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| Damien Gayet. Expected local topology of random complex submanifolds. 2022. hal-03580841v2

HAL Id: hal-03580841

<https://hal.science/hal-03580841v2>

Preprint submitted on 21 Feb 2022

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Expected local topology of random complex submanifolds

DAMIEN GAYET

February 21, 2022

Abstract

Let $n \geq 2$ and $r \in \{1, \dots, n-1\}$ be integers, M be a compact smooth Kähler manifold of complex dimension n , E be a holomorphic vector bundle with complex rank r and equipped with an hermitian metric h_E , and L be an ample holomorphic line bundle over M equipped with a metric h with positive curvature form. For any $d \in \mathbb{N}$ large enough, we equip the space of holomorphic sections $H^0(M, E \otimes L^d)$ with the natural Gaussian measure associated to h_E , h and its curvature form. Let $U \subset M$ be an open subset with smooth boundary. We prove that the average of the $(n-r)$ -th Betti number of the vanishing locus in U of a random section s of $H^0(M, E \otimes L^d)$ is asymptotic to $\binom{n-1}{r-1} d^n \int_U c_1(L)^n$ for large d . On the other hand, the average of the other Betti numbers are $o(d^n)$. The first asymptotic recovers the classical deterministic global algebraic computation. Moreover, such a discrepancy in the order of growth of these averages is new and contrasts with all known other smooth Gaussian models, in particular the real algebraic one. We prove a similar result for the affine complex Bargmann-Fock model.

MATHEMATICS SUBJECT CLASSIFICATION 2010: 32L05 (Holomorphic bundles and generalizations) ; 60G15 (Gaussian processes)

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1 Introduction

The goal of this article is to understand the statistics of the local topology of random complex submanifolds, for projective manifolds and the affine complex space.

Projective manifolds. Let n be a positive integer, M be a compact smooth complex manifold of complex dimension n , and L be an ample holomorphic line bundle over M . Let h be a Hermitian metric on L with positive curvature form $c_1(L) = \omega$, that is locally

$$\omega = \frac{1}{2i\pi} \partial\bar{\partial} \log \|s\|_h^2, \quad (1.1)$$

where s is any local non-vanishing section of L . Then, (M, ω) becomes a Kähler manifold, and by the Kodaira theorem, it can be embedded in a projective space. For any large enough degree $d \geq 1$, and any generic holomorphic section $s \in H^0(M, L^d)$, denote by $Z_s \subset M$ the smooth vanishing locus of s . The famous hyperplane Lefschetz theorem asserts, in particular, that [14]

$$\forall 0 \leq i \leq n-2, \quad b_i(Z_s) = b_i(M).$$

For instance, if $M = \mathbb{C}P^n$, then for $i \leq n-2$, $b_i(Z_s) = 0$ if i is odd and $b_i(Z_s) = 1$ if i is even. On the other hand,

$$\frac{1}{d^n} b_{n-1}(Z_s) \xrightarrow{d \rightarrow \infty} \int_M \omega^n. \quad (1.2)$$

Of course, there is no local (deterministic) version of the Lefschetz theorem. Indeed, if U is an open subset of M , the intersection of U with Z_s can be empty or can have a topological complexity bigger than the one of Z_s . In particular for $n \geq 2$, Z_s is connected but its intersection with U can be disconnected. There is even no bound for the number of components of it, since we can twist U for that. However, for a fixed U defined by algebraic inequalities, the following bound exists:

Theorem 1.1 ([20, Theorem 3]) *Let $U \subset \mathbb{C}P^n \setminus \{Z_0 = 0\}$ be an open subset defined by real algebraic inequalities. Then, there exists a constant C_U depending only on the number and the degree of the defining polynomials of U , such that for any generic r -uple of homogeneous complex polynomials $s = (s_1, \dots, s_r) \in (\mathbb{C}_d^{hom})^r$ of degree d ,*

$$\sum_{i=0}^{2n-2r} b_i(Z_s \cap U) \leq C_U d^{2n}. \quad (1.3)$$

Now, if the section s is taken at random, one could hope that for fixed U , not necessarily defined by polynomials, the average topology of $Z_s \cap U$ reflects in some way the Lefschetz theorem and with further hope, the asymptotic (1.2) as well. In this paper, we prove that these two intuitions are true, in the following more general classical setting. In addition to (L, h) , let (E, h_E) be a holomorphic vector bundle of rank r and equipped with a Hermitian metric h_E . Since L is ample, for d large enough, the space of holomorphic sections $H^0(M, E \otimes L^d)$ is non-trivial. Then, a natural scalar product associated to this setting is the following:

$$\forall (s, t) \in (H^0(M, E \otimes L^d))^2, \quad \langle s, t \rangle = \int_M h_E \otimes h_{L^d}(s, t) \frac{\omega^n}{n!}, \quad (1.4)$$

where h_{L^d} is the metric over L^d induced by h^d . A natural probability measure μ_d over this space is the Gaussian one associated to this Hermitian product. In other terms, for any Borelian $A \subset H^0(M, E \otimes L^d)$,

$$\mu_d(A) = \int_A e^{-\frac{1}{2}\|s\|^2} \frac{ds}{(2\pi)^{N_d}}, \quad (1.5)$$

where $\|\cdot\|$ denotes the norm associated to the Hermitian product (1.4), N_d the (complex) dimension of $H^0(M, E \otimes L^d)$ and ds the Lebesgue measure. Notice that if $(S_i)_{i \in \{1, \dots, N_d\}}$ is an orthonormal basis for this scalar product, then $s = \sum_{i=1}^{N_d} a_i S_i$ is random for μ_d when the coefficients $a_i \in \mathbb{C}$ are i.i.d standard complex Gaussians, that is $\Re a_i$ and $\Im a_i$ are independent standard Gaussians.

Example 1.2 For $M = \mathbb{C}P^n$, $E = \mathbb{C}^r$ equipped with its standard Hermitian metric, $L = \mathcal{O}(1)$ equipped with the Fubini-Studi metric, then s consists in r independent copies of random polynomials

$$\forall 1 \leq i \leq r, s_i([Z]) = \sum_{i_0 + \dots + i_n = d} a_{i_0 \dots i_n} \sqrt{\frac{(n+d)!}{n! i_0! \dots i_n!}} Z_0^{i_0} \dots Z_n^{i_n},$$

where the $(a_i)_i$ are independent standard complex Gaussian variables.

Our main result is the following:

Theorem 1.3 Let $n \geq 2$ and $1 \leq r \leq n-1$ be integers, M be a compact smooth Kähler manifold and (L, h) be an ample complex line bundle over M , with positive curvature form ω , (E, h_E) be a holomorphic rank r vector bundle and let $U \subset M$ be a open subset with smooth boundary. Then

$$\begin{aligned} \forall i \in \{0, \dots, 2n-2r\} \setminus \{n-r\}, \frac{1}{d^n} \mathbb{E} b_i(Z_s \cap U) &\xrightarrow{d \rightarrow \infty} 0 \\ \frac{1}{d^n} \mathbb{E} b_{n-r}(Z_s \cap U) &\xrightarrow{d \rightarrow \infty} \binom{n-1}{r-1} \int_U \omega^n. \end{aligned}$$

Here the probability measure is the Gaussian one given by (1.5). These asymptotics hold when $U = M$ as well.

Of course, when $U = M$, the topological type of Z_s does not depend on the random section s . Markov's inequality implies the following corollary.

Corollary 1.4 Under the hypotheses of Theorem 1.3, for any $\varepsilon > 0$,

$$\limsup_{d \rightarrow +\infty} \mu_d \left\{ s \in H^0(M, E \otimes L^d) \mid b_{n-r}(Z_s \cap U) \geq \frac{d^n}{\varepsilon} \binom{n-1}{r-1} \int_U \omega^n \right\} \leq \varepsilon,$$

where μ_d is defined by (1.5).

Note that the Gaussian measure can be replaced by the round metric on the sphere $\mathbb{S}H^0(M, E \otimes L^d)$, where the metric is defined by (1.4). Hence, this corollary can be seen as a deterministic result about the volume of certain subsets of topological interest in this sphere.

Example 1.5 Under the standard setting of Example 1.2, $\int_{\mathbb{C}P^n} \omega_{FS}^n = 1$, so that

$$\frac{1}{d^n} \mathbb{E} b_{n-r}(Z_s \cap U) \xrightarrow{d \rightarrow \infty} \binom{n-1}{r-1} \frac{\text{vol}(U)}{\text{vol}(\mathbb{C}P^n)}.$$

Remark 1.6 1. Theorem 1.3 provides the first explicit asymptotic for one mean Betti numbers of the nodal set of a smooth Gaussian field. Former explicit asymptotics were proven [9] in a real context for high level random sets (in particular, not the zero one). For $r = 1$, it is striking that the asymptotic average local behaviour reflects exactly the global asymptotic estimate given by (1.2).

2. For $U = M$, Theorem 1.3 has a deterministic corollary. Indeed, any divisor of given degree is diffeomorphic to the another of the same degree. Hence, for $r = 1$ the second assertion of Theorem 1.3 is equivalent to the asymptotic (1.2). For higher codimensions r , [11, Corollary 3.5.2] shows that

$$\frac{1}{d^n} \chi(Z_s) \xrightarrow{d \rightarrow \infty} \binom{n-1}{r-1} \int_M \omega^n. \quad (1.6)$$

Again, (1.6) is implied by Theorem 1.3.

3. We emphasize that these qualitatively different asymptotics are new. In particular constrasts with the real situation [11, Corollary 1.2.2] and all known others smooth Gaussian models like [13] (see also [25]). In these latter cases, all Betti numbers grow like L^n , where $1/L$ is the natural scale of the model, $1/\sqrt{d}$ in this one. This is especially true for the number of connected components, see [21]. Here, the scale is $d^{-\frac{1}{2}}$, however only the $(n-r)$ -th Betti number grows like d^n .

4. In [8], it was proved that for any compact smooth real hypersurface Σ of \mathbb{R}^n , for any open subset $U \subset M$, with uniform probability, a uniform proportion of the $(n-1)$ -homology in $Z_s \cap U$ can be represented by Lagrangians submanifolds diffeomorphic to \mathcal{L} .

5. In [1, Theorem 5 (2)] (see also [5, Theorem A]), it is shown that as far as (local) topology of $Z_s \cap \mathbb{R}P^n$ is only concerned, a random real polynomial s of degree d can be replaced, with high probability, by a polynomial of degree slightly greater than \sqrt{d} . In fact, this statement holds for complex polynomials on a ball in the complementary of a complex hypersurface as well. Using Milnor's bound (1.3), this replacement allows to get a similar estimate as Corollary 1.4 when U is defined algebraically. The decay is almost exponential in this case.

6. In [2, Proposition 6], the author proved that (deterministic) Donaldson hypersurfaces, which are zeros of sections with vanish transversally with a controlled derivative, satisfy such local topology estimate for the $(n-r)$ -th Betti number. Theorem 1.3 shows a further evidence that Donaldson hypersurfaces have common features with random ones. For instance, the current of integration over Z_s fills out uniformly M for large degrees d in both contexts, see [6] and [22].

The complex Bargmann-Fock field. Finally, we prove an affine version in the universal limit for holomorphic sections, namely the complex Bargmann-Fock field. The Bargmann-Fock field is defined by

$$\forall z \in \mathbb{C}^n, f(z) = \sum_{(i_1, \dots, i_n) \in \mathbb{N}^n} a_{i_0, \dots, i_n} \sqrt{\frac{\pi^{i_1 + \dots + i_n}}{i_1! \dots i_n!}} z_1^{i_1} \dots z_n^{i_n} e^{-\frac{1}{2}\pi \|z\|^2}, \quad (1.7)$$

where the a_I 's are independent normal complex Gaussian random variables. The strange presence of π will be explained below.

Theorem 1.7 *Let $n \geq 2$ and $1 \leq r \leq n - 1$ be integers, $f : \mathbb{C}^n \rightarrow \mathbb{C}^r$ be r independent copies of the Bargmann-Fock field, and $U \subset \mathbb{C}^n$ be an open subset with compact smooth boundary. Then,*

$$\forall i \in \{0, \dots, 2n - 2r\} \setminus \{n - r\}, \quad \frac{1}{R^{2n}} \mathbb{E}b_i(Z_f \cap RU) \xrightarrow{R \rightarrow +\infty} 0$$

$$\frac{1}{R^{2n}} \mathbb{E}b_{n-r}(Z_f \cap RU) \xrightarrow{R \rightarrow +\infty} n! \binom{n-1}{r-1} \text{vol}(U).$$

The volume is the standard one.

Remark 1.8 1. *Again, compared to the other known results, the order of magnitude of the mean number of connected components is not the natural one, that is R^{2n} , see [21] for instance.*

2. *Theorem 1.7 (and Theorem 1.3) was guessed by the author for the following geometric reasons, which we present for $n = 2$ and $r = 1$: because of the maximum principle, if a complex curve in \mathbb{C}^2 locally touches a real hyperplane H , being (locally) on one side of H , then C is affine and $C \subset H$. Now, if $p : U \subset \mathbb{C}^2 \rightarrow \mathbb{R}$ is Morse, for any $R > 0$, let $p_R = p(\cdot/R)$. Then for large $R > 0$, the level sets of p_R are locally closer and closer to be planar, so that there should be less and less random curves touching them from the interior, that is there are less and less critical points of $p|_{Z_f}$ of index 0, compared to critical points of index 1. Morse theory should then imply the result.*

Related results. The study of the statistics of the Betti numbers, or even the diffeomorphism type, of a random smooth submanifold (of positive dimension) is now a well-developed subdomain of random geometry, with current links to percolation. We refer to [9] for a historical account of this topic. The results were proven mainly in the real algebraic and Riemannian semiclassical settings. Both models share a common feature: the Betti numbers grow (with the parameter, degree or eigenvalue) like the inverse of the scale to a power equal to the dimension of the ambient manifold. In both cases, the covariance of the model is the spectral kernel, for which estimates exist.

The local study of the geometry of random *complex* submanifolds of positive dimension began with [22], under the hypotheses of Theorem 1.3, with $r = 1$. It was proven that the average current of integration over Z_s tends to the curvature form of the line bundle, when d grows to infinity. Since the topology of the complex hypersurfaces depend only on the degree, a crucial difference with the real setting, the topology of random complex hypersurfaces seemed less promising. Our paper [8] showed that local random (symplectic) topology is interesting as well, and even can provide new deterministic results.

A lot of results about critical points of random sections has been done. In this complex algebraic context, it seems to begin with [7]. In [10], the restriction of a Lefschetz pencil to the complex random hypersurface was used in order to get topological estimates through Morse theory, which is the spirit of the present paper. We refer to [11, §1.3] for further references. The following result is close to the present work:

Theorem 1.9 (*[12, Theorem 1.3] for $r = 1$, [11, Theorem 3.5.1] for any r) Under the hypotheses of Theorem 1.3, let $p : M \dashrightarrow \mathbb{C}P^1$ be a Lefschetz pencil. Then,*

$$\frac{1}{d^n} \mathbb{E}\#(U \cap \text{Crit}(p|_{Z_s})) \xrightarrow{d \rightarrow \infty} \binom{n-1}{r-1} \int_U \omega^n.$$

This result holds in particular for any local holomorphic map. A similar real version of Theorem 1.9 was proven as well. In the real setting, the authors used the weak Morse inequalities in order to get an upper bound for the average Betti numbers of Z_s . Lower bounds of the same order of magnitude (in the degree) were estimated by the barrier method.

As in [9], in the present paper we use the *strong* Morse inequalities, and moreover we use this theory on manifolds with boundary, which implies to take in account the critical points of the restriction of the function to the boundary. Joint with the weak ones, strong Morse inequalities allow us to get the proper estimate of the mean middle Betti number given by Theorem 1.3. On the contrary to the real setting, in our complex setting strong Morse inequalities help, because the complex Hessian of a holomorphic function has a symmetric signature, which implies that all mean critical points of $p|_{Z_s}$ have the wrong order of magnitude, except when the index is the middle one, that is $n-r$, see Theorem 4.4.

The method to prove Theorem 4.4 is different than the one used for Theorem 1.9, but both provide, on the one hand, a Kac-Rice formula (both based, at the end, on the coarea formula), and on the other hand, an estimate of it when the degree goes to infinity. In [12] and [11], the authors used explicit peak sections to compute the average, and the aforementioned parts were mixed. In this paper we wanted to clearly separate the two parts of the proof: one part which is based on a general Kac-Rice formula as Corollary 3.4, and one part which depends on the particular model, real, holomorphic or mixed (on the boundary of the open set U). This can be done because the second part only needs informations about the covariance function. For projective manifolds, this is the Bergman kernel, see section 4.2. The peak sections is a way to recover the needed informations, see [24]. In [22], the Szegö kernel was used, based on Zelditch's semiclassical way [26] of proving Tian's theorem. For the Riemannian setting like in [13], the covariance is the spectral kernel and Hörmander estimates can be used.

Holomorphic percolation. Theorem 1.7 raises a natural question related to percolation theory: is there a Russo-Seymour-Welsh phenomenon for the complex Bargmann-Fock field? In its simplest non-trivial form, this question is the following:

Let $B, B' \subset \mathbb{S}^3 \subset \mathbb{C}^2$ two disjoint closed smooth 3-balls lying in the unit sphere, and let f be the complex Bargmann-Fock field over \mathbb{C}^2 see (1.7). Is it true that

$$\liminf_{R \rightarrow +\infty} \mathbb{P}(\exists \text{ a connected component of } \{f = 0\} \cap R\mathbb{B}^4 \text{ joining } RB \text{ to } RB') > 0 ?$$

The analog for the real Bargmann-Fock over \mathbb{R}^2 is true, see [3]. We emphasize that the holomorphic situations contrasts in many ways with the real setting. Firstly, there is no bounded component of $\{f = 0\}$ in the complex case and with probability one, there is a unique component of Z_f . Secondly, none of the classical tools in percolation theory does hold in this holomorphic context, in particular duality and FKG property. Besides, the isotropy of the field and the absence of bounded components imply that with uniform positive probability there exists a component of $\{f = 0\} \cap R\mathbb{B}$ from \mathbb{B} to RB . In the real setting, this probability tends to 0. Finally, note that a similar question can be asked for complex algebraic submanifolds:

Let $U \subset \mathbb{C}P^2$ be a smooth ball in the projective plane, $B, B' \subset \partial U$ two disjoint closed smooth 3-balls lying in the boundary of U , and let $s \in H^0(\mathbb{C}P^2, \mathcal{O}(d))$ be a random polynomial of degree d . Is it true that

$$\liminf_{d \rightarrow +\infty} \mathbb{P}(\exists \text{ a connected component of } \{s = 0\} \cap U \text{ joining } B \text{ to } B') > 0 ?$$

The real analog has been proven in [4].

Ideas of the proof of Theorem 1.3. Let $U \subset M$ be an open set with compact smooth boundary and $p : \bar{U} \rightarrow \mathbb{R}$ be a smooth Morse function in the sense of Definition 4.8, that is p Morse on U , its restriction to ∂U is Morse and p has no critical point on ∂U . Let Z be a complex smooth submanifold of U with boundary in ∂U , such that $p|_Z$ is Morse in the latter sense. Then, by Morse theory for manifolds with boundary, for any $0 \leq i \leq \dim_{\mathbb{R}} Z$, the i -th Betti number of Z is less or equal to the number of critical points of $p|_Z$ and $p|_{\partial Z}$ of index i , see Theorem 4.12. Besides, from the strong Morse inequalities, we can estimate the i -th Betti number of Z if the critical points of index different than i are far smaller.

We apply this to Z_s the zero set of a random holomorphic section s of degree d . Note that the natural scale for the natural measure is $1/\sqrt{d}$. Hence, in every ball of this radius, the geometry of Z_s should be independent of d . This implies in particular that on a manifold of real dimension m , the average of geometric or analytic observables like the number of critical points of $p|_{Z_s}$ should grow like $d^{\frac{m}{2}}$.

The general Kac-Rice formula given by Corollary 3.4 applied to our projective situation allows us to estimate the number of critical points in the interior of U , see Theorem 4.1, and a factor d^m emerges, as guessed by the previous heuristic arguments. Here we use the fact that the covariance function of s is the Bergman kernel and that this kernel has a universal rescaled limit, see Theorem 4.5. Now, the integral in the Kac-Rice formula involves the determinant of a random matrix provided by a perturbation of the Hessian of s (restricted to the tangent space of Z_s), where the perturbation decreases with d . At the limit, the matrix is non zero only for middle index, since the Hessian has complex symmetries. On the contrary, the mean number of critical points of middle index has a precise non-trivial asymptotic, see Theorem 4.4.

We need also to control the number of critical points of the restriction of p to the boundary of U and of Z_s . We use the Kac-Rice formula in this mixed case as well, see Proposition 4.10. As guessed, a factor $d^{m-\frac{1}{2}}$ emerges. Both estimates and Morse theory finish up the proof of Theorem 1.3.

Structure of the article. In section 2, we prove various deterministic lemmas in order to prepare the main Kac-Rice formula computing the mean of critical points. This formula is established in section 3. In section 4, we apply this formula in order to prove Theorem 1.3 and Theorem 1.7.

Acknowledgements. The author thanks Thomas Letendre for his expertise of the book [18] and Michele Ancona for discussions about the subject of the article and his paper [1].

2 Deterministic geometric preliminaries

The general setting of this paper is a real manifold M of dimension n and a real vector bundle E over M . Let also $p : M \rightarrow \mathbb{R}$ be a Morse function. For any generic smooth section of E , we will look at the critical points of $p|_{\{s=0\}}$, which are the points $x \in M$ where the tangent space of the vanishing locus Z_s of s lies in $\ker dp(x)$. For this reason, we must understand the geometry of $\ker \nabla s$ as an element of the Grassmannian bundle $\text{Grass}(n-r, TM)$ or $\text{Grass}(n-r, \ker dp)$, where ∇ denotes any covariant connection on E . In this section, we provide various simple lemmas which will be used in the main results. To make the computations easier, M and E will be endowed with metrics.

2.1 Kernel and Grassmannians

The following Lemma is classical:

Lemma 2.1 *Let n be an integer and (V, g) be a finite dimensional real vector space of dimension n equipped with a scalar product g . For any integer $0 \leq m \leq n$, the Grassmannian $\text{Grass}(m, V)$ of m -planes of V is a smooth manifold and for any $K \in \text{Grass}(m, V)$, $T_K \text{Grass}(m, V)$ is canonically (with respect to g and K) identified with $\mathcal{L}(K, K^\perp)$. In particular, $T_K \text{Grass}(m, V)$ inherits the natural metric on $\mathcal{L}(K, K^\perp)$ induced by g . If V is complex and g is Hermitian, then the same holds, replacing the real Grassmannian by the complex Grassmannian $\text{Grass}_{\mathbb{C}}$, and $\mathcal{L}(K, K^\perp)$ by the complex linear maps $\mathcal{L}^{\mathbb{C}}(K, K^\perp)$.*

Remark 2.2 *Note that for $f \in \mathcal{L}(K, K^\perp)$, if A denotes the matrix of f in any g -orthonormal basis of K and K^\perp , then the squared norm of f induced by g equals $\text{Tr}(AA^*)$.*

Lemma 2.3 *Let $1 \leq r \leq n$, (V, g) be an Euclidean vector space, and E be a real vector space, of respective dimensions n and r . Let $\alpha_0 \in \mathcal{L}_{\text{onto}}(V, E)$ and $K = \ker \alpha_0 \in \text{Grass}(n - r, V)$. Then, there exists a neighborhood $U \subset \mathcal{L}_{\text{onto}}(V, E)$ of α_0 and a smooth map $\varphi : U \rightarrow \mathcal{L}(K, K^\perp)$ such that $\varphi(\alpha_0) = 0$ and*

$$\forall \alpha \in U, \ker \alpha = (\text{Id}|_K + \varphi(\alpha))(K).$$

Moreover, for any $\beta \in \mathcal{L}(E, V)$, $d\varphi(\alpha_0)(\beta) = -(\alpha_0|_{K^\perp})^{(-1)}\beta|_K$. The same holds in the complex-Hermitian setting.

Proof. Let

$$\begin{aligned} F : \mathcal{L}(V, E) \times \mathcal{L}(K, K^\perp) &\rightarrow \mathcal{L}(K, E) \\ (\alpha, f) &\mapsto \alpha \circ (\text{Id}|_K + f). \end{aligned}$$

Then, F is smooth and $F(\alpha_0, 0) = 0$. The partial differential in f at $(\alpha_0, 0)$ writes

$$\forall g \in \mathcal{L}(K, K^\perp), d_f F(\alpha_0, 0)(g) = \alpha_0 \circ g = \alpha_0|_{K^\perp} \circ g \in \mathcal{L}(K, E).$$

This partial differential is an isomorphism because α_0 is onto, so that $\alpha_0|_{K^\perp} \in \mathcal{L}(K^\perp, E)$ is an isomorphism. Note that the partial differential in α satisfies

$$\beta \in \mathcal{L}(V, E), d_\alpha F(\alpha_0, 0)(\beta) = \beta|_K.$$

Hence, by the implicit function theorem, there are two open neighborhoods $U \subset \mathcal{L}(V, E)$ and $W \subset \mathcal{L}(K, K^\perp)$ of α_0 and 0 respectively, and a smooth function $\varphi : U \rightarrow W$ such that

$$\forall (\alpha, f) \in U \times W, F(\alpha, f) = 0 \Leftrightarrow f = \varphi(\alpha).$$

Besides, $d\varphi(\alpha_0) = -(d_f F(\alpha_0, 0))^{(-1)} \circ d_\alpha F(\alpha_0, 0)$, hence the result. \square

Let (V, g) and (E, h) as in Lemma 2.3. Define

$$\begin{aligned} \kappa : \mathcal{L}_{\text{onto}}(V, E) &\rightarrow \text{Grass}(n - r, V) \\ \alpha &\mapsto \ker \alpha. \end{aligned} \tag{2.1}$$

The following lemma computes the derivative of κ .

Lemma 2.4 *Let (V, g) and (E, h) be two real vector spaces as in Lemma 2.3. Then, κ defined by (2.1) is smooth and for any $\alpha_0 \in \mathcal{L}_{\text{onto}}(V, E)$, in the chart given by Lemma 2.1,*

$$\begin{aligned} d\kappa(\alpha_0) : \mathcal{L}(V, E) &\rightarrow T_{\ker \alpha_0} \text{Grass}(n-r, V) \simeq \mathcal{L}(\ker \alpha_0, \ker^\perp \alpha_0) \\ \beta &\mapsto -(\alpha_0|_{K^\perp})^{-1} \beta|_K. \end{aligned}$$

Proof. Let $\alpha_0 \in \mathcal{L}_{\text{onto}}(V, E)$. By Lemma 2.3, Locally, for any α close enough to α_0 ,

$$\kappa(\alpha) = (\text{Id}|_{\ker \alpha_0} + \varphi(\alpha))(\ker \alpha_0).$$

The second assertion of Lemma 2.3 concludes. \square

2.2 The field and its geometry

Let $n \geq 2$ and $1 \leq r \leq n-1$ be integers, (M, g) be a smooth Riemannian manifold of dimension $n \geq 1$, and $E \rightarrow M$ be a smooth real vector bundle of rank r . Define

$$F = E \oplus T^*M \otimes E.$$

Let $p : M \rightarrow \mathbb{R}$ be a Morse function, and $W \subset F$ be the subset of F defined by

$$W = \{(x, 0, \alpha) \in F, dp(x) \neq 0, \alpha \text{ is onto and } \ker \alpha \subset \ker dp(x)\}.$$

Note that W projects onto $M \setminus \text{Crit}(p)$. Since p is Morse, $\text{Crit}(p)$ is a discrete set in M without any accumulation point. If M is compact, $\text{Crit}(p)$ is finite. We will use later that for any C^1 section s of E , then $x \in M$ is critical for the restriction of p on $\{s = 0\}$ at x is equivalent to $(x, s(x), \nabla s(x)) \in W$, see (3.1) below. For any $x \in M \setminus \text{Crit}(p)$, let

$$W_x = \{(0, \alpha) \in F_x, (x, 0, \alpha) \in W\}. \quad (2.2)$$

Lemma 2.5 *Let M, E, p, F and W be defined as above. Then,*

1. W is a smooth submanifold of F of codimension n ;
2. W intersects transversally the fibres of F ;
3. for any $x \in M \setminus \text{Crit}(p)$, W_x is a smooth submanifold of F_x of codimension n , and

$$\forall (0, \alpha) \in W_x, T_{(0, \alpha)} W_x = \{(0, \beta) \in F_x, dp(x)(\alpha|_{\ker^\perp \alpha})^{-1} \beta|_{\ker \alpha} = 0\}. \quad (2.3)$$

Proof. Let $(x_0, 0, \alpha_0) \in W$ and $K = \ker \alpha_0 \subset T_{x_0}M$. Let O be a neighborhood of x_0 such that M can be locally identified with $T_{x_0}M$ by a chart over O , and $E|_O$ can be identified with $O \times E_{x_0}$ via a trivialization. Then, $F|_O$ can be identified with $O \times (E_{x_0} \oplus T_{x_0}^*M \otimes E_{x_0})$. By Lemma 2.3, there exists $U \subset \mathcal{L}(T_{x_0}M, E_{x_0})$ a neighborhood of α_0 and a smooth map $\varphi : U \rightarrow \mathcal{L}(K, K^\perp)$, such that for any $\alpha \in U$, $\ker \alpha = (\text{Id} + \varphi(\alpha))|_K$. Now, define the smooth map

$$\begin{aligned} \Phi : F|_O &\rightarrow E_{x_0} \times K^* \\ (x, s, \alpha) &\mapsto \left[s, dp(x) (\text{Id} + \varphi(\alpha))|_K \right]. \end{aligned} \quad (2.4)$$

Then, $W \cap O = \Phi^{-1}(0)$. Moreover, by Lemma 2.3 again, for all $(v, t, \beta) \in T_{x_0}M \times E_{x_0} \times \mathcal{L}(T_{x_0}M, E_{x_0})$,

$$d\Phi(x_0, 0, \alpha_0)(v, t, \beta) = \left[t, d^2p(x_0)(v)|_K - dp(x_0)(\alpha|_{K^\perp})^{-1} \beta|_K \right]. \quad (2.5)$$

Since $dp(x) \neq 0$, $d\Phi$ is onto, so that W is a smooth submanifold of F of codimension n . The third assertion of the lemma is an immediate consequence of (2.5). For the second assertion of the lemma, let $(x, 0, \alpha) \in F \cap W$. Then, by (2.5), $(v, t, \beta) \in F_x \cap T_{(x, 0, \alpha)} W$ iff $(v, t) = 0$ and $dp(x)(\alpha|_{\ker^\perp \alpha})^{-1} \beta|_{\ker \alpha} = 0$, that is $(0, \beta) \in T_{(0, \alpha)} W_x$. Hence, $F \pitchfork W$. \square

2.3 Two Jacobians

In this paragraph, the setting is the same as in the latter one, with the novelty that the vector bundle E is endowed with a Euclidean metric h_E on its fibres. We compute two Jacobians which will be needed for the coarea formula used in the main Kac-Rice formula Corollary 3.4. For any $(x, \alpha) \in TM^* \otimes E$, such that α is onto and $dp(x)(\alpha|_{\ker^\perp \alpha})^{-1} \neq 0 \in E_x^*$, define

$$\mu(x, \alpha) = \ker dp(x)(\alpha|_{\ker^\perp \alpha})^{-1} \subset E_x. \quad (2.6)$$

Let $\varepsilon(x, \alpha)$ be one of the two unit vector in $\mu(x, \alpha)^\perp \subset E_x$, and $K = \ker \alpha$. The following decomposition will help:

$$(\alpha|_{K^\perp})^{-1} = \begin{pmatrix} \mathbb{R}\varepsilon & \mu \\ (\alpha|_{K^\perp})|_{\mathbb{R}\varepsilon}^{-1} & 0 \\ * & (\alpha|_{K^\perp})|_{\mu}^{-1} \end{pmatrix} \begin{matrix} \ker^\perp dp(x) \\ \ker dp(x) \cap K^\perp \end{matrix}. \quad (2.7)$$

Note that for any $x \in M \setminus \text{Crit}(p)$, by (2.3),

$$T_{(0, \alpha)}W_x = \mathcal{L}(\ker \alpha, \mu(x, \alpha)) \oplus \mathcal{L}(\ker^\perp \alpha, E_x)$$

Definition 2.6 (see [23, C.1]) *Let M, N be two Riemannian manifolds and $\kappa : M \rightarrow N$ be a C^1 map. Then, $J_x \kappa$ denotes the normal Jacobian, that is the determinant in orthonormal basis of $d\kappa(x)|_{\ker^\perp d\kappa(x)}$.*

In the following, for any $x \in M$, let $\kappa : \mathcal{L}_{\text{onto}}(T_x M, E_x) \rightarrow \text{Grass}(n - r, T_x M)$ defined by (2.1) for $V = T_x M$ and $E = E_x$. By an abuse of notation, we denote also κ the map $E_x \times \mathcal{L}_{\text{onto}}(T_x M, E_x) \ni (0, \alpha) \mapsto \kappa(\alpha)$.

Lemma 2.7 *For any $x \in M \setminus \text{Crit}(p)$, let $\kappa|_{W_x} : W_x \ni (0, \alpha) \mapsto \ker \alpha \subset \ker dp(x)$. Then, for all $(0, \alpha) \in W_x$, $J_{(0, \alpha)}(\kappa|_{W_x}) = \left| \det \alpha|_{\ker^\perp \alpha \cap \ker dp(x)} \right|^{n-r}$.*

Proof. Firstly, by Lemma 2.5,

$$T_{(0, \alpha)}W_x \ker(\beta \mapsto \langle \varepsilon(x, \alpha), \beta|_K \rangle),$$

where ε has been defined above. Since

$$d(\kappa|_{W_x})(0, \alpha) = (d\kappa(0, \alpha))|_{T_{(0, \alpha)}W_x},$$

from Lemma 2.4 we infer that $J_{(0, \alpha)}(\kappa|_{W_x}) = \left| \det(\alpha|_{K^\perp})|_{\mu(x, \alpha)}^{-1} \right|^{n-r}$, where μ has been defined by (2.6). Since $\alpha|_{K^\perp}$ induces an isomorphism between $K^\perp \cap \ker dp(x)$ and $\mu(x, \alpha)$, we obtain the result. \square

Lemma 2.8 *Fix $x \in M \setminus \text{Crit}(p)$ and $K \in \text{Grass}(n - r, \ker dp(x))$. Let*

$$\begin{aligned} g : \kappa^{-1}(K) &\rightarrow \text{Grass}(r - 1, E_x) \\ \alpha &\mapsto \mu(x, \alpha) = \ker \left(dp(x) \circ (\alpha|_{K^\perp})^{-1} \right). \end{aligned}$$

Then, for all α , $J_{(0, \alpha)}g = \left| \det \alpha|_{\ker^\perp \alpha \cap \ker dp(x)} \right|$.

Proof. Firstly, the map

$$\nu : \alpha \in \kappa^{-1}(K) \mapsto (\alpha|_{K^\perp})^{-1} \in \mathcal{L}(E_x, K^\perp)$$

is smooth, and for any $\alpha \in \kappa^{-1}(K)$,

$$\forall \beta \in T_\alpha \kappa^{-1}(K), \quad d\nu(\alpha)(\beta) = -(\alpha|_{K^\perp})^{-1} \beta|_{K^\perp} (\alpha|_{K^\perp})^{-1} \in \mathcal{L}(E_x, K^\perp).$$

Moreover, by Lemma 2.1, the differential of

$$\begin{aligned} \kappa_E : E_x^* \setminus \{0\} &\rightarrow \text{Grass}(r-1, E_x) \\ f &\mapsto \ker f \end{aligned}$$

satisfies that, for any $f \in E_x^* \setminus \{0\}$ and $h \in E_x^*$, $d\kappa_E(f)h = -(f|_{\ker^\perp f})^{-1} h|_{\ker f}$, so that for any $\alpha \in \kappa^{-1}(K)$ and any $\beta \in T_\alpha \kappa^{-1}K$,

$$dg(\alpha)\beta = \left(dp(x)(\alpha|_{K^\perp})|_{\mathbb{R}\varepsilon(x,\alpha)}^{-1} \right)^{-1} dp(x)(\alpha|_{K^\perp})^{-1} \beta|_{K^\perp} (\alpha|_{K^\perp})|_{\mu(x,\alpha)}^{-1} \in \mathcal{L}(\mu(x,\alpha), \mathbb{R}\varepsilon(x,\alpha)),$$

hence $J_\alpha g = |\det(\alpha|_{K^\perp})|_{\mu(x,\alpha)}^{-1}$. Since $\alpha|_{K^\perp}$ induces an isomorphism between $K^\perp \cap \ker dp(x)$ and $\mu(x,\alpha)$, we obtain the result. \square

3 The mean number of induced critical points

In the first part of this section, we provide two results. The first one, Proposition 3.3, is a Kac-Rice formula for the mean number of critical points of the restriction of a Morse function to the vanishing locus of a random section of some vector field. It is an application of the general Kac-Rice formula given by Theorem 3.2. The second result, Corollary 3.4, is a more explicit and computable Kac-Rice formula which will be used in the applications of section 4. In the second part of the section, we adapt the formula in a holomorphic context, see Corollary 3.6.

3.1 The general formula

Let M, E, F and W be as in section 2, and let ∇ be a smooth connection on E . For any section $s \in C^1(M, E)$, let $Z_s := \{x \in M, s(x) = 0\}$ and $X \in C^0(M, F)$ defined by

$$\begin{aligned} X : M &\rightarrow F = E \oplus T^*M \otimes E \\ x &\mapsto [x, s(x), \nabla s(x)]. \end{aligned} \quad (3.1)$$

Note that for any $x \in M$, $X(x) \in W$ if and only if $x \in Z_s$ and x is a critical point of $p|_{Z_s}$. For any random section $s \in C^2(M, E)$, we are interested in the subset of M :

$$\text{Crit}_i^p(s) = \{x \in M, X(x) \in W \text{ and } \text{Ind } \nabla^2(p|_{Z_s})(x) = i\}, \quad (3.2)$$

where ∇ is any connection over Z_s . Note that $\text{Ind } \nabla^2(p|_{Z_s})(x)$ is well defined for any $x \in X^{-1}(W)$ because in this case $d(p|_{Z_s})(x) = 0$. Hence, $x \in X^{-1}(W)$ if and only if p lies in Z_s and the restriction of f to Z_s at p is critical and its index equals i . For any $s \in C^2(M, E)$ and $x \in X^{-1}(W)$, define also

$$\pi(x, \alpha) := \nabla^2 p(x)|_{\ker \alpha} - dp(x) \left(\alpha|_{\ker^\perp \alpha} \right)^{-1} \nabla^2 s(x)|_{\ker \alpha} \in \text{Sym}^2(\ker \nabla s(x), E), \quad (3.3)$$

where $\alpha = \nabla s(x)$. Here, $\nabla^2 p$ denotes the covariant derivative of dp for the Levi-Civita connexion associated to g . However, the formulas will not depend on the choice of this particular connexion.

Lemma 3.1 *Assume that $p : M \rightarrow \mathbb{R}$ is a Morse function. Let $s \in C^2(M, E)$ be a section of E , $x \in X^{-1}(W)$. Then,*

$$\text{Ind}(p|_{Z_s}(x)) = \text{Ind}(\pi(x, \nabla s(x))).$$

Proof. Let $s \in C^2(M, E)$, $x_0 \in Z_s$ and $K = \ker \nabla s(x_0)$. Assume that $\dim K = n - r$. We choose coordinates near x_0 so that M is identified with $T_{x_0}M$. By the implicit function theorem, locally near $x_0 = (0, 0) \in K \oplus K^\perp$, and Z_s is the graph over K of a C^2 map $f : K \rightarrow K^\perp$ with $f(0) = 0$ and $df(0) = 0$. Since locally $\forall z \in K, s(z, f(z)) = 0$, we obtain

$$\nabla_z s + \nabla_y s \circ df = 0,$$

where ∇_z and ∇_y denote the partial covariant derivatives along K and K^\perp respectively. so that $\nabla_z^2 s(0, 0) + \nabla_y s(0, 0) \circ d^2 f(0) = 0$. Now let $p_0 : K \rightarrow \mathbb{R}$ define locally by

$$\forall z \in K, p_0(z) = p(z, f(z)).$$

Note that if $K \subset \ker dp(z)$, then $\text{Ind} d^2 p(z) = \text{Ind} \nabla^2 p|_{Z_s}$. Now

$$dp_0 = d_z p + d_y p \circ df,$$

so that $d^2 p_0(0) = d_z^2 p(0, 0) + d_y p \circ d^2 f(0)$. Replacing $d^2 f(0)$ by its value above, we obtain the result. \square

We will use the following general Kac-Rice formula.

Theorem 3.2 ([23, Theorem 3.3]) *Let n be a positive integer, M be a smooth manifold of dimension n , $F \rightarrow M$ be a smooth vector bundle and $X \in \Gamma(M, F)$ be a non-degenerate smooth Gaussian random section. Let $W \subset F$ be a smooth submanifold of codimension n such that for every $x \in M$, $W_x := W \cap F_x$. Let the total space of F be endowed with a Riemannian metric that is Euclidean on fibers. Then for any Borel subset $A \subset M$*

$$\mathbb{E} \#\{x \in A \cap X^{-1}(W)\} = \int_{x \in A} \int_{q \in W_x} \mathbb{E} \left(J_x X \frac{\sigma_q(X, W)}{\sigma_q(F_x, W)} \middle| X(x) = q \right) \rho_{X(x)}(q) d\text{vol}(q) d\text{vol}(x),$$

where $\rho_{X(x)}(q)$ is the density of $X(x)$ at q and besides, $\sigma_q(X, W)$, $\sigma_q(F_x, W)$ denote the ‘‘angles’’ made by $T_q W$ with, respectively, $d_x X(T_x M)$ and $T_q F_x$, see [23, Definition B.2].

The random section $X \in \Gamma(M, F)$ is said to be *non-degenerate* [23, Definition 3.1] if for any $x \in M$, $\text{supp } X(x) = F_x$. We will not explain here the terms σ_q , because by the proof of [23, Lemma 7.2], locally

$$J_x X \frac{\sigma_q(X, W)}{\sigma_q(F_x, W)} = \frac{\delta_x(\Phi \circ X)}{J_x(\Phi|_{F_x})}, \quad (3.4)$$

where $\Phi : F \rightarrow \mathbb{R}^n$ is a local defining function for W , that is $W = \Phi^{-1}(0)$, where δ_p denotes the Jacobian density [23, (A.1)] and where J_x is the normal Jacobian, see Definition 2.6.

The following Proposition 3.3 is an application of the general Kac-Rice formula above, namely a Kac-Rice formula for the number of induced critical points of the restriction of a Morse function on random nodal sets. We need some notations. Let $s \in C^2(M, F)$, $x \in X^{-1}(W)$ and $\alpha = \nabla s(x)$. Recall that $\varepsilon(x, \alpha) \in E_x$ denotes a unit vector of $\ker^\perp dp(x)(\alpha|_{K^\perp})^{-1} \subset E_x$, that $\pi(x, \alpha)$ is defined by (3.3) and Crit_i^p by (3.2). Also, let $h(x)$ be one of the two unit vector in $\ker^\perp dp(x) \subset T_x M$.

Proposition 3.3 *Let $n \geq 2$ and $1 \leq r \leq n - 1$ be integers, (M, g) be a Riemannian manifold, $(E, h) \rightarrow M$ be a rank r smooth Euclidean vector bundle and $s \in \Gamma(M, E)$ be a non-degenerate Gaussian smooth field. Let $p : M \rightarrow \mathbb{R}$ be a smooth Morse function. Then, for any $i \in \{0, \dots, n - r\}$ and any Borel subset $A \subset M$,*

$$\begin{aligned} \mathbb{E} [\#(A \cap \text{Crit}_i^p)] &= \int_{x \in A} \int_{\substack{\alpha \in \mathcal{L}_{\text{onto}}(T_x M, E_x) \\ \ker \alpha \subset \ker dp(x)}} \left| \det \alpha|_{\ker^\perp \alpha} \right| \\ &\quad \mathbb{E} \left[\mathbf{1}_{\{\text{Ind}(\pi(x, \alpha)) = i\}} \left| \det \left(\langle \nabla^2 s(x)|_{\ker \alpha}, \varepsilon(x, \alpha) \rangle \right. \right. \right. \\ &\quad \left. \left. \left. - \langle \alpha(h(x)), \varepsilon(x, \alpha) \rangle \frac{\nabla^2 p(x)|_{\ker \alpha}}{\|dp(x)\|} \right) \right| \mid s(x) = 0, \nabla s(x) = \alpha \right] \\ &\quad \rho_{X(x)}(0, \alpha) d\text{vol}(\alpha) d\text{vol}(x), \end{aligned}$$

where $\rho_{X(x)}$ is the Gaussian density of $X(x)$ and the determinants are computed in orthonormal basis. Moreover, this integral is finite if $\text{vol}(A)$ is finite.

Proof. We use Theorem 3.2, using locally (3.4). So let $(x_0, 0, \alpha_0) \in W$. Locally and in coordinates, using the local defining function Φ for W given by (2.4),

$$\forall x \in O, \Phi(X(x)) = [s(x), dp(x)(\text{Id} + \varphi(\nabla s(x)))] \in E_{x_0} \times (\ker \alpha_0)^*,$$

where ∇ still denotes the connection ∇ through the trivialization. Hence,

$$\begin{aligned} \forall v \in T_x M, d(\Phi \circ X)(x)(v) &= \left[ds(x)(v), d^2 p(x)(v)(\text{Id} + \varphi(\nabla s(x)))|_{\ker \alpha_0} \right. \\ &\quad \left. - dp(x)(\nabla s(x)|_{\ker \nabla f(x)^\perp})^{-1} d\nabla s(x)(v)|_{\ker \alpha_0} \right]. \end{aligned}$$

In particular, for any $x \in X^{-1}(W)$ and any $v \in T_x M$, if $K = \ker \nabla s(x)$,

$$d(\Phi \circ X)(x)(v) = \left[\nabla_v s(x), \nabla_v dp(x)|_K - dp(x)(\nabla s(x)|_{K^\perp})^{-1} \nabla_v \nabla s(x)|_K \right].$$

Now, decomposing $T_x M$ as $T_x M = K^\perp \oplus K$, since $\nabla s(x)|_K = 0$, computing the determinant of this differential gives

$$\delta_x(\Phi \circ X) = \left| \det \nabla s(x)|_{K^\perp} \right| \left| \det \left(\nabla dp(x)|_K - dp(x)(\nabla s(x)|_{K^\perp})^{-1} \nabla^2 s(x)|_K \right) \right|.$$

Now, let us compute $J_x \Phi|_{F_x}$. For this, recall that

$$\forall (s, \alpha) \in F_x, \Phi(s, \alpha) = [s, dp(x)(\text{Id} + \varphi(\alpha))],$$

so that

$$\forall (0, \alpha) \in W_x, \forall (t, \beta) \in T_p F_x, d(\Phi|_{F_x})(0, \alpha)(t, \beta) = [t, -dp(x)(\alpha|_{K^\perp})^{-1} \beta|_K].$$

Since $J_{(0, \alpha)}(\beta \mapsto \langle \beta|_K, \varepsilon(x, \alpha) \rangle) = 1$, we get that

$$J_{(0, \alpha)}(\Phi|_{F_x}) = |dp(x)(\alpha|_{K^\perp})^{-1} \varepsilon(x, \alpha)|^{n-r}.$$

Moreover,

$$dp(x)(\alpha|_{K^\perp})^{-1} \nabla^2 s(x)|_K = \left(dp(x)(\alpha|_{K^\perp})^{-1} \varepsilon(x, \alpha) \right) \langle \nabla^2 s(x)|_K, \varepsilon(x, \alpha) \rangle.$$

By Lemma 2.5, W intersects the fibres of F transversally, so that Theorem 3.2 applies. Replacing the integrand in the theorem by (3.4), we obtain the formula. Finally, by the decomposition (2.7),

$$dp(x)(\alpha|_{\ker^\perp \alpha})^{-1} \varepsilon(x, \alpha) = \|dp(x)\| \langle \alpha(h(x)), \varepsilon(x, \alpha) \rangle^{-1}.$$

Since p is Morse, for any critical point $x \in \text{Crit}(p)$, there exists a constant C_x such that $\|dp(y)\| \geq C_x \|y - x\|$. Hence, the pole in the integration over U created by x has order $n - r$, which is integrable, see also [11, Remark 3.3.3]. \square

In order to provide an effective formula in concrete settings, we add further parameters. For any $x \in M$, recall that $h(x) \in T_x M$ be one of the two unit vectors in $\ker^\perp dp(x)$. Moreover, for any real hyperplane $\mu \subset E_x$, let $\varepsilon(\mu)$ be a unit vector in μ^\perp . Recall that Crit_i^p is defined by (3.2).

Corollary 3.4 *Assume the hypotheses of Proposition 3.3 are satisfied. Let $A \subset M$ be a Borel subset. Then, for any $i \in \{0, \dots, n - r\}$,*

$$\begin{aligned} \mathbb{E}[\#(A \cap \text{Crit}_i^p)] &= \int_{x \in A} \int_{\substack{K \in \text{Grass}(n-r, \ker dp(x)) \\ \mu \in \text{Grass}(r-1, E_x)}} \int_{\substack{\alpha \in TM_x^* \otimes E_x \\ \ker \alpha = K \\ \ker dp(x)(\alpha|_{K^\perp})^{-1} = \mu}} \\ &\quad \left| \det(\alpha|_{K^\perp \cap \ker dp(x)}) \right|^{n-r+2} |\langle \alpha(h(x)), \varepsilon(\mu) \rangle| \\ &\quad \mathbb{E} \left[\mathbf{1}_{\{\text{Ind}(\pi(x, \alpha)) = i\}} \left| \det \left(\langle \nabla^2 s(x)|_K, \varepsilon(\mu) \rangle \right. \right. \right. \\ &\quad \left. \left. \left. - \langle \alpha(h(x)), \varepsilon(\mu) \rangle \frac{\nabla^2 p(x)|_K}{\|dp(x)\|} \right) \right| \mid s(x) = 0, \nabla s(x) = \alpha \right] \\ &\quad \rho_{X(x)}(0, \alpha) d\text{vol}(\alpha) d\text{vol}(\mu) d\text{vol}(K) d\text{vol}(x), \end{aligned}$$

where $\pi(x, \alpha)$ is given by (3.3) and ρ_X denotes the density of X .

Proof. In the formula given by Proposition 3.3, we handle first the determinant of $\alpha|_{K^\perp}$. Since $(\alpha|_{K^\perp})^{-1}(\mu) = \ker dp(x) \cap K^\perp$, if $h(x)$ is a unit vector in $\ker^\perp dp(x)$, then

$$|\det \alpha|_{K^\perp}| = |\det \alpha|_{K^\perp \cap \ker dp(x)}| |\langle \alpha(h(x)), \varepsilon(\mu) \rangle|. \quad (3.5)$$

We then apply two times the coarea formula (see for instance [23, Theorem C.3] from which we borrow the notations) for the integral in α . The first formula is applied with the map $\kappa|_{W_x} : W_x \rightarrow \text{Grass}(n - r, \ker dp(x))$, where κ is defined by (2.7). By Lemma 2.7, its Jacobian satisfies, for any $(0, \alpha) \in W_x$, $J_{(0, \alpha)}(\kappa|_{W_x}) = \left| \det \alpha|_{\ker^\perp \alpha \cap \ker dp(x)} \right|^{n-r}$. The second coarea formula is applied with $K \in \text{Grass}(n - r, \ker dp(x))$ fixed, with the function $g : \kappa^{-1}(K) \rightarrow \text{Grass}(r - 1, E_x)$ defined in Lemma 2.8. Then, By the latter, for all α , $J_{(0, \alpha)}g = |\det \alpha|_{\ker^\perp \alpha \cap \ker dp(x)}|$. We obtain the result. Together with (3.5), we obtain the desired formula. \square

3.2 The holomorphic setting

In this paragraph, let $n \geq 2$ and $1 \leq r \leq n - 1$ be integers, M be a complex smooth manifold of complex dimension n , endowed with a Hermitian metric g . Let $(E, h_E) \rightarrow M$ be a holomorphic Hermitian vector bundle of rank r , and $s \in \Gamma(M, E)$ be a holomorphic Gaussian field. In section 4, M will be either a compact projective manifold and E the

tensor product of a fixed vector bundle tensored by the high powers of an ample line bundle, or M will be the affine complex space and E the trivial complex vector bundle of rank r . Let ∇ be the Chern connection for E , that is the unique holomorphic and metric connection on E , see [14]. In this complex case, the real setting of paragraph 3.1 adapts formally, changing the field \mathbb{R} into \mathbb{C} . In particular, we define

$$F = E \oplus \mathcal{L}^{\mathbb{C}}(TM, E).$$

However specific changes must be also done. Let $p : M \rightarrow \mathbb{R}$ be a smooth Morse function. Then, for any holomorphic section s of E and any $x \in Z_s$,

$$\ker \nabla s(x) \subset \ker dp(x) \Leftrightarrow \ker \nabla s(x) \subset \ker \pi_{\mathbb{C}}(x),$$

where $\pi_{\mathbb{C}}(x)$ denotes the complexification of $dp(x)$, that is $\pi_{\mathbb{C}}(x) \in \mathcal{L}^{\mathbb{C}}(T_x M, \mathbb{C})$ and $dp(x) = \Re \pi_{\mathbb{C}}(x)$. Then, we use that for any complex subspace $K \subset T_x M$ and any $\alpha \in \mathcal{L}^{\mathbb{C}}(K, E_x)$, the real determinant (computed in orthonormal basis) of the associated real map $\alpha_{\mathbb{R}}$ equals

$$|\det \alpha_{\mathbb{R}}| = |\det \alpha|^2. \quad (3.6)$$

As in the real case, the Gaussian holomorphic field s is said to be *non-degenerate* if for any $x \in M$,

$$s \mapsto (s(x), \nabla s(x)) \in E \times \mathcal{L}^{\mathbb{C}}(T_x M, E_x)$$

is onto. As before, we define

$$W = \{(x, 0, \alpha) \in F, dp(x) \neq 0, \alpha \text{ onto and } \ker \alpha \subset \ker \pi_{\mathbb{C}}(x)\},$$

and W_x as the fibre of W over x . For any $K \in \text{Grass}_{\mathbb{C}}(n-r, T_x M)$ and $\mu \in \text{Grass}_{\mathbb{C}}(r-1, E_x)$, let

$$W(x, K, \mu) := \{\alpha \in \mathcal{L}_{\text{onto}}^{\mathbb{C}}(TM_x, \otimes E_x) \mid \ker \alpha = K, \ker \pi_{\mathbb{C}}(\alpha|_{K^\perp})^{-1} = \mu\}.$$

Lemma 3.5 *Under the hypotheses above, $W(x, K, \mu)$ is a submanifold of W_x of complex dimension $r^2 - (r-1)$.*

For any $s \in H^0(M, E)$ and $x \in X^{-1}(W)$, define also

$$\pi(x, \alpha) := \nabla^2 \pi_{\mathbb{C}}(x)|_{\ker \alpha} - \pi_{\mathbb{C}}(x) \left(\alpha|_{\ker^\perp \alpha} \right)^{-1} \nabla^2 s(x)|_{\ker \alpha} \in \text{Sym}^2(\ker \nabla s(x), E_x), \quad (3.7)$$

where $\alpha = \nabla s(x)$. Lastly, for any $x \in M$, denote by $h(x) \in T_x M$ any unit vector in $(\ker \pi_{\mathbb{C}}(x))^\perp \subset T_x M$, and for any complex hyperplane $\mu \subset E_x$, let $\varepsilon(\mu)$ be a unit vector in $\mu^\perp \subset E_x$. Recall that Crit_i^p is defined by (3.2).

Theorem 3.6 *Let (M, g) be a complex manifold, $(E, h_E) \rightarrow M$ be a holomorphic Hermitian vector bundle, and $s \in \Gamma(M, E)$ be a non-degenerate holomorphic Gaussian field. Let $A \subset M$ any Borel subset. Then,*

$$\begin{aligned} \mathbb{E} [\#(A \cap \text{Crit}_i^p)] &= \int_{x \in A} \int_{\substack{K \in \text{Grass}_{\mathbb{C}}(n-r, \ker \pi_{\mathbb{C}}(x)) \\ \mu \in \text{Grass}_{\mathbb{C}}(r-1, E_x)}} \int_{\substack{\alpha \in \mathcal{L}^{\mathbb{C}}(T_x M, E_x) \\ \ker \alpha = K \\ \ker \pi_{\mathbb{C}}(\alpha|_{K^\perp})^{-1} = \mu}} \\ &\quad \left| \det(\alpha|_{K^\perp \cap \ker \pi_{\mathbb{C}}(x)}) \right|^{2(n-r+2)} |\langle \alpha(h), \varepsilon(\mu) \rangle|^2 \\ &\quad \mathbb{E} \left[\mathbf{1}_{\{\text{Ind}(\pi(x, \alpha))=i\}} \left| \det_{\mathbb{R}} \left(\langle \nabla^2 s(x)|_K, \varepsilon(\mu) \rangle \right. \right. \right. \\ &\quad \left. \left. \left. - \langle \alpha(h(x)), \varepsilon(\mu) \rangle \frac{\nabla \pi_{\mathbb{C}}(x)|_K}{\|\pi_{\mathbb{C}}(x)\|} \right) \right| \mid s(x) = 0, \nabla s(x) = \alpha \right] \\ &\quad \rho_{X(x)}(0, \alpha) d\text{vol}(\alpha) d\text{vol}(\mu) d\text{vol}(K) d\text{vol}(x), \end{aligned}$$

where $\pi(x, \alpha)$ is given by (3.7) and ρ_X is the density of X . Moreover, the integral is finite if $\text{vol}(A)$ is finite.

Proof. The proof is formally the same as the one of Corollary 3.4, using the rules mentioned above, so we omit it. \square

4 Applications

In this section we apply Theorem 3.6 to the complex Bargmann-Fock field on \mathbb{C}^n and then to the projective setting. Finally, we apply Proposition 3.3 to the boundary case, which is a mixed between complex and the real setting and is needed for the main Theorems 1.3 and 1.7.

4.1 The Bargmann-Fock field

Recall that the Bargmann-Fock field is defined by

$$\forall z \in \mathbb{C}^n, f(z) = \sum_{(i_1, \dots, i_n) \in \mathbb{N}^n} a_{i_0, \dots, i_n} \sqrt{\frac{\pi^{i_1 + \dots + i_n}}{i_1! \dots i_n!}} z_1^{i_1} \dots z_n^{i_n} e^{-\frac{1}{2}\pi \|z\|^2}, \quad (4.1)$$

where the a_I 's are independent normal complex Gaussian random variables. The associated covariant function equals

$$\forall z, w \in \mathbb{C}^n, \mathcal{P}(z, w) := \mathbb{E}(f(z)\overline{f(w)}) = \exp\left(-\frac{\pi}{2}(\|z\|^2 + \|w\|^2 - 2\langle z, w \rangle_{\mathbb{C}^n})\right). \quad (4.2)$$

Even if the kernel \mathcal{P} is not invariant under translation or rotations, the law of Z_f is, see [15, Proposition 2.3.4]. The a priori superfluous presence of π is in fact consistent with the projective situation. Indeed, the affine Bargmann-Fock is the universal local limit of the projective model, see Theorem 4.5. In order to unify the setting, we consider here that $M = \mathbb{C}^n$ and $L = \mathbb{C}^n \times \mathbb{C}$ with its standard Hermitian metric. Then

$$\mathcal{P}(z, w) \in L_z \otimes L_w^*.$$

Note that the trivial connection on L has vanishing curvature. Hence, let ∇_0 be the metric connection for this setting:

$$\nabla_0 1 = \frac{1}{2}\pi(\bar{\partial} - \partial)\|z\|^2, \quad (4.3)$$

whereas the dual connection ∇^* on L^* satisfies

$$\nabla_0^* 1^* = -\frac{1}{2}\pi(\bar{\partial} - \partial)\|z\|^2, \quad (4.4)$$

where 1^* is the dual of 1 . Notice that 1 is no longer a holomorphic section for this connection, but the (peak) section (see [24], [6])

$$\sigma_0 := \exp\left(-\frac{1}{2}\pi\|z\|^2\right)$$

is, since $\nabla_0^{(0,1)}\sigma_0 = (-\frac{1}{2}\pi\bar{\partial}\|z\|^2 + \frac{1}{2}\pi\bar{\partial}\|z\|^2)\sigma_0 = 0$. The connection ∇_0 is then the Chern connection for the trivial metric and this holomorphic structure. This implies that the

section \mathcal{P} is holomorphic in z , and antiholomorphic in w . Moreover, the curvature of ∇_0 equals

$$\mathcal{R}_0 = \bar{\partial}\partial \log \|\sigma_0\|^2 = \pi \partial \bar{\partial} \|z\|^2,$$

and the curvature form equals

$$\frac{i}{2\pi} \mathcal{R}_0 = \frac{i}{2} \sum_{i=1}^n dz_i \wedge \overline{dz_i}$$

which is the standard symplectic form ω_0 over \mathbb{R}^{2n} . Now, almost surely an instance f of the Bargmann Fock Gaussian field is a holomorphic section for the standard complex structure and the connection with standard curvature form given by (4.3).

Let $E = \mathbb{C}^n \times \mathbb{C}^r$ endowed with its trivial metric and let $f = (f_i)_{i=1, \dots, r}$ be r independent copies of the Bargmann-Fock field. Then, f is a random section of $E \otimes L$, and its covariance function equals $\mathcal{P} \text{Id}_{\mathbb{C}^r}$. In the following theorem, recall that Crit_i^p is defined by (3.2), where we use the connexion ∇_0^r acting on sections of $E \otimes L$. By an abuse of notation, we continue to use ∇_0 for ∇_0^r .

Theorem 4.1 *Let $1 \leq r \leq n$ be integers, $f : \mathbb{C}^n \rightarrow \mathbb{C}^r$ be r independent copies of the Bargmann-Fock field (4.1), and $U \subset \mathbb{C}^n$ be an open subset of finite volume. Let $p : U \rightarrow \mathbb{R}$ be a smooth Morse function. Then,*

$$\begin{aligned} \forall 0 \leq i \leq 2n - 2r \setminus \{n - r\}, \quad \frac{1}{R^{2n}} \mathbb{E} \#(RU \cap \text{Crit}_i^p) &\xrightarrow{R \rightarrow +\infty} 0 \\ \frac{1}{R^{2n}} \mathbb{E} \#(RU \cap \text{Crit}_{n-r}^p) &\xrightarrow{R \rightarrow +\infty} n! \binom{n-1}{r-1} \text{vol}(U), \end{aligned}$$

where vol denotes the volume for the standard metric on \mathbb{C}^n .

We postpone the proof of this theorem after the projective case, since the latter is similar but more complicated. In both cases, we will need the following lemma:

Lemma 4.2 *Let \mathcal{P} the Bargmann-Fock covariance (4.2), and ∇_0 the connection defined by (4.3) and (4.4). Then, for any $z \in \mathbb{C}^n$,*

$$\begin{aligned} \nabla_{0 \ z, \bar{w}}^{(1,0), (0,1)} \mathcal{P}(z, z) &= \pi \sum_{i=1}^n dz_i \otimes \overline{dw_i} \\ \text{and } \nabla_{0 \ z^2, \bar{w}^2}^{(1,0)^2, (0,1)^2} \mathcal{P}(z, z) &= \pi^2 \sum_{i,j,k,\ell=1}^n (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) dz_i \otimes dz_j \otimes \overline{dw_k} \otimes \overline{dw_\ell}. \end{aligned}$$

Proof. This is a straightforward consequence of the definition of ∇_0 and \mathcal{P} . \square

We will need the following covariance matrix for Hessians:

$$\Sigma_{GOE} = \left(\delta_{(ij)(kl)} + \delta_{(ji)(kl)} \right)_{\substack{1 \leq i \leq j \leq n \\ 1 \leq k \leq l \leq n}} \in M_{\frac{n(n+1)}{2}}(\mathbb{C}). \quad (4.5)$$

Corollary 4.3 *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^r$ be r independent copies of the Bargmann-Fock field. Then, for any $x \in \mathbb{C}^n$,*

$$\text{Cov}(f(x), \nabla_0 f(x), \nabla_0^2 f(x)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi \text{Id}_{\mathbb{C}^n} & 0 \\ 0 & 0 & \pi^2 \Sigma_{GOE} \end{pmatrix} \otimes \text{Id}_{\mathbb{C}^r}.$$

Proof. This is an immediate consequence of Lemma 4.2. \square

4.2 The complex projective case

Let $n \geq 2$, $1 \leq r \leq n - 1$ be integers, M be a compact smooth complex manifold of dimension n equipped with a holomorphic Hermitian vector bundle (E, h_E) of rank r and an ample holomorphic line bundle (L, h) . Assume that h has a positive curvature form ω , see (1.1). Let ∇ be the Chern connection of $E \otimes L^d$. Recall that Crit_i^p is defined by (3.2).

Theorem 4.4 *Let M , (E, h_E) , (L, h) , ω as above and let $U \subset M$ be a 0-codimension submanifold with finite volume. Then*

$$\begin{aligned} \forall i \in \{0, \dots, 2n - 2r\} \setminus \{n - r\}, \quad \frac{1}{d^n} \mathbb{E} \#(U \cap \text{Crit}_i^p) &\xrightarrow{d \rightarrow \infty} 0 \\ \frac{1}{d^n} \mathbb{E} \#(U \cap \text{Crit}_{n-r}^p) &\xrightarrow{d \rightarrow \infty} \binom{n-1}{r-1} \int_U \omega^n. \end{aligned}$$

The probability measure μ_d used for the average is defined by (1.5).

Theorem 1.9 ([11, Theorem 3.5.1]) implies this result for the squared modulus of a Lefschetz pencil $p : M \dashrightarrow \mathbb{C}P^1$. Indeed, since p is holomorphic (outside its singular locus), $p|_{Z_s}$ is critical if and only if $|p|_{Z_s}^2$ is, and in the latter case the index equals $n - r$.

Bergman and Bargmann-Fock. The covariance function for the Gaussian field generated by the holomorphic sections $s \in H^0(M, E \otimes L^d)$ is

$$\forall z, w \in M, \quad E_d(z, w) = \mathbb{E} [s(z) \otimes (s(w))^*] \in (E \otimes L^d)_z \otimes (E \otimes L^d)_w^*,$$

where E^* is the (complex) dual of E and

$$\forall w \in M, \quad \forall s, t \in (E \otimes L^d)_w, \quad s^*(t) = h_E \otimes h_{L^d}(s, t).$$

The covariance E_d is the *Bergman kernel*, that is the kernel of the orthogonal projector from $L^2(M, E \otimes L^d)$ onto $H^0(M, E \otimes L^d)$. This fact can be seen through the equations

$$\forall z, w \in M, \quad E_d(z, w) = \sum_{i=1}^{N_d} S_i(z) \otimes S_i^*(w),$$

where $(S_i)_i$ is an orthonormal basis of $H^0(M, E \otimes L^d)$ for the Hermitian product (1.4). Recall that the metric g is induced by the curvature form ω and the complex structure. It is now classical that the Bergman kernel has a universal rescaled (at scale $1/\sqrt{d}$) limit, the Bargmann-Fock kernel \mathcal{P} defined by (4.2). Theorem 4.5 below quantifies this phenomenon. For this, we need to introduce local trivializations and charts. Let $x \in M$ and $R > 0$ such that $2R$ is less than the radius of injectivity of M at x . Then the exponential map based at x induces a chart near x with values in $B_{T_x M}(0, 2R)$. The parallel transport provides a trivialization

$$\varphi_x : B_{T_x M}(0, 2R) \times (E \otimes L^d)_x \rightarrow (E \otimes L^d)|_{B_{T_x M}(0, 2R)}$$

which induces a trivialization of $(E \otimes L^d) \boxtimes (E \otimes L^d)^*|_{B_{T_x M}(0, 2R)^2}$. Under this trivialization, the Bergman kernel E_d becomes a map from $T_x M^2$ with values into $\text{End}((E \otimes L^d)_x)$.

Theorem 4.5 ([19, Theorem 1]) *Under the hypotheses of Theorem 1.3, let $m \in \mathbb{N}$. Then, there exist $C > 0$, such that for any $k \in \{0, \dots, m\}$, for any $x \in M$, $\forall z, w \in B_{T_x M}(0, \frac{1}{\sqrt{d}})$,*

$$\left\| D_{(z,w)}^k \left(\frac{1}{d^n} E_d(z, w) - \mathcal{P}(z\sqrt{d}, w\sqrt{d}) \text{Id}_{(E \otimes L^d)_x} \right) \right\| \leq C d^{\frac{k}{2}-1}.$$

The original reference is a little more intricated, see [17, Proposition 3.4] for the present simplification. We will also need the following lemma:

Lemma 4.6 *Under the local trivializations given before, at x (the center of the chart) the two equalities hold:*

$$\nabla = \nabla_0 + O\left(\frac{1}{\sqrt{d}}\right) \text{ and } \nabla^2 = (\nabla_0)^2 + O\left(\frac{1}{\sqrt{d}}\right).$$

Proof. The conjunction of [18, Lemma 1.6.6] and [18, (4.1.103)] implies that

$$\nabla = \nabla_0 + O\left(\frac{1}{\sqrt{d}}\right) + O(\|z - x\|^3),$$

which gives the first estimate. The second one is implied by the first one and by the fact that the Levi-Civita connection associated to g is trivial at x , because the coordinates on M are normal at x . \square

Corollary 4.7 *Under the hypotheses and trivializations above near $x \in M$, in any orthonormal basis of $T_x M$,*

$$\text{Cov}(s, \nabla s, \nabla^2 s)|_x = d^n \begin{pmatrix} (1 + O(\frac{1}{d})) & O(\frac{1}{\sqrt{d}}) & O(1) \\ O(\frac{1}{\sqrt{d}}) & \pi d I_n (1 + O(\frac{1}{d})) & O(\sqrt{d}) \\ O(1) & O(\sqrt{d}) & \pi^2 d^2 \Sigma_{GOE} (1 + O(\frac{1}{d})) \end{pmatrix} \text{Id}_{(E \otimes L^d)_x},$$

where $I_n \in M_n(\mathbb{R})$ and Σ_{GOE} is defined by (4.5). Moreover, for any $\alpha \in T_x^* M \otimes E_x$,

$$\langle \langle \nabla^2 s, \varepsilon \rangle \mid s = 0, \nabla s = \alpha \rangle \sim N\left(O\left(\frac{\|\alpha\|}{\sqrt{d}}\right), \Sigma\right),$$

where

$$\Sigma := \pi^2 d^{n+2} \Sigma_{GOE} \text{Id}_{(E \otimes L^d)_x} (1 + O(\frac{1}{d})).$$

The constants involved in the error terms do not depend on α .

Proof. The first assertion is a direct consequence of Theorem 4.5, Lemma 4.6 and Corollary 4.3. The second one is deduced from the classical regression formula and from

$$(\text{Cov}(s, \nabla s))^{-1} = \frac{1}{d^n} \begin{pmatrix} (1 + O(\frac{1}{d})) & O(\frac{1}{d^{\frac{3}{2}}}) \\ O(\frac{1}{d^{\frac{3}{2}}}) & \frac{1}{\pi d} (1 + O(\frac{1}{d})) \end{pmatrix} \text{Id}_{(E \otimes L^d)_z}. \quad (4.6)$$

\square

Proof of Theorem 4.4. We want to apply Theorem 3.6. First, from Corollary 4.7 we get that for any $x \in M$ and any $\alpha \in \mathcal{L}^{\mathbb{C}}(T_x M, E_x)$,

$$\rho_{X(x)}(0, \alpha) = \frac{(1 + O(\frac{1}{d}))}{(2\pi)^{r+nr} (d^n)^r (\pi d^{n+1})^{nr}} \exp\left(-\frac{1}{2} \frac{1}{\pi d^{n+1}} (1 + O(\frac{1}{d})) \|\alpha\|^2\right).$$

Now, if $K = \ker \alpha$, $\mu = \ker dp(x)(\alpha|_{K^\perp})^{-1}$ and $\varepsilon(\mu) \in \mu^\perp$ has a norm equal to 1, let Let

$$(\beta, a, b) = \frac{1}{\sqrt{\pi d^{n+1}}} \left(\alpha|_{K_{\mathbb{C}}^\perp \cap \ker \pi_{\mathbb{C}}}, \langle \alpha|_{\ker^\perp \pi_{\mathbb{C}}}, \varepsilon \rangle, \pi_\mu^\perp \alpha|_{\ker^\perp \pi_{\mathbb{C}}} \right), \quad (4.7)$$

where π_μ^\perp denotes the orthogonal projection (for h_E) onto μ . Using Lemma 3.5 for the transformation of $d\text{vol}(\alpha)$, the term

$$\left| \det \alpha|_{K_\mathbb{C}^\perp \cap \ker \pi_\mathbb{C}} \right|^{2(n-r+2)} |\langle \alpha(h(x)), \varepsilon \rangle|^2 d\text{vol}(\alpha)|_{W(x,K,\mu)\rho_X(x)}(0, \alpha)$$

in the integral of Theorem 3.6 equals

$$(1 + O(\frac{1}{d})) \frac{(2\pi)^{r-1} (\pi d^{n+1})^{(r-1)(n-r+2)+1+r^2-(r-1)} (2\pi)^{(r-1)^2+1}}{(2\pi)^{r+nr} d^{nr} (\pi d^{n+1})^{nr}} \frac{|\det \beta|^{2(n-r+2)} |a|^2 d\text{vol}(\beta, a, b)}{(2\pi)^{(r-1)^2} 2\pi (2\pi)^{r-1}} \exp\left(-\frac{1}{2}(1 + O(\frac{1}{d}))(\|\beta\|^2 + |a|^2 + \|b\|^2)\right).$$

Note that

$$\int_{a \in \mathbb{C}} |a|^2 e^{-\frac{1}{2}|a|^2} da = 4\pi.$$

By Corollary 4.7, the field X defined by (3.1) is non-degenerate for d large enough. Hence, we can apply Theorem 3.6. Let $Y = \frac{1}{\sqrt{\pi^2 d^{n+2}}} \nabla^2 s(x)$. Then, the average in the formula provided by Theorem 3.6 is now equal to $(\pi^2 d^{n+2})^{n-r}$ times

$$\mathbb{E} \left[\mathbf{1}_{\{\text{Ind}(\pi(x,\alpha))=i\}} \left| \det_{\mathbb{R}} \left(\langle Y|_K, \varepsilon \rangle - a \frac{\nabla \pi_\mathbb{C}(x)|_K (1 + O(\frac{1}{d}))}{\|\pi_\mathbb{C}(x)\| \pi^{\frac{3}{2}} d^{\frac{2n+3}{2}}} \right) \right| \mid s(x) = 0, \nabla s(x) = \alpha \right]. \quad (4.8)$$

Recall that $\pi(x, \alpha)$ defined by (3.3). Besides, by Corollary 4.7,

$$\left(\langle Y|_K, \varepsilon \rangle \mid s(x) = 0, \nabla s(x) = \alpha \right) \sim N \left(O\left(\frac{\|(\beta, a, b)\|}{d^{\frac{3}{2}}}\right), \Sigma_{GOE}^{n-r} (1 + O(\frac{1}{d})) \right),$$

where the constants are independent of α and ε , and where Σ_{GOE}^{n-r} denotes the covariance matrix Σ_{GOE} defined by (4.5) in dimension $n-r$. When d grows to infinity, the average (4.8) is uniformly bounded above by an integrable map, since the pole generated by $\|\pi_\mathbb{C}(x)\|$ is integrable. Consequently, the dominated convergence theorem implies that

$$\frac{1}{d^n \text{vol}(U)} \mathbb{E} \#(\text{Crit}_i^p \cap U) \xrightarrow{d \rightarrow \infty} \frac{2^{r^2-nr-2r+2}}{\pi^{(r-1)(n-r+1)}} \text{vol}(\text{Grass}_\mathbb{C}(n-r, n-1)) \text{vol}(\text{Grass}_\mathbb{C}(r-1, r)) \mathbb{E} \left(|\det \beta|^{2(n-r+2)} \right) \mathbb{E} \left(\mathbf{1}_{\{\text{Ind } A=i\}} |\det A|^2 \right),$$

where $A \in M_{n-r}(\mathbb{C})$ has covariance Σ_{GOE}^{n-r} and where we used the determinant equality (3.6). Note that we passed from the real determinant $\det_{\mathbb{R}}$ to the complex one for the random complex matrix A . Since its index is always $n-r$, all the averages divided by d^n for $i \neq n-r$ converge to 0. The computations of the expectations and volume are given in [11, Remark 3.1.1, proof of Theorem 3.5.1]:

$$\begin{aligned} \text{vol}(\text{Grass}_\mathbb{C}(n-r, n-1)) &= \pi^{(n-r)(r-1)} \frac{\prod_{j=1}^{r-1} \Gamma(j)}{\prod_{j=n-r+1}^{n-1} \Gamma(j)} \\ \text{vol}(\text{Grass}_\mathbb{C}(r-1, r)) &= \pi^{r-1} \frac{1}{\Gamma(r)} \\ \mathbb{E} |\det \beta|^{2(n-r+2)} &= 2^{(r-1)(n-r+2)} \frac{\prod_{j=n-r+3}^{n+1} \Gamma(j)}{\prod_{j=1}^{r-1} \Gamma(j)} \\ \mathbb{E} (|\det Y|^2) &= 2^{n-r} (n-r+1)! \end{aligned}$$

The powers of 2 in the latter equalities come from different choices of the measures, more precisely our choice of the half in the exponentials. Hence,

$$\frac{1}{d^n \text{vol}(U)} \mathbb{E} \#(\text{Crit}_{n-r}^p \cap U) \xrightarrow{d \rightarrow \infty} n! \binom{n-1}{r-1}.$$

□

We give now a sketch proof of the affine case.

Proof of Theorem 4.1. Let $f = (f_1, \dots, f_r) \in \mathbb{C}^r$ be the random Bargmann-Fock field. For any $R > 0$, let $p_R = p(\frac{\cdot}{R})$, so that the associated complexification $\pi_{R, \mathbb{C}}$ of $dp_R(x)$ satisfies $\pi_{R, \mathbb{C}}(x) = \frac{1}{R} \pi_{\mathbb{C}}(\frac{x}{R})$. Note that p_R is a Morse function on RU . By Corollary 4.3, the field X defined by 3.1 is non-degenerate, so that we can apply Theorem 3.6 on the open set RU . By the independance of the triplet $(f, \nabla_0 f, \nabla_0^2 f)$, the conditional expectation in Theorem 3.6 equals

$$\mathbb{E} \left[\mathbf{1}_{\{\text{Ind}(\pi(x, \alpha))=i\}} \left| \det \left(\langle \nabla_0^2 f(x)|_K, \varepsilon \rangle - \frac{1}{R} \frac{d\pi_{\mathbb{C}}(\frac{x}{R})|_K}{\|\pi_{\mathbb{C}}(\frac{x}{R})\|} \langle \alpha(h(x)), \varepsilon(\mu) \rangle \right) \right] \right].$$

Recall that $\pi(x, \alpha)$ defined by (3.3). We make the change of variables $(\beta, a, b) = \frac{1}{\sqrt{\pi}} \alpha|_{K^\perp}$ (as (4.7)) and $Y = \frac{1}{\pi} \nabla_0^2 f|_K$, and then the change of variables $y = x/R$. By Corollary 4.3, we obtain

$$\begin{aligned} \frac{1}{R^{2n} \text{vol}(U)} \mathbb{E} \#(RU \cap \text{Crit}_i^p) &\xrightarrow{R \rightarrow +\infty} \frac{2^{r^2 - nr - 2r + 2}}{\pi^{(r-1)(n-r+1)}} \text{vol}(\text{Grass}_{\mathbb{C}}(n-r, n-1)) \\ &\quad \text{vol}(\text{Grass}_{\mathbb{C}}(r-1, r)) \\ &\quad \mathbb{E} \left(|\det \beta|^{2(n-r+2)} \right) \mathbb{E} \left(\mathbf{1}_{\{\text{Ind } A=i\}} |\det A|^2 \right). \end{aligned}$$

we conclude as in the projective case. □

4.3 The boundary case

In this paragraph, we apply Proposition 3.3 to estimate the mean number of critical points of the restriction of p on the boundary of Z_s inside ∂U , where $U \subset M$ is an open set with smooth boundary and M is complex. We begin by a description of the mixed complex geometry on the boundary of U .

Complex geometry on the boundary. In the sequel, for any $x \in M$ and any real subspace $L \subset T_x M$, we denote by $L_{\mathbb{C}}$ the largest complex subspace in L . Let $U \subset M$ be a codimension 0 open set with smooth boundary ∂U .

Definition 4.8 *Let Z be a smooth manifold of dimension m , with C^2 boundary, and $p : Z \rightarrow \mathbb{R}$ a smooth function. Then, p is said to be Morse if there is no critical point on ∂Z , if p is Morse and if $p|_{\partial Z}$ is Morse as well.*

Let $p : M \rightarrow \mathbb{R}$ be a Morse function, such that $p|_U$ is Morse in the sens of Definition 4.8. Let $H = \ker p_\partial \subset T\partial U$. For any $x \in \partial U$ which is not a critical point of p_∂ , $\dim H = 2n - 2$. Moreover, either $H = H_{\mathbb{C}}$ and in this case $\dim_{\mathbb{C}} H = n - 1$, or $\dim_{\mathbb{C}} H_{\mathbb{C}} = n - 2$. The first situation is non-generic, but our result holds in this case as well. We define

$$F_\partial = E|_{\partial U} \oplus \mathcal{L}^{\mathbb{C}}(TM, E)|_{T(\partial U)},$$

$$W = \{(x, 0, \alpha) \in F_\partial, \alpha \text{ onto and } \ker \alpha \subset \ker dp_\partial(x)\},$$

W_x its fiber over $x \in M$, and

$$X(x) = (x, s(x), \nabla_\partial s(x)) \in F_\partial,$$

where $\nabla_\partial = \nabla|_{T\partial U}$ denotes the restriction of the Chern connection ∇ on E to the tangent space of the boundary of U . For any $x \in \partial U$ and any $\alpha \in \mathcal{L}_{\text{onto}}^{\mathbb{C}}(T_x M, E_x)$, let

$$K = K(x, \alpha) = \ker \alpha \cap T\partial U.$$

If $K \neq \ker \alpha$, then $\dim_{\mathbb{R}} K = 2n - 2r - 1$ and $\dim_{\mathbb{C}} K_{\mathbb{C}} = n - r - 1$. Assume now that $K \subset H$. Then, $K_{\mathbb{C}} \subset H_{\mathbb{C}}$. Let $g \in H$ be a (one of the two) unit vector such that

$$K = K_{\mathbb{C}} \oplus \mathbb{R}g.$$

Note that $\ker \alpha = K \oplus \mathbb{R}Jg$, where J denotes the complex structure $J : TM \rightarrow TM$. Now, $\dim_{\mathbb{R}} K^\perp = 2r$, where \perp stands for the metric on $T\partial U$. Moreover,

$$\dim_{\mathbb{R}}(K^\perp \cap H) = 2r - 1$$

so that $\dim_{\mathbb{C}}(K^\perp \cap H)_{\mathbb{C}} = r - 1$. Let $v \in H$ a unit vector such that

$$K^\perp \cap H = (K^\perp \cap H)_{\mathbb{C}} \oplus \mathbb{R}v.$$

Note that $K^\perp = (K^\perp \cap H)_{\mathbb{C}} + \mathbb{R}v + \mathbb{R}h$, where $h \in H^\perp \setminus \{0\}$. Now, let

$$\mu = \ker dp_\partial(x)(\alpha|_{K^\perp})^{-1} = \alpha(K^\perp \cap H) \subset E_x.$$

and $\mu_{\mathbb{C}} = \alpha((K^\perp \cap H)_{\mathbb{C}})$. Finally, let ε be a unit vector in $\mu^\perp \subset E_x$.

Lemma 4.9 *Under the setting above, for any $x \in \partial U$, the real dimension of W_x equals $2nr - 2n + 2r + 1$.*

Proof. For any $(x, 0, \alpha) \in W$,

$$T_{(0, \alpha)} W_x = \{(0, \beta) \in E_x \times \mathcal{L}^{\mathbb{C}}(T_x M, E_x), \ker dp_\partial(x)(\alpha|_{K^\perp})^{-1} \beta|_K = 0\}.$$

Since $\beta|_{K_{\mathbb{C}}}$ is a complex linear map, its image in E_x is a complex subspace, so that $(0, \beta) \in W_x$ if and only if

$$\beta(K_{\mathbb{C}}) \subset \mu_{\mathbb{C}} \text{ and } \langle \beta|_{\mathbb{R}g}, \varepsilon \rangle = 0.$$

Since $\dim_{\mathbb{C}} \alpha((K^\perp \cap H)_{\mathbb{C}}) = r - 1$, the real dimension of W_x equals

$$\dim_{\mathbb{R}} W_x = 2r^2 + 2(n - r - 1)(r - 1) + (2r - 1),$$

where the first term equals $\dim_{\mathbb{R}} \mathcal{L}^{\mathbb{C}}(\ker^\perp \alpha, E_x)$ (here \perp stands for $T_x M$), the second equals $\dim_{\mathbb{R}} \mathcal{L}^{\mathbb{C}}(K_{\mathbb{C}}, \mu_{\mathbb{C}})$ and the third equals $\dim_{\mathbb{R}} \{\beta \in \mathcal{L}^{\mathbb{C}}(g^{\mathbb{C}}, E_x), \langle \beta, \varepsilon \rangle = 0\}$, where $g^{\mathbb{C}} = \mathbb{R}g + \mathbb{R}Jg$ denotes the complex line generated by g . \square

The projective case. We first specialize this setting to the projective setting of Theorem 1.3. Recall that the natural scale for the random sections of degree d is $d^{-\frac{1}{2}}$. Since the dimension of ∂U is $2n - 1$, we can guess that the average number of critical points of $p|_{\partial U \cap Z_s}$ should be bounded by $O(d^{\frac{2n-1}{2}})$.

Proposition 4.10 *Let $n \geq 2$ and $1 \leq r \leq n - 1$ be integers, M be a compact smooth Kähler manifold and (L, h) be an ample complex line bundle over M , with curvature form ω , (E, h_E) be a holomorphic rank r vector bundle and let $U \subset M$ be a 0-codimension submanifold with smooth boundary. Then, for any Borel subset $A \subset \partial U$,*

$$\forall 0 \leq i \leq 2n - 2r - 1, \frac{1}{d^{n-\frac{1}{2}}} \mathbb{E} \# \text{Crit}_i^{p|_{\partial U}} = O_{d \rightarrow \infty}(1).$$

Here the probability measure is the Gaussian one given by (1.5).

Proof. Since we only need a bound for the averages and not their exact asymptotics, we apply Theorem 3.3 which is easier to handle with than Corollary 3.4. By Theorem 3.3, we have that for any Borel subset $A \subset \partial U$,

$$\begin{aligned} \mathbb{E} [\#(A \cap \text{Crit}_i^{p\partial})] &= \int_{x \in A} \int_{\substack{\alpha \in \mathcal{L}_{\text{onto}}^{\mathbb{C}}(T_x M, E_x)|_{T_x \partial U} \\ \ker \alpha \subset \ker dp_{\partial}(x)}} \left| \det \alpha|_{\ker^{\perp} \alpha} \right| \\ &\mathbb{E} \left[\mathbf{1}_{\{\text{Ind}(\pi(x, \alpha))=i\}} \left| \det \left(\langle \nabla_{\partial}^2 s(x)|_{\ker \alpha}, \varepsilon(x, \alpha) \rangle \right. \right. \right. \\ &\left. \left. \left. - \frac{\nabla^2 p_{\partial}(x)|_{\ker \alpha}}{\|dp_{\partial}(x)\|} \langle \alpha(h(x)), \varepsilon(x, \alpha) \rangle \right) \right| \mid s(x) = 0, \nabla_{\partial} s(x) = \alpha \right] \\ &\rho_{X(x)}(0, \alpha) d\text{vol}(\alpha) d\text{vol}(x), \end{aligned} \quad (4.9)$$

where $\rho_{X(x)}$ is the Gaussian density of $X(x)$ and \perp refers to the orthogonality in $T\partial U$. As in the proof for projective manifold case, in equation (4.9) we perform the change of variables $\beta = d^{\frac{n+1}{2}} \alpha$ and $Y = d^{\frac{n+2}{2}} \nabla^2 s$. Then, thanks to Lemma 4.9 which provides the power of d which pops up from $\text{vol}(\alpha)$, the average equals $d^{n-\frac{1}{2}}$ times a multiple integral which converges to a convergent integral independent of d . \square

The affine setting. For the Bargmann-Fock field, we have the similar proposition:

Proposition 4.11 *Let $1 \leq r \leq n$ be integers, $f : \mathbb{C}^n \rightarrow \mathbb{C}^r$ be r independent copies of the Bargmann-Fock field (4.1), $U \subset \mathbb{C}^n$ be an open subset with smooth boundary, and $p : \bar{U}$ be a smooth Morse function, in the sense of Definition 4.11. Then,*

$$\forall 0 \leq i \leq 2n - 2r - 1, \frac{1}{R^{2n-1}} \mathbb{E} \# \text{Crit}_i^{p|_{\partial(RU)}} = O_{R \rightarrow +\infty}(1). \quad (4.10)$$

Proof. This is very similar to the projective setting. \square

4.4 Proof of the main theorems

Theorem 1.3 is a simple consequence of Theorem 4.4 and Proposition 4.10. Indeed, Morse inequalities for manifolds with boundary hold:

Theorem 4.12 (see [16, Theorem A]). *Under the setting of Definition 4.8, assume that \bar{Z} is compact. For any $i \in \{0, \dots, m-1\}$, let N_i be the number of boundary critical points of $p|_{\partial Z}$ of index i , such that p increases in the direction of Z . Then,*

- (weak Morse inequalities) $\forall 0 \leq i \leq m$, $b_i(Z) \leq \#\text{Crit}_i^p + N_i$.
- (strong Morse inequalities) $\forall 0 \leq i \leq m$,

$$\sum_{k=0}^i (-1)^{i-k} b_k(Z) \geq \sum_{k=0}^i (-1)^{i-k} (\#\text{Crit}_k^p + N_k).$$

We will apply these Morse inequalities to the random nodal sets $Z_s \cap U$.

Proof of Theorem 1.3. By [12, Lemma 2.8], almost surely the restriction $p|_{Z_s}$ is Morse in the latter sense. The proof of this lemma extends to $p|_{Z_s \cap U}$, so that we can apply Theorem 4.12 to $Z_s \cap U$, for almost all s . Hence,

$$\forall 0 \leq i \leq 2n - 2r, \mathbb{E}b_i(Z_s \cap U) \leq \mathbb{E}(\#\text{Crit}_i^p) + \mathbb{E}(\#\text{Crit}_i^{p|_{\partial U}}). \quad (4.11)$$

By (4.11), Theorem 4.4 and Proposition 4.10, we obtain

$$\forall 0 \leq i \leq 2n - 2r \setminus \{n - r\}, \mathbb{E}b_i(Z_s \cap U) = o(d^n)$$

and $\mathbb{E}b_{n-r}(Z_s \cap U) \leq \mathbb{E}(\#\text{Crit}_{n-r}^p) + o(d^n)$. On the other hand, the two assertions of Theorem 4.12 and Proposition 4.10 imply that

$$\mathbb{E}b_{n-r}(Z_s \cap U) \geq \mathbb{E}(\#\text{Crit}_{n-r}^p) - o(d^n),$$

so that by Theorem 4.4, $\mathbb{E}b_{n-r} = d^n \binom{n-1}{r-1} \int_U \omega + o(d^n)$, which is the result. \square

Lemma 4.13 ([25, Lemma 3.2]) *Under the hypotheses of Theorem 1.7, there exists a map $p: \mathbb{C}^n \rightarrow \mathbb{R}$ such that for almost all instance of the Bargmann-Fock field, $p|_{Z_s \cap \bar{U}}$ is Morse as well in the sense of Definition 4.8.*

Proof. This is proven (in a more general setting) in the proof of [25, Lemma 3.2] for a manifold without boundary. The argument extends immediatly to manifolds with C^2 boundary. \square

Proof of Theorem 1.7. The proof is similar to the one of Theorem 1.3, using Theorem 4.1 and Proposition 4.11. \square

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