can be any of the following equations:

$$D_\lambda u = 0, \quad (38)$$

$$Du - F(x, u) = 0, \quad (39)$$

$$D_\lambda u - F(x, u) = 0, \quad (40)$$

where the right-hand side of (39) is anti-monogenic and the right-hand side (40) is anti-meta-monogenic.

The initial value problem (34), (35) can be rewritten as (see [35])

$$u(t, x) = \varphi(x) + \int_0^t \mathcal{F}(\tau, x, u(\tau, x), \partial_j u(\tau, x)) d\tau. \quad (41)$$

Consequently, the solution of the initial value problem (34), (35) is a fixed point of the operator

$$Tu(t, x) = \varphi(x) + \int_0^t \mathcal{F}(\tau, x, u(\tau, x), \partial_j u(\tau, x)) d\tau. \quad (42)$$

and vice versa. To apply a fixed point theorem as the Contraction Mapping Principle, the operator (42) should map a certain Banach space B of functions

solutions of equation (37) into itself. Since the operator \mathcal{F} also depends on the derivatives $\partial_{x_i}u$, that map exists in case the derivatives $\partial_{x_i}(Tu(t, x))$ do exist and can be estimated accordingly. Therefore, one has to restrict the operator to a space of solutions functions of equation (37) for which the derivatives $\partial_{x_i}u$ of a function u belonging to this space can be estimated by u itself. This space is the so-called associated space and the estimates for the derivatives $\partial_{x_i}u$ can be attained by using the so-called interior estimate.

We will assume that the operator \mathcal{F} defined by the right-side hand of (34) is associated to the space of functions that satisfy the equation $\mathcal{G}u = 0$. Concerning operators associated to function spaces see [10, 27] and the references therein.

The necessary interior estimates in the sup-norm were obtained in Section 3 and they have the form given by (24), (31) and (33) corresponding to the three cases studied (38), (39) and (40) respectively. Without loss of generality, we can consider those estimates in the form

$$\|\partial_i u\|_K \leq \frac{\text{Const.}}{\text{dist}(K, \partial\Omega)} \|u\|_\Omega, \quad (43)$$

where K is a compact subset of $\Omega \in \mathbb{R}^{n+1}$ and $\|u\|_\Omega = \sup_{x \in \bar{\Omega}} |u(x)|$.

Now we consider an exhaustion of the bounded domain Ω by a family of subdomains Ω_s , $0 < s < s_0$. The exhaustion can be chosen in such a way that each point x of Ω lies on the boundary $\partial\Omega_{s(x)}$ of a uniquely determined domain $\Omega_{s(x)}$ of the exhaustion. Then $s_0 - s(x)$ is a measure of the distance of a point x of Ω from the boundary $\Omega_{s(x)}$ (see [25]).

Next we consider the conical set

$$M = \{(t, x) : x \in \Omega, 0 \leq t < \eta(s_0 - s(x))\},$$

where the parameter η describes the height of the conical set. The base M is the given domain Ω and its lateral surface is defined by $t = \eta(s_0 - s(x))$.

Let B_s be the Banach space of all functions u in Ω_s solutions of equation (37), equipped with the norm

$$\|u\|_s = \max_A (\sup_{\Omega_s} |u_A(x)|), \quad (44)$$

where $A \in \Gamma_n$ and let $B_*(M)$ be the set of all functions $u = u(t, x)$ which are continuous in M . $B_*(M)$, endowed with the norm

$$\|u\|_* = \sup_M \|u(t, x)\|_{s(x)} d(t, x),$$

where $d(t, x) = s_0 - s(x) - \frac{t}{\eta}$, becomes a Banach space (see [20]).

Applying estimate (43) for $K = \Omega_{s'}$ and $\Omega = \Omega_{s''}$ and using the definition of the norm (44), we get for a function belonging to the solution space of equation (37) and $0 < s' < s'' < s_0$, the following estimate in Ω :

$$\|\partial_i u\|_{s'} \leq \frac{\text{Const.}}{\text{dist}(\Omega_{s'}, \partial\Omega_{s''})} \|u\|_{s''}. \quad (45)$$

Now we consider the following space:

$$B_*^D(M) = \{u \in B_*(M) : \mathcal{G}(u(t, x)) = 0, \text{ for each fixed } t\}.$$

Applying the Weierstrass approximation theorem, we can see that $B_*^D(M)$ is a closed subset of the Banach space $B_*(M)$. Hence $B_*^D(M)$ is also a Banach space.

Since \mathcal{F} is associated to the solution space of equation $\mathcal{G}(u(t, x)) = 0$, then the operator T maps $B_*^D(M)$ into itself.

Using (45) and assuming that \mathcal{F} satisfies a Lipschitz condition with respect to u and its first order derivatives, we have the estimate

$$\|Tu - T\tilde{u}\|_* \leq \eta C \|u - \tilde{u}\|_*,$$

where C is a constant depending on the coefficients of \mathcal{F} and involving the constant of the interior estimate given by (43) and the Lipschitz constants. Taking $\eta < \frac{1}{C}$, we obtain that T is a contractive operator. Therefore the problem (34)-(35) has an unique solution in $B_*^D(M)$.

In consequence, the following theorem has been proved:

Theorem 5. *Let \mathcal{F} be the operator defined by the right-hand side of equation (34) satisfying a Lipschitz condition with respect to u and its first order derivatives. If \mathcal{F} is associated to the solution space of equation $\mathcal{G}(u(t, x)) = 0$, then the initial value problem (34)-(35) is solvable provided the initial function satisfies the equation $\mathcal{G}(u(t, x)) = 0$.*

Note that the solution is obtained by construction, using the method of successive approximations, i.e., the solution is the limit of the sequence $\{u_n\}_{n=0}^\infty$ defined by:

$$u_0(t, x) = \varphi(x) + \int_0^t C(\tau, x) d\tau \quad (46)$$

and

$$u_{n+1}(t, x) = u_0(t, x) + \int_0^t (\mathcal{F}'u_n)(\tau, x)d\tau, \quad (47)$$

where $\mathcal{F}'u = \mathcal{F}u - C(t, x)$ (see [9, 17]).

5 Concluding remarks

Interior estimates in the L_p norm for monogenic functions and generalized monogenic functions satisfying a differential equation with an anti-monogenic right-hand side, with values in a Clifford type algebra $\mathcal{A}_n(2, \alpha, \gamma_{ij})$ can be found in [27]. Concerning interior estimates in other norms or other differential equations represented by $\mathcal{G}u = 0$ for functions in Clifford type algebras, the research continues in progress.

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