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Faster change of order algorithm for Gröbner bases under shape and stability assumptions

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ABSTRACT

Solving polynomial systems whose solution set is finite is usually done in two main steps: compute a Gröbner basis for the degree reverse lexicographic order, and perform a change of order to find the lexicographic Gröbner basis. The second step is generally considered as better understood, in terms of algorithms and complexity. Yet, after two decades of progress on the first step, it turns out that the change of order now takes a large part of the solving time for many instances, including those that are generic or reached after applying a random change of variables.

Like the fastest known change of order algorithms, this work focuses on the latter situation, where the ideal defined by the system satisfies structural properties. First, the ideal has a *shape* lexicographic Gröbner basis. Second, the set of leading terms with respect to the degree reverse lexicographic order has a *stability* property; in particular, the multiplication matrix of the smallest variable is computed for free from the input Gröbner basis.

The current fastest algorithms rely on the sparsity of this multiplication matrix to find its minimal polynomial efficiently using Wiedemann’s approach. This paper starts from the observation that this sparsity is a consequence of an algebraic structure, which can be exploited to represent the matrix concisely as a univariate polynomial matrix. We show that the Hermite normal form of that matrix yields the sought lexicographic Gröbner basis, under assumptions which cover the shape position case. This leads to an improved complexity bound for the second step. The practical benefit is also confirmed via implementations based on the state-of-the-art software libraries `msolve` and `PML`.

1 INTRODUCTION

Context. Modeling problems from biology, coding theory, combinatorics, robotics or aerospace engineering fundamentally relies on the exact solving of polynomial systems over a finite field or over the rational numbers. The topical method to solve a system of polynomial equations $f_1 = \dots = f_m = 0$ in variables x_1, \dots, x_n over a field \mathbb{K} is to compute a Gröbner basis, for a lexicographic order, of the ideal \mathcal{I} spanned by f_1, \dots, f_m . In this paper the variable x_n will play a special role, hence we will denote it by y : the variables are x_1, \dots, x_{n-1}, y . Whenever $m = n$, for generic polynomials f_1, \dots, f_n of respective degrees d_1, \dots, d_n , the solution set consists in finitely many $D = d_1 \cdots d_n$ points, counted with multiplicity, over an algebraic closure of \mathbb{K} . Furthermore, the ideal spanned by f_1, \dots, f_n is

in *shape position* (see the *shape lemma* in [21, Lem. 1.4], and Eq. (1)). This ensures that the lexicographic Gröbner basis gives easy access to the solutions of the system.

While computing Gröbner bases is exponential in n in general, specific ones can be easier to compute than others. Indeed, the traditional strategy to find a lexicographic Gröbner basis uses two steps. First, compute a Gröbner basis \mathcal{G} of the ideal \mathcal{I} for a degree-refining order, using an algorithm such as Buchberger’s, or Faugère’s F_4 or F_5 [9, 13, 14]. Then, apply a change of order to obtain the lexicographic Gröbner basis, using the FGLM algorithm [16], or its faster Sparse-FGLM variant [17, 18] in the *shape* position case. The latter also requires the prior computation of a *multiplication matrix* easily obtained from \mathcal{G} under some *stability* assumption (see below).

In many examples, the change of order is in fact the bottleneck of this strategy, even for instances that satisfy the shape and stability properties, such as Katsura- n (see [8, Tbl. 1]). Faster polynomial system solving requires a faster change of order algorithm.

Prior results. The original FGLM algorithm [16] takes as input two monomial orders \preceq_1 and \preceq_2 , and a \preceq_1 -Gröbner basis of a zero-dimensional ideal \mathcal{I} , and returns the reduced \preceq_2 -Gröbner basis of \mathcal{I} using $O(nD^3)$ operations in \mathbb{K} . This algorithm may be viewed as a two-step process: first compute the multiplication matrix of each of the n variables in $\mathbb{K}[x_1, \dots, x_{n-1}, y]/\mathcal{I}$ with respect to the \preceq_1 -monomial basis; then use these multiplication matrices to compute the sought \preceq_2 -Gröbner basis. The latter step can be done in $O(nD^\omega \log D)$ operations in \mathbb{K} [37, Thm. 1.7] while computing the matrices currently requires, to reach the same complexity, some stability of the \preceq_1 -leading terms of elements of \mathcal{I} [37, Thm. 1.9].

For the sake of presentation, in the rest of the introduction we make the mild assumption $n \leq D$.

In [15], the authors consider the special case where \preceq_1 is the degree reverse lexicographic order \preceq_{drl} and \preceq_2 is the lexicographic one \preceq_{lex} , with $y \preceq_{\text{drl}} x_k$ and $y \preceq_{\text{lex}} x_k$ for $1 \leq k \leq n-1$. They show that the multiplication matrix of y can be obtained for free from the \preceq_{drl} -Gröbner basis, assuming either genericity or a random linear change of variables and using results from [34]. One can see that the assumptions are exploited for the *stability* they bring to the ideal of \preceq_{drl} -leading terms of \mathcal{I} , as described precisely in Definition 2.1. Adding the shape position assumption, this multiplication matrix alone suffices to recover the \preceq_{lex} -Gröbner basis, which is done in time $O(D^\omega)$ [15]. Furthermore, even if \mathcal{I} is not in shape position, the shape lemma [4, 21] ensures that its radical ideal is in shape position, allowing the authors to compute the \preceq_{lex} -Gröbner basis of the radical instead within the same complexity.

In [17, 18], the authors follow on from the same assumptions: shape and stability. They observe that the multiplication matrix is sparse, and study how to exploit this for faster computations thanks

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to Wiedemann’s approach [46]. Precisely, the matrix has about tD nonzero entries out of D^2 , where the parameter t is the number of polynomials in the \preceq_{drl} -Gröbner basis whose leading monomial is divisible by y . This leads to a complexity estimate of $O(tD^2)$ operations in \mathbb{K} , which improves upon $O(D^\omega)$ when t is small compared to D . This provides significant practical benefit for the change of order step of polynomial system solving in many cases [18, Tbl. 2], and this is the approach used by the state-of-the-art change of order implementation in `msolve` [8].

Contributions. More precisely, the multiplication matrix has t dense columns, and $D - t$ sparse columns which come from the $D \times D$ identity matrix [18, Sec. 5] [43, Sec. 4]. This is a well-known matrix structure in \mathbb{K} -linear algebra, called a *shifted form* in [38, 39], and studied in particular in the context of the computation of the characteristic polynomial or the Frobenius normal form of a matrix over \mathbb{K} (see [44, Sec. 9.1] and Section 3.3).

Here, rather than focusing of the resulting sparsity, we exploit the algebraic structure itself, and relate it to operations in a $\mathbb{K}[y]$ -submodule of \mathcal{I} . Following a classical construction in [44, Sec. 9.1], instead of the multiplication matrix M which is in $\mathbb{K}^{D \times D}$, we consider a univariate polynomial matrix P in $\mathbb{K}[y]^{t \times t}$ whose average column degree is D/t . This polynomial matrix can be seen as a “compression” of M , or more precisely of the characteristic matrix $yI_D - M$, with smaller matrix dimension but larger degrees.

Our main result is that, if P is known and the lexicographic Gröbner basis satisfies some assumption which covers the shape position case, then this Gröbner basis can be directly retrieved from the Hermite normal form of P (Theorem 4.1). We also prove that the matrix P can be computed for free from some part of a border basis of \mathcal{I} in general (Theorem 5.2) and, as a consequence under the stability assumption, from the input Gröbner basis (Corollary 5.4). Observe that both structural assumptions, of being *stable* and *shape*, are used independently. In particular it is expected that in some situations where the stability assumption is not satisfied, P may still be obtained efficiently, and then its Hermite normal form yields the lexicographic Gröbner basis if \mathcal{I} is in shape position.

The Hermite normal form can be computed deterministically in $O(t^{\omega-1}D)$ operations in \mathbb{K} [31], which dominates the overall complexity of the change of order. Compared to the previous $O(tD^2)$, the speed-up factor is of the order of $t^{2-\omega}D$. The least favorable situation is $t \sim D$ with a speed-up of $D^{3-\omega}$. The most favorable one is when $t \ll D$, the extreme case being $t \sim n$. We give explicit complexity gains for families of polynomial systems for which closed formulas or asymptotic estimates for t and D are known [7, 18].

Finally, we study the practical performance of the new approach. For this, we designed an efficient implementation of the Hermite normal form, which follows the approach of [31] but tailored to the matrices P encountered here, which have specific degree shapes. This implementation relies on the Polynomial Matrix Library [28] (PML) and on NTL [42]. We show that it outperforms both the existing change of order algorithm in the current version of `msolve` [8], and an implementation of a block-Wiedemann approach in NTL.

Structure of the paper. In Section 2, we recall basic notation and facts on Gröbner bases, and define the above-mentioned structural assumptions. Section 3 is devoted to the definition, from \mathcal{I} and

some chosen monomials, of a $\mathbb{K}[y]$ -submodule of \mathcal{I} : the matrix P will be a basis of that module, and we relate this construction to the multiplication matrix of y . Section 4 establishes the connexion between the reduced \preceq_{lex} -Gröbner basis of \mathcal{I} and the Hermite normal form of P . Section 5 shows how to compute P from a Gröbner basis. Finally, in Section 6, we discuss complexity results and report on practical performance of the new approach.

2 NOTATION AND PRELIMINARIES

Consider the polynomial ring $\mathcal{R} = \mathbb{K}[x_1, \dots, x_{n-1}, y]$. For a nonzero polynomial $f \in \mathcal{R} \setminus \{0\}$, the *support* of f , denoted by $\text{supp}(f)$, is the collection of all monomials appearing in f , with a nonzero coefficient. For a set of polynomials $S \subset \mathcal{R}$, the ideal generated by S in \mathcal{R} is denoted by $\langle S \rangle$.

Monomial orders, normal forms. For the definition of a *monomial order* \preceq on \mathcal{R} , we refer to [10, Chap. 2, §2]. We recall that \preceq is a total order on the set of monomials, and we write \prec for the corresponding strict order. Here, monomial orders are such that $y < x_{n-1} < \dots < x_1$. In particular, we will use the *lexicographic* order \preceq_{lex} , and the *degree reverse lexicographic* order \preceq_{drl} . We denote by $\text{lt}_{\preceq}(f)$ the \preceq -leading term of a nonzero polynomial $f \in \mathcal{R}$, and by $\text{lt}_{\preceq}(S)$ the set of \preceq -leading terms of all nonzero elements of a set $S \subseteq \mathcal{R}$.

For a monomial order \preceq and an ideal $\mathcal{I} \subset \mathcal{R}$, consider the set \mathcal{B} of monomials in \mathcal{R} that are not in the ideal of leading terms $\langle \text{lt}_{\preceq}(\mathcal{I}) \rangle$. This set \mathcal{B} is called the \preceq -*monomial basis* of \mathcal{R}/\mathcal{I} : it is a basis of \mathcal{R}/\mathcal{I} as a \mathbb{K} -vector space [5, Prop. 6.52]. For a polynomial $f \in \mathcal{R}$, the \preceq -*normal form* of f with respect to \mathcal{I} , denoted by $\text{nf}_{\preceq, \mathcal{I}}(f)$, is the unique polynomial whose support is in \mathcal{B} and such that $f - \text{nf}_{\preceq, \mathcal{I}}(f) \in \mathcal{I}$.

Gröbner bases, shape position. For the notion of (reduced) \preceq -*Gröbner bases* of ideals in \mathcal{R} , we refer to [10, Chap. 2]. By definition, for a \preceq -Gröbner basis \mathcal{G} of \mathcal{I} , we have $\langle \text{lt}_{\preceq}(\mathcal{G}) \rangle = \langle \text{lt}_{\preceq}(\mathcal{I}) \rangle$ and the \preceq -monomial basis \mathcal{B} is also the set of monomials that are not multiples of an element in $\text{lt}_{\preceq}(\mathcal{G})$.

A proper ideal $\mathcal{I} \subset \mathcal{R}$ is *zero-dimensional* [5, Def. 6.46] if, and only if, \mathcal{R}/\mathcal{I} has finite dimension D as a \mathbb{K} -vector space [5, Thm. 6.54]. In that case, following [4], \mathcal{I} is said to be in *shape position* if its reduced \preceq_{lex} -Gröbner basis has the form

$$\mathcal{G}_{\text{lex}} = \{h(y), x_{n-1} - g_{n-1}(y), \dots, x_1 - g_1(y)\}, \quad (1)$$

where g_1, \dots, g_{n-1}, h are in $\mathbb{K}[y]$. By properties of reduced Gröbner bases, this implies $\deg g_i < \deg h$ for $1 \leq i < n$. Then, \mathcal{R}/\mathcal{I} is isomorphic to $\mathbb{K}[y]/\langle h(y) \rangle$ as an \mathcal{R} -module (equipping this quotient with the multiplication $x_i \cdot f = g_i(y)f$ for $1 \leq i < n$), and the \preceq_{lex} -monomial basis is $(1, y, \dots, y^{D-1})$ for $D = \deg h = \dim_{\mathbb{K}}(\mathcal{R}/\mathcal{I})$.

Stability assumption. This assumption, mentioned in Section 1, concerning the stability of the ideal of \preceq_{drl} -leading terms of \mathcal{I} , is defined as follows.

Definition 2.1. For a set S of monomials in \mathcal{R} , the statement $S(S)$ is: “for any monomial $\mu \in S$ such that y divides μ , the monomial $\frac{x_i}{y}\mu$ belongs to S for all $i \in \{1, \dots, n-1\}$ ”.

This is directly related to classical notions of stability of sets of monomials and of monomial ideals [26, Sec. 4.2.2, 6.3 and 7.2.2], which arise notably through the Borel-fixedness of generic initial

ideals [3, 20]. The next lemma states that when considering the monomials in a monomial ideal, the above statement holds if, and only if, it holds for the minimal generating monomials of that ideal.

LEMMA 2.2. *Let \mathcal{J} be a monomial ideal of \mathcal{R} , and $\{\mu_1, \dots, \mu_s\}$ be its minimal generating set. Let S be the set of monomials in \mathcal{J} . Then, $\mathcal{S}(S)$ is equivalent to $\mathcal{S}(\mu_1, \dots, \mu_s)$.*

We do not prove this, as this is a direct consequence of [37, Lem. 2.2]. For our purpose, we are mostly interested in the case $\mathcal{J} = \text{lt}_{\leq}(\mathcal{I})$ for some monomial order \leq and some ideal \mathcal{I} . The above lemma shows that, if $\text{lt}_{\leq}(\mathcal{I})$ is known (for example via the \leq -leading terms of a \leq -Gröbner basis), then it is straightforward to check whether $\mathcal{S}(\text{lt}_{\leq}(\mathcal{I}))$ holds.

Example 2.3. Consider the ideal \mathcal{I} of $\mathbb{F}_{29}[x_1, x_2, y]$ generated by

$$\begin{aligned} & \mathbf{x}_2^2 + 12x_1y + 26x_2y + 5y^2 + 9x_1 + 6x_2 + 8y + 6, \\ & x_1x_2 + 10x_2^2 + 10x_1y + 9y^2 + 2x_1 + 14x_2 + y + 13, \\ & \mathbf{x}_1^2 + 7x_1x_2 + 27x_2^2 + 15x_1y + 24x_2y + 3y^2 + 4x_1 + 28x_2 + 18y + 26. \end{aligned}$$

Its reduced \leq_{drl} -Gröbner basis \mathcal{G}_{drl} consists of the polynomials

$$\begin{aligned} & \mathbf{y}^4 + 3y^3 + 15x_1y + 23x_2y + 3y^2 + 26x_2 + 22y, \\ & \mathbf{x}_2y^2 + 5x_1y + 28x_2y + 3y^2 + 19x_1 + 15x_2 + 17, \\ & \mathbf{x}_1y^2 + 18y^3 + 24x_1y + 27x_2y + 19y^2 + 2x_1 + 9y + 3, \\ & \mathbf{x}_2^2 + 12x_1y + 26x_2y + 5y^2 + 9x_1 + 6x_2 + 8y + 6, \\ & x_1x_2 + 6x_1y + x_2y + 17y^2 + 28x_1 + 12x_2 + 8y + 11, \\ & \mathbf{x}_1^2 + x_1y + 10x_2y + 2y^2 + 3x_1 + 16x_2 + 21. \end{aligned}$$

Observe that it has $t = 3$ polynomials whose \leq_{drl} -leading terms, in boldface font, are multiples of y . The \leq_{drl} -monomial basis \mathcal{B} of \mathcal{R}/\mathcal{I} is the set of monomials not in

$$\langle \text{lt}_{\leq_{\text{drl}}}(\mathcal{G}_{\text{drl}}) \rangle = \langle y^4, x_2y^2, x_1y^2, x_2^2, x_1x_2, x_1^2 \rangle,$$

that is, $\mathcal{B} = (1, y, y^2, y^3, x_2, x_2y, x_1, x_1y)$.

Finally, we verify that the stability property $\mathcal{S}(\text{lt}_{\leq_{\text{drl}}}(\mathcal{I}))$ holds. As noted in Lemma 2.2, it is sufficient to check that for each minimal generator μ of $\langle \text{lt}_{\leq_{\text{drl}}}(\mathcal{I}) \rangle$ such that y divides μ , the monomials $\frac{x_1}{y}\mu$ and $\frac{x_2}{y}\mu$ remain in $\text{lt}_{\leq_{\text{drl}}}(\mathcal{I})$. Thus we consider $\mu \in \{x_2y^2, x_1y^2, y^4\}$, and it is easily verified that $x_1x_2y, x_2^2y, x_1^2y, x_1x_2y, x_1y^3, x_2y^3$ are all in $\text{lt}_{\leq_{\text{drl}}}(\mathcal{I})$. \square

3 RESTRICTING TO A $\mathbb{K}[y]$ -MODULE

The algorithmic approach in this paper makes use of a $\mathbb{K}[y]$ -module denoted by $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$, which is defined from an ideal \mathcal{I} and a set of monomials \mathcal{T} . This module is a subset of \mathcal{I} , which is sufficient to recover the \leq_{lex} -Gröbner basis of \mathcal{I} in the shape position case, and which allows us to benefit from efficient algorithms for matrices over $\mathbb{K}[y]$.

3.1 General definitions and properties

For general definitions and properties of modules, we refer to [11, Chap. 10]. Roughly, *free* modules are those which admit a basis, and since $\mathbb{K}[y]$ is commutative, all bases of a free $\mathbb{K}[y]$ -module have

the same cardinality which is called the *rank* of the module [11, Sec. 10.3].

For modules over a principal ideal domain such as $\mathbb{K}[y]$, we refer to [11, Chap. 12]. In particular, if \mathcal{N} is a free $\mathbb{K}[y]$ -module of rank $t \in \mathbb{N}$ and \mathcal{M} is a $\mathbb{K}[y]$ -submodule of \mathcal{N} , then \mathcal{M} is free and its rank ρ is at most t [11, Sec. 12.1, Thm. 4]. As a result, \mathcal{M} has a basis of cardinality ρ , which can be represented as a matrix in $\mathbb{K}[y]^{\rho \times t}$. This matrix has full row rank, and its rows are the basis elements (see also Eq. (4) in Section 3.2). Furthermore \mathcal{M} has a unique basis in a specific form, at the core of this work: the *Hermite normal form* [25, 29]. When $\rho = t$, a matrix $P = [p_{ij}] \in \mathbb{K}[y]^{t \times t}$ is in Hermite normal form if:

- P is lower triangular;
- the diagonal entries of P are monic;
- in each column of P , the diagonal entry has greater degree than the other entries, i.e. $\deg(p_{ij}) < \deg(p_{jj})$ for $i \neq j$.

A typical example of ambient module is $\mathcal{N} = \mathbb{K}[y]^t$. Here we will also consider the $\mathbb{K}[y]$ -module $\mathcal{N} = \mathcal{R}_{\mathcal{T}} \subset \mathcal{R}$, defined as follows. Let $\mathcal{T} = (\mu_1, \dots, \mu_t)$ be a list of pairwise distinct monomials in $\mathbb{K}[x_1, \dots, x_{n-1}]$, and consider the set of monomials

$$\mathcal{T}^* = \{y^e \mu \mid \mu \in \mathcal{T}, e \geq 0\}$$

of y -multiples of a monomial in \mathcal{T} . Then we define

$$\mathcal{R}_{\mathcal{T}} = \{f \in \mathcal{R} \mid \text{supp}(f) \in \mathcal{T}^*\}, \quad (2)$$

which is a free $\mathbb{K}[y]$ -module of rank t , with basis given by \mathcal{T} .

Hereafter, for a finite set of polynomials $S \subset \mathcal{R}$, the $\mathbb{K}[y]$ -module generated by S will be denoted by $\langle S \rangle$.

Example 3.1. Let $\mathcal{T} = (1, x_{n-1}, \dots, x_1)$. Then

$$\mathcal{R}_{\mathcal{T}} = \langle 1, x_{n-1}, \dots, x_1 \rangle = \mathbb{K}[y] + x_{n-1}\mathbb{K}[y] + \dots + x_1\mathbb{K}[y]$$

is a $\mathbb{K}[y]$ -submodule of \mathcal{R} of rank n . \square

3.2 A module associated to the ideal

For pairwise distinct monomials $\mathcal{T} = (\mu_1, \dots, \mu_t)$ in $\mathbb{K}[x_1, \dots, x_{n-1}]$, we consider the $\mathbb{K}[y]$ -module $\mathcal{R}_{\mathcal{T}}$ defined in Section 2; it is free of rank t , with basis \mathcal{T} . Then, for any ideal \mathcal{I} of \mathcal{R} , let

$$\mathcal{M}_{\mathcal{T}, \mathcal{I}} = \mathcal{I} \cap \mathcal{R}_{\mathcal{T}}. \quad (3)$$

By construction, we have the inclusion of ideals $\langle \mathcal{M}_{\mathcal{T}, \mathcal{I}} \rangle \subseteq \mathcal{I}$.

Example 3.2. Let \mathcal{T} be as in Example 3.1. Then $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ is the set of polynomials in \mathcal{I} which have degree at most 1 in each of the variables x_{n-1}, \dots, x_1 , and

- for $\mathcal{I} = \langle x_1^2 \rangle$, $\mathcal{M}_{\mathcal{T}, \mathcal{I}} = \{0\}$ and $\langle \mathcal{M}_{\mathcal{T}, \mathcal{I}} \rangle \subsetneq \mathcal{I}$;
- for $\mathcal{I} = \langle x_1 - 1 \rangle$, $\mathcal{M}_{\mathcal{T}, \mathcal{I}} = (x_1 - 1)\mathbb{K}[y]$ and $\langle \mathcal{M}_{\mathcal{T}, \mathcal{I}} \rangle \subsetneq \mathcal{I}$;
- for a zero-dimensional ideal \mathcal{I} , if \mathcal{I} is in shape position then $\langle \mathcal{M}_{\mathcal{T}, \mathcal{I}} \rangle = \mathcal{I}$ (see Lemma 5.1). \square

The case of equality $\langle \mathcal{M}_{\mathcal{T}, \mathcal{I}} \rangle = \mathcal{I}$ is of particular interest: it ensures that no information is lost when restricting to polynomials with monomial support in \mathcal{T}^* . Our aim is to compute objects related to \mathcal{I} , such as its \leq_{lex} -Gröbner basis, using only computations in the smaller submodule $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$. The motivation behind this idea is that many efficient tools are known for computing with $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$, thanks to the matrix representation explained below.

As seen in Section 3.1, as a $\mathbb{K}[y]$ -submodule of $\mathcal{R}_{\mathcal{T}}$, $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ is free of rank ρ , with $\rho \leq t$, and any basis of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ is a collection of

ρ polynomials $\{P_1, \dots, P_\rho\} \subset \mathcal{R}_{\mathcal{T}}$. Such a basis can be represented as a matrix

$$P = \begin{bmatrix} P_{11} & \cdots & P_{1t} \\ \vdots & \vdots & \vdots \\ P_{\rho 1} & \cdots & P_{\rho t} \end{bmatrix} \in \mathbb{K}[y]^{\rho \times t} \quad (4)$$

of rank ρ , whose row i is formed by the univariate polynomials $P_{i1}, \dots, P_{it} \in \mathbb{K}[y]$ such that $P_i = P_{i1}\mu_1 + \cdots + P_{it}\mu_t$.

Example 3.3 (following on from Example 2.3). Take \mathcal{T} as the set of monomials in \mathcal{B} which are not multiples of y , that is, $\mathcal{T} = (1, x_2, x_1)$; observe that the cardinality of \mathcal{T} is $t = 3$. As noted in Example 3.1, $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ is then the set of polynomials in \mathcal{I} which have degree at most 1 in x_1 and in x_2 . This is the case for $t = 3$ polynomials of \mathcal{G}_{drl} :

$$\begin{aligned} & y^4 + 3y^3 + 15x_1y + 23x_2y + 3y^2 + 26x_2 + 22y, \\ & x_2y^2 + 5x_1y + 28x_2y + 3y^2 + 19x_1 + 15x_2 + 17, \\ & x_1y^2 + 18y^3 + 24x_1y + 27x_2y + 19y^2 + 2x_1 + 9y + 3. \end{aligned}$$

Hence these polynomials are in $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$; note they are exactly the polynomials of \mathcal{G}_{drl} whose \leq_{drl} -leading terms are multiples of y . In Section 5 we will prove that, since $\mathcal{S}(\text{lt}_{\leq_{\text{drl}}}(\mathcal{I}))$ is satisfied (see Example 2.3), these polynomials form a basis of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$.

Representing these polynomials on the basis \mathcal{T} of $\mathcal{R}_{\mathcal{T}}$, we obtain the following matrix in $\mathbb{F}_{29}[y]^{t \times t}$:

$$P = \begin{bmatrix} y^4 + 3y^3 + 3y^2 + 22y & 23y + 26 & 15y \\ 3y^2 + 17 & y^2 + 28y + 15 & 5y + 19 \\ 18y^3 + 19y^2 + 9y + 3 & 27y & y^2 + 24y + 2 \end{bmatrix}.$$

Note that this matrix is directly read off from \mathcal{G}_{drl} . \square

We end this section by showing that if \mathcal{I} is zero-dimensional, then the bases $P \in \mathbb{K}[y]^{\rho \times t}$ of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ are square ($\rho = t$), nonsingular matrices. This is implied by the first item of the following lemma, thanks to the fact that a zero-dimensional ideal contains a univariate polynomial in each variable [5, Lem. 6.50]. For completeness, we also give a partial converse property in the second item.

LEMMA 3.4. *With the above notation,*

- If there exists a nonzero univariate $h \in \mathcal{I} \cap \mathbb{K}[y]$, then $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ has rank $\rho = t$ as a $\mathbb{K}[y]$ -module.
- If $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ has rank $\rho = t$ as a $\mathbb{K}[y]$ -module and $1 \in \mathcal{T}$, then there exists a nonzero univariate $h \in \mathcal{I} \cap \mathbb{K}[y]$.

PROOF. We already observed that $\rho \leq t$. *First item:* assuming the existence of h , the set $g\mathcal{R}_{\mathcal{T}} = \{hf \mid f \in \mathcal{R}_{\mathcal{T}}\}$ is a $\mathbb{K}[y]$ -module of rank t , having $h\mathcal{T}$ as a basis. Since $h \in \mathcal{I}$, $h\mathcal{R}_{\mathcal{T}}$ is contained in $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$, which implies $t \leq \rho$ as recalled in Section 2. Hence $\rho = t$. *Second item:* assuming $\rho = t$, we define $P \in \mathbb{K}[y]^{t \times t}$ as in Eq. (4), from a basis $\{P_1, \dots, P_\rho\} \subset \mathcal{R}_{\mathcal{T}}$ of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$. Then, we let $h = \det(P) \in \mathbb{K}[y]$, which is nonzero since P is nonsingular. By assumption, there exists $j \in \{1, \dots, t\}$ such that $\mu_j = 1$. Then, by Cramer's rule, there are $u_1, \dots, u_t \in \mathbb{K}[y]$ such that $[u_1 \cdots u_t]P = [0 \cdots 0 \ h \ 0 \cdots 0]$ with h at the j th position. By construction of P , this means $u_1P_1 + \cdots + u_tP_t = h\mu_j = h$, hence $h \in \mathcal{I}$. \square

3.3 Link with the multiplication matrix

In Example 3.3, the basis $P \in \mathbb{K}[y]^{t \times t}$ of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ can be seen as a compact representation of the operator of multiplication by y in

\mathcal{R}/\mathcal{I} . The more classical representation uses a *multiplication matrix*, which is the matrix of this operator expressed on the \leq -monomial basis. We have seen that in the case of Examples 2.3 and 3.3, the \leq_{drl} -monomial basis is $\mathcal{B} = (1, y, y^2, y^3, x_2, x_2y, x_1, x_1y)$. Then this multiplication matrix is

$$M = \left[\begin{array}{cccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 7 & 26 & 26 & 3 & 6 & 0 & 14 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 12 & 0 & 26 & 0 & 14 & 1 & 10 & 24 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 26 & 20 & 10 & 11 & 0 & 2 & 27 & 5 \end{array} \right] \in \mathbb{K}^{D \times D}.$$

The choice of ordering of \mathcal{B} makes the following structure obvious: this matrix has companion blocks on the diagonal, and its other blocks have zeroes everywhere but possibly on the last row. Note how the basis P from Example 3.3 can be built by replacing each block by a single polynomial in $\mathbb{K}[y]$ (recall here $\mathbb{K} = \mathbb{F}_{29}$):

- companion blocks are replaced by their respective characteristic polynomials, for example the first companion block becomes $y^4 - (26y^3 + 26y^2 + 7y) = y^4 + 3y^3 + 3y^2 + 22y$;
- other blocks are replaced by the opposite of the polynomial given by the last row, for example the block (3, 1) yields $-(11y^3 + 10y^2 + 20y + 26) = 18y^3 + 19y^2 + 9y + 3$.

Both this type of structure for matrices over a field and the corresponding compact representation as univariate polynomial matrices have been studied, in particular concerning questions of matrix similarity. For example, the Frobenius normal form of M corresponds to the Smith normal form of P [44, Thm. 9.1], whereas the shifted Hessenberg form of M corresponds to the Hermite normal form of P [44, Thm. 9.5 and Lem. 9.7]. More recently, such matrix structures were instrumental in the design of fast algorithms for the Frobenius normal form of a matrix over a field [38, 39].

However, to our knowledge, in the context of Gröbner basis change of order, this structure of the multiplication matrix had only been exploited through the sparsity it brings, in order to rely on (block-)Wiedemann techniques [18, 27, 43].

4 RETRIEVING LEXICOGRAPHIC GRÖBNER BASES FROM HERMITE NORMAL FORMS

From a matrix P as in Eq. (4), whose rows in $\mathbb{K}[y]^{1 \times t}$ represent a basis of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$, one can compute the reduced Gröbner basis of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ with respect to a chosen monomial order on $\mathbb{K}[y]^{1 \times t}$; see [12, Chap. 15] for Gröbner bases of submodules of a free module with basis. Here this ambient free module is $\mathcal{R}_{\mathcal{T}} \simeq \mathbb{K}[y]^{1 \times t}$, with $\mathbb{K}[y]$ univariate: specific terminology and computational tools exist. In particular, classical reduced Gröbner bases are the Hermite normal form [25] (corresponding to the position-over-term order [30]), the Popov normal form [41] (corresponding to the term-over-position order [30]), and shifted variants of the latter [6] (corresponding to term-over-position orders with weights [35, Sec. 1.3.4]). The definition of Hermite normal forms was given in Section 2.

However, these Gröbner bases of the submodule $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ do not necessarily correspond to Gröbner bases of the ideal \mathcal{I} , even when

$\langle \mathcal{M}_{\mathcal{T}, \mathcal{I}} \rangle = \mathcal{I}$. The next result states that, under the stability assumption, there is a correspondence between the lexicographic Gröbner basis of \mathcal{I} and the basis of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ in Hermite normal form. (A link of this kind is not new [32, Sec. 5] [45, Sec. 7], yet we were not able to find a statement similar to the next one in the literature.)

THEOREM 4.1. *Let \mathcal{I} be a zero-dimensional ideal of \mathcal{R} and let \mathcal{G}_{lex} be the reduced \leq_{lex} -Gröbner basis of \mathcal{I} . Let $\mathcal{T} = (\mu_1, \dots, \mu_t)$ be pairwise distinct monomials in $\mathbb{K}[x_1, \dots, x_{n-1}]$, sorted increasingly according to \leq_{lex} . Define $\mathcal{R}_{\mathcal{T}}$ and $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ as in Eqs. (2) and (3). Let $H \in \mathbb{K}[y]^{t \times t}$ be the basis of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ in Hermite normal form.*

Assuming $\mathcal{G}_{\text{lex}} \subseteq \mathcal{R}_{\mathcal{T}}$, then \mathcal{G}_{lex} can be read off from the rows of H . Explicitly, let f be an element of \mathcal{G}_{lex} and let i be the unique integer in $\{1, \dots, t\}$ such that $\text{lm}_{\leq_{\text{lex}}}(f) = y^e \mu_i$ for some $e \geq 0$. Then the i th row of H has the form $[f_1 \ \dots \ f_i \ 0 \ \dots \ 0] \in \mathbb{K}[y]^{1 \times t}$, with $\deg(f_i) = e$ and $f = f_1 \mu_1 + \dots + f_i \mu_i$.

PROOF. In this proof, \leq stands for the lexicographic order \leq_{lex} .

Let f be an element of \mathcal{G}_{lex} . Since $\mathcal{G}_{\text{lex}} \subseteq \mathcal{R}_{\mathcal{T}}$, every monomial of f belongs to $\mathcal{T}^* = \{y^e \mu_j \mid 1 \leq j \leq t, e \geq 0\}$. In particular, $\text{lm}_{\leq}(f) = y^e \mu_i$ for some i in $\{1, \dots, t\}$ and $e \geq 0$ (and i is unique since the μ_j 's are pairwise distinct).

Since $f \in \mathcal{R}_{\mathcal{T}}$, and \mathcal{T} is a basis of $\mathcal{R}_{\mathcal{T}}$ as a $\mathbb{K}[y]$ -module, there is a unique $[f_1 \ \dots \ f_t] \in \mathbb{K}[y]^{1 \times t}$ such that $f = f_1 \mu_1 + \dots + f_t \mu_t$.

Let $j \in \{i+1, \dots, t\}$. We are going to prove $f_j = 0$. Recall that $y < x_k$ for $1 \leq k \leq n-1$, and that the monomial μ_j only involves the variables x_1, \dots, x_{n-1} . Besides, \mathcal{T} being sorted increasingly ensures $\mu_i < \mu_j$. Hence $y^e \mu_i < y^d \mu_j$ for any $d \geq 0$: having $f_j \neq 0$ would contradict $\text{lm}_{\leq}(f) = y^e \mu_i$.

Thus $f = f_1 \mu_1 + \dots + f_i \mu_i$, and $\text{lm}_{\leq}(f) = y^e \mu_i$ ensures $\deg(f_i) = e$. It remains to show that the i th row of H is equal to $[f_1 \ \dots \ f_i \ 0 \ \dots \ 0]$.

We first show that the i th diagonal entry of H has degree $e = \deg(f_i)$. On the one hand, the i th row of H corresponds to a nonzero polynomial in \mathcal{I} whose \leq -leading term is $y^d \mu_i$. Then, having $d < e$ would mean that $\text{lt}_{\leq}(f)$ is a strict multiple of the \leq -leading term of some element of \mathcal{I} , which contradicts the definition of \mathcal{G}_{lex} . Thus $d \geq e$. On the other hand, f is in $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ and therefore corresponds to a vector in the $\mathbb{K}[y]$ -row space of H whose rightmost nonzero entry is at index i . Then, the triangularity of H implies that

$$[f_1 \ \dots \ f_i \ 0 \ \dots \ 0] = [\lambda_1 \ \dots \ \lambda_i \ 0 \ \dots \ 0]H$$

for some $\lambda_1, \dots, \lambda_i \in \mathbb{K}[y]$ and $\lambda_i \neq 0$. Using the triangularity again, we obtain $e = \deg(f_i) = \deg(\lambda_i) + d$, hence $d \leq e$. This yields $e = d$.

Let $d_1, \dots, d_{i-1} \in \mathbb{N}$ be the degrees of the first $i-1$ diagonal entries of H . In this paragraph we show that, to conclude the proof, it is enough to prove $\deg(f_j) < d_j$ for $1 \leq j < i$. Indeed, as seen above, the vector $[f_1 \ \dots \ f_i \ 0 \ \dots \ 0]$ is in the $\mathbb{K}[y]$ -row space of H , and has rightmost nonzero entry f_i at index i , which has the same degree as the i th diagonal entry of H and is monic by definition of a reduced \leq -Gröbner basis. Thus, if $\deg(f_j) < d_j$ for $1 \leq j < i$, then this vector must be equal to the i th row of H , by uniqueness of the Hermite normal form: otherwise one could replace the i th row of H by this vector and get a different Hermite normal form for the same $\mathbb{K}[y]$ -module.

Let $1 \leq j < i$. We are going to prove $\deg(f_j) < d_j$. The j th row of H yields a polynomial in \mathcal{I} with \leq -leading term $y^{d_j} \mu_j$, hence $y^{d_j} \mu_j \in \langle \text{lt}_{\leq}(\mathcal{I}) \rangle$. At the same time, since \mathcal{G}_{lex} is reduced, $\text{lt}_{\leq}(f) =$

$y^e \mu_i$ is the only monomial appearing in f which is in $\langle \text{lt}_{\leq}(\mathcal{I}) \rangle$. In particular, defining $d = \deg(f_j)$, the monomial $y^d \mu_j$ of f is not a multiple of or equal to $y^{d_j} \mu_j$, hence $d < d_j$. \square

Example 4.2 (following on from Example 3.3). Computing the Hermite normal form of the basis matrix P from Example 3.3 yields

$$H = \begin{bmatrix} y^8 + 26y^7 + 8y^6 + 17y^5 + 19y^4 + y^3 + 28y^2 + 20y + 18 & 0 & 0 \\ 28y^7 + 23y^6 + 17y^5 + 25y^4 + 24y^3 + 17y^2 + 14y + 4 & 1 & 0 \\ 6y^7 + 13y^6 + 22y^5 + 12y^4 + 28y^3 + 24y^2 + 26y + 14 & 0 & 1 \end{bmatrix}.$$

This is the basis of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ in Hermite normal form; recall that here $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ is the $\mathbb{K}[y]$ -submodule of polynomials in \mathcal{I} which have the form $p_1(y) + p_2(y)x_2 + p_3(y)x_1$. This basis gives the \leq_{lex} -Gröbner basis of \mathcal{I} :

$$\begin{aligned} & y^8 + 26y^7 + 8y^6 + 17y^5 + 19y^4 + y^3 + 28y^2 + 20y + 18, \\ & x_2 + 28y^7 + 23y^6 + 17y^5 + 25y^4 + 24y^3 + 17y^2 + 14y + 4, \\ & x_1 + 6y^7 + 13y^6 + 22y^5 + 12y^4 + 28y^3 + 24y^2 + 26y + 14. \quad \square \end{aligned}$$

Suppose the basis H of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ in Hermite normal form is known. If $\text{lt}_{\leq}(\mathcal{G}_{\text{lex}})$ is known as well, which is the case under the shape position assumption, then Theorem 4.1 indicates precisely which rows of H give the polynomials of \mathcal{G}_{lex} , without any further computation.

REMARK 4.3. *Even when $\text{lt}_{\leq_{\text{lex}}}(\mathcal{G}_{\text{lex}})$ is unknown, \mathcal{G}_{lex} is easily found from H . Indeed, H yields polynomials h_1, \dots, h_t in $\mathcal{M}_{\mathcal{T}, \mathcal{I}} \subseteq \mathcal{I}$, and Theorem 4.1 ensures that they include the polynomials of \mathcal{G}_{lex} . Thus $\{h_1, \dots, h_t\}$ is a \leq_{lex} -Gröbner basis of \mathcal{I} , and filtering out from it the polynomials which are not in \mathcal{G}_{lex} is easily done and computationally cheap, by following the classical procedure for transforming a non-minimal Gröbner basis into a minimal one. Explicitly:*

- let $v_i = \text{lt}_{\leq_{\text{lex}}}(h_i)$ for $1 \leq i \leq t$;
- find the indices $1 \leq i_1 < \dots < i_s \leq t$ such that $\{v_{i_1}, \dots, v_{i_s}\}$ is a minimal generating set of the monomial ideal $\langle v_1, \dots, v_t \rangle$;
- then $\mathcal{G}_{\text{lex}} = \{h_{i_1}, \dots, h_{i_s}\}$.

Note that the uniqueness of the indices i_1, \dots, i_s is ensured by the fact that v_1, \dots, v_t are pairwise distinct by construction.

5 CONSTRUCTING A BASIS OF THE MODULE FROM A KNOWN GRÖBNER BASIS

There are two missing ingredients in order to use the above framework to compute \mathcal{G}_{lex} . First, the assumptions of Theorem 4.1 must be satisfied. Second, we need an efficient method to compute the basis H of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ in Hermite normal form; for this, known methods require the knowledge of some basis $P \in \mathbb{K}[y]^{t \times t}$ of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$.

The assumption that \mathcal{I} is zero-dimensional will be guaranteed from our context. The main constraint is therefore the choice of \mathcal{T} in order to ensure that $\mathcal{G}_{\text{lex}} \subseteq \mathcal{R}_{\mathcal{T}}$ is satisfied. This relates to the more general equality $\langle \mathcal{M}_{\mathcal{T}, \mathcal{I}} \rangle = \mathcal{I}$, via the following characterization: $\langle \mathcal{M}_{\mathcal{T}, \mathcal{I}} \rangle = \mathcal{I}$ if, and only if, there exists a generating set of \mathcal{I} formed by polynomials in $\mathcal{R}_{\mathcal{T}}$. Obviously, taking \mathcal{T} large enough ensures $\mathcal{G}_{\text{lex}} \subseteq \mathcal{R}_{\mathcal{T}}$; yet a larger set \mathcal{T} also means a larger matrix dimension t and thus more expensive computations to find P and deduce H .

Now, focusing on the shape position case as explained in Section 1, all monomials occurring in \mathcal{G}_{lex} are either in $\{x_{n-1}, \dots, x_1\}$ or in $\{y^e \mid e \geq 0\}$. Hence the following lemma.

LEMMA 5.1. *Using notation from Theorem 4.1, assume the ideal \mathcal{I} is zero-dimensional and in shape position. If $\{1, x_{n-1}, \dots, x_1\} \subseteq \mathcal{T}$, then $\mathcal{G}_{\text{lex}} \subset \mathcal{R}_{\mathcal{T}}$.*

Therefore, in the shape position case, the condition $\mathcal{G}_{\text{lex}} \subseteq \mathcal{R}_{\mathcal{T}}$ is easily satisfied, and the main missing ingredient is an efficient method for computing P . The next theorem shows that, for any monomial order \preceq , the knowledge of some \preceq -border basis of \mathcal{I} [33] directly provides a suitable set \mathcal{T} and a corresponding basis $P \in \mathbb{K}[y]^{t \times t}$ of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$. Furthermore, this matrix P has a particular degree pattern related to the \preceq -monomial basis of \mathcal{I} .

In Corollary 5.4 we deduce that, under the stability assumption $\mathcal{S}(\text{lt}_{\preceq}(\mathcal{I}))$, the knowledge of the reduced \preceq -Gröbner basis of \mathcal{I} is enough to find \mathcal{T} and P . Then, it will only remain to find the Hermite normal form of P , which is the sought basis H : the efficient computation of H from P is discussed in Section 6.1.

THEOREM 5.2. *Let \preceq be a monomial order such that $y < x_i$ for $1 \leq i \leq n-1$. Let \mathcal{I} be a zero-dimensional ideal in \mathcal{R} and let \mathcal{B} be the \preceq -monomial basis of \mathcal{R}/\mathcal{I} . Let $\mathcal{T} = (\mu_1, \dots, \mu_t)$ be the monomials in \mathcal{B} which are not divisible by y , i.e. $\mathcal{T} = \mathcal{B} \cap \mathbb{K}[x_1, \dots, x_{n-1}]$. Then,*

$$\{\mu \in \mathcal{B} \mid y\mu \notin \mathcal{B}\} = \{y^{e_i-1}\mu_i \mid 1 \leq i \leq t\} \quad (5)$$

for some $e_1, \dots, e_t \in \mathbb{Z}_{>0}$ with $e_1 + \dots + e_t = \dim_{\mathbb{K}}(\mathcal{R}/\mathcal{I})$, and

$$\mathcal{P} = \{y^{e_i}\mu_i - \text{nf}_{\preceq, \mathcal{I}}(y^{e_i}\mu_i) \mid 1 \leq i \leq t\}$$

is a basis of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ as a $\mathbb{K}[y]$ -module.

Furthermore, representing \mathcal{P} as a matrix $P \in \mathbb{K}[y]^{t \times t}$ whose i th row contains the coefficients of $y^{e_i}\mu_i - \text{nf}_{\preceq, \mathcal{I}}(y^{e_i}\mu_i)$ on the basis \mathcal{T} of $\mathcal{R}_{\mathcal{T}}$, then $P = \text{diag}(y^{e_1}, \dots, y^{e_t}) + R$ where $R \in \mathbb{K}[y]^{t \times t}$, with the j th column of R of degree less than e_j for $1 \leq j \leq t$.

PROOF. First note that, since \mathcal{I} is zero-dimensional, \mathcal{B} is finite and therefore \mathcal{T} is finite as well. Concerning the identity in Eq. (5) we first observe that, since \mathcal{B} is finite and is the complement of $\text{lt}_{\preceq}(\mathcal{I})$, for each $i \in \{1, \dots, t\}$ there is a unique $e \in \mathbb{Z}_{>0}$ such that $y^{e-1}\mu_i \in \mathcal{B}$ and $y^e\mu_i \notin \mathcal{B}$. Conversely, for any $\mu \in \mathcal{B}$ such that $y\mu \notin \mathcal{B}$, the integer $e = 1 + \max\{j \in \mathbb{N} \mid y^j \text{ divides } \mu\}$ satisfies $y^{1-e}\mu \in \mathcal{T}$, hence $\mu = y^{e-1}\mu_i$ for some i . This shows Eq. (5).

Now, concerning \mathcal{P} , its elements are in \mathcal{I} by definition of the \preceq -normal form (see Section 2), hence $\mathcal{P} \subseteq \mathcal{M}_{\mathcal{T}, \mathcal{I}}$. Furthermore \mathcal{P} has cardinality t , which is the cardinality of \mathcal{T} and therefore the rank of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ (see Lemma 3.4). Thus, to prove that \mathcal{P} is a basis of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ it is sufficient to show that any polynomial f in $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ is a \mathcal{R} -linear combination of \mathcal{P} , that is, $f \in \langle \mathcal{P} \rangle$. Since $f \in \mathcal{M}_{\mathcal{T}, \mathcal{I}}$,

$$\begin{aligned} f &\in \text{Span}_{\mathbb{K}}(\{y^e\mu_i \mid 1 \leq i \leq t, e \in \mathbb{N}\}) \\ &= \text{Span}_{\mathbb{K}}(\mathcal{B} \cup \{y^{e_i+k}\mu_i \mid 1 \leq i \leq t, k \in \mathbb{N}\}). \end{aligned}$$

On the other hand, as showed in Lemma 5.3, $y^{e_i+k}\mu_i - b_{i,k} \in \langle \mathcal{P} \rangle$ for some $b_{i,k} \in \text{Span}_{\mathbb{K}}(\mathcal{B})$, for all $1 \leq i \leq t$ and $k \in \mathbb{N}$. Altogether, this implies that $f = b + p$, for some $b \in \text{Span}_{\mathbb{K}}(\mathcal{B})$ and $p \in \langle \mathcal{P} \rangle$. Since $\langle \mathcal{P} \rangle \subseteq \mathcal{M}_{\mathcal{T}, \mathcal{I}}$, we have $f - p \in \mathcal{M}_{\mathcal{T}, \mathcal{I}} \subseteq \mathcal{I}$ and therefore $b = \text{nf}_{\preceq, \mathcal{I}}(f) = 0$. Hence $f \in \langle \mathcal{P} \rangle$.

Finally, consider the matrix representation P of \mathcal{P} . As seen in Section 3.2, the i th row of P is the vector $[p_1 \ \dots \ p_t] \in \mathbb{K}[y]^{1 \times t}$ such that $y^{e_i}\mu_i - \text{nf}_{\preceq, \mathcal{I}}(y^{e_i}\mu_i) = p_1\mu_1 + \dots + p_t\mu_t$. Therefore all monomials of $p_1\mu_1 + \dots + p_t\mu_t - y^{e_i}\mu_i$ are in \mathcal{B} . By definition of the e_j 's, it follows that $\deg(p_j) < e_j$ for all $j \neq i$, and $\deg(p_i - y^{e_i}) < e_i$.

This shows that the j th column of $R = P - \text{diag}(y^{e_1}, \dots, y^{e_t})$ has degree less than e_j , for $1 \leq j \leq t$. \square

The next lemma uses notation from Theorem 5.2.

LEMMA 5.3. *For all $k \in \mathbb{N}$ and $i \in \{1, \dots, t\}$, $y^{e_i+k}\mu_i - b_{i,k} \in \langle \mathcal{P} \rangle$, where we have defined $b_{i,k} = \text{nf}_{\preceq, \mathcal{I}}(y^{e_i+k}\mu_i) \in \text{Span}_{\mathbb{K}}(\mathcal{B})$.*

PROOF. We prove this by induction on k , noting that this property holds for $k = 0$ by definition of \mathcal{P} . Now, consider $k \in \mathbb{Z}_{>0}$ and suppose the property holds for all integers up to $k-1$. Let $i \in \{1, \dots, t\}$. By induction hypothesis there exists $p \in \langle \mathcal{P} \rangle$ such that $y^{e_i+k-1}\mu_i = b_{i,k-1} + p$. Then $y^{e_i+k}\mu_i = yb_{i,k-1} + yp$, with $yp \in \langle \mathcal{P} \rangle$ and therefore $b_{i,k} = \text{nf}_{\preceq, \mathcal{I}}(y^{e_i+k}\mu_i) = \text{nf}_{\preceq, \mathcal{I}}(yb_{i,k-1})$.

It remains to prove that $yb_{i,k-1}$ is the sum of an element of $\langle \mathcal{P} \rangle$ and one of $\text{Span}_{\mathbb{K}}(\mathcal{B})$ (the latter must then be $b_{i,k}$ by uniqueness). This follows from the facts that

$$yb_{i,k-1} \in \text{Span}_{\mathbb{K}}(y\mathcal{B}) \subseteq \text{Span}_{\mathbb{K}}(\mathcal{B} \cup \{y^{e_j}\mu_j \mid 1 \leq j \leq t\}),$$

and that the elements of \mathcal{P} are $\{y^{e_j}\mu_j - b_{j,0} \mid 1 \leq j \leq t\}$ with $b_{j,0} \in \text{Span}_{\mathbb{K}}(\mathcal{B})$; indeed these imply more precisely that $yb_{i,k-1}$ is the sum of an element of $\text{Span}_{\mathbb{K}}(\mathcal{P})$ and one of $\text{Span}_{\mathbb{K}}(\mathcal{B})$. \square

The next result is a direct consequence of Theorem 5.2.

COROLLARY 5.4. *Using notation from Theorem 5.2, assume further $\mathcal{S}(\text{lt}_{\preceq}(\mathcal{I}))$, let \mathcal{G} be the reduced \preceq -Gröbner basis of \mathcal{I} , and let f_1, \dots, f_s be the elements of \mathcal{G} whose \preceq -leading term is divisible by y . Then $\{f_1, \dots, f_s\}$ is a basis of $\mathcal{M}_{\mathcal{T}, \mathcal{I}}$ as a $\mathbb{K}[y]$ -module.*

PROOF. It suffices to prove that, thanks to $\mathcal{S}(\text{lt}_{\preceq}(\mathcal{I}))$, we have

$$\{y^{e_j}\mu_j \mid 1 \leq j \leq t\} = \{\text{lt}_{\preceq}(f_i) \mid 1 \leq i \leq s\};$$

then $\mathcal{P} = \{f_1, \dots, f_s\}$ follows (and in particular $s = t$).

To prove this identity, we first observe that for $1 \leq i \leq s$, the monomial $\text{lt}_{\preceq}(f_i)$ is divisible by y and does not belong to \mathcal{B} , whereas $y^{-1}\text{lt}_{\preceq}(f_i)$ belongs to \mathcal{B} . Therefore

$$y^{-1}\text{lt}_{\preceq}(f_i) \in \{\mu \in \mathcal{B} \mid y\mu \notin \mathcal{B}\} = \{y^{e_j-1}\mu_j \mid 1 \leq j \leq t\},$$

and $\text{lt}_{\preceq}(f_i) \in \{y^{e_j}\mu_j \mid 1 \leq j \leq t\}$. Hence $f_i \in \mathcal{P}$.

Conversely, for $1 \leq j \leq t$, we want to prove that $y^{e_j}\mu_j = \text{lt}_{\preceq}(f_i)$ for some $i \in \{1, \dots, s\}$. By construction, the monomial $y^{e_j}\mu_j$ is in $\text{lt}_{\preceq}(\mathcal{P}) \subseteq \text{lt}_{\preceq}(\mathcal{I})$. Thus $y^{e_j}\mu_j$ is divisible by $\text{lt}_{\preceq}(f)$ for some $f \in \mathcal{G}$. If $\text{lt}_{\preceq}(f)$ is not divisible by y , then $\text{lt}_{\preceq}(f) \in \text{lt}_{\preceq}(\mathcal{I})$. It follows that $f = f_i$ for some $i \in \{1, \dots, s\}$. Thus $y^{e_j}\mu_j = \mu \text{lt}_{\preceq}(f_i)$ for some monomial μ , which may only involve the variables x_1, \dots, x_{n-1} since $y^{e_j-1}\mu_j$ is in \mathcal{B} and thus cannot be a multiple of $\text{lt}_{\preceq}(f_i)$. If $\mu \neq 1$, there exists $k \in \{1, \dots, n-1\}$ such that x_k divides μ . By $\mathcal{S}(\text{lt}_{\preceq}(\mathcal{I}))$, the monomial $\frac{x_k}{y}\text{lt}_{\preceq}(f_i)$ is in $\text{lt}_{\preceq}(\mathcal{I})$, hence $\frac{\mu}{x_k} \frac{x_k}{y}\text{lt}_{\preceq}(f_i)$ is in $\text{lt}_{\preceq}(\mathcal{I})$ as well. Yet this contradicts the fact that the latter monomial is $y^{-1}\mu \text{lt}_{\preceq}(f_i) = y^{e_j-1}\mu_j$, which is in \mathcal{B} . Thus $\mu = 1$ and then the proof is complete. \square

6 COMPLEXITY AND PERFORMANCE

6.1 Hermite normal form computation

Here we assume that the basis $P \in \mathbb{K}[y]^{t \times t}$ from Theorem 5.2 is known, and we list the best known methods for the computation of its Hermite normal form H .

matrix-matrix product $M \cdot [v_1 \dots v_k]$ versus performing several matrix-vector products Mv_i for $1 \leq i \leq k$.

In our experiments on the block-Wiedemann approach, a block-size k in the range between 64 and 128 appeared as a good compromise. When k is below 64, the benefit from matrix-matrix products remains limited. On the other hand, when k is above 128, although there could still be some gain by further increasing the matrix dimension, this is counter-balanced by the cost of the second step which starts to be non-negligible. This second step is a matrix fraction reconstruction, performed via an approximant basis of a $(2k) \times k$ matrix at order $2D/k$, for which we used PML [28].

Note that, although block-Wiedemann approaches are often used for benefiting from multi-threaded or parallel computations, here only single-threaded performance is considered, and we keep the design of an optimized, multi-threaded implementation of our new change of order algorithm as a future perspective. Indeed, we expect it to also benefit from multi-threading, since the dominant part of its computations consists of multiplication and Gaussian elimination of large-dimension matrices over \mathbb{K} .

We summarize our comparison in Table 2. All computations were performed on a single thread on a computer equipped with INTEL® XEON® GOLD CPU 6246R v4 @ 3.40GHz and 1.5TB of RAM.

The base field is $\mathbb{K} = \mathbb{Z}/p\mathbb{Z}$ with a 30-bit prime modulus p . This choice comes from the fact that many application areas require Gröbner bases computations over large fields. This is the case for problems in multivariate cryptography and number theory [1, 2, 40]. Furthermore, large computations over the rationals $\mathbb{K} = \mathbb{Q}$ boil down to solving several instances over $\mathbb{K} = \mathbb{Z}/p\mathbb{Z}$, through the Chinese Remainder Theorem. We consider the 30-bit prime fields as a base case in this setting, since it allows us both to choose sufficiently many primes for large instances, and to avoid bad primes with higher probability than e.g. 16-bit prime fields. It also seems to be the base case used in computer algebra software like Macaulay2, Maple and Singular and in the state-of-the-art change of order implementation in `msolve` which we compare to.

We observe that our implementation of the new algorithm is always faster than both other implementations, and that the gap is increasing with the size of the instances. For large instances, the speed-up factor is close to 5.

Let us notice that the block-Wiedemann approach also outperforms `msolve` for large sizes, as might be expected, yet only by a very small margin. One explanation can be that NTL does not seem to use AVX2 vectorization techniques for matrix multiplication over a 30-bit prime field, whereas `msolve` does for its matrix-vector products. Investigating this is a future perspective, and incorporating AVX2 may lead to further accelerations for the block-Wiedemann approach, but also for the new algorithm which makes an intensive use of the multiplication of matrices over \mathbb{K} when multiplying univariate polynomial matrices.

Let us recall that when computing over the rationals, the common strategy through Chinese Remainder Theorem is to use F_4 with a tracer for the computation of \mathcal{G}_{drl} modulo each prime: perform a full F_4 algorithm modulo the first prime and *learn* which polynomials are used in the construction of each matrix and remove those that reduce to 0, and those used only in these reductions to 0. This allows one to minimize the computations modulo the subsequent primes. As a practical consequence, FGLM used to be slower than F_4 -tracer,

but this is not the case anymore, we have reestablished a kind of balance. With the above perspective we expect the change of order step to be consistently faster than the F_4 -tracer step. Furthermore, even when computing over a 30-bit prime field, with only F_4 and no tracer, FGLM could take more than 25% of the total time, sometimes even close to 40% (see `rand (4,7)` or `(4,8)`), now it is closer to negligible (often below or close to 10%).

system	$O(D)$	$O(t)$	$O(t^{2-\omega}D)$
<code>rand (n, d)</code> [18, Cor. 5.10]	d^n	$\frac{d^{n-1}}{\sqrt{n}}$	$(d^{n-1})^{3-\omega} n^{\frac{\omega-2}{2}}$
<code>crit (n, 2, p)</code> [7, Thm. 2]	$2^p \binom{n-1}{p-1}$	$\frac{2^p}{\sqrt{p}} \binom{n-2}{p-1}$	$(2^p \binom{n-2}{p-1})^{3-\omega} p^{\frac{\omega-2}{2}}$
<code>crit (n, d, p)</code> [7, Thm. 2]	$d^n \binom{n-1}{p-1}$	$\frac{d^{n-1}}{\sqrt{n-p}} \binom{n-2}{p-1}$	$(d^{n-1} \binom{n-2}{p-1})^{3-\omega}$
<code>symdet</code> $(3, d, m, m-2)$ [19, Prop. 9]	m^3	$m^2 d$	$\frac{m^{7-2\omega}}{d^{\omega-2}}$
<code>symdet</code> $(6, d, m, m-1)$ [19, Prop. 12]	$m^6 d^6$	$m^5 d^5$	$(md)^{16-5\omega}$
<code>symdet</code> $(\binom{m}{2}, d, m, 1)$ [19, Prop. 14]	$2^m d^{\binom{m}{2}}$	$\frac{2^m}{m} d^{\binom{m}{2}-1}$	$(2^m d^{\binom{m}{2}})^{3-\omega} m^{\omega-2}$

Table 1: Simplified asymptotic gain of complexity compared to Sparse-FGLM [17, 18] for systems of polynomial equations of degree d in n variables.

n, d	D	t	Step 1: $\mathcal{G}_{\text{drl}} \approx P$		Step 2: $\mathcal{G}_{\text{lex}} \approx H$		
			<code>msolve</code>	<code>msolve</code>	NTL	PML	
			F_4	F_4 -tr	Wied.	bl-Wied.	HNF
2, 11	2048	462	11.6	1.1	1.2	1.7	0.8
2, 12	4096	924	115.9	8.3	6.5	14.5	5.3
2, 13	8192	1716	970	62	103.6	110	34.8
2, 14	16384	3432	7921	460	1011	880	240
2, 15	32768	6435	61381	3193	7844	6691	1665
2, 16	65536	12870	482515	24523	58744	52709	11359
3, 8	6561	1107	122.6	12.8	23.6	44.7	15.1
3, 9	19683	3139	3552.7	361	1302	1163	314
3, 10	59049	8953	95052	8664	34844	29974	6709
4, 6	4096	580	9.9	2.2	4	8.8	3.5
4, 7	16384	2128	876	128	575	545	157
4, 8	65536	8092	57237	6977	36454	33452	7231

Table 2: Timings in seconds for random square systems in n variables and degree d , over a prime field $\mathbb{Z}/p\mathbb{Z}$ with a 30-bit modulus. F_4 is the algorithm of [13]. “ F_4 -tr” is the tracer algorithm after the initial learn phase [8, Sec. 4.3]. “Wied.” and “bl-Wied.” are the change of order comparison points described in Section 6.3. They use respectively the Wiedemann-based Sparse-FGLM algorithm [18] and a folklore block-Wiedemann variant (see e.g. [27, 43]). “HNF” is the new algorithm, based on Hermite normal form, as described in Section 6.1.

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