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Friedrich Wehrung

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# PROJECTIVE CLASSES AS IMAGES OF ACCESSIBLE FUNCTORS

FRIEDRICH WEHRUNG

ABSTRACT. We are dealing with projective classes (in short PC) over first-order vocabularies with no restrictions on the (possibly infinite) arities of relation or operation symbols. We verify that  $\text{PC}(\mathcal{L}_{\infty\lambda}) = \text{RPC}(\mathcal{L}_{\infty\lambda})$  for any infinite cardinal  $\lambda$ , and that if  $\lambda$  is singular, then  $\text{PC}(\mathcal{L}_{\infty\lambda}) = \text{PC}(\mathcal{L}_{\infty\lambda^+})$ . If  $\lambda$  is regular, then a class of structures over a  $\lambda$ -ary vocabulary is  $\text{PC}(\mathcal{L}_{\infty\lambda})$ -definable iff it is the image of a  $\lambda$ -continuous functor on a  $\lambda$ -accessible category; we also provide separating counterexamples for the non  $\lambda$ -ary case. We prove that many PC classes of structures, previously known not to be closed under elementary equivalence over any  $\mathcal{L}_{\infty\lambda}$ , are not even co-PC over  $\mathcal{L}_{\infty\infty}$ . Those classes arise from diverse contexts including convex  $\ell$ -subgroup lattices of lattice-ordered groups, ideal lattices of rings, nonstable  $K_0$ -theory of rings, coordinatization of sectionally complemented modular lattices, and real spectra of commutative unital rings. For example, the class of posets of finitely generated two-sided ideals of all unital rings is PC but not co-PC over  $\mathcal{L}_{\infty\infty}$ . We also provide a negative solution to a problem, raised in 2011 by Gillibert and the author, asking whether essential surjectivity of a “well-behaved” functor on objects entails its essential surjectivity on diagrams indexed by arbitrary finite posets.

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## 1. INTRODUCTION

**1.1. Intractability results.** There are numerous results stating that a given class  $\mathcal{C}$ , of structures over a given vocabulary (with possibly infinite arities)  $\mathbf{v}$ , is “intractable” in the sense that  $\mathcal{C}$  is not closed under  $\mathcal{L}_{\infty\lambda}$ -elementary equivalence for any infinite cardinal  $\lambda$  (due to a classical result of Chang [4, Proposition 6], this is equivalent to saying that  $\mathcal{C}$  is not the class of models of any class of  $\mathcal{L}_{\infty\lambda}$ -sentences). Many of those results take advantage of  $\mathcal{C}$  being the image of a functor  $\Phi$  defined on a category  $\mathcal{S}$  and taking values in all structures over a given vocabulary, with both  $\mathcal{S}$  and  $\Phi$  “well-behaved” in some sense. Good behavior is here expressed by stating that the category  $\mathcal{S}$  and the functor  $\Phi$  are both *accessible* as defined in Makkai and Paré [22] (cf. Subsection 2.2 for formal definitions). Examples, originating in various references including Goodearl [11], Jónsson [17], Keimel [20], Mundici [25], investigated in Mellor and Tressl [24], Lenzi and Di Nola [21], Wehrung [35, 39], include the following:

- (1)  $\mathcal{S}$  is a large enough accessible category of lattice-ordered groups (in short  $\ell$ -groups) and  $\Phi$  sends every object  $G$  of  $\mathcal{S}$  to its lattice  $\text{Cs}_c G$  of finitely generated convex  $\ell$ -subgroups (resp., its semilattice  $\text{Id}_c G$  of finitely generated  $\ell$ -ideals);
- (2)  $\mathcal{S}$  is a large enough accessible category of rings and  $\Phi$  sends every object  $R$  of  $\mathcal{S}$  to its semilattice  $\text{Id}_c R$  of finitely generated two-sided ideals;
- (3)  $\mathcal{S}$  is a large enough accessible category of unital rings and  $\Phi$  sends every object  $R$  of  $\mathcal{S}$  to the monoid  $\text{V}(R)$  of all isomorphism classes of finitely generated projective right  $R$ -modules;
- (4)  $\mathcal{S}$  is a large enough accessible category of von Neumann regular rings, possibly with additional structure, and  $\Phi$  sends every object  $R$  of  $\mathcal{S}$  to the lattice  $\mathbb{L}(R)$  of its principal right ideals;
- (5)  $\mathcal{S}$  is the category of all commutative unital rings and  $\Phi$  sends every object of  $\mathcal{S}$  to the Stone dual of its real spectrum.

The meaning of “large enough” varies from one context to the other. The corresponding results, described in Section 10, all state a level of intractability that takes the following general form, involving the concept of a *PC-definable class* which we shall discuss in more detail shortly (cf. Subsection 1.2).

**General Intractability Scheme (GIS).** *Classes  $\mathcal{C}_0$  and  $\mathcal{C}_1$  of structures are constructed such that  $\mathcal{C}_0$  is  $\text{PC}(\mathcal{L}_{\infty\infty})$ -definable,  $\mathcal{C}_0 \subseteq \mathcal{C}_1$ , and there is no  $\text{co-PC}(\mathcal{L}_{\infty\infty})$ -definable class  $\mathcal{C}$  such that  $\mathcal{C}_0 \subseteq \mathcal{C} \subseteq \mathcal{C}_1$ .*

**1.2. PC and RPC classes.** We are dealing with classes of structures over (one-sorted) first-order languages, with infinite arities allowed. Standard such classes are provided by  $\sigma$ -complete Boolean algebras or Banach algebras. A less standard such class is provided by Tarski's *cardinal algebras* [32].

For a logic  $\mathcal{L}$  (a formal definition of that concept will not be needed here), recall that a class  $\mathcal{C}$  of  $\mathbf{v}$ -structures is

- *projective over  $\mathcal{L}$*  (in short  $\text{PC}(\mathcal{L})$ ) if it is the class of  $\mathbf{v}$ -reducts of all models of some  $\mathcal{L}$ -sentence over a larger vocabulary  $\mathbf{w}$ ;
- *relatively projective over  $\mathcal{L}$*  (in short  $\text{RPC}(\mathcal{L})$ ) if it is the class of relativizations to  $\mathbf{U}$  of  $\mathbf{v}$ -reducts of all models of some  $\mathcal{L}$ -sentence over a larger vocabulary  $\mathbf{w}$  containing a unary predicate symbol  $\mathbf{U}$ .

(See Definition 3.1 for more detail.) A class is *co-projective* (in abbreviation  $\text{co-PC}$ ) if its complement is projective. A typical example of a  $\text{co-PC}$ , non  $\text{PC}$  (over  $\mathcal{L}_{\infty\infty}$ ) class of structures is the class of all complete Boolean algebras (cf. Cole and Dickmann [5, Theorem 3]). At that point a remark is in order.

*Remark 1.1.* The equation  $\text{PC}(\mathcal{L}_{\infty\omega}) = \text{RPC}(\mathcal{L}_{\infty\omega})$ , stated in Oikkonen [26, Corollary 1.3], is extended in Oikkonen [27, Remark 1.1], with a sketch of proof, to the logics  $\mathcal{L}_{\kappa\lambda}$ , where  $\kappa$  and  $\lambda$  are regular cardinals such that  $\alpha^\beta < \kappa$  whenever  $\alpha < \kappa$  and  $\beta < \lambda$ . However, the given argument, stated there for relational vocabularies, does not extend to vocabularies containing infinitary operation symbols *a priori*:

- the possible presence of many operation symbols, even within a single  $\mathcal{L}_{\kappa\lambda}$ -sentence<sup>1</sup>, runs counter to the possibility of a suitable Löwenheim-Skolem type Theorem as required in the argument of [27, Remark 1.1];
- the possible presence of operation symbols with large arity prevents, in Definition 3.1, the  $\mathbf{v}$ -closedness of  $\mathbf{U}^{\mathbf{N}}$  from being expressible by an  $\mathcal{L}_{\infty\lambda}$ -sentence.

We shall clarify the interaction between those concepts, in light of accessible categories, by establishing, in full generality, the following results:

- $\text{PC}(\mathcal{L}_{\infty\lambda}) = \text{RPC}(\mathcal{L}_{\infty\lambda})$ , for any infinite cardinal  $\lambda$  (cf. Corollary 8.7).
- $\text{PC}(\mathcal{L}_{\infty\lambda}) = \text{PC}(\mathcal{L}_{\infty\lambda^+})$ , for any singular cardinal  $\lambda$  (cf. Theorem 8.6).
- For any regular cardinal  $\lambda$ , every  $\text{PC}(\mathcal{L}_{\infty\lambda})$ -definable class of structures, over a vocabulary  $\mathbf{v}$  in which all operations have arity smaller than  $\lambda$ , is the image of a faithful<sup>2</sup>  $\lambda$ -continuous functor from a  $\lambda$ -accessible category to  $\mathbf{v}$ -structures (cf. Theorem 6.9). The assumption that all operations in  $\mathbf{v}$  have arity smaller than  $\lambda$  cannot be dispensed with (cf. Example 5.7).
- For any regular cardinal  $\lambda$ , any  $\lambda$ -accessible category  $\mathcal{S}$ , and any  $\lambda$ -continuous functor  $\Phi$  from  $\mathcal{S}$  to all structures over a given  $\lambda$ -ary vocabulary  $\mathbf{v}$ , the image of  $\Phi$  is  $\text{PC}(\mathcal{L}_{\infty\lambda})$ -definable (cf. Theorem 7.1). The assumption that  $\mathbf{v}$  be  $\lambda$ -ary cannot be dispensed with (cf. Example 8.8).

<sup>1</sup>Consider the sentence  $(\forall x)(x \neq f(\mathbf{g}_\xi(x) \mid \xi < \theta))$  for large  $\theta$ .

<sup>2</sup>Throughout this work it appears that faithfulness, of the representing functor, marks the advantage of  $\text{PC}$  over  $\text{RPC}$ ; see, in particular, Theorem 6.9.

The latter two results above can also be viewed as extensions, from “elementary” to “projective”, of Adámek and Rosický [1, Theorem 5.44]; see also Adámek and Rosický [1, Theorem 5.35].

In the course of establishing those results, we will observe that *every*  $\text{PC}(\mathcal{L}_{\infty\lambda})$ -definable class of structures, over a vocabulary  $\mathbf{v}$ , is the class of  $\mathbf{v}$ -reducts of models of a sentence  $\mathbf{E}$  over a vocabulary  $\mathbf{w}$  containing  $\mathbf{v}$  such that  $\mathbf{w} \setminus \mathbf{v}$  is  $\lambda$ -ary; moreover,  $\mathbf{E}$  can be taken a conjunction of universal  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ -sentences (cf. Theorem 4.11 and Proposition 5.4).<sup>3</sup>

**1.3. GIS and images of accessible functors.** In the context of Item (2) in Subsection 1.1 above, our General Intractability Scheme GIS yields that the class of posets of finitely generated ideals of unital rings (resp., unital von Neumann regular rings) is not  $\text{co-PC}(\mathcal{L}_{\infty\infty})$ -definable (cf. Theorem 10.3). More generally, in GIS, all interesting instances of  $\mathcal{C}$  and  $\mathcal{C}_0$  turn out to be images of accessible functors, thus  $\text{PC}(\mathcal{L}_{\infty\infty})$ -definable (thus lending some optimality to the scheme). The proof of GIS then runs as follows. If  $\mathcal{C}$  were  $\text{co-PC}(\mathcal{L}_{\infty\infty})$ -definable, then Tuuri’s Interpolation Theorem from [34] would imply the existence of a sentence, in the infinitely deep language  $\mathcal{M}_{\infty\infty}$ , whose class of models is intermediate between  $\mathcal{C}_0$  and  $\mathcal{C}$ , thus between  $\mathcal{C}_0$  and  $\mathcal{C}_1$ . By a result of Karttunen [19], that class of models is closed under a suitable concept of back-and-forth equivalence  $\simeq_{\lambda}$  (cf. Definition 9.1) for large enough  $\lambda$ . This would in turn contradict earlier results by the author [39], employing an extension of the concept of “condensate” introduced in Gillibert and Wehrung [10], and entailing the existence of  $\simeq_{\lambda}$ -equivalent models, one in  $\mathcal{C}_0$  and the other in the complement of  $\mathcal{C}_1$ .

In the specific context of  $\ell$ -ideal lattices of  $\ell$ -groups, we extend, in Section 11, this method to an *a priori* non  $\text{co-PC}$  lattice theoretical condition, that we call *Ploščica’s Condition*, valid in  $\text{Id}_c G$  for every Abelian  $\ell$ -group  $G$  (cf. Proposition 11.5). In particular, we verify, under the Generalized Continuum Hypothesis GCH, that Ploščica’s Condition, together with all currently known  $\mathcal{L}_{\infty\infty}$ -sentences satisfied by all lattices of finitely generated  $\ell$ -ideals in Abelian  $\ell$ -groups, is still not sufficient to characterize the latter objects (we get a counterexample of cardinality  $\aleph_4$ ).

The construction, in Tůma and Wehrung [33], of a diagram  $\mathcal{D}_{\boxtimes}$  of finite Boolean  $(\vee, 0)$ -semilattices that cannot be lifted by any diagram of congruence lattices of (among others) majority algebras<sup>4</sup>, raised the hope for a negative solution to the well known open question asking whether every distributive algebraic lattice is the congruence lattice of a majority algebra. That hope was also supported by the condensate construction from Gillibert and Wehrung [10], introduced as a means to turn essential surjectivity, of a given functor, from *objects* to *diagrams indexed by finite lattices*. However, the above-mentioned diagram  $\mathcal{D}_{\boxtimes}$  is indexed by a variant of the six-element bounded poset  $P_0$  represented on the left hand side of Figure 12.2; *that poset is not a lattice*. This was the original motivation for [10, Problem 1], which asked for an extension of [10, Theorem 3.4.2] (called there CLL) to arbitrary bounded posets. In Section 12 we produce a counterexample for that problem. Roughly speaking, this counterexample consists of a faithful, finitely

<sup>3</sup>The latter result would not follow from some hypothetical infinitary translation of Skolem normal form reduction *a priori*: for one thing, even the less demanding prenex normal form reduction may fail in  $\mathcal{L}_{\infty\lambda}$  (cf. Dickmann [6, Fact 1.1.1]).

<sup>4</sup>A *majority algebra* is a set  $X$  endowed with a ternary operation  $m$  such that  $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$  whenever  $x, y \in X$ .

accessible functor  $\Phi$  from a (finitary, congruence-distributive) variety  $\mathcal{V}$  of algebras to the category  $\mathbf{DSLat}^0$  of all distributive  $(\vee, 0)$ -semilattices with  $(\vee, 0)$ -homomorphisms, such that  $\Phi$  is essentially surjective on objects but not (in a strong sense) on  $P_0$ -indexed diagrams.

## 2. BASIC CONCEPTS

**2.1. Sets, ordinals, cardinals, posets.** Like for many works involving proper classes, a natural axiomatic environment for this work is Bernays-Gödel set theory (cf. Jech [14, p. 70]) with the axiom of choice for sets. The cardinality of a set  $X$  will be denoted by  $\text{card } X$ . For a cardinal  $\kappa$ , a set  $X$ , or, more generally, a structure with universe  $X$ , is  $\kappa$ -small if  $\text{card } X < \kappa$ ; we denote by  $[X]^{<\kappa}$  the set of all  $\kappa$ -small subsets of  $X$ . We denote set exponentiation by  $(X, Y) \mapsto {}^X Y$  and cardinal exponentiation by  $(\alpha, \beta) \mapsto \alpha^\beta$ . We set  $\alpha^{\text{cs}} \stackrel{\text{def}}{=} \sup\{\alpha^\gamma \mid \gamma < \kappa\}$  whenever  $\alpha$  and  $\kappa$  are cardinals. We denote by  $\text{dom } f$  and  $\text{rng } f$  the domain and the range of a function  $f$ , respectively. Disjoint unions will be denoted in the form  $X \sqcup Y$ .

For every element  $a$  in a poset  $P$ , we shall denote by  $P \downarrow a$ , or  $\downarrow a$  if  $P$  is understood, the set  $\{x \in P \mid x \leq a\}$ . A subset  $X$  of  $P$  is a *lower subset* of  $P$  if  $P \downarrow x \subseteq X$  whenever  $x \in X$ . If, in addition,  $P = (P, \vee, 0)$  is a join-semilattice with least element  $0$  (in abbreviation a  $(\vee, 0)$ -semilattice), a nonempty lower subset  $I$  of  $P$  is an *ideal* if it closed under finite joins. Then  $I$  induces a join-congruence  $\equiv_I$  on  $P$ , by letting  $x \equiv_I y$  hold if there exists  $z \in I$  such that  $x \vee z = y \vee z$ . The quotient  $(\vee, 0)$ -semilattice is then denoted by  $P/I$ . A poset  $(P, \leq)$  is a *tree* if it has a least element and every principal ideal  $P \downarrow a$ , where  $a \in P$ , is well-ordered.

**2.2. Categories and functors.** For more details we refer the reader to Adámek and Rosický [1].

Every category  $\mathcal{S}$  will be identified with its class of arrows (aka morphisms), and we denote by  $\text{Ob } \mathcal{S}$  its class of objects (aka identities). If  $\varphi: A \rightarrow B$  is a morphism in  $\mathcal{S}$ , we write  $A = \mathbf{d}(\varphi)$  and  $B = \mathbf{r}(\varphi)$ . Commutative diagrams, indexed by a poset  $P$ , will be denoted in the form  $\vec{S} = (S_{p_1}, \sigma_{p_1, p_2} \mid p_1 \leq p_2 \text{ in } P)$  (where each  $\sigma_{p_1, p_2}: S_{p_1} \rightarrow S_{p_2}$ ). A cocone above  $\vec{S}$  will then be denoted in the form  $(S, \sigma_p \mid p \in P)$ , with the usual commutation relations (e.g.,  $\sigma_{p_1} = \sigma_{p_2} \circ \sigma_{p_1, p_2}$  whenever  $p_1 \leq p_2$  in  $P$ ) assumed. Colimit cocones will be written in the form

$$(S, \sigma_p \mid p \in P) = \varinjlim (S_{p_1}, \sigma_{p_1, p_2} \mid p_1 \leq p_2 \text{ in } P) \quad \text{within } \mathcal{S}$$

(this notation defines  $(S, \sigma_p \mid p \in P)$  up to isomorphism).

Let  $\lambda$  be a regular cardinal. A subcategory  $\mathcal{S}^\dagger$  of a category  $\mathcal{S}$  is  $\lambda$ -directed colimit-dense if every object of  $\mathcal{S}$  is the colimit, within  $\mathcal{S}$ , of a  $\lambda$ -directed commutative diagram of  $\mathcal{S}^\dagger$ . The category  $\mathcal{S}$  is  $\lambda$ -accessible if it has all  $\lambda$ -directed colimits and it has a  $\lambda$ -directed colimit-dense full small subcategory of  $\lambda$ -presentable objects; then we denote by  $\mathbf{Pres}_\lambda \mathcal{S}$  any set of representatives, relatively to isomorphy, for the  $\lambda$ -presentable objects of  $\mathcal{S}$ , and we may take  $\mathcal{S}^\dagger = \mathbf{Pres}_\lambda \mathcal{S}$ . If, in addition,  $\mathcal{S}$  is cocomplete, we say that  $\mathcal{S}$  is *locally  $\lambda$ -presentable*. A functor  $\Phi: \mathcal{S} \rightarrow \mathcal{T}$  is  $\lambda$ -continuous (resp.,  $\lambda$ -accessible) if it preserves  $\lambda$ -directed colimits (resp., it is  $\lambda$ -continuous and  $\mathcal{S}$  and  $\mathcal{T}$  are both  $\lambda$ -accessible categories). We will omit the prefix “ $\lambda$ -” if it is existentially quantified over  $\lambda$ , and replace it by “finitely” in case  $\lambda = \omega$ : for instance, “accessible” means “ $\lambda$ -accessible for some  $\lambda$ ” and “finitely accessible” means “ $\omega$ -accessible”.

Slightly deviating from the terminology in use in Gillibert and Wehrung [10] and Wehrung [39], we define, for any functor  $\Phi: \mathcal{A} \rightarrow \mathcal{B}$ ,

- the *range* of  $\Phi$ ,  $\text{rng } \Phi \stackrel{\text{def}}{=} \{\Phi(A) \mid A \in \text{Ob } \mathcal{A}\}$ ,
- the *image* of  $\Phi$ ,  $\text{im } \Phi = \Phi(\mathcal{A}) \stackrel{\text{def}}{=} \{B \in \text{Ob } \mathcal{B} \mid (\exists A \in \text{Ob } \mathcal{A})(B \cong \Phi(A))\}$ .

Recall that  $\Phi$  is *essentially surjective* (on objects) if  $\mathcal{B} = \Phi(\mathcal{A})$ .

We denote by **Set** the category of all sets with maps.

**2.3. Infinitary languages.** We are dealing with first-order structures over vocabularies with possibly infinite arities (cf. Adámek and Rosický [1, Ch. 5]). A vocabulary  $\mathbf{v}$  consists of a set of relation and operation symbols, with which it is occasionally identified, together with the arity map  $\mathbf{s} \mapsto \text{ar}(\mathbf{s})$ , usually (but not always, see Sections 4 and 6) with values in ordinals, nonzero on relation symbols. We denote by  $\text{Var}$  the set of all variables, by  $\#\mathbf{v}$  the number of symbols in  $\mathbf{v}$ , and by  $\mathbf{v}_{\text{cst}}$ ,  $\mathbf{v}_{\text{ope}}$ , and  $\mathbf{v}_{\text{rel}}$  the sub-vocabularies of  $\mathbf{v}$  consisting of the constant symbols, operation symbols, and relation symbols, respectively. Further, we set  $\overline{\text{ar}}(\mathbf{s}) \stackrel{\text{def}}{=} \text{card var}(\mathbf{s})$  for every symbol  $\mathbf{s}$  of  $\mathbf{v}$ . For a cardinal  $\lambda$ , the vocabulary  $\mathbf{v}$  is  $\lambda$ -*ary* if  $\overline{\text{ar}}(\mathbf{s}) < \lambda$  for every symbol  $\mathbf{s}$  of  $\mathbf{v}$ . Syntactical objects such as formulas or symbols of variable, relation, operation will usually<sup>5</sup> be denoted in sans serif fonts (e.g.,  $E, x, R, f$ ), while their semantic counterparts, if defined, will be denoted in math fonts (e.g.  $x, R, f$ ). We shall denote by  $\text{var}(\mathbf{s})$  (resp.,  $\text{fvar}(\mathbf{s})$ ) the set of all variables (resp., free variables) occurring in a term or formula  $\mathbf{s}$ .

We denote by  $\infty$  an extra object set on top of all cardinal numbers. Whenever  $\lambda \leq \kappa \leq \infty$ , the *preformulas* of the infinitary language  $\mathcal{L}_{\kappa\lambda}(\mathbf{v})$  are obtained by closing the atomic formulas of  $\mathbf{v}$  under negations (denoted in the form  $\neg E$ ), disjunctions of less than  $\kappa$  formulas (denoted in the form  $\bigvee_{i \in I} E_i$ ), and existential quantifiers over  $\lambda$ -small sets of variables (denoted in the form  $(\exists X)E$ ). *Formulas* are preformulas with less than  $\lambda$  free variables. Conjunctions and universal quantifiers are defined as usual, *via*  $\bigwedge = \neg \bigvee \neg$  and  $\forall = \neg \exists \neg$ , respectively. This choice of  $\neg$ ,  $\bigvee$ , and  $\exists$  as primitive connectives affects the definition of a subformula, as for example  $\exists x \neg E(x)$  is a subformula of  $\forall x E(x)$ . This is of direct importance in the definition of the sentences  $\widehat{E}$  introduced in Notation 5.1. Indexed sets of variables will be denoted in the form  $\vec{x}_{(I)}$  where  $\vec{x} = (x_i \mid i \in I)$  is a one-to-one map from  $I$  into the variable symbols. Quantifiers, if applied to indexed sets of variables, will be denoted in the form  $Q \vec{x}_{(I)}$  (or just  $Qx$  if  $I$  is a singleton), where  $Q$  is either  $\exists$  or  $\forall$ .

The universe  $M$  of a  $\mathbf{v}$ -structure  $\mathbf{M} = (M, \dots)$  will be denoted by  $|\mathbf{M}|$ . For a sub-vocabulary  $\mathbf{u}$  of  $\mathbf{v}$ , we denote by  $\mathbf{M}|_{\mathbf{u}}$  the  $\mathbf{u}$ -reduct of  $\mathbf{M}$ . The interpretation of a symbol  $\mathbf{s}$  of  $\mathbf{v}$  in  $\mathbf{M}$  will be denoted by  $\mathbf{s}^{\mathbf{M}}$ . Homomorphisms, embeddings, and isomorphisms of  $\mathbf{v}$ -structures are defined the usual way; for example, a  $\mathbf{v}$ -homomorphism  $\varphi: \mathbf{M} \rightarrow \mathbf{N}$  is an embedding iff it is one-to-one and  $\vec{x} \in R^{\mathbf{M}} \Leftrightarrow \varphi \vec{x} \in R^{\mathbf{N}}$  whenever  $R \in \mathbf{v}_{\text{rel}}$  and  $\vec{x} \in \text{ar}^{(R)} M$ .

We denote by  $\mathbf{Str}(\mathbf{v})$  the category whose objects are the  $\mathbf{v}$ -structures and whose arrows are the  $\mathbf{v}$ -homomorphisms. The category  $\mathbf{Str}(\mathbf{v})$  is locally presentable (cf. Adámek and Rosický [1, Theorem 5.30]). We will often need the description of  $\lambda$ -directed colimits, in  $\mathbf{Str}(\mathbf{v})$ , given as follows.

<sup>5</sup>With the exception of the unary predicate symbol  $U$  involved in the definition of a relatively projective class in Definition 3.1.

**Lemma 2.1** (Folklore). *Let  $\lambda$  be a regular cardinal and let  $\mathbf{v}$  be a vocabulary with  $\mathbf{v}_{\text{ope}}$   $\lambda$ -ary. Let  $P$  be a  $\lambda$ -directed poset and let  $\vec{M} = (M_p, \varphi_{p,q} \mid p \leq q \text{ in } P)$  be a  $P$ -indexed commutative diagram in  $\mathbf{Str}(\mathbf{v})$ . Then a cocone  $(M, \varphi_p \mid p \in P)$  above  $\vec{M}$  is the colimit of  $\vec{M}$  iff*

- (i)  $M = \bigcup \{\varphi_p[M_p] \mid p \in P\}$ ;
- (ii) for all  $p \in P$  and all  $x, y \in M_p$ ,  $\varphi_p(x) = \varphi_p(y)$  implies that there exists  $q \geq p$  such that  $\varphi_{p,q}(x) = \varphi_{p,q}(y)$ ;
- (iii)  $R^M = \bigcup \{\varphi_p R^{M_p} \mid p \in P\}$  whenever  $R \in \mathbf{v}_{\text{rel}}$ . If  $\overline{\text{ar}}(\mathbf{R}) < \lambda$ , then this amounts to verifying that for all  $p \in P$  and  $(x_\xi \mid \xi \in \text{ar}(\mathbf{R})) \in {}^{\text{ar}(\mathbf{R})}M_p$ ,
 
$$(\varphi_p(x_\xi) \mid \xi \in \text{ar}(\mathbf{R})) \in R^M \implies (\exists q \geq p)((\varphi_{p,q}(x_\xi) \mid \xi \in \text{ar}(\mathbf{R})) \in R^{M_q}).$$

### 3. PROJECTIVE AND RELATIVELY PROJECTIVE CLASSES

The following encompasses the classical definitions of PC and RPC classes (cf. Ebbinghaus [7]) stated for vocabularies without restrictions on arities.

**Definition 3.1.** Let  $\mathcal{L}$  be a logic of the form either  $\mathcal{L}_{\infty\lambda}$ , for an infinite cardinal  $\lambda$ , or  $\mathcal{L}_{\infty\infty}$ , and let  $\mathbf{v}$  be a vocabulary. A class  $\mathcal{C}$  of  $\mathbf{v}$ -structures is

- *abstract* if every isomorphic copy of a member of  $\mathcal{C}$  belongs to  $\mathcal{C}$ ;
- *small-abstract* if it is the class of isomorphic copies of all members of a set of  $\mathbf{v}$ -structures;
- *projective over  $\mathcal{L}$* , in short  $\text{PC}(\mathcal{L})$ -*definable* or just  $\text{PC}(\mathcal{L})$ , if there are a vocabulary  $\mathbf{w}$  extending  $\mathbf{v}$  and a sentence  $E$  from  $\mathcal{L}(\mathbf{w})$  such that

$$\mathcal{C} = \{N \upharpoonright \mathbf{v} \mid N \in \mathbf{Mod}_{\mathbf{w}}(E)\}; \quad (3.1)$$

- *relatively projective over  $\mathcal{L}$* , in short  $\text{RPC}(\mathcal{L})$ -*definable* or just  $\text{RPC}(\mathcal{L})$ , if there are a vocabulary  $\mathbf{w}$  extending  $\mathbf{v}$ , a unary predicate symbol  $U$  in  $\mathbf{w}$ , and a sentence  $E$  of  $\mathcal{L}(\mathbf{w})$  such that

$$\mathcal{C} = \{U^N \upharpoonright \mathbf{v} \mid N \in \mathbf{Mod}_{\mathbf{w}}(E), U^N \text{ is } \mathbf{v}\text{-closed}\} \quad (3.2)$$

(where “ $\mathbf{v}$ -closed” means closed, within  $N$ , under all operations from  $\mathbf{v}_{\text{ope}}$ . Note that the closedness of  $U^N$  under  $\mathbf{v}$  cannot be expressed by an  $\mathcal{L}_{\infty\lambda}$ -sentence unless  $\mathbf{v}_{\text{ope}}$  is  $\lambda$ -ary; in the latter case, that closedness statement can of course be incorporated into  $E$ );

- *co-PC*( $\mathcal{L}$ ) if its complement  $\mathbf{Str}(\mathbf{v}) \setminus \mathcal{C}$  is  $\text{PC}(\mathcal{L})$ .

As discussed in Ebbinghaus [7], both concepts of PC and RPC can be defined for most logics. However, as illustrated in Oikkonen [27], the resulting concepts are sensitive to the restrictions put on the vocabularies  $\mathbf{v}$  and  $\mathbf{w}$ . That possible ambiguity is addressed in Definition 3.1 by removing all the possible restrictions on  $\mathbf{v}$  and  $\mathbf{w}$ . In particular, even in case  $\mathcal{L} = \mathcal{L}_{\infty\lambda}$ ,  $\mathbf{v}$  may be  $\lambda$ -ary whereas  $\mathbf{w}$  is not. The resulting remaining ambiguity in the definitions of PC and RPC will be addressed in Theorem 4.11: *if  $\mathbf{v}$  is  $\lambda$ -ary, then we may take  $\mathbf{w}$   $\lambda$ -ary as well.*

Of course, every PC class is also RPC. It is well known that the converse fails for various logics, including classical first-order logic  $\mathcal{L}_{\omega\omega}$ ; see for example Oikkonen [26]. We will see (cf. Corollary 8.7) that the equality  $\text{PC}(\mathcal{L}_{\infty\lambda}) = \text{RPC}(\mathcal{L}_{\infty\lambda})$  holds in full generality, over any vocabulary.

The well known fact that the isomorphism class of any model is  $\text{PC}(\mathcal{L}_{\infty\omega})$  is stated in Oikkonen [26, page 260]. We include that observation in the following lemma, along with proofs, for convenience.

**Lemma 3.2.** *The following statements holds, for any vocabulary  $\mathbf{v}$ :*

- (1) *Let  $\lambda$  be an infinite cardinal and let  $(\mathcal{C}_i \mid i \in I)$  be a collection, indexed by a set  $I$ , of classes of  $\mathbf{v}$ -structures. If each  $\mathcal{C}_i$  is  $\text{PC}(\mathcal{L}_{\infty\lambda})$ , then so are  $\bigcup_{i \in I} \mathcal{C}_i$  and  $\bigcap_{i \in I} \mathcal{C}_i$ .*
- (2) *Every small-abstract class of  $\mathbf{v}$ -structures is both  $\text{PC}(\mathcal{L}_{\infty\omega})$  (via a single universal  $\mathcal{L}_{\infty\omega}$ -sentence) and  $\text{co-PC}(\mathcal{L}_{\infty\omega})$ .*

*Proof.* *Ad (1).* We may assume that there are vocabularies  $\mathbf{v}_i \supseteq \mathbf{v}$ , pairwise intersecting in  $\mathbf{v}$ , with  $\mathcal{L}_{\infty\lambda}(\mathbf{v}_i)$ -sentences  $\mathbf{E}_i$  such that each  $\mathcal{C}_i = \{\mathbf{N} \upharpoonright_{\mathbf{v}} \mid \mathbf{N} \in \text{Mod}_{\mathbf{v}_i}(\mathbf{E}_i)\}$ .

Setting  $\mathbf{v}^* \stackrel{\text{def}}{=} \bigcup_{i \in I} \mathbf{v}_i$ ,  $\bigcup_{i \in I} \mathcal{C}_i$  is the class of  $\mathbf{v}$ -reducts of all models of the  $\mathbf{v}^*$ -sentence  $\bigvee_{i \in I} \mathbf{E}_i$  and  $\bigcap_{i \in I} \mathcal{C}_i$  is the class of  $\mathbf{v}$ -reducts of all models of  $\bigwedge_{i \in I} \mathbf{E}_i$ .

*Ad (2).* By (1) (and its proof) it suffices to settle the case of the isomorphism class  $\mathcal{C}$  of a single  $\mathbf{v}$ -structure  $\mathbf{M}$ . Letting  $\mathcal{C}$  be a set of constant symbols, outside  $\mathbf{v}$ , indexing  $M$ ,  $\mathcal{C}$  is the class of  $\mathbf{v}$ -reducts of the universal  $\mathcal{L}_{\infty\omega}(\mathbf{v} \cup \mathcal{C})$ -sentence obtained as the conjunction of the diagram of  $\mathbf{M}$  (each element of  $M$  being indexed by the corresponding element of  $\mathcal{C}$ ) and the universal  $\mathcal{L}_{\infty\omega}(\mathcal{C})$ -sentence stating that every element belongs to  $\mathcal{C}$ . Thus  $\mathcal{C}$  is  $\text{PC}(\mathcal{L}_{\infty\omega})$ -definable.

Denote by  $\leq$  a binary relation symbol. There are  $\mathcal{L}_{\infty\omega}(\leq)$ -sentences  $\mathbf{E}_\alpha$ , ranging over all ordinals  $\alpha$ , such that a structure  $(P, \leq)$  satisfies  $\mathbf{E}_\alpha$  iff  $\leq$  is a linear order and  $(\alpha, \leq)$  embeds into  $(P, \leq)$  as a lower subset (cf. Dickmann [6, p. 319]). It follows that for every cardinal  $\kappa$ , the class of all  $\mathbf{v}$ -structures of cardinality at least  $\kappa$  is the class of all  $\mathbf{v}$ -reducts of the  $\mathcal{L}_{\infty\omega}(\mathbf{v} \cup \{\leq\})$ -sentence  $\mathbf{E}_\kappa$ ; whence it is  $\text{PC}(\mathcal{L}_{\infty\omega})$ .

Letting  $\kappa > \text{card } M$ , the complement of  $\mathcal{C}$  is the union of the class of all  $\mathbf{v}$ -structures of cardinality at least  $\kappa$  (which is  $\text{PC}(\mathcal{L}_{\infty\omega})$ ) as observed in the paragraph above) and the small-abstract class of all  $\kappa$ -small structures outside  $\mathcal{C}$  (which is  $\text{PC}(\mathcal{L}_{\infty\omega})$  by the above); whence it is  $\text{PC}(\mathcal{L}_{\infty\omega})$ . Now apply (1).  $\square$

#### 4. A FIRST CASE OF CONJUGACY: PARTIAL $\lambda$ -ARY TRUNCATING

For an infinite cardinal  $\lambda$  and a vocabulary  $\mathbf{v}$ , any  $\text{PC}(\mathcal{L}_{\infty\lambda})$  class of  $\mathbf{v}$ -structures is the class of  $\mathbf{v}$ -reducts of all models of some  $\mathcal{L}_{\infty\lambda}$ -sentence in a larger vocabulary  $\mathbf{w}$  (cf. Definition 3.1). No restriction is set on the arities of the symbols in either  $\mathbf{v}$  or  $\mathbf{w}$  *a priori*. We shall verify in this section that  $\mathbf{w}$  may be taken in such a way that  $\mathbf{w} \setminus \mathbf{v}$  is  $\lambda$ -ary; a similar result holds for RPC (cf. Theorem 4.11).

Our basic idea is quite straightforward: to every  $\lambda$ -ary *term*  $s$  (resp.,  $\lambda$ -ary *atomic formula*  $\mathbf{E}$ ) of  $\mathbf{w}$ , with root outside  $\mathbf{v}$ , we shall associate a new  $\lambda$ -ary *operation symbol*  $\dot{s}$  (resp., *relation symbol*  $\dot{\mathbf{E}}$ ), meant to interpret the original  $s$  (resp.,  $\mathbf{E}$ ). The arity of  $\dot{s}$  (resp.,  $\dot{\mathbf{E}}$ ) is the set of all variables of  $s$  (resp.,  $\mathbf{E}$ ) — so it is usually not an ordinal. This way, every  $\mathbf{w}$ -structure  $\mathbf{M}$  is “conjugate” to a unique structure  $\mathbf{N}$ , with the same universe, over the new vocabulary  $\tilde{\mathbf{w}} = \text{col}_\lambda(\mathbf{w}, \mathbf{v})$  (cf. Lemma 4.7). The  $\tilde{\mathbf{w}}$ -structures  $\mathbf{N}$  arising, in this way, from some  $\mathbf{w}$ -structure  $\mathbf{M}$  are exactly those satisfying a “coherence sentence”  $\mathbf{G}$  (cf. Lemma 4.8). The translation process from  $\mathbf{w}$  to  $\tilde{\mathbf{w}}$  does not affect symbols from  $\mathbf{v} \cup \mathbf{w}_{\text{cst}}$ .

**Notation 4.1.** For a vocabulary  $\mathbf{w}$  and an infinite cardinal  $\lambda$ , we shall denote by  $\text{T}_\lambda(\mathbf{w})$  (resp.,  $\text{A}_\lambda(\mathbf{w})$ ) the set of all terms (resp., atomic formulas) of  $\mathbf{w}$  on less than  $\lambda$  variables. Hence,  $\text{A}_\lambda(\mathbf{w})$  is the set of all atomic formulas in  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ .

**Definition 4.2.** We set  $\tilde{s} \stackrel{\text{def}}{=} s$  whenever  $s$  is either a variable or a constant symbol from  $\mathbf{w}$ . In case  $s = f(s_\xi \mid \xi \in \text{ar}(f)) \in \text{T}_\lambda(\mathbf{w})$  is non-constant, we define a term  $\tilde{s}$ ,

over a possibly larger vocabulary, by

$$\tilde{s} \stackrel{\text{def}}{=} \begin{cases} f(\tilde{s}_\xi \mid \xi \in \text{ar}(f)), & \text{if } f \in \mathbf{v}, \\ \dot{s}(u \mid u \in \text{var}(s)) & \text{(usually written as } \dot{s}), \text{ otherwise,} \end{cases} \quad (4.1)$$

where, in the second case (i.e.,  $f \notin \mathbf{v}$ ),  $\dot{s}$  is a new operation symbol with arity  $\text{var}(s)$ .

Let  $E \in A_\lambda(\mathbf{w})$ . If  $E$  has the form  $s = t$ , where  $s, t \in T_\lambda(\mathbf{w})$ , then we define  $\tilde{E}$  as  $\tilde{s} = \tilde{t}$ . If  $E$  does not have that form (let us then say that  $E$  is *relational*), then  $E = R(s_\xi \mid \xi \in \text{ar}(R))$  for  $R \in \mathbf{w}_{\text{rel}}$  and terms  $s_\xi \in T_\lambda(\mathbf{w})$  with less than  $\lambda$  variables altogether; we then define an atomic formula  $\tilde{E}$ , with the same variables as  $E$  but over a possibly larger vocabulary, by

$$\tilde{E} \stackrel{\text{def}}{=} \begin{cases} R(\tilde{s}_\xi \mid \xi \in \text{ar}(R)), & \text{if } R \in \mathbf{v}, \\ \dot{E}(u \mid u \in \text{var}(E)) & \text{(usually written as } \dot{E}), \text{ otherwise,} \end{cases} \quad (4.2)$$

where, in the second case (i.e.,  $R \notin \mathbf{v}$ ),  $\dot{E}$  is a new relation symbol with arity  $\text{var}(E)$ .

The common vocabulary  $\tilde{\mathbf{w}} = \text{col}_\lambda(\mathbf{w}, \mathbf{v})$  for all those syntactical objects is

$\mathbf{v} \cup \mathbf{w}_{\text{cst}} \cup \{\dot{s} \mid s \text{ as in the second case of (4.1)}\} \cup \{\dot{E} \mid E \text{ as in the second case of (4.2)}\}$ .

In particular,  $\mathbf{w} \cap \tilde{\mathbf{w}} = \mathbf{v} \cup \mathbf{w}_{\text{cst}}$ .

Since all additional symbols  $\dot{s}$  and  $\dot{E}$  have  $\lambda$ -small arity, we obtain:

**Lemma 4.3.** *In the context above,  $\text{var}(\tilde{s}) = \text{var}(s)$  and  $\text{var}(\tilde{E}) = \text{var}(E)$ . Moreover,  $\tilde{\mathbf{w}} \setminus \mathbf{v}$  is  $\lambda$ -ary.*

**Notation 4.4** (Coherence sentence). Denote by  $G$  the conjunction of all sentences of the following form:

$$\left( \forall_{(\text{var}(s))} \vec{x} \right) \left( \forall_{(\text{var}(t))} \vec{y} \right) \left( \bigwedge_{\xi \in \text{ar}(f)} (\tilde{s}_\xi(\vec{x}) = \tilde{t}_\xi(\vec{y})) \Rightarrow \tilde{s}(\vec{x}) = \tilde{t}(\vec{y}) \right), \quad (4.3)$$

ranging over all pairs  $(s, t)$  of terms in  $T_\lambda(\mathbf{w})$  of the form  $s = f(s_\xi \mid \xi \in \text{ar}(f))$  and  $t = f(t_\xi \mid \xi \in \text{ar}(f))$  (i.e., with the same “root”  $f$ ), and

$$\left( \forall_{(\text{var}(E))} \vec{x} \right) \left( \forall_{(\text{var}(F))} \vec{y} \right) \left[ \left( \tilde{E}(\vec{x}) \wedge \bigwedge_{\xi \in \text{ar}(f)} (\tilde{s}_\xi(\vec{x}) = \tilde{t}_\xi(\vec{y})) \right) \Rightarrow \tilde{F}(\vec{y}) \right], \quad (4.4)$$

ranging over all pairs  $(E, F)$  of atomic formulas in  $A_\lambda(\mathbf{w})$  of the form  $E = R(s_\xi \mid \xi \in \text{ar}(R))$  and  $F = R(t_\xi \mid \xi \in \text{ar}(R))$  (i.e., with the same “root”  $R$ ). By construction,  $G$  is an  $\mathcal{L}_{\infty\lambda}(\tilde{\mathbf{w}})$ -sentence.

Here and elsewhere, in order to avoid clutter, we will often allow for dummy variables, for example writing  $\tilde{s}_\xi(\vec{x})$  instead of  $\tilde{s}_\xi(\vec{x} \upharpoonright_{\text{var}(s_\xi)})$ , and so on.

We shall now establish a correspondence between  $\mathbf{w}$ -structures and  $\tilde{\mathbf{w}}$ -structures.

**Definition 4.5.** Let  $M$  be a  $\mathbf{w}$ -structure and let  $N$  be a  $\tilde{\mathbf{w}}$ -structure. We say that  $(M, N)$  is a *conjugate pair*, or, equivalently, that  $M$  is *left conjugate to*  $N$ , respectively that  $N$  is *right conjugate to*  $M$ , if the following statements hold:

- (1)  $M \upharpoonright_{\mathbf{v} \cup \mathbf{w}_{\text{cst}}} = N \upharpoonright_{\mathbf{v} \cup \mathbf{w}_{\text{cst}}}$  (in particular,  $M = N$ );
- (2) For every non-constant term  $s = f(s_\xi \mid \xi \in \text{ar}(f)) \in T_\lambda(\mathbf{w})$  with  $f \notin \mathbf{v}$ ,

$$(\forall \vec{a} \in \text{var}(s) M) (\dot{s}^N(\vec{a}) = s^M(\vec{a})). \quad (4.5)$$

(3) For every atomic formula  $E = R(s_\xi \mid \xi \in \text{ar}(R)) \in A_\lambda(\mathbf{w})$  with  $R \notin \mathbf{v}$ ,

$$(\forall \vec{a} \in \text{var}^{(E)}M)(\vec{a} \in \dot{E}^N \iff \mathbf{M} \models E(\vec{a})). \quad (4.6)$$

**Lemma 4.6.** *The following statements hold, for any conjugate pair  $(\mathbf{M}, \mathbf{N})$ :*

- (1) *The equation  $s^{\mathbf{M}}(\vec{a}) = \tilde{s}^{\mathbf{N}}(\vec{a})$  holds, for all  $s \in T_\lambda(\mathbf{w})$  and all  $\vec{a} \in \text{var}^{(s)}M$ .*
- (2) *The equivalence  $\mathbf{N} \models \tilde{E}(\vec{a}) \iff \mathbf{M} \models E(\vec{a})$  holds, for all  $E \in A_\lambda(\mathbf{w})$  and all  $\vec{a} \in \text{var}^{(E)}M$ .*
- (3)  *$\mathbf{N} \models G$  (cf. Notation 4.4).*

*Proof.* Ad (1). We argue by induction on the term  $s$ . The result is obvious if  $s$  is either a constant (cf. Definition 4.5(1)) or a variable. Let  $s = f(s_\xi \mid \xi \in \text{ar}(f))$  be non-constant. If  $f \in \mathbf{v}$ , then  $f^{\mathbf{M}} = f^{\mathbf{N}}$  (cf. Definition 4.5(1)) so

$$\begin{aligned} \tilde{s}^{\mathbf{N}}(\vec{a}) &= f^{\mathbf{M}}(\tilde{s}_\xi^{\mathbf{N}}(\vec{a}) \mid \xi \in \text{ar}(f)) && \text{(cf. (4.1))} \\ &= f^{\mathbf{M}}(s_\xi^{\mathbf{M}}(\vec{a}) \mid \xi \in \text{ar}(f)) && \text{(by induction hypothesis)} \\ &= s^{\mathbf{M}}(\vec{a}). \end{aligned}$$

If  $f \notin \mathbf{v}$ , then a direct application of (4.5) yields the relation  $\tilde{s}^{\mathbf{N}}(\vec{a}) = s^{\mathbf{M}}(\vec{a})$ .

Ad (2). By (1) above, the given statement is obvious in case  $E$  has the form  $s = t$ . Now let  $E = R(s_\xi \mid \xi \in \text{ar}(R))$ . If  $R \in \mathbf{v}$ , then  $R^{\mathbf{M}} = R^{\mathbf{N}}$  (cf. Definition 4.5(1)) so

$$\begin{aligned} \mathbf{N} \models \tilde{E}(\vec{a}) &\Leftrightarrow (\tilde{s}_\xi^{\mathbf{N}}(\vec{a}) \mid \xi \in \text{ar}(R)) \in R^{\mathbf{M}} && \text{(cf. (4.2))} \\ &\Leftrightarrow (s_\xi^{\mathbf{M}}(\vec{a}) \mid \xi \in \text{ar}(R)) \in R^{\mathbf{M}} && \text{(by (1) above)} \\ &\Leftrightarrow \mathbf{M} \models E(\vec{a}). \end{aligned}$$

If  $R \notin \mathbf{v}$ , then a direct application of (4.6) yields that  $\mathbf{N} \models \tilde{E}(\vec{a})$  iff  $\mathbf{M} \models E(\vec{a})$ .

Ad (3). We start with the sentences of the form (4.3). In that context, let  $\vec{a} \in \text{var}^{(s)}M$  and  $\vec{b} \in \text{var}^{(t)}M$  such that  $\tilde{s}_\xi^{\mathbf{N}}(\vec{a}) = \tilde{t}_\xi^{\mathbf{N}}(\vec{b})$  whenever  $\xi \in \text{ar}(f)$ . By (1) above, this means that  $s_\xi^{\mathbf{M}}(\vec{a}) = t_\xi^{\mathbf{M}}(\vec{b})$  whenever  $\xi \in \text{ar}(f)$ . By applying  $f^{\mathbf{M}}$ , we get  $s^{\mathbf{M}}(\vec{a}) = t^{\mathbf{M}}(\vec{b})$ , thus, by (1) above,  $\tilde{s}^{\mathbf{N}}(\vec{a}) = \tilde{t}^{\mathbf{N}}(\vec{b})$ , as required.

We now deal with sentences of the form (4.4). In that context, let  $\vec{a} \in \text{var}^{(E)}M$  and  $\vec{b} \in \text{var}^{(F)}M$  such that  $\mathbf{N} \models \tilde{E}(\vec{a})$  whereas  $\tilde{s}_\xi^{\mathbf{N}}(\vec{a}) = \tilde{t}_\xi^{\mathbf{N}}(\vec{b})$  whenever  $\xi \in \text{ar}(R)$ . By (1) and (2) above, this means that  $\mathbf{M} \models E(\vec{a})$  whereas  $s_\xi^{\mathbf{M}}(\vec{a}) = t_\xi^{\mathbf{M}}(\vec{b})$  whenever  $\xi \in \text{ar}(R)$ . Therefore,  $\mathbf{M} \models F(\vec{b})$ , thus, by (2) above,  $\mathbf{N} \models \tilde{F}(\vec{b})$ , as required.  $\square$

**Lemma 4.7.** *Every  $\mathbf{w}$ -structure is left conjugate to a unique  $\tilde{\mathbf{w}}$ -structure.*

*Proof.* The missing interpretations  $\dot{s}^{\mathbf{N}}$  and  $\dot{E}^{\mathbf{N}}$  are necessarily given by (4.5) and (4.6), respectively. They define the  $\tilde{\mathbf{w}}$ -structure  $\mathbf{N}$  uniquely.  $\square$

**Lemma 4.8.** *Every  $\tilde{\mathbf{w}}$ -structure  $\mathbf{N}$ , satisfying the statement  $\mathbf{G}$ , is right conjugate to some  $\mathbf{w}$ -structure  $\mathbf{M}$ .*

*Proof.* We need to define the missing interpretations in  $\mathbf{M}$  of all symbols in  $\mathbf{w} \setminus (\mathbf{v} \cup \mathbf{w}_{\text{cst}})$ , in such a way that all instances of (4.5) and (4.6) hold. We start with operation symbols in  $\mathbf{w} \setminus \mathbf{v}$ .

**Claim 1.** *For every  $f \in \mathbf{w}_{\text{ope}}$ , there exists a map  $f^{\mathbf{M}}: \text{ar}(f)M \rightarrow M$ , equal to  $f^{\mathbf{N}}$  in case  $f \in \mathbf{v}_{\text{ope}}$ , such that for every term of the form  $s = f(s_\xi \mid \xi \in \text{ar}(f))$  in  $T_\lambda(\mathbf{w})$  and every  $\vec{a} \in \text{var}^{(s)}M$ ,*

$$f^{\mathbf{M}}(\tilde{s}_\xi^{\mathbf{N}}(\vec{a}) \mid \xi \in \text{ar}(f)) = \tilde{s}^{\mathbf{N}}(\vec{a}). \quad (4.7)$$

*Proof of Claim.* In order to prove that (4.7) correctly defines a partial function  $f^M$ , we need to verify that for all terms of the form  $\mathbf{s} = f(\mathbf{s}_\xi \mid \xi \in \text{ar}(f))$  and  $\mathbf{t} = f(\mathbf{t}_\xi \mid \xi \in \text{ar}(f))$  in  $\mathbf{T}_\lambda(\mathbf{w})$ , all  $\vec{a} \in \text{var}(\mathbf{s})M$ , and all  $\vec{b} \in \text{var}(\mathbf{t})M$ , the conjunction over all  $\xi \in \text{ar}(f)$  of  $\tilde{\mathbf{s}}_\xi^N(\vec{a}) = \tilde{\mathbf{t}}_\xi^N(\vec{b})$  implies  $\tilde{\mathbf{s}}^N(\vec{a}) = \tilde{\mathbf{t}}^N(\vec{b})$ . This follows from our assumption that  $\mathbf{N}$  satisfies  $\mathbf{G}$ .

Now suppose that  $f \in \mathbf{v}_{\text{ope}}$ . Then  $\tilde{\mathbf{s}} = f(\tilde{\mathbf{s}}_\xi \mid \xi \in \text{ar}(f))$  (cf. (4.1)), thus

$$\tilde{\mathbf{s}}^N(\vec{a}) = f^N(\tilde{\mathbf{s}}_\xi^N(\vec{a}) \mid \xi \in \text{ar}(f)) . \quad \square \text{ Claim 1.}$$

This yields a  $(\mathbf{v} \cup \mathbf{w}_{\text{ope}})$ -structure, specializing the intended  $\mathbf{M}$  thus which we will also temporarily denote by  $\mathbf{M}$ , satisfying all equations (4.7).

**Claim 2.** *The relation  $\mathbf{s}^M(\vec{a}) = \tilde{\mathbf{s}}^N(\vec{a})$  holds, for every  $\mathbf{s} \in \mathbf{T}_\lambda(\mathbf{w})$  and every  $\vec{a} \in \text{var}(\mathbf{s})M$ .*

*Proof of Claim.* We argue by induction on the term  $\mathbf{s}$ . The given conclusion is trivial if  $\mathbf{s}$  is either a variable or a constant. Let  $\mathbf{s} = f(\mathbf{s}_\xi \mid \xi \in \text{ar}(f))$  be non-constant. Then

$$\begin{aligned} \mathbf{s}^M(\vec{a}) &= f^M(\mathbf{s}_\xi^M(\vec{a}) \mid \xi \in \text{ar}(f)) \\ &= f^M(\tilde{\mathbf{s}}_\xi^N(\vec{a}) \mid \xi \in \text{ar}(f)) && \text{(by our induction hypothesis)} \\ &= \tilde{\mathbf{s}}^N(\vec{a}) && \text{(cf. (4.7)).} \end{aligned} \quad \square \text{ Claim 2.}$$

Now let  $\mathbf{s} = f(\mathbf{s}_\xi \mid \xi \in \text{ar}(f))$  be non-constant with  $f \notin \mathbf{v}$ , and let  $\vec{a} \in \text{var}(\mathbf{s})M$ . Using Claim 2, we get  $\mathbf{s}^M(\vec{a}) = \tilde{\mathbf{s}}^N(\vec{a})$ . By (4.1),  $\tilde{\mathbf{s}}^N(\vec{a}) = \dot{\mathbf{s}}^N(\vec{a})$ . Therefore, all statements of the form (4.5) hold.

It remains to construct interpretations  $\mathbf{R}^M$ , for  $\mathbf{R} \in \mathbf{w}_{\text{rel}} \setminus \mathbf{v}$ , in such a way that all statements of the form (4.6) hold. This means that whenever  $\mathbf{E} = \mathbf{R}(\mathbf{s}_\xi \mid \xi \in \text{ar}(\mathbf{R}))$  in  $\mathbf{A}_\lambda(\mathbf{w})$  and  $\vec{a} \in \text{var}(\mathbf{E})M$ , the equivalence

$$(\mathbf{s}_\xi^M(\vec{a}) \mid \xi \in \text{ar}(\mathbf{R})) \in \mathbf{R}^M \iff \vec{a} \in \dot{\mathbf{E}}^N$$

holds. In order for those to yield a consistent definition of  $\mathbf{R}^M$ , we need to verify that whenever  $\mathbf{F} = \mathbf{R}(\mathbf{t}_\xi \mid \xi \in \text{ar}(\mathbf{R})) \in \mathbf{A}_\lambda(\mathbf{w})$  and  $\vec{b} \in \text{var}(\mathbf{F})M$  such that

$$\mathbf{s}_\xi^M(\vec{a}) = \mathbf{t}_\xi^M(\vec{b}) \quad \text{whenever } \xi \in \text{ar}(\mathbf{R}), \quad (4.8)$$

the equivalence  $\vec{a} \in \dot{\mathbf{E}}^N \iff \vec{b} \in \dot{\mathbf{F}}^N$  — in other words (cf. (4.2)),  $\mathbf{N} \models (\tilde{\mathbf{E}}(\vec{a}) \iff \tilde{\mathbf{F}}(\vec{b}))$  — holds. Now from (4.8) and Claim 2 it follows that

$$\tilde{\mathbf{s}}_\xi^N(\vec{a}) = \tilde{\mathbf{t}}_\xi^N(\vec{b}) \quad \text{whenever } \xi \in \text{ar}(\mathbf{R}). \quad (4.9)$$

Since  $\mathbf{N}$  satisfies  $\mathbf{G}$ , it also satisfies the desired  $\tilde{\mathbf{E}}(\vec{a}) \iff \tilde{\mathbf{F}}(\vec{b})$ .  $\square$

**Notation 4.9.** We extend the definition of  $\tilde{\mathbf{E}}$  (cf. (4.2)) from all formulas in  $\mathbf{A}_\lambda(\mathbf{w})$  to all formulas in  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ , by setting  $\mathbf{E} = \neg\mathbf{F} \Rightarrow \tilde{\mathbf{E}} = \neg\tilde{\mathbf{F}}$ ;  $\mathbf{E} = \bigvee_{i \in I} \mathbf{E}_i \Rightarrow \tilde{\mathbf{E}} = \bigvee_{i \in I} \tilde{\mathbf{E}}_i$ ;  $\mathbf{E} = (\exists \mathbf{X})\mathbf{F} \Rightarrow \tilde{\mathbf{E}} = (\exists \mathbf{X})\tilde{\mathbf{F}}$ .

A straightforward induction argument, based on Lemma 4.6(2), then yields the following.

**Lemma 4.10.** *Let  $(\mathbf{M}, \mathbf{N})$  be a conjugate pair. Then the equivalence  $\mathbf{N} \models \tilde{\mathbf{E}}(\vec{a}) \iff \mathbf{M} \models \mathbf{E}(\vec{a})$  holds, for all  $\mathbf{E} \in \mathcal{L}_{\infty\lambda}(\mathbf{w})$  and all  $\vec{a} \in \text{fvar}(\mathbf{E})M$ .*

**Theorem 4.11.** *Let  $\mathbf{v}$  be a vocabulary and let  $\lambda$  be an infinite cardinal. Then a class  $\mathcal{C}$  of  $\mathbf{v}$ -structures is  $\text{PC}(\mathcal{L}_{\infty\lambda})$  (resp.,  $\text{RPC}(\mathcal{L}_{\infty\lambda})$ ) iff there are a vocabulary  $\mathbf{w}$  containing  $\mathbf{v}$  (resp.,  $\mathbf{v} \cup \{U\}$  for a unary predicate  $U$ ), such that  $\mathbf{w} \setminus \mathbf{v}$  is  $\lambda$ -ary, together with a sentence  $E$  in  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ , such that  $\mathcal{C} = \{N \upharpoonright \mathbf{v} \mid N \in \text{Mod}_{\mathbf{w}}(E)\}$  (resp.,  $\mathcal{C} = \{U^N \upharpoonright \mathbf{v} \mid N \in \text{Mod}_{\mathbf{w}}(E), U^N \text{ v-closed}\}$ ). In particular, if  $\mathbf{v}$  is  $\lambda$ -ary, then so is  $\mathbf{w}$ .*

*Proof.* We present the argument for RPC; the argument for PC is similar. By definition, there are a unary predicate symbol  $U$ , a vocabulary  $\mathbf{w} \supseteq \mathbf{v} \cup \{U\}$ , and a sentence  $E$  in  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$  such that

$$\mathcal{C} = \{U^M \upharpoonright \mathbf{v} \mid M \in \text{Mod}_{\mathbf{w}}(E), U^M \text{ is } \mathbf{v}\text{-closed}\}$$

Set  $\tilde{\mathbf{w}} \stackrel{\text{def}}{=} \text{col}_{\lambda}(\mathbf{w}, \mathbf{v} \cup \{U\})$ . Since  $\tilde{\mathbf{w}} \setminus \mathbf{v}$  is  $\lambda$ -ary, it thus suffices to prove that  $\mathcal{C} = \mathcal{C}'$  where

$$\mathcal{C}' \stackrel{\text{def}}{=} \{U^N \upharpoonright \mathbf{v} \mid N \in \text{Mod}_{\tilde{\mathbf{w}}}(\tilde{E} \wedge G), U^N \text{ is } \mathbf{v}\text{-closed}\}$$

Let  $A \in \mathcal{C}$ . By definition,  $A = U^M \upharpoonright \mathbf{v}$  for some  $M \in \text{Mod}_{\mathbf{w}}(E)$  such that  $U^M$  is  $\mathbf{v}$ -closed. By Lemma 4.7,  $M$  has a right conjugate  $N \in \text{Str}(\tilde{\mathbf{w}})$ . By Lemmas 4.6(3) and 4.10,  $N$  satisfies  $\tilde{E} \wedge G$ ; whence  $A = U^M \upharpoonright \mathbf{v} = U^N \upharpoonright \mathbf{v} \in \mathcal{C}'$ . Conversely, let  $A \in \mathcal{C}'$ . Then  $A = U^N \upharpoonright \mathbf{v}$  for some  $N \in \text{Mod}_{\tilde{\mathbf{w}}}(\tilde{E} \wedge G)$  such that  $U^N$  is  $\mathbf{v}$ -closed. By Lemma 4.8,  $N$  has a left conjugate  $M$ . By Lemma 4.10 and since  $N \models \tilde{E}$ , we get  $M \models E$ ; whence  $A = U^N \upharpoonright \mathbf{v} = U^M \upharpoonright \mathbf{v} \in \mathcal{C}$ .  $\square$

## 5. AN $\mathcal{L}_{\infty\lambda}$ VERSION OF SKOLEM NORMAL FORM

We shall begin this section by verifying that the  $\text{PC}(\mathcal{L}_{\infty\lambda})$  (or  $\text{RPC}(\mathcal{L}_{\infty\lambda})$ ) character of a given class of  $\mathbf{v}$ -structures can be witnessed by a *conjunction of universal  $\mathcal{L}_{\infty\lambda}$ -sentences*. (This contrasts with the usual Skolem normal form reduction in first-order logic, which produces a single universal sentence.) This will lead, under the assumption that  $\mathbf{v}_{\text{ope}}$  be  $\lambda$ -ary, to a representation of any  $\text{RPC}(\mathcal{L}_{\infty\lambda})$  class as the image of a  $\lambda$ -continuous functor on a  $\lambda$ -accessible category (cf. Theorem 5.5).

Up to Lemma 5.3 we shall fix an infinite cardinal  $\lambda$  and a vocabulary  $\mathbf{w}$ .

**Notation 5.1.** For any formula  $E \in \mathcal{L}_{\infty\lambda}(\mathbf{w})$ , we construct inductively a quantifier-free formula  $\bar{E} \in \mathcal{L}_{\infty\lambda}$ , over a vocabulary containing  $\mathbf{w}$ , as follows. If  $E$  is atomic, let  $\bar{E} \stackrel{\text{def}}{=} E$ . If  $E = \bigvee_{i \in I} E_i$ , let  $\bar{E} \stackrel{\text{def}}{=} \bigvee_{i \in I} \bar{E}_i$ . If  $E = \neg F$ , let  $\bar{E} \stackrel{\text{def}}{=} \neg \bar{F}$ . If  $E(\vec{x})$  is  $(\exists \vec{y})F(\vec{x}, \vec{y})$  (call such formulas  $\exists$ -*prefixed*), let  $\bar{E}(\vec{x}) \stackrel{\text{def}}{=} \bar{F}(\vec{x}, \vec{f}_{\bar{E}}(\vec{x}))$ , where  $\vec{f}_{\bar{E}}(\vec{x}) \stackrel{\text{def}}{=} (f_{\bar{E},j}(\vec{x}) \mid j \in J)$  with new operation symbols  $f_{\bar{E},j}$  (meant as Skolem functions)<sup>6</sup>. Observe that  $\bar{E}$  is quantifier-free, with the same free variables as  $E$ . Denote by  $\mathbf{w}[E]$  the union of its vocabulary with  $\mathbf{w}$ .

For any  $\mathcal{L}_{\infty\lambda}$ -formula  $E(\vec{x}) = (\exists \vec{y})F(\vec{x}, \vec{y})$ , denote by  $\text{sk}_E$  the following universal  $\mathcal{L}_{\infty\lambda}$ -sentence of  $\mathbf{w}[E]$ :

$$(\forall \vec{x})(\forall \vec{y}) \left( \bar{F}(\vec{x}, \vec{y}) \Rightarrow \bar{F}(\vec{x}, \vec{f}_{\bar{E}}(\vec{x})) \right). \quad (\text{sk}_E)$$

The conjunction  $\hat{E}$  of the  $\text{sk}_F$ , over all  $\exists$ -prefixed subformulas  $F$  of  $E$ , is thus a conjunction of universal sentences from  $\mathcal{L}_{\infty\lambda}(\mathbf{w}[E])$ .

<sup>6</sup>Strictly speaking,  $I = \text{fvar}(E)$  and  $J = \text{fvar}(F) \setminus I$ . For better readability, we keep the notation involving indexed sets of variables.

A standard existence argument for Skolem expansions yields the following.

**Lemma 5.2.** *Every  $\mathbf{w}$ -structure expands to some  $\mathbf{w}[E]$ -structure satisfying  $\widehat{E}$ .*

**Lemma 5.3.** *Let  $E \in \mathcal{L}_{\infty\lambda}(\mathbf{w})$  and let  $F$  be a subformula of  $E$ . Then every  $\mathbf{w}[E]$ -structure satisfies<sup>7</sup> the implication  $\widehat{E} \Rightarrow (F \Leftrightarrow \overline{F})$ .*

*Proof.* By induction on  $F$ . The atomic case and the  $\bigvee, \neg$  steps are all straightforward (we use the observation that  $\models (\widehat{G} \Rightarrow \widehat{F})$  whenever  $F$  is a subformula of  $G$ ). Let  $F(\vec{x})$  be  $(\exists \vec{y})G(\vec{x}, \vec{y})$ , let  $M \in \mathbf{Mod}_{\mathbf{w}[E]}(\widehat{E})$ , and let  $\vec{a} \in {}^I M$ .

Suppose first that  $M \models \overline{F}(\vec{a})$ . This means that  $M \models \overline{G}(\vec{a}, \vec{b})$  where  $\vec{b} \stackrel{\text{def}}{=} \vec{f}_F^M(\vec{a})$ . By induction hypothesis,  $M \models G(\vec{a}, \vec{b})$ ; whence  $M \models F(\vec{a})$ .

Suppose, conversely, that  $M \models F(\vec{a})$ , that is,  $M \models G(\vec{a}, \vec{b})$  for some  $\vec{b}$ . By induction hypothesis,  $M \models \overline{G}(\vec{a}, \vec{b})$ . Since  $M \models \text{sk}_F$ , it follows that  $M \models \overline{G}(\vec{a}, \vec{f}_F(\vec{a}))$ ; that is,  $M \models \overline{F}(\vec{a})$ .  $\square$

**Proposition 5.4.** *Let  $\mathbf{v}$  be a vocabulary and let  $\lambda$  be an infinite cardinal. Then a class  $\mathcal{C}$  of  $\mathbf{v}$ -structures is  $\text{RPC}(\mathcal{L}_{\infty\lambda})$  (resp.,  $\text{PC}(\mathcal{L}_{\infty\lambda})$ ) iff there are a vocabulary  $\mathbf{w}$  containing  $\mathbf{v}$ , with  $\mathbf{w} \setminus \mathbf{v}$   $\lambda$ -ary and a unary predicate symbol  $U$  for the RPC case, together with a conjunction  $E$  of universal  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ -sentences, such that*

$$\mathcal{C} = \{U^M \upharpoonright_{\mathbf{v}} \mid M \in \mathbf{Mod}_{\mathbf{w}}(E), U^M \text{ } \mathbf{v}\text{-closed}\}, \quad (5.1)$$

(resp.,  $\mathcal{C} = \{M \upharpoonright_{\mathbf{v}} \mid M \in \mathbf{Mod}_{\mathbf{w}}(E)\}$ ).

*Proof.* We deal with the statement on RPC; the argument for PC is similar. By Theorem 4.11, there exists a vocabulary  $\mathbf{w}$  containing  $\mathbf{v} \cup \{U\}$ , with  $\mathbf{w} \setminus \mathbf{v}$   $\lambda$ -ary, together with a sentence  $E$  in  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ , such that (5.1) holds. Note that  $\mathbf{w}[E] \setminus \mathbf{w}$  consists only of operations with  $\lambda$ -small arities (viz. the  $f_{F,j}$ ). By Lemma 5.2, every  $\mathbf{w}$ -structure  $M$  expands to some  $N \in \mathbf{Mod}_{\mathbf{w}[E]}(\widehat{E})$ . Further, by Lemma 5.3,  $N \models \overline{E}$  iff  $N \models E$ , equivalently  $M \models E$ . Therefore,

$$\mathcal{C} = \{U^N \upharpoonright_{\mathbf{v}} \mid N \in \mathbf{Mod}_{\mathbf{w}[E]}(\widehat{E} \wedge \overline{E}), U^N \text{ } \mathbf{v}\text{-closed}\}.$$

Observe that  $\overline{E}$  is universal, whereas  $\widehat{E}$  is a conjunction of universal sentences.  $\square$

**Theorem 5.5.** *Let  $\lambda$  be a regular cardinal, let  $\mathbf{v}$  be a vocabulary with  $\mathbf{v}_{\text{ope}}$   $\lambda$ -ary, and let  $\mathcal{C}$  be an  $\text{RPC}(\mathcal{L}_{\infty\lambda})$  class of  $\mathbf{v}$ -structures. Then there are a  $\lambda$ -accessible category  $\mathcal{S}$  and a  $\lambda$ -continuous functor  $\Phi: \mathcal{S} \rightarrow \mathbf{Str}(\mathbf{v})$  such that  $\mathcal{C} = \text{im } \Phi = \text{rng } \Phi$ . Moreover,  $\mathcal{S}$  can be taken a full subcategory, closed under  $\lambda$ -directed colimits, of  $\mathbf{Str}(\mathbf{w})$  with  $\mathbf{w}$ -embeddings, for some vocabulary  $\mathbf{w}$  extending  $\mathbf{v}$  with  $\mathbf{w} \setminus \mathbf{v}$   $\lambda$ -ary.*

*Note.* We will see in Theorem 6.9 that the functor  $\Phi$  can be taken faithful.

*Proof.* By Proposition 5.4, there are a vocabulary  $\mathbf{w}$ , containing  $\mathbf{v} \cup \{U\}$  with  $\mathbf{w} \setminus \mathbf{v}$   $\lambda$ -ary, together with a conjunction  $E$  of universal  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ -sentences, such that (5.1) holds. Set  $\mathcal{S} \stackrel{\text{def}}{=} \mathbf{Mod}_{\mathbf{w}}(E)$ , with  $\mathbf{w}$ -embeddings as morphisms.

We claim that  $\mathcal{S}$  is  $\lambda$ -accessible. First observe that since  $E$  is a conjunction of universal sentences of  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ ,  $\mathbf{w}_{\text{ope}}$  is  $\lambda$ -ary, and by Lemma 2.1,  $\mathcal{S}$  has all  $\lambda$ -directed colimits, which are, up to isomorphism, the  $\lambda$ -directed unions. Since every member of  $\mathcal{S}$  is the  $\lambda$ -directed union of its  $\lambda$ -generated substructures (i.e., those generated by

<sup>7</sup>With the usual convention that  $\models F(\vec{x})$  means  $\models (\forall \vec{x})F(\vec{x})$ .

a  $\lambda$ -small subset), every  $\lambda$ -presentable member of  $\mathcal{S}$  is  $\lambda$ -generated. Let, conversely,  $\mathbf{A}$  be a  $\lambda$ -generated submodel of a  $\lambda$ -directed union  $\mathbf{M} = \bigcup_{p \in P} \mathbf{M}_p$  in  $\mathcal{S}$ . Owing to the facts that  $\mathbf{A}$  is  $\lambda$ -generated,  $M = \bigcup_{p \in P} M_p$ , and  $P$  is  $\lambda$ -directed, there exists  $p \in P$  such that  $A \subseteq M_p$ . We shall prove that  $\mathbf{R}^{\mathbf{A}} = \text{ar}^{(\mathbf{R})} A \cap \mathbf{R}^{M_p}$  whenever  $\mathbf{R} \in \mathbf{w}_{\text{rel}}$ . For all  $q \geq p$ ,  $\mathbf{M}_p$  is a submodel of  $\mathbf{M}_q$  thus  $\mathbf{R}^{M_p} = \text{ar}^{(\mathbf{R})} M_p \cap \mathbf{R}^{M_q}$ . Since  $\mathbf{R}^M = \bigcup_{q \in P} \mathbf{R}^{M_q}$ , it follows that  $\mathbf{R}^{M_p} = \text{ar}^{(\mathbf{R})} M_p \cap \mathbf{R}^M$ . Therefore,  $\mathbf{R}^{\mathbf{A}} = \text{ar}^{(\mathbf{R})} A \cap \mathbf{R}^M = \text{ar}^{(\mathbf{R})} A \cap \mathbf{R}^{M_p}$ , as desired. This proves that the  $\lambda$ -presentable members of  $\mathcal{S}$  are exactly its  $\lambda$ -generated members. Since those are obviously  $\lambda$ -directed colimit-dense in  $\mathcal{S}$ , it follows that  $\mathcal{S}$  is  $\lambda$ -accessible.

For any  $\mathbf{N} \in \text{Ob } \mathcal{S}$ , define  $\Phi(\mathbf{N}) \stackrel{\text{def}}{=} U^{\mathbf{N}} \downarrow_{\mathbf{v}}$ . For any arrow  $\alpha: \mathbf{M} \hookrightarrow \mathbf{N}$  within  $\mathcal{S}$  (so  $\alpha$  is a  $\mathbf{w}$ -embedding), define  $\Phi(\alpha)$  as the induced map from  $U^{\mathbf{M}}$  into  $U^{\mathbf{N}}$ . By Lemma 2.1,  $\Phi$  preserves  $\lambda$ -directed colimits. By (5.1),  $\mathcal{C} = \text{rng } \Phi = \text{im } \Phi$ .  $\square$

*Remark 5.6.* We do not know whether  $\mathbf{v}_{\text{ope}}$  being  $\lambda$ -ary implies that  $\mathbf{Str}(\mathbf{v})$  is  $\lambda$ -accessible, especially in presence of relation symbols with arity beyond  $\lambda$ . If this were the case, then the qualifier “ $\lambda$ -continuous”, applied to the functor  $\Phi$ , could be improved to “ $\lambda$ -accessible” in the statement of Theorem 5.5. On the other hand, the functor  $\Phi$  constructed in the proof of Theorem 5.5 is  $\lambda$ -accessible when viewed as a functor from  $\mathcal{S}$  to the category of all  $\mathbf{v}$ -structures *with  $\mathbf{v}$ -embeddings*.

The following example shows that the assumption that  $\mathbf{v}_{\text{ope}}$  be  $\lambda$ -ary cannot be dispensed with in the statement of Theorem 5.5.

**Example 5.7** (The Idempotent). *Let  $\lambda$  be a regular cardinal and let the vocabulary  $\mathbf{u}_\lambda$  consist of the single operation symbol  $f$  with  $\text{ar}(f) = \lambda$ . Set  $\mathcal{J}_\lambda \stackrel{\text{def}}{=} \mathbf{Mod}_{\mathbf{u}_\lambda}(\mathbf{J}_\lambda)$  where  $\mathbf{J}_\lambda$  is the  $\mathcal{L}_{\infty\lambda}(\mathbf{u}_\lambda)$ -sentence  $(\forall x)(f(x, x, \dots) = x)$ . Then  $\mathcal{J}_\lambda$  is not the image of any  $\lambda$ -continuous functor from any  $\lambda$ -accessible category to  $\mathbf{Str}(\mathbf{u}_\lambda)$ .*

*Proof.* We shall in fact prove the following stronger statement (note the formal analogy with part of the statement of the forthcoming Theorem 7.1):

*Let  $\mathcal{S}$  be a category, with a subcategory  $\mathcal{S}^\dagger$  whose objects form a set, and let  $\Phi: \mathcal{S} \rightarrow \mathbf{Str}(\mathbf{u}_\lambda)$  be a functor such that*

- (1) *every object  $S$  of  $\mathcal{S}$  is a colimit of a  $\lambda$ -directed commutative diagram  $\vec{S}$  in  $\mathcal{S}^\dagger$  such that  $\Phi(S) = \varinjlim \Phi(\vec{S})$ ;*
- (2)  *$\mathcal{S}$  has all colimits from  $\mathcal{S}^\dagger$  indexed by the poset  $\lambda$ , and those colimits are preserved by  $\Phi$ ;*
- (3)  *$\text{im } \Phi$  is contained in  $\mathcal{J}_\lambda$ .*

*Then  $\text{im } \Phi$  is small-abstract.*

(This indeed implies the statement of Example 5.7: take  $\mathcal{S}^\dagger \stackrel{\text{def}}{=} \mathbf{Pres}_\lambda \mathcal{S}$ .)

Suppose, to the contrary, that  $\mathcal{S}$ ,  $\mathcal{S}^\dagger$ , and  $\Phi$  satisfy (1)–(3) above, with  $\text{im } \Phi$  not small-abstract. Let  $\kappa$  be an upper bound for all the cardinalities of  $|\Phi(S)|$  for  $S \in \text{Ob } \mathcal{S}^\dagger$ , with  $\kappa \geq \lambda$ . Since  $\text{im } \Phi$  is not small-abstract, there exists an object  $S$  of  $\mathcal{S}$  such that  $(M, f) \stackrel{\text{def}}{=} \Phi(S) \in \text{rng } \Phi$  with  $\kappa^\lambda < \text{card } M$ . The statement that  $(M, f) \in \mathcal{J}_\lambda$  means that  $f$  is idempotent, that is,  $f(x, x, \dots) = x$  whenever  $x \in M$ . By our assumption (1), there exists a  $\lambda$ -directed colimit cocone, within  $\mathcal{S}$ , of the form  $(S, \sigma_p \mid p \in P) = \varinjlim (S_p, \sigma_{p,q} \mid p \leq q \text{ in } P)$ , which is preserved by  $\Phi$ ; whence, setting  $(M_p, f_p) \stackrel{\text{def}}{=} \Phi(S_p)$ ,  $\sigma_p \stackrel{\text{def}}{=} \Phi(\sigma_p)$ , and  $\sigma_{p,q} \stackrel{\text{def}}{=} \Phi(\sigma_{p,q})$ , we get

$$((M, f), \sigma_p \mid p \in P) = \varinjlim ((M_p, f_p), \sigma_{p,q} \mid p \leq q \text{ in } P) \quad \text{within } \mathbf{Str}(\mathbf{u}_\lambda). \quad (5.2)$$

The set  $\overline{M} \stackrel{\text{def}}{=} \bigcup_{p \in P} \sigma_p[M_p]$  is contained in  $M$ .

**Claim 1.**  $M$  is the closure of  $\overline{M}$  under  $f$ .

*Proof of Claim.* Denote by  $M'$  the closure of  $\overline{M}$  under  $f$  and by  $f'$  the domain-range restriction of  $f$  from  ${}^\lambda M'$  to  $M'$ . Then each domain-range restriction  $\sigma'_p: M_p \rightarrow M'$  of  $\sigma_p$  is an f-homomorphism from  $(M_p, f_p)$  to  $(M', f')$ , so those maps define a cocone above  $((M_p, f_p), \sigma_{p,q} \mid p \leq q \text{ in } P)$ , and so the universal property of the colimit yields a unique f-homomorphism  $\varphi: M \rightarrow M'$  such that each  $\varphi \circ \sigma_p = \sigma'_p$ . Denoting by  $\varepsilon: (M', f') \hookrightarrow (M, f)$  the inclusion embedding, it follows that each  $\varepsilon \circ \varphi \circ \sigma_p = \varepsilon \circ \sigma'_p = \sigma_p$ ; whence  $\varepsilon \circ \varphi = \text{id}_M$ . Since  $\varepsilon$  is one-to-one, it is thus bijective; that is,  $M = M'$ .  $\square$  Claim 1.

**Claim 2.**  $\text{card } \overline{M} > \kappa^\lambda$ .

*Proof of Claim.* If  $\text{card } \overline{M} \leq \kappa^\lambda$ , then, since  $f$  has arity  $\lambda$  and by Claim 1, we get  $\text{card } M \leq \kappa^\lambda$ , a contradiction.  $\square$  Claim 2.

**Claim 3.** *There exists a strictly increasing  $\lambda$ -sequence  $(p_\xi \mid \xi < \lambda)$  of elements in  $P$  such that the  $\lambda$ -sequence  $(\sigma_{p_\xi}[M_{p_\xi}] \mid \xi < \lambda)$  is strictly increasing with respect to set inclusion.*

*Proof of Claim.* Let  $\alpha < \lambda$  and suppose having performed the construction up to level  $\alpha$ , getting  $(p_\xi \mid \xi < \alpha)$ . Since  $\text{card}(\bigcup_{\xi < \alpha} \sigma_{p_\xi}[M_{p_\xi}]) \leq \kappa \leq \kappa^\lambda$ , it follows from Claim 2 that  $\sigma_{p_\alpha}[M_{p_\alpha}] \not\subseteq \bigcup_{\xi < \alpha} \sigma_{p_\xi}[M_{p_\xi}]$  for some  $p_\alpha \in P$ . Since  $P$  is  $\lambda$ -directed,  $p_\alpha$  may be taken an upper bound of  $\{p_\xi \mid \xi < \alpha\}$ . Then  $\sigma_{p_\xi}[M_{p_\xi}] \subsetneq \sigma_{p_\alpha}[M_{p_\alpha}]$  whenever  $\xi < \alpha$ .  $\square$  Claim 3.

Renaming the  $p_\xi$  obtained in Claim 3, we may assume that  $p_\xi = \xi$  for all  $\xi < \lambda$  (so  $\lambda$  is a sub-poset of  $P$ ). Pick  $a_\xi \in \sigma_{\xi+1}[M_{\xi+1}] \setminus \sigma_\xi[M_\xi]$ , for each  $\xi < \lambda$ . Setting  $D \stackrel{\text{def}}{=} \bigcup_{\alpha < \lambda} {}^\lambda \sigma_\alpha[M_\alpha]$ , it follows that  $\vec{a} \stackrel{\text{def}}{=} (a_\xi \mid \xi < \lambda)$  belongs to  ${}^\lambda \overline{M} \setminus D$ . Moreover, for each  $\alpha < \lambda$  and  $\vec{x} \in {}^\lambda \sigma_\alpha[M_\alpha]$ ,  $f \sigma_\alpha(\vec{x}) = \sigma_\alpha f_\alpha(\vec{x}) \in \overline{M}$ ; so  $f[D] \subseteq \overline{M}$ .

We can thus endow the set  $N \stackrel{\text{def}}{=} \overline{M} \sqcup \{\infty\}$ , for an extra element  $\infty \notin N$ , with the map  $g: {}^\lambda N \rightarrow N$  defined by

- $g(\vec{x}) = f(\vec{x})$  whenever  $\vec{x} \in D$  (in that case  $g(\vec{x}) \in \overline{M}$ );
- $g(\infty, \infty, \dots) \stackrel{\text{def}}{=} a_0$ ;
- $g(\vec{x}) = \infty$  in all other cases.

It follows from our Assumption (2) that the colimit

$$(S', \sigma'_\xi \mid \xi < \lambda) = \varinjlim (S_\xi, \sigma_{\xi, \eta} \mid \xi \leq \eta < \lambda)$$

exists in  $\mathcal{S}$ , and further, setting  $(M', f') \stackrel{\text{def}}{=} \Phi(S')$  and  $\sigma'_\xi \stackrel{\text{def}}{=} \Phi(\sigma'_\xi)$ ,

$$((M', f'), \sigma'_\xi \mid \xi < \lambda) \stackrel{\text{def}}{=} \varinjlim ((M_\xi, f_\xi), \sigma_{\xi, \eta} \mid \xi \leq \eta < \lambda). \quad (5.3)$$

Since  $(M', f') = \Phi(S') \in \text{rng } \Phi \subseteq \mathcal{J}_\lambda$  (cf. (3)), the map  $f'$  is idempotent. Moreover, for each  $\xi < \lambda$ , the domain-range restriction  $\tau_\xi: (M_\xi, f_\xi) \rightarrow (N, g)$  of  $\sigma'_\xi$  is an f-homomorphism; so those maps define a cocone above  $((M_\xi, f_\xi), \sigma_{\xi, \eta} \mid \xi \leq \eta < \lambda)$  within  $\mathbf{Str}(\mathbf{u}_\lambda)$ . By the universal property of the colimit (5.3), there exists a unique f-homomorphism  $\psi: (M', f') \rightarrow (N, g)$  such that each  $\psi \circ \sigma'_\xi = \tau_\xi$ .

For each  $\xi < \lambda$ , pick  $u_\xi \in M_{\xi+1}$  such that  $a_\xi = \sigma_{\xi+1}(u_\xi)$ ; then  $a'_\xi \stackrel{\text{def}}{=} \sigma'_{\xi+1}(u_\xi)$  belongs to  $M'$ , and so does  $t \stackrel{\text{def}}{=} f'(a'_\xi \mid \xi < \lambda)$ . Now each  $\psi(a'_\xi) = \psi\sigma'_{\xi+1}(u_\xi) = \tau_{\xi+1}(u_\xi) = \sigma_{\xi+1}(u_\xi) = a_\xi$ , whence  $\psi(t) = g(a_\xi \mid \xi < \lambda) = \infty$ . It follows that

$$g(\infty, \infty, \dots) = g(\psi(t), \psi(t), \dots) = \psi(f'(t, t, \dots)) = \psi(t) = \infty;$$

a contradiction since  $g(\infty, \infty, \dots) = a_0$ .  $\square$

## 6. ANOTHER CASE OF CONJUGACY: FROM RPC TO PC IN THE REGULAR CASE

In general, a vocabulary  $\mathbf{v}$  may have operation symbols of arity  $\lambda$  or larger. This makes it harder to write an RPC class, of  $\mathbf{v}$ -structures, in PC form: the statement that  $U^N$  be  $\mathbf{v}$ -closed cannot *a priori* be incorporated into  $\mathbf{E}$  in (3.2). Similarly, the statement that a relation, with arity  $\lambda$  or greater, is the graph of a function, cannot be expressed by an  $\mathcal{L}_{\infty\lambda}$ -sentence. Nonetheless, we shall see that this difficulty can be circumvented, enabling us to prove (cf. Theorem 6.8) that  $\text{PC}(\mathcal{L}_{\infty\lambda}) = \text{RPC}(\mathcal{L}_{\infty\lambda})$  for any regular cardinal  $\lambda$ .

Until Lemma 6.7 we shall fix an infinite cardinal  $\lambda$  together with a unary predicate symbol  $U$  and vocabularies  $\mathbf{v}, \mathbf{w}$  such that  $\mathbf{v} \cup \{U\} \subseteq \mathbf{w}$ . Set

$$\tilde{\mathbf{w}} \stackrel{\text{def}}{=} \mathbf{v} \sqcup \{ \dot{s} \mid s \in T_\lambda(\mathbf{w}) \setminus (\mathbf{v}_{\text{cst}} \cup \text{Var}) \} \sqcup \{ \dot{E} \mid E \in A_\lambda(\mathbf{w}) \text{ relational} \} \sqcup \{ J_{s,t} \mid s, t \in T_\lambda(\mathbf{w}) \},$$

where each  $\text{ar}(\dot{s}) \stackrel{\text{def}}{=} \text{var}(s)$ , each  $\text{ar}(\dot{E}) \stackrel{\text{def}}{=} \text{var}(E)$ , and each  $J_{s,t}$  is a relation symbol with arity a disjoint union of  $\text{var}(s)$  and  $\text{var}(t)$  (we will thus use the shortcut  $J_{s,t}(\vec{x}, \vec{y})$  instead of  $J_{s,t}(\vec{z})$  for a suitable concatenation  $\vec{z}$  of  $\vec{x}$  and  $\vec{y}$ ). Partly following (4.1) and (4.2), for  $s \in T_\lambda(\mathbf{w})$  and  $E \in A_\lambda(\mathbf{w})$ , the terms  $\tilde{s} \in T_\lambda(\tilde{\mathbf{w}})$  and atomic formulas  $\tilde{E} \in A_\lambda(\tilde{\mathbf{w}})$ , with the same variables as  $s$  and  $E$ , respectively, are now defined as follows:

$$\tilde{s} \stackrel{\text{def}}{=} \begin{cases} s, & \text{if } s \in \mathbf{v}_{\text{cst}} \cup \text{Var}, \\ \dot{s}(u \mid u \in \text{var}(s)) & \text{(usually written as } \dot{s}), \text{ otherwise,} \end{cases} \quad (6.1)$$

$$\tilde{E} \stackrel{\text{def}}{=} \begin{cases} J_{s,t}(\vec{x}, \vec{y}), & \text{if } E \text{ has the form } s(\vec{x}) = t(\vec{y}), \\ \dot{E}(u \mid u \in \text{var}(E)) & \text{(usually written as } \dot{E}), \text{ otherwise.} \end{cases} \quad (6.2)$$

The following definition may be viewed as an adaptation, to the present context, of the “ $\kappa$ -partitions” introduced in Oikkonen [26, Notation 2.1].

**Definition 6.1.** A  $\mathbf{w}$ -structure  $M$  is a  $(\lambda, U)$ -*distension* of a  $\mathbf{v}$ -structure  $A$  if  $A = U^M \upharpoonright_{\mathbf{v}}$  and  $M = \{s^M(\vec{a}) \mid s \in T_\lambda(\mathbf{w}) \text{ and } \vec{a} \in \text{var}(s)A\}$ .

Observe that in the context of Definition 6.1,  $U^M$  is necessarily  $\mathbf{v}$ -closed in  $N$ .

**Notation 6.2** (Coherence sentence). Denote by  $H$  the conjunction of all  $\mathcal{L}_{\infty\lambda}(\tilde{\mathbf{w}})$ -sentences of the following form (where  $z, z_1, z_2$  are variables,  $s, s_i, t \in T_\lambda(\mathbf{w})$ ,

$f \in \mathbf{w}_{\text{ope}}, R \in \mathbf{w}_{\text{rel}}$ :

$$(\forall_{(\text{var}(\mathbf{s}))} \vec{x}) J_{\mathbf{s}, \mathbf{s}}(\vec{x}, \vec{x}); \quad (6.3)$$

$$(\forall_{(\text{var}(\mathbf{s}))} \vec{x}) (\forall_{(\text{var}(\mathbf{t}))} \vec{y}) (J_{\mathbf{s}, \mathbf{t}}(\vec{x}, \vec{y}) \Rightarrow J_{\mathbf{t}, \mathbf{s}}(\vec{y}, \vec{x})); \quad (6.4)$$

$$(\forall_{(\text{var}(\mathbf{s}_1))} \vec{x}_1) (\forall_{(\text{var}(\mathbf{s}_2))} \vec{x}_2) (\forall_{(\text{var}(\mathbf{s}_3))} \vec{x}_3) \left( (J_{\mathbf{s}_1, \mathbf{s}_2}(\vec{x}_1, \vec{x}_2) \wedge J_{\mathbf{s}_2, \mathbf{s}_3}(\vec{x}_2, \vec{x}_3)) \Rightarrow J_{\mathbf{s}_1, \mathbf{s}_3}(\vec{x}_1, \vec{x}_3) \right); \quad (6.5)$$

$$J_{\mathbf{c}, \mathbf{z}}(\mathbf{c}), \quad \text{whenever } \mathbf{c} \in \mathbf{v}_{\text{cst}}; \quad (6.6)$$

$$(\forall_{(\text{var}(\mathbf{s}))} \vec{x}) (\forall_{(\text{var}(\mathbf{t}))} \vec{y}) (J_{\mathbf{s}, \mathbf{t}}(\vec{x}, \vec{y}) \Rightarrow \tilde{\mathbf{s}}(\vec{x}) = \tilde{\mathbf{t}}(\vec{y})); \quad (6.7)$$

$$(\forall \mathbf{x}) J_{\mathbf{z}_1, \mathbf{z}_2}(\mathbf{x}, \mathbf{x}); \quad (6.8)$$

$$(\forall_{(\text{var}(\mathbf{s}))} \vec{x}) (\forall_{(\text{var}(\mathbf{t}))} \vec{y}) \left( \left( \bigwedge_{\xi \in \text{ar}(f)} J_{\mathbf{s}_\xi, \mathbf{t}_\xi}(\vec{x}, \vec{y}) \right) \Rightarrow J_{\mathbf{s}, \mathbf{t}}(\vec{x}, \vec{y}) \right),$$

provided  $\mathbf{s} = f(\mathbf{s}_\xi \mid \xi \in \text{ar}(f))$  and  $\mathbf{t} = f(\mathbf{t}_\xi \mid \xi \in \text{ar}(f))$  in  $\mathbf{T}_\lambda(\mathbf{w})$ ; (6.9)

(Here and elsewhere we allow for dummy variables in notations such as  $J_{\mathbf{s}_\xi, \mathbf{t}_\xi}(\vec{x}, \vec{y})$ .)

$$(\forall_{(\text{var}(\mathbf{E}))} \vec{x}) (\forall_{(\text{var}(\mathbf{F}))} \vec{y}) \left( \left( \tilde{\mathbf{E}}(\vec{x}) \wedge \bigwedge_{\xi \in \text{ar}(\mathbf{R})} J_{\mathbf{s}_\xi, \mathbf{t}_\xi}(\vec{x}, \vec{y}) \right) \Rightarrow \tilde{\mathbf{F}}(\vec{y}) \right),$$

provided  $\mathbf{E} = \mathbf{R}(\mathbf{s}_\xi \mid \xi \in \text{ar}(\mathbf{R}))$  and  $\mathbf{F} = \mathbf{R}(\mathbf{t}_\xi \mid \xi \in \text{ar}(\mathbf{R}))$  in  $\mathbf{A}_\lambda(\mathbf{w})$ ; (6.10)

$$(\forall_{(\text{var}(\mathbf{s}))} \vec{x}) \left( \left( \bigwedge_{\xi \in \text{ar}(f)} J_{\mathbf{s}_\xi, \mathbf{z}}(\vec{x}, \tilde{\mathbf{s}}_\xi(\vec{x})) \right) \Rightarrow \right.$$

$$\left. \left( \tilde{\mathbf{s}}(\vec{x}) = f(\tilde{\mathbf{s}}_\xi(\vec{x}) \mid \xi \in \text{ar}(f)) \wedge J_{\mathbf{s}, \mathbf{z}}(\vec{x}, \tilde{\mathbf{s}}(\vec{x})) \right) \right),$$

provided  $\mathbf{s} = f(\mathbf{s}_\xi \mid \xi \in \text{ar}(f))$  in  $\mathbf{T}_\lambda(\mathbf{w})$  with  $f \in \mathbf{v}_{\text{ope}}$ ; (6.11)

$$(\forall_{(\text{var}(\mathbf{E}))} \vec{x}) \left( \left( \bigwedge_{\xi \in \text{ar}(\mathbf{R})} J_{\mathbf{s}_\xi, \mathbf{z}}(\vec{x}, \tilde{\mathbf{s}}_\xi(\vec{x})) \right) \Rightarrow \right.$$

$$\left. \left( \tilde{\mathbf{E}}(\vec{x}) \Leftrightarrow \mathbf{R}(\tilde{\mathbf{s}}_\xi(\vec{x}) \mid \xi \in \text{ar}(\mathbf{R})) \right) \right),$$

provided  $\mathbf{E} = \mathbf{R}(\mathbf{s}_\xi \mid \xi \in \text{ar}(f))$  in  $\mathbf{A}_\lambda(\mathbf{w})$  with  $\mathbf{R} \in \mathbf{v}_{\text{rel}}$ , (6.12)

$$(\forall_{(\text{var}(\mathbf{s}))} \vec{x}) \left( \tilde{\mathbf{U}}\mathbf{s}(\vec{x}) \Leftrightarrow J_{\mathbf{s}, \mathbf{z}}(\vec{x}, \tilde{\mathbf{s}}(\vec{x})) \right). \quad (6.13)$$

**Definition 6.3.** Let  $\mathbf{M}$  be a  $\mathbf{w}$ -structure and let  $\mathbf{A}'$  be a  $\tilde{\mathbf{w}}$ -structure. We say that  $(\mathbf{M}, \mathbf{A}')$  is a *conjugate pair*, or, equivalently, that  $\mathbf{M}$  is *left conjugate to*  $\mathbf{A}'$ , respectively that  $\mathbf{A}'$  is *right conjugate to*  $\mathbf{M}$ , if the following statements hold:

- (1)  $\mathbf{A}' \upharpoonright_{\mathbf{v}} = \mathbf{U}^{\mathbf{M}} \upharpoonright_{\mathbf{v}}$  (denote that  $\mathbf{v}$ -structure by  $\mathbf{A}$ );
- (2)  $M = \{\mathbf{s}^{\mathbf{M}}(\vec{a}) \mid \mathbf{s} \in \mathbf{T}_\lambda(\mathbf{w}) \text{ and } \vec{a} \in \text{var}(\mathbf{s})A\}$ ;
- (3) for all  $\mathbf{s}, \mathbf{t} \in \mathbf{T}_\lambda(\mathbf{w})$ ,  $\vec{a} \in \text{var}(\mathbf{s})A$ , and  $\vec{b} \in \text{var}(\mathbf{t})A$ ,  $\mathbf{s}^{\mathbf{M}}(\vec{a}) = \mathbf{t}^{\mathbf{M}}(\vec{b})$  implies that  $\tilde{\mathbf{s}}^{\mathbf{A}'}(\vec{a}) = \tilde{\mathbf{t}}^{\mathbf{A}'}(\vec{b})$ ;
- (4) for every relational  $\mathbf{E} \in \mathbf{A}_\lambda(\mathbf{w})$ ,  $\tilde{\mathbf{E}}^{\mathbf{A}'} = \{\vec{a} \in \text{var}(\mathbf{E})A \mid \mathbf{M} \models \mathbf{E}(\vec{a})\}$ ;

(5) for all  $s, t \in T_\lambda(\mathbf{w})$ , all  $\vec{a} \in \text{var}(s)A$ , and all  $\vec{b} \in \text{var}(t)A$ ,  $\mathbf{A}' \models J_{s,t}(\vec{a}, \vec{b})$  iff  $\mathbf{M} \models s(\vec{a}) = t(\vec{b})$ .

**Lemma 6.4.** *For every conjugate pair  $(\mathbf{M}, \mathbf{A}')$  and  $\mathbf{A} \stackrel{\text{def}}{=} \mathbf{A}' \upharpoonright_{\mathbf{v}}$ ,  $\mathbf{M}$  is a  $(\lambda, U)$ -distension of  $\mathbf{A}$  and  $\mathbf{A}'$  satisfies  $\mathbf{H}$ .*

*Proof.* The statement that  $\mathbf{M}$  is a  $(\lambda, U)$ -distension of  $\mathbf{A}$  trivially follows from items (1) and (2) of Definition 6.3. By letting  $t$  be a variable in (3), we obtain

(3') For all  $s \in T_\lambda(\mathbf{w})$  and all  $\vec{a} \in \text{var}(s)A$ ,  $s^{\mathbf{M}}(\vec{a}) \in A$  implies that  $\tilde{s}^{\mathbf{A}'}(\vec{a}) = s^{\mathbf{M}}(\vec{a})$ .

The verification of the statements (6.3)–(6.13) in  $\mathbf{A}'$  breaks down into the following observations:

- (6.3)–(6.5): use Definition 6.3(5);
- (6.6): using Definition 6.3(1), we get  $c^{\mathbf{A}} = c^{\mathbf{M}}$  whenever  $c \in \mathbf{v}_{\text{cst}}$ ;
- (6.7): use (5) and (3) from Definition 6.3;
- (6.8): use Definition 6.3(5);
- (6.9): Definition 6.3(5);
- (6.10): use (5) and (4) from Definition 6.3;
- (6.11): suppose that  $\mathbf{A}' \models J_{s_\xi, z}(\vec{a}, \tilde{s}_\xi(\vec{a}))$  for all  $\xi \in \text{ar}(f)$ . By (3') and Definition 6.3(5), this means that the element  $b_\xi \stackrel{\text{def}}{=} s_\xi^{\mathbf{M}}(\vec{a})$  belongs to  $A$  and  $b_\xi = \tilde{s}_\xi^{\mathbf{A}'}(\vec{a})$ , whenever  $\xi \in \text{ar}(f)$ . Since  $f \in \mathbf{v}_{\text{ope}}$ , the element  $b \stackrel{\text{def}}{=} s^{\mathbf{M}}(\vec{a}) = f^{\mathbf{M}}(b_\xi \mid \xi \in \text{ar}(f)) = f^{\mathbf{A}}(b_\xi \mid \xi \in \text{ar}(f))$  belongs to  $A$ . By (3') and Definition 6.3(5),  $b = \tilde{s}^{\mathbf{A}'}(\vec{a})$  and  $\mathbf{A}' \models J_{s,z}(\vec{a}, b)$ .
- (6.12): a similar argument as for (6.11), using (3') and Definition 6.3(4).
- (6.13):  $\mathbf{A}' \models \widetilde{U}_s(\vec{a})$  iff  $\mathbf{M} \models U_s(\vec{a})$  iff  $s^{\mathbf{M}}(\vec{a}) \in A$  (use (4) and (1) from Definition 6.3).  $\square$

**Lemma 6.5.** *The following statements hold, for any  $\mathbf{v}$ -structure  $\mathbf{A}$ :*

- (1) *Every  $(\lambda, U)$ -distension  $\mathbf{M}$  of  $\mathbf{A}$  has a right conjugate  $\mathbf{A}'$ .*
- (2) *Every expansion  $\mathbf{A}'$  of  $\mathbf{A}$  to  $\widetilde{\mathbf{w}}$ , satisfying  $\mathbf{H}$ , has a left conjugate  $\mathbf{M}$ .*

*Proof.* Ad (1). We need to define the interpretations in  $\mathbf{A}'$  of all symbols in  $\widetilde{\mathbf{w}} \setminus \mathbf{v}$ . Pick  $o \in A$ .

- For every  $s \in T_\lambda(\mathbf{w}) \setminus (\mathbf{v}_{\text{cst}} \cup \text{Var})$  and every  $\vec{a} \in \text{var}(s)A$ , we set

$$\dot{s}^{\mathbf{A}'}(\vec{a}) \stackrel{\text{def}}{=} \begin{cases} s^{\mathbf{M}}(\vec{a}), & \text{if } s^{\mathbf{M}}(\vec{a}) \in A; \\ o, & \text{otherwise.} \end{cases}$$

- For every relational  $E \in \mathbf{A}_\lambda(\mathbf{w})$ , we set

$$\dot{E}^{\mathbf{A}'} \stackrel{\text{def}}{=} \left\{ \vec{a} \in \text{var}(E)A \mid \mathbf{M} \models E(\vec{a}) \right\}.$$

- For all  $s, t \in T_\lambda(\mathbf{w})$ , we set

$$J_{s,t}^{\mathbf{A}'} \stackrel{\text{def}}{=} \left\{ (\vec{a}, \vec{b}) \in \text{var}(s)A \times \text{var}(t)A \mid \mathbf{M} \models s(\vec{a}) = t(\vec{b}) \right\}.$$

The verifications of (1)–(5) in Definition 6.3 are straightforward.

Ad (2). Setting  $\mathcal{M} \stackrel{\text{def}}{=} \{(s, \vec{a}) \mid s \in T_\lambda(\mathbf{w}) \text{ and } \vec{a} \in \text{var}(s)A\}$ , we define a binary relation  $\sim$  on  $\mathcal{M}$  by the rule

$$(s, \vec{a}) \sim (t, \vec{b}) \quad \text{if } \mathbf{A}' \models J_{s,t}(\vec{a}, \vec{b}), \quad \text{for all } (s, \vec{a}), (t, \vec{b}) \in \mathcal{M}. \quad (6.14)$$

Since  $\mathbf{A}'$  satisfies all statements (6.3)–(6.5),  $\sim$  is an equivalence relation on  $\mathcal{M}$ . Denoting the  $\sim$ -equivalence class of  $(\mathbf{s}, \vec{a})$  by  $[\mathbf{s}, \vec{a}]$ , we set

$$M \stackrel{\text{def}}{=} \mathcal{M}/\sim = \{[\mathbf{s}, \vec{a}] \mid (\mathbf{s}, \vec{a}) \in \mathcal{M}\}.$$

To avoid clutter, we will frequently extend the notation  $[\mathbf{s}, \vec{a}]$  to pairs  $(\mathbf{s}, \vec{a})$  where  $\vec{a} \in {}^X A$  with  $\text{var}(\mathbf{s}) \subseteq X$ , by setting  $[\mathbf{s}, \vec{a}] \stackrel{\text{def}}{=} [\mathbf{s}, \vec{a}|_{\text{var}(\mathbf{s})}]$ . We further set  $\varepsilon(a) \stackrel{\text{def}}{=} [z, a]$  (for  $a \in A$ ), which is independent of the variable  $z$  by (6.8), whenever  $a \in A$ . By (6.7) for  $\mathbf{s}, \mathbf{t}$  variables,  $\varepsilon$  is one-to-one. We set  $\mathbf{c}^M \stackrel{\text{def}}{=} [c, \emptyset]$  whenever  $c \in \mathbf{w}_{\text{cst}}$ . By (6.6),  $\varepsilon(\mathbf{c}^A) = \mathbf{c}^M$  whenever  $c \in \mathbf{w}_{\text{cst}}$ .

**Claim 1.** *For every  $(\mathbf{s}, \vec{a}) \in \mathcal{M}$ ,  $[\mathbf{s}, \vec{a}] \in \varepsilon A$  iff  $\mathbf{A}' \models J_{\mathbf{s}, z}(\vec{a}, \vec{s}(\vec{a}))$ .*

*Proof of Claim.* If  $[\mathbf{s}, \vec{a}] \in \varepsilon A$ , then there exists  $b \in A$  such that  $[\mathbf{s}, \vec{a}] = [z, b]$ , that is,  $\mathbf{A}' \models J_{\mathbf{s}, z}(\vec{a}, b)$ . By (6.7) with  $\mathbf{t}$  a variable,  $b = \vec{s}^{\mathbf{A}'}(\vec{a})$ , whence  $\mathbf{A}' \models J_{\mathbf{s}, z}(\vec{a}, \vec{s}(\vec{a}))$ . The argument can be followed backwards.  $\square$  Claim 1.

**Claim 2.** *For every  $f \in \mathbf{w}_{\text{ope}}$ , there exists a map  $f^M : \text{ar}(f)M \rightarrow M$  such that for all  $\vec{x} = (x_\xi \mid \xi \in \text{ar}(f)) \in \text{ar}(f)M$ ,*

$$f^M(\vec{x}) = \begin{cases} [\mathbf{s}, \vec{a}], & \text{if } \mathbf{s} = f(\mathbf{s}_\xi \mid \xi \in \text{ar}(f)) \in T_\lambda(\mathbf{w}), \vec{a} \in {}^{\text{var}(\mathbf{s})} A, \text{ each } x_\xi = [\mathbf{s}_\xi, \vec{a}], \\ \varepsilon f^A(\vec{b}), & \text{if } f \in \mathbf{v}_{\text{ope}} \text{ and } \vec{x} = \varepsilon(\vec{b}) \text{ for some } \vec{b} \in \text{ar}(f)A. \end{cases}$$

*Proof of Claim.* The case where  $f \in \mathbf{v}_{\text{cst}}$  has already been covered above (via  $\varepsilon(\mathbf{c}^A) = \mathbf{c}^M$ ). Suppose from now on that  $f \notin \mathbf{v}_{\text{cst}}$ . If  $\mathbf{s} = f(\mathbf{s}_\xi \mid \xi \in \text{ar}(f))$  and  $\mathbf{t} = f(\mathbf{t}_\xi \mid \xi \in \text{ar}(f))$  in  $T_\lambda(\mathbf{w})$ , with each  $x_\xi = [\mathbf{s}_\xi, \vec{a}] = [\mathbf{t}_\xi, \vec{b}]$ , so  $\mathbf{A}' \models J_{\mathbf{s}_\xi, \mathbf{t}_\xi}(\vec{a}, \vec{b})$ , then, by (6.9),  $\mathbf{A}' \models J_{\mathbf{s}, \mathbf{t}}(\vec{a}, \vec{b})$ , that is,  $[\mathbf{s}, \vec{a}] = [\mathbf{t}, \vec{b}]$ . Now if  $f \in \mathbf{v}_{\text{ope}}$ ,  $\mathbf{s} = f(\mathbf{s}_\xi \mid \xi \in \text{ar}(f)) \in T_\lambda(\mathbf{w})$ , and each  $x_\xi = [\mathbf{s}_\xi, \vec{a}] = \varepsilon(b_\xi)$ , then, by the proof of Claim 1, each  $b_\xi = \vec{s}_\xi^{\mathbf{A}'}(\vec{a})$ , so, setting  $b \stackrel{\text{def}}{=} \vec{s}^{\mathbf{A}'}(\vec{a})$ , it follows from (6.11) that  $b = f^A(\vec{b})$  and  $\mathbf{A}' \models J_{\mathbf{s}, z}(\vec{a}, b)$ ; whence  $[\mathbf{s}, \vec{a}] = [z, b] = \varepsilon(b)$ .  $\square$  Claim 2.

**Claim 3.** *Let  $R \in \mathbf{w}_{\text{rel}}$ , set*

$$R^* \stackrel{\text{def}}{=} \{([\mathbf{s}_\xi, \vec{a}] \mid \xi \in \text{ar}(R)) \mid E \stackrel{\text{def}}{=} R(\mathbf{s}_\xi \mid \xi \in \text{ar}(R)) \in T_\lambda(\mathbf{w}), \\ \vec{a} \in {}^{\text{var}(E)} A, \text{ and } \mathbf{A}' \models \tilde{E}(\vec{a})\},$$

and  $R^M \stackrel{\text{def}}{=} \begin{cases} \varepsilon R^A \cup R^*, & \text{if } R \in \mathbf{v}_{\text{rel}}, \\ R^*, & \text{otherwise.} \end{cases}$ . Then the following statements hold:

- (1) *If  $R \in \mathbf{v}_{\text{rel}}$ , then  $R^M \cap \text{ar}(R)\varepsilon A = \varepsilon R^A$ .*
- (2) *Whenever  $E = R(\mathbf{s}_\xi \mid \xi \in \text{ar}(R)) \in A_\lambda(\mathbf{w})$  and  $\vec{a} \in {}^{\text{var}(E)} A$ ,*  
 $([\mathbf{s}_\xi, \vec{a}] \mid \xi \in \text{ar}(R)) \in R^M$  *iff  $\mathbf{A}' \models \tilde{E}(\vec{a})$ .*

*Proof of Claim.* Ad (1). It suffices to verify that  $R^* \cap \text{ar}(R)\varepsilon A \subseteq \varepsilon R^A$ . Let  $E = R(\mathbf{s}_\xi \mid \xi \in \text{ar}(R))$  in  $T_\lambda(\mathbf{w})$  and  $\vec{a} \in {}^{\text{var}(E)} A$  such that  $\mathbf{A}' \models \tilde{E}(\vec{a})$ . Moreover, let  $\vec{b} = (b_\xi \mid \xi \in \text{ar}(R)) \in \text{ar}(R)A$  such that each  $[\mathbf{s}_\xi, \vec{a}] = [z, b_\xi]$ ; that is,  $\mathbf{A}' \models J_{\mathbf{s}_\xi, z}(\vec{a}, b_\xi)$ . By the proof of Claim 1, each  $b_\xi = \vec{s}_\xi^{\mathbf{A}'}(\vec{a})$ . By (6.12) and since  $\mathbf{A}' \models \tilde{E}(\vec{a})$ , it follows that  $\vec{b} \in R^A$ , as desired.

Ad (2). Let  $([\mathbf{s}_\xi, \vec{a}] \mid \xi \in \text{ar}(R)) \in R^M$ . If  $([\mathbf{s}_\xi, \vec{a}] \mid \xi \in \text{ar}(R)) \in R^*$  then there are  $F = R(\mathbf{t}_\xi \mid \xi \in \text{ar}(R)) \in A_\lambda(\mathbf{w})$  and  $\vec{b} \in {}^{\text{var}(F)} A$  such that  $\mathbf{A}' \models \tilde{F}(\vec{b})$  and

each  $[s_\xi, \vec{a}] = [t_\xi, \vec{b}]$ ; that is,  $\mathbf{A}' \models J_{s_\xi, t_\xi}(\vec{a}, \vec{b})$ . The desired conclusion then follows from (6.10). Suppose now that  $R \in \mathbf{v}_{\text{rel}}$  and  $([s_\xi, \vec{a}] \mid \xi \in \text{ar}(R)) \in \varepsilon R^{\mathbf{A}}$ . There exists  $\vec{b} \in R^{\mathbf{A}}$  such that each  $[s_\xi, \vec{a}] = [z, b_\xi]$ ; so  $\mathbf{A}' \models J_{s_\xi, z}(\vec{a}, \vec{s}_\xi(\vec{a}))$  and  $b_\xi = \tilde{s}_\xi^{\mathbf{A}'}(\vec{a})$ . Since  $\mathbf{A}' \models R(b_\xi \mid \xi \in \text{ar}(R))$ , it follows from (6.12) that  $\mathbf{A}' \models \tilde{E}(\vec{a})$ .

Conversely, if  $\mathbf{A}' \models \tilde{E}(\vec{a})$ , then  $([s_\xi, \vec{a}] \mid \xi \in \text{ar}(R)) \in R^* \subseteq R^{\mathbf{M}}$ .  $\square$  Claim 3.

In particular, Claims 2 and 3 ensure that the map  $\varepsilon$  is a  $\mathbf{v}$ -embedding from  $\mathbf{A}$  into the  $\mathbf{w}$ -structure  $\mathbf{M}$  thus defined.

**Claim 4.**  $\varepsilon \mathbf{A} = U^{\mathbf{M}} \upharpoonright_{\mathbf{v}}$ .

*Proof of Claim.* Since  $\varepsilon$  is a  $\mathbf{v}$ -embedding, it suffices to verify that  $[s, \vec{a}] \in \varepsilon \mathbf{A}$  iff  $[s, \vec{a}] \in U^{\mathbf{M}}$  whenever  $(s, \vec{a}) \in \mathcal{M}$ . By Claim 1,  $[s, \vec{a}] \in \varepsilon \mathbf{A}$  is equivalent to  $\mathbf{A}' \models J_{s, z}(\vec{a}, \vec{s}(\vec{a}))$ . The statement  $[s, \vec{a}] \in U^{\mathbf{M}}$  means that  $\mathbf{A}' \models \tilde{E}(\vec{a})$  where  $E \stackrel{\text{def}}{=} U(s)$ . By (6.13), the two statements are equivalent.  $\square$  Claim 4.

**Claim 5.**  $s^{\mathbf{M}}(\varepsilon \vec{a}) = [s, \vec{a}]$  whenever  $(s, \vec{a}) \in \mathcal{M}$ .

*Proof of Claim.* If  $s = c \in \mathbf{w}_{\text{cst}}$  then the desired relation follows from the definition  $c^{\mathbf{M}} = [c, \emptyset]$ . For nonconstant  $s$ , argue by induction on  $s$  and apply Claim 2.  $\square$  Claim 5.

By Claims 4 and 5, items (1) and (2) of Definition 6.3 are both satisfied; that is,  $\mathbf{M}$  is a  $(\lambda, U)$ -distension of  $\mathbf{A}$  (or, strictly speaking, of its isomorphic copy  $\varepsilon \mathbf{A}$ ).

Let us verify Definition 6.3(3). Suppose that  $s^{\mathbf{M}}(\varepsilon \vec{a}) = t^{\mathbf{M}}(\varepsilon \vec{b})$ . By Claim 5,  $(s, \vec{a}) \sim (t, \vec{b})$ , that is,  $\mathbf{A}' \models J_{s, t}(\vec{a}, \vec{b})$ . Using (6.7), it follows that  $\tilde{s}^{\mathbf{A}'}(\varepsilon \vec{a}) = \tilde{t}^{\mathbf{A}'}(\varepsilon \vec{b})$ .

Now Definition 6.3(4). For all  $\vec{a} \in \text{var}^{(E)} A$ ,  $\mathbf{M}$  satisfies  $E(\varepsilon \vec{a})$  iff the vector  $(s_\xi^{\mathbf{M}}(\varepsilon \vec{a}) \mid \xi \in \text{ar}(R))$  belongs to  $R^{\mathbf{M}}$ . Since each  $s_\xi^{\mathbf{M}}(\varepsilon \vec{a}) = [s_\xi, \vec{a}]$  (cf. Claim 5) and by Claim 3, this holds iff  $\mathbf{A}' \models \tilde{E}(\vec{a})$ .

For Definition 6.3(5), apply Claim 5 together with (6.14).  $\square$

**Notation 6.6.** Suppose that  $\lambda$  is regular. We extend the assignment  $E \mapsto \tilde{E}$ , originally defined on all atomic formulas of  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ , as follows.

- If  $E$  is  $\bigvee_{i \in I} E_i$ , then  $\tilde{E}$  is  $\bigvee_{i \in I} \tilde{E}_i$ .
- If  $E$  is  $\neg F$ , then  $\tilde{E}$  is  $\neg \tilde{F}$ .
- Let  $E(\vec{x})$  be  $(\exists \vec{y}) F(\vec{x}, \vec{y})$ , where  $\text{card } J < \lambda$ . For every  $t \in {}^J T_\lambda(\mathbf{w})$ , setting

$\text{var}(t) \stackrel{\text{def}}{=} \bigcup_{j \in J} \text{var}(t_j)$  (note that *since*  $\lambda$  is regular, that set is  $\lambda$ -small), we denote by  $F^t(\vec{x}, \vec{u})$  the formula  $F(\vec{x}, (t_j(\vec{u}) \mid j \in J))$ , then define  $\tilde{E}(\vec{x})$  as

$$\bigvee_{t \in {}^J T_\lambda(\mathbf{w})} (\exists \vec{u} \underset{(\text{var}(t))}{}) \tilde{F}^t(\vec{x}, \vec{u}).$$

**Lemma 6.7.** Let  $\lambda$  be regular, let  $(\mathbf{M}, \mathbf{A}')$  be a conjugate pair, and set  $\mathbf{A} \stackrel{\text{def}}{=} \mathbf{A}' \upharpoonright_{\mathbf{v}}$ . Then for every formula  $E(\vec{x}) \in \mathcal{L}_{\infty\lambda}(\mathbf{w})$  and every  $\vec{a} \in {}^I A$ ,  $\mathbf{M} \models E(\vec{a})$  iff  $\mathbf{A}' \models \tilde{E}(\vec{a})$ .

*Proof.* The case where  $E$  is atomic follows from the definition of conjugacy (cf. Definition 6.3, especially items (4) and (5)). The induction steps corresponding to disjunction and negation are trivial. Now let  $E(\vec{x})$  be  $(\exists \vec{y}) F(\vec{x}, \vec{y})$ , where  $I$

and  $J$  are both  $\lambda$ -small, and let  $\vec{a} \in {}^I A$ . If  $\mathbf{M} \models \mathbf{E}(\vec{a})$  then there exists  $\vec{y} \in {}^J M$  such that  $\mathbf{M} \models \mathbf{F}(\vec{a}, \vec{y})$ . By Definition 6.3(2), for each  $j \in J$  there are  $t_j \in \mathsf{T}_\lambda(\mathbf{w})$  and  $\vec{u}_j \in \text{var}(t_j)A$  such that  $y_j = t_j^{\mathbf{M}}(\vec{u}_j)$ . Since  $\lambda$  is regular,  $\sum_{j \in J} \text{card var}(t_j) < \lambda$  and we may thus assume, after reindexing, that each  $y_j = t_j^{\mathbf{M}}(\vec{u})$  for a single vector  $\vec{u} \in \text{var}(\mathbf{t})A$  with  $\text{card var}(\mathbf{t}) < \lambda$ ; whence  $\mathbf{M} \models \mathbf{F}^{\mathbf{t}}(\vec{a}, \vec{u})$ . By our induction hypothesis,  $\mathbf{A}' \models \tilde{\mathbf{F}}^{\mathbf{t}}(\vec{a}, \vec{u})$ ; whence  $\mathbf{A}' \models \tilde{\mathbf{E}}(\vec{a})$ . This argument can be followed backwards, thus yielding the desired equivalence.  $\square$

**Theorem 6.8.** *Let  $\lambda$  be a regular cardinal. Then  $\text{PC}(\mathcal{L}_{\infty\lambda}) = \text{RPC}(\mathcal{L}_{\infty\lambda})$ .*

*Proof.* Let  $\mathbf{v}$  be a vocabulary and let  $\mathcal{C}$  be an  $\text{RPC}(\mathcal{L}_{\infty\lambda})$  class of  $\mathbf{v}$ -structures. By Proposition 5.4, there are a vocabulary  $\mathbf{w}$  containing  $\mathbf{v}$ , with  $\mathbf{w} \setminus \mathbf{v}$   $\lambda$ -ary and a unary predicate symbol  $\mathbf{U}$ , together with a conjunction  $\mathbf{E}$  of universal  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ -sentences, such that (3.2) holds.

**Claim.**

$$\mathcal{C} = \{U^{\mathbf{M}} \upharpoonright_{\mathbf{v}} \mid \mathbf{M} \in \mathbf{Mod}_{\mathbf{w}}(\mathbf{E}), \\ \mathbf{M} \text{ is a } (\lambda, \mathbf{U})\text{-distension of } U^{\mathbf{M}}, U^{\mathbf{M}} \text{ } \mathbf{v}\text{-closed}\}. \quad (6.15)$$

*Proof of Claim.* We prove the nontrivial containment. Let  $\mathbf{A} \in \mathcal{C}$ . By (3.2),  $\mathbf{A} = U^{\mathbf{N}} \upharpoonright_{\mathbf{v}}$  for some  $\mathbf{N} \in \mathbf{Mod}_{\mathbf{w}}(\mathbf{E})$  such that  $U^{\mathbf{N}}$  is  $\mathbf{v}$ -closed. Set

$$M \stackrel{\text{def}}{=} \{s^{\mathbf{N}}(\vec{a}) \mid s \in \mathsf{T}_\lambda(\mathbf{w}), \vec{a} \in \text{var}(\mathbf{s})A\}.$$

Since some operations of  $\mathbf{v}$  may have arity  $\lambda$  or larger, we cannot ensure that  $M$  be  $\mathbf{v}$ -closed within  $\mathbf{N}$  *a priori*. However, picking  $o \in A$ , we may define an alternate interpretation of each  $f \in \mathbf{w}_{\text{ope}}$  on  $M$  by setting  $f^{\mathbf{M}}(\vec{x}) \stackrel{\text{def}}{=} f^{\mathbf{N}}(\vec{x})$  if either there are  $\mathbf{s} = f(\mathbf{s}_\xi \mid \xi < \text{ar}(f)) \in \mathsf{T}_\lambda(\mathbf{w})$  and  $\vec{a} \in \text{var}(\mathbf{s})A$  with each  $x_\xi = s_\xi^{\mathbf{N}}(\vec{a})$ , in which case  $f^{\mathbf{N}}(\vec{x}) = s^{\mathbf{N}}(\vec{a}) \in M$ , or  $\vec{x} \in \text{ar}(f)A$ , in which case  $f^{\mathbf{N}}(\vec{x}) = f^{\mathbf{A}}(\vec{x}) \in A$ ; and  $f^{\mathbf{M}}(\vec{x}) \stackrel{\text{def}}{=} o$  in all other cases. Setting  $\mathbf{R}^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{R}^{\mathbf{N}}$  whenever  $\mathbf{R} \in \mathbf{w}_{\text{rel}}$ ,  $\mathbf{M}$  is now a  $(\lambda, \mathbf{U})$ -distension of  $\mathbf{A}$ . Since  $s^{\mathbf{M}}(\vec{a}) = s^{\mathbf{N}}(\vec{a})$  whenever  $s \in \mathsf{T}_\lambda(\mathbf{w})$  and  $\vec{a} \in \text{var}(\mathbf{s})A$ , and since  $\mathbf{M}$  is a  $\mathbf{w}_{\text{rel}}$ -substructure of  $\mathbf{N}$ , every universal  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ -sentence satisfied by  $\mathbf{N}$  is also satisfied by  $\mathbf{M}$ . Since  $\mathbf{N} \models \mathbf{E}$ , it follows that  $\mathbf{M} \models \mathbf{E}$ .  $\square$  Claim.

Now set  $\mathcal{C}' \stackrel{\text{def}}{=} \{\mathbf{A}' \upharpoonright_{\mathbf{v}} \mid \mathbf{A}' \in \mathbf{Mod}_{\tilde{\mathbf{w}}}(\tilde{\mathbf{E}} \wedge \mathbf{H})\}$ . In order to complete the proof of Theorem 6.8 it suffices to verify that  $\mathcal{C} = \mathcal{C}'$ .

By the Claim above, every  $\mathbf{A} \in \mathcal{C}$  is  $U^{\mathbf{M}} \upharpoonright_{\mathbf{v}}$  for some  $(\lambda, \mathbf{U})$ -distension  $\mathbf{M}$  of  $\mathbf{A}$  satisfying  $\mathbf{E}$ . By Lemma 6.5,  $\mathbf{M}$  has a right conjugate  $\mathbf{A}'$ . By Lemma 6.7,  $\mathbf{A}' \models \tilde{\mathbf{E}}$ . By Lemma 6.4,  $\mathbf{A}' \models \mathbf{H}$ . Hence,  $\mathbf{A} \in \mathcal{C}'$ , thus completing the proof that  $\mathcal{C} \subseteq \mathcal{C}'$ .

Conversely, every  $\mathbf{A} \in \mathcal{C}'$  is  $\mathbf{A}' \upharpoonright_{\mathbf{v}}$  for some  $\mathbf{A}' \in \mathbf{Mod}_{\tilde{\mathbf{w}}}(\tilde{\mathbf{E}} \wedge \mathbf{H})$ . By Lemma 6.5,  $\mathbf{A}'$  has a left conjugate  $\mathbf{M}$ , which is necessarily a  $(\lambda, \mathbf{U})$ -distension of  $\mathbf{A}$ . By Lemma 6.7,  $\mathbf{M} \models \mathbf{E}$ . Therefore,  $\mathbf{A} = U^{\mathbf{M}} \upharpoonright_{\mathbf{v}} \in \mathcal{C}$ , thus completing the proof that  $\mathcal{C}' \subseteq \mathcal{C}$ .  $\square$

Now that we know that  $\text{PC}(\mathcal{L}_{\infty\lambda}) = \text{RPC}(\mathcal{L}_{\infty\lambda})$ , we can take into account the projectivity of the class  $\mathcal{C}$  in Theorem 5.5. We obtain an additional faithfulness information on  $\Phi$ .

**Theorem 6.9.** *Let  $\lambda$  be a regular cardinal, let  $\mathbf{v}$  be a vocabulary with  $\mathbf{v}_{\text{ope}}$   $\lambda$ -ary, and let  $\mathcal{C}$  be an  $\text{RPC}(\mathcal{L}_{\infty\lambda})$  class (equivalently, a  $\text{PC}(\mathcal{L}_{\infty\lambda})$  class) of  $\mathbf{v}$ -structures. Then there are a  $\lambda$ -accessible category  $\mathcal{S}$  and a faithful  $\lambda$ -continuous*

functor  $\Phi: \mathcal{S} \rightarrow \mathbf{Str}(\mathbf{v})$  such that  $\mathcal{C} = \text{im } \Phi = \text{rng } \Phi$ . Moreover,  $\mathcal{S}$  can be taken a full subcategory, closed under  $\lambda$ -directed colimits, of  $\mathbf{Str}(\mathbf{w})$  with  $\mathbf{w}$ -embeddings, for some vocabulary  $\mathbf{w}$  extending  $\mathbf{v}$  with  $\mathbf{w} \setminus \mathbf{v}$   $\lambda$ -ary.

*Proof.* We imitate the proof of Theorem 5.5, incorporating the PC = RPC information. By Theorem 4.11, there exists a vocabulary  $\mathbf{w}$  containing  $\mathbf{v}$ , with  $\mathbf{w} \setminus \mathbf{v}$   $\lambda$ -ary, together with a conjunction  $E$  of universal sentences from  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ , such that  $\mathcal{C} = \{N|_{\mathbf{v}} \mid N \in \mathbf{Mod}_{\mathbf{w}}(E)\}$ . Define  $\mathcal{S}$  as in the proof of Theorem 5.5, and let  $\Phi: \mathcal{S} \rightarrow \mathbf{Str}(\mathbf{v})$ ,  $N \mapsto N|_{\mathbf{v}}$  be the forgetful functor. As in the proof of Theorem 5.5,  $\mathcal{S}$  is  $\lambda$ -accessible,  $\Phi$  is  $\lambda$ -continuous, and  $\mathcal{C} = \text{rng } \Phi = \text{im } \Phi$ . Moreover, the functor  $\Phi$  is obviously faithful.  $\square$

## 7. RPC CLASSES AS IMAGES OF CONTINUOUS FUNCTORS

The main aim of this section is to provide a converse to the results of Section 5, by proving (cf. Theorem 7.1) that the image of any  $\lambda$ -accessible functor  $\Phi: \mathcal{S} \rightarrow \mathbf{Str}(\mathbf{v})$ , for a  $\lambda$ -accessible category  $\mathcal{S}$  and a  $\lambda$ -ary vocabulary  $\mathbf{v}$ , is  $\text{RPC}(\mathcal{L}_{\infty\lambda})$  (thus, due to Theorem 6.8,  $\text{PC}(\mathcal{L}_{\infty\lambda})$ ). We prove something a bit more general.

**Theorem 7.1.** *Let  $\lambda$  be a regular cardinal, let  $\mathbf{v}$  be a  $\lambda$ -ary vocabulary, let  $\mathcal{S}$  be a category, and let  $\Phi: \mathcal{S} \rightarrow \mathbf{Str}(\mathbf{v})$  be a functor. We suppose that  $\mathcal{S}$  has a  $\lambda$ -directed colimit-dense small subcategory  $\mathcal{S}^\dagger$ , such that all colimits of  $\lambda$ -directed diagrams of  $\mathcal{S}^\dagger$  exist in  $\mathcal{S}$  and are preserved by  $\Phi$ . Then the image of  $\Phi$  is  $\text{RPC}(\mathcal{L}_{\infty\lambda})$ -definable, via an extension of  $\mathbf{v}$  by a unary predicate  $U$  together with a collection of binary predicates. Hence it is also  $\text{PC}(\mathcal{L}_{\infty\lambda})$ -definable.*

*Proof.* Our assumptions imply immediately that  $\text{im } \Phi$  is the class of all  $\mathbf{v}$ -structures  $M$  for which there are a commutative diagram  $(S_{p_1}, \sigma_{p_1, p_2} \mid p_1 \leq p_2 \text{ in } P)$  in  $\mathcal{S}^\dagger$ , with  $P$   $\lambda$ -directed, and a colimit cocone

$$(M, \sigma_p \mid p \in P) = \varinjlim (\Phi(S_{p_1}), \Phi(\sigma_{p_1, p_2}) \mid p_1 \leq p_2 \text{ in } P) \quad \text{within } \mathbf{Str}(\mathbf{v}). \quad (7.1)$$

To that end, let the vocabulary  $\mathbf{w}$  consist of  $\mathbf{v}$  together with the following collection of additional relation symbols:

- a unary predicate  $U$  and a binary relation symbol  $\leq$ ;
- binary relation symbols  $D_\varphi$ , indexed by morphisms  $\varphi \in \mathcal{S}^\dagger$ ;
- binary relation symbols  $F_{S, u}$ , for  $S \in \text{Ob } \mathcal{S}^\dagger$  and  $u \in |\Phi(S)|$ .

Let us describe the semantic interpretation of each of those additional relation symbols, and write down the corresponding sentence in  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ .

- $U$  should be interpreted by the original model  $M$ , and its complement (meant as the  $P$  in (7.1)) should be a  $\lambda$ -directed poset under the interpretation of  $\leq$ . The required  $\mathcal{L}_{\infty\lambda}$ -sentences are thus:

$$U \text{ is nonempty, and closed under all operations in } \mathbf{v}; \quad (7.2)$$

$$\leq \text{ is a } \lambda\text{-directed partial ordering on the complement of } U. \quad (7.3)$$

Let us, from now on, use the abbreviations  $P(p)$ ,  $(\forall^U x)E$ ,  $(\exists^U x)E$ ,  $(\forall^P p)E$ ,  $(\exists^P p)E$ , for  $\neg U(p)$ ,  $(\forall x)(U(x) \Rightarrow E)$ ,  $(\exists x)(U(x) \wedge E)$ ,  $(\forall p)(\neg U(p) \Rightarrow E)$ , and  $(\exists p)(\neg U(p) \wedge E)$  respectively.

- The relation symbols  $D_\varphi$ , for  $\varphi \in \mathcal{S}^\dagger$ , capture the labeling of the commutative diagram  $(S_{p_1}, \sigma_{p_1, p_2} \mid p_1 \leq p_2 \text{ in } P)$ . For  $p, q \in P$ , the relation

$D_\varphi(p, q)$  expresses the statement  $\varphi = \sigma_{p, q}$ . The required  $\mathcal{L}_{\infty\lambda}$ -sentences are thus:

$$(\forall^P \mathbf{p}, \mathbf{q})(D_\varphi(\mathbf{p}, \mathbf{q}) \Rightarrow \mathbf{p} \leq \mathbf{q}); \quad (7.4)$$

$$(\forall^P \mathbf{p}, \mathbf{q})\left(\mathbf{p} \leq \mathbf{q} \Rightarrow \bigvee_{\varphi \in \mathcal{S}^\dagger} D_\varphi(\mathbf{p}, \mathbf{q})\right); \quad (7.5)$$

$$(\forall^P \mathbf{p}, \mathbf{q}) \neg (D_{\varphi_1}(\mathbf{p}, \mathbf{q}) \wedge D_{\varphi_2}(\mathbf{p}, \mathbf{q})), \quad \text{whenever } \varphi_1 \neq \varphi_2; \quad (7.6)$$

$$(\forall^P \mathbf{p}, \mathbf{q})\left(D_\varphi(\mathbf{p}, \mathbf{q}) \Rightarrow (D_{\mathbf{d}(\varphi)}(\mathbf{p}, \mathbf{p}) \wedge D_{\mathbf{r}(\varphi)}(\mathbf{q}, \mathbf{q}))\right); \quad (7.7)$$

$$(\forall^P \mathbf{p}, \mathbf{q}, \mathbf{r})\left((D_\varphi(\mathbf{p}, \mathbf{q}) \wedge D_\psi(\mathbf{q}, \mathbf{r})) \Rightarrow D_{\psi \circ \varphi}(\mathbf{p}, \mathbf{r})\right). \quad (7.8)$$

- The relation symbols  $F_{S, u}$ , for  $S \in \text{Ob } \mathcal{S}^\dagger$  and  $u \in |\Phi(S)|$ , capture the representation of  $\mathbf{M}$  as the colimit in (7.1). More precisely, the relation  $F_{S, u}(p, x)$  expresses the conjunction of the statements  $p \in P$ ,  $S = S_p$ , and  $x = \sigma_p(u)$ . The corresponding  $\mathcal{L}_{\infty\lambda}$ -statements are then obtained by translating the conditions stated in Lemma 2.1:

$$(\forall \mathbf{p})(\forall \mathbf{x})(F_{S, u}(\mathbf{p}, \mathbf{x}) \Rightarrow (P(\mathbf{p}) \wedge U(\mathbf{x}))); \quad (7.9)$$

$$(\forall^P \mathbf{p})(D_S(\mathbf{p}, \mathbf{p}) \Leftrightarrow (\exists \mathbf{x})F_{S, u}(\mathbf{p}, \mathbf{x})); \quad (7.10)$$

$$(\forall^P \mathbf{p})(\forall \mathbf{x}, \mathbf{y})\left((F_{S, u}(\mathbf{p}, \mathbf{x}) \wedge F_{S, u}(\mathbf{p}, \mathbf{y})) \Rightarrow \mathbf{x} = \mathbf{y}\right); \quad (7.11)$$

$$(\forall^P \mathbf{p}, \mathbf{q})(\forall \mathbf{x})\left((F_{\mathbf{d}(\varphi), u}(\mathbf{p}, \mathbf{x}) \wedge D_\varphi(\mathbf{p}, \mathbf{q})) \Rightarrow F_{\mathbf{r}(\varphi), \Phi(\varphi)(u)}(\mathbf{q}, \mathbf{x})\right) \quad (7.12)$$

$$(\forall^U \mathbf{x})(\exists^P \mathbf{p}) \bigvee_{S \in \text{Ob } \mathcal{S}^\dagger, u \in |\Phi(S)|} F_{S, u}(\mathbf{p}, \mathbf{x}); \quad (7.13)$$

$$(\forall^P \mathbf{p})(\forall_{(\text{ar}(\mathbf{R}))} \vec{x})\left(\bigwedge_{\beta \in \text{ar}(\mathbf{R})} F_{S, u_\beta}(\mathbf{p}, \mathbf{x}_\beta) \Rightarrow R(\mathbf{x}_\beta \mid \beta \in \text{ar}(\mathbf{R}))\right) \\ \text{for } \mathbf{R} \in \mathbf{v}_{\text{rel}}, \text{ provided } (u_\beta \mid \beta \in \text{ar}(\mathbf{R})) \in \mathbf{R}^{\Phi(S)}; \quad (7.14)$$

$$(\forall^P \mathbf{p})(\forall_{(\text{ar}(\mathbf{f}))} \vec{x})(\forall \mathbf{x})\left(\left(F_{S, u}(\mathbf{p}, \mathbf{x}) \wedge \bigwedge_{\beta \in \text{ar}(\mathbf{f})} F_{S, u_\beta}(\mathbf{p}, \mathbf{x}_\beta)\right) \Rightarrow \right. \\ \left. \mathbf{x} = \mathbf{f}(\mathbf{x}_\beta \mid \beta \in \text{ar}(\mathbf{f}))\right) \\ \text{for } \mathbf{f} \in \mathbf{v}_{\text{ope}}, \text{ provided } u = \mathbf{f}^{\Phi(S)}(u_\beta \mid \beta \in \text{ar}(\mathbf{f})); \quad (7.15)$$

$$(\forall^P \mathbf{p})(\forall \mathbf{x})\left(\left(F_{S, u_1}(\mathbf{p}, \mathbf{x}) \wedge F_{S, u_2}(\mathbf{p}, \mathbf{x})\right) \Rightarrow \right. \\ \left. \bigvee_{\substack{\varphi: S \rightarrow T \text{ within } \mathcal{S}^\dagger, \\ \Phi(\varphi)(u_1) = \Phi(\varphi)(u_2)}} (\exists^P \mathbf{q})D_\varphi(\mathbf{p}, \mathbf{q})\right); \quad (7.16)$$

$$\begin{aligned}
(\forall^P \mathbf{p})(\forall_{(\text{ar}(\mathbf{R}))} \vec{x}) & \left( \left( \bigwedge_{\beta \in \text{ar}(\mathbf{R})} F_{S, u_\beta}(\mathbf{p}, \mathbf{x}_\beta) \wedge R(\mathbf{x}_\beta \mid \beta \in \text{ar}(\mathbf{R})) \right) \Rightarrow \right. \\
& \bigvee_{\substack{\varphi: S \rightarrow T \text{ within } \mathcal{S}^\dagger, \\ (\Phi(\varphi)(u_\beta) \mid \beta \in \text{ar}(\mathbf{R})) \in \mathbf{R}^{\Phi(T)}}} (\exists^P \mathbf{q}) D_\varphi(\mathbf{p}, \mathbf{q}) \Big) \\
& \text{for } \mathbf{R} \in \mathbf{v}_{\text{rel}}. \quad (7.17)
\end{aligned}$$

The conjunction  $\Lambda$ , of all possible sentences (7.2)–(7.17), is an  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$ -sentence. Therefore, in order to complete the proof that  $\mathcal{C}$  is  $\text{RPC}(\mathcal{L}_{\infty\lambda})$ , it suffices to verify that a  $\mathbf{v}$ -structure  $\mathbf{M}$  belongs to  $\mathcal{C}$  iff it is  $\mathbf{U}^{\mathbf{N}} \downarrow_{\mathbf{v}}$  for some  $\mathbf{w}$ -structure  $\mathbf{N}$  satisfying  $\Lambda$ .

Suppose first that  $\mathbf{M} = \mathbf{U}^{\mathbf{N}} \downarrow_{\mathbf{v}}$  for some  $\mathbf{w}$ -structure  $\mathbf{N}$  satisfying  $\Lambda$ . We shall construct a  $\lambda$ -directed colimit representation of the form (7.1). Since  $\mathbf{N}$  satisfies both (7.2) and (7.3), the interpretation  $\leq$  of  $\leq$  in  $\mathbf{N}$  is a  $\lambda$ -directed partial ordering on  $P \stackrel{\text{def}}{=} N \setminus M$ . Moreover, each  $D_\varphi \stackrel{\text{def}}{=} D_\varphi^{\mathbf{N}}$  is contained in  $\leq$  (use (7.4)), whereas each  $F_{S, u} \stackrel{\text{def}}{=} F_{S, u}^{\mathbf{N}}$  is contained in  $P \times M$  (use (7.9)).

Let  $p, q \in P$  with  $p \leq q$ . Due to (7.5) and (7.6), there are unique  $\sigma_{p, q}, S_p, S_q \in \mathcal{S}^\dagger$  such that  $(p, q) \in D_{\sigma_{p, q}}$ ,  $(p, p) \in D_{S_p}$ , and  $(q, q) \in D_{S_q}$ . Due to (7.7), both  $S_p = \mathbf{d}(\sigma_{p, p})$  and  $S_q = \mathbf{r}(\sigma_{q, q})$  are objects in  $\mathcal{S}^\dagger$ . Hence  $\sigma_{p, q}: S_p \rightarrow S_q$ . If  $p = q$  then  $\sigma_{p, q} = S_p$  is itself an identity in  $\mathcal{S}^\dagger$ . Whenever  $p \leq q \leq r$  in  $P$ , it follows from (7.8) that  $\sigma_{p, r} = \sigma_{q, r} \circ \sigma_{p, q}$ . Therefore,  $(S_p, \sigma_{p, q} \mid p \leq q \text{ in } P)$  is a commutative diagram.

Let  $p \in P$  and set  $S \stackrel{\text{def}}{=} S_p$ ; so  $(p, p) \in D_S$ . Let  $u \in |\Phi(S)|$ . Due to (7.10) and (7.11), there exists a unique  $\sigma_p(u) \in M$  such that  $(p, \sigma_p(u)) \in F_{S, u}$ . This defines a map  $\sigma_p: |\Phi(S)| \rightarrow M$ .

We claim that  $\sigma_p$  is a  $\mathbf{v}$ -homomorphism  $\Phi(S) \rightarrow \mathbf{M}$ . Let  $\mathbf{R} \in \mathbf{v}_{\text{rel}}$ , let  $(u_\beta \mid \beta \in \text{ar}(\mathbf{R})) \in \mathbf{R}^{\Phi(S)}$ , and set  $x_\beta \stackrel{\text{def}}{=} \sigma_p(u_\beta)$  for each  $\beta \in \text{ar}(\mathbf{R})$ . Then each  $(p, x_\beta) \in F_{S, u_\beta}$ , thus, due to (7.14),  $(x_\beta \mid \beta \in \text{ar}(\mathbf{R})) \in \mathbf{R}^{\mathbf{M}}$ . Likewise, let  $\mathbf{f} \in \mathbf{v}_{\text{ope}}$  and let  $u \stackrel{\text{def}}{=} \mathbf{f}^{\Phi(S)}(u_\beta \mid \beta \in \text{ar}(\mathbf{f}))$ . Set  $x \stackrel{\text{def}}{=} \sigma_p(u)$  and  $x_\beta \stackrel{\text{def}}{=} \sigma_p(u_\beta)$  for each  $\beta \in \text{ar}(\mathbf{f})$ . Then  $(p, x) \in F_{S, u}$  whereas each  $(p, x_\beta) \in F_{S, u_\beta}$ . Due to (7.15), it follows that  $x = \mathbf{f}^{\mathbf{M}}(x_\beta \mid \beta \in \text{ar}(\mathbf{f}))$ . This completes the proof of our claim that  $\sigma_p$  is a  $\mathbf{v}$ -homomorphism.

Let  $p \leq q$  in  $P$ . Set  $S \stackrel{\text{def}}{=} S_p$ ,  $T \stackrel{\text{def}}{=} S_q$ , and  $\varphi \stackrel{\text{def}}{=} \sigma_{p, q}$ . Let  $u \in |\Phi(S)|$  and set  $x \stackrel{\text{def}}{=} \sigma_p(u)$ ,  $y \stackrel{\text{def}}{=} \sigma_q(\Phi(\varphi)(u))$ . Then  $(p, q) \in D_\varphi$  and  $(p, x) \in F_{S, u}$ . Due to (7.12),  $(q, x) \in F_{T, \Phi(\varphi)(u)}$ ; that is,  $x = y$ . This completes the proof that  $\sigma_p = \sigma_q \circ \Phi(\sigma_{p, q})$ .

Therefore,  $(\mathbf{M}, \sigma_p \mid p \in P)$  is a cocone above  $(\Phi(S_p), \Phi(\sigma_{p, q}) \mid p \leq q \text{ in } P)$  within  $\mathbf{Str}(\mathbf{v})$ .

We shall now verify, using Lemma 2.1, that it is a colimit cocone.

Let  $x \in M$ . Due to (7.13), there are  $p \in P$  and  $u \in |\Phi(S_p)|$  such that  $(p, x) \in F_{S_p, u}$ ; whence  $x = \sigma_p(u)$ . Therefore,  $M = \bigcup \{ \sigma_p[|\Phi(S_p)|] \mid p \in P \}$ .

Let  $p \in P$ ,  $S \stackrel{\text{def}}{=} S_p$ , and  $u_1, u_2 \in |\Phi(S)|$  such that  $\sigma_p(u_1) = \sigma_p(u_2)$  — denote by  $x$  that element of  $M$ . Then  $(p, x)$  belongs to each  $F_{S, u_i}$ , thus, due to (7.16), there exists  $q \in P$  such that  $p \leq q$  and  $\Phi(\sigma_{p, q})(u_1) = \Phi(\sigma_{p, q})(u_2)$ .

Similarly, let  $\mathbf{R} \in \mathbf{v}_{\text{rel}}$  and  $(u_\beta \mid \beta \in \text{ar}(\mathbf{R})) \in \text{ar}(\mathbf{R})|\Phi(S)|$  such that  $(\sigma_p(u_\beta) \mid \beta \in \text{ar}(\mathbf{R})) \in \mathbf{R}^{\mathbf{M}}$ . Set  $x_\beta \stackrel{\text{def}}{=} \sigma_p(u_\beta)$  for  $\beta \in \text{ar}(\mathbf{R})$ . Since each  $(p, x_\beta) \in F_{S, u_\beta}$  and due to (7.17), there exists  $q \in P$ , with  $p \leq q$ , such that  $(\sigma_{p, q}(u_\beta) \mid$

$\beta \in \text{ar}(\mathbf{R}) \in \mathbf{R}^{\Phi(S_q)}$ . Owing to Lemma 2.1, this completes the proof that

$$(\Phi(S_p), \sigma_p \mid p \in P) = \varinjlim (\Phi(S_p), \Phi(\sigma_{p,q}) \mid p \leq q \text{ in } P).$$

Therefore,  $\mathbf{M}$  belongs to  $\text{im } \Phi$ .

Let, conversely,  $\mathbf{M} \in \text{im } \Phi$ , as witnessed by a  $\lambda$ -directed commutative diagram  $(S_{p_1}, \sigma_{p_1, p_2} \mid p_1 \leq p_2 \text{ in } P)$  in  $\mathcal{S}^\dagger$  and a colimit cocone within  $\mathcal{S}$  as in (7.1). We may assume that  $P \cap M = \emptyset$ . Define the  $\mathbf{w}$ -structure  $\mathbf{N}$ , with universe  $N \stackrel{\text{def}}{=} M \sqcup P$ , as follows:

- $U^{\mathbf{N}} \stackrel{\text{def}}{=} M$ ;
- The binary relation symbol  $\leq$  is interpreted as the partial ordering  $\leq$  of  $P$ ;
- for every  $\mathbf{R} \in \mathbf{v}_{\text{rel}}$ ,  $\mathbf{R}^{\mathbf{N}} \stackrel{\text{def}}{=} \mathbf{R}^M$ ;
- for every  $\mathbf{f} \in \mathbf{v}_{\text{ope}}$ ,  $\mathbf{f}^{\mathbf{N}}$  is any map from  $\text{ar}(\mathbf{f})N$  to  $M$  extending  $\mathbf{f}^M$ ;
- for every  $\varphi \in \mathcal{S}^\dagger$ , we set

$$D_\varphi \stackrel{\text{def}}{=} \{(p, q) \in P \times P \mid p \leq q \text{ and } \sigma_{p,q} = \varphi\};$$

- for every  $S \in \text{Ob } \mathcal{S}^\dagger$  and  $u \in |\Phi(S)|$ , we set

$$F_{S,u} \stackrel{\text{def}}{=} \{(p, \sigma_p(u)) \mid p \in P, S = S_p\}.$$

We verify that those interpretations satisfy all instances of (7.2)–(7.17) within  $\mathbf{N}$ .

For (7.2)–(7.4) this is obvious. For (7.5) and (7.6) we must take  $\varphi = \varphi_1 = \varphi_2 \stackrel{\text{def}}{=} \sigma_{p,q}$ . For (7.7) we just observe that  $\sigma_{p,q}$  is an arrow from  $\sigma_{p,p} = S_p$  to  $\sigma_{q,q} = S_q$ . For (7.8), we apply the equality  $\sigma_{p,r} = \sigma_{q,r} \circ \sigma_{p,q}$ . For (7.9) this follows from the containment  $F_{S,u} \subseteq P \times M$ . For (7.10) and (7.11), just observe that each  $\sigma_p$  is a function defined on  $|\Phi(S_p)|$ . For (7.12),  $\varphi$  is necessarily  $\sigma_{p,q}$  so we just need to apply the equality  $\sigma_p = \sigma_q \circ \Phi(\sigma_{p,q})$ . For (7.13) this follows from Lemma 2.1(i). The statements (7.14) and (7.15) follow from  $\sigma_p$  being a  $\mathbf{v}$ -homomorphism. Finally, (7.16) and (7.17) follows from items (ii) and (iii) of Lemma 2.1, respectively.

The final part of the statement of Theorem 7.1, that  $\text{im } \Phi$  is  $\text{PC}(\mathcal{L}_{\infty\lambda})$ -definable, now follows from Theorem 6.8.  $\square$

**Corollary 7.2.** *Let  $\lambda$  be a regular cardinal and let  $\mathbf{v}$  be a  $\lambda$ -ary vocabulary. Then for any small subcategory  $\mathcal{T}$  of  $\mathbf{Str}(\mathbf{v})$ , the class  $\mathcal{T}_{(\lambda)}$  of all colimits of  $\lambda$ -directed commutative diagrams from  $\mathcal{T}$  is  $\text{PC}(\mathcal{L}_{\infty\lambda})$ -definable.*

*Proof.* View  $\mathcal{T}_{(\lambda)}$  as a full subcategory of  $\mathbf{Str}(\mathbf{v})$  and apply Theorem 7.1 to the inclusion functor from  $\mathcal{S} \stackrel{\text{def}}{=} \mathcal{T}_{(\lambda)}$  into  $\mathbf{Str}(\mathbf{v})$ , with  $\mathcal{S}^\dagger \stackrel{\text{def}}{=} \mathcal{T}$ .  $\square$

Note that the objects of  $\mathcal{T}$  may not be  $\lambda$ -presentable in  $\mathbf{Str}(\mathbf{v})$ . Besides, in contrast to the context of Adámek and Rosický [1, Theorem 2.26],  $\mathcal{T}_{(\lambda)}$  is only a class of objects (as opposed to a category).

**Examples 7.3.** Consider the following small subcategories  $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$  of the category  $\mathbf{Set}_{\neq \emptyset}$  of all nonempty sets (viewed as  $\mathbf{Str}(\emptyset)$ ):

- $\mathcal{S}_0$  is the category of all finite subsets of  $\omega$  with all inclusion maps;
- $\mathcal{S}_1$  is the category of all finite subsets of  $\omega$  with all one-to-one maps;
- $\mathcal{S}_2$  is the category with the unique object  $\omega$  and its identity map.

Then  $(\mathcal{S}_0)_{(\omega)}$  is the class of all at most countable sets,  $(\mathcal{S}_1)_{(\omega)}$  is the class of all sets, and  $(\mathcal{S}_2)_{(\omega)}$  is the class of all countably infinite sets.

**Corollary 7.4.** *Let  $\lambda$  be a regular cardinal, let  $\mathbf{v}$  be a  $\lambda$ -ary vocabulary, let  $\mathcal{S}$  be a  $\lambda$ -accessible category, and let  $\Phi: \mathcal{S} \rightarrow \mathbf{Str}(\mathbf{v})$  be a  $\lambda$ -continuous functor. Then the image of  $\Phi$  is  $\text{PC}(\mathcal{L}_{\infty\lambda})$ -definable.*

*Proof.* Apply Theorem 7.1 with  $\mathcal{S}^\dagger \stackrel{\text{def}}{=} \mathbf{Pres}_\lambda \mathcal{S}$ . □

Bringing together Theorem 5.5 and Corollary 7.4, we obtain the following.

**Theorem 7.5.** *Let  $\lambda$  be a regular cardinal and let  $\mathbf{v}$  be a  $\lambda$ -ary vocabulary. For any class  $\mathcal{C}$  of  $\mathbf{v}$ -structures, the following are equivalent:*

- (i)  $\mathcal{C}$  is  $\text{PC}(\mathcal{L}_{\infty\lambda})$ ;
- (ii)  $\mathcal{C}$  is  $\text{RPC}(\mathcal{L}_{\infty\lambda})$ ;
- (iii) *there are a  $\lambda$ -accessible category  $\mathcal{S}$  and a  $\lambda$ -continuous functor  $\Phi: \mathcal{S} \rightarrow \mathbf{Str}(\mathbf{v})$  such that  $\mathcal{C} = \text{im } \Phi$ .*

Although, by Theorem 6.8, the condition that  $\mathbf{v}$  be  $\lambda$ -ary is not needed for the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 7.5, this is not the case for both implications (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) (cf. Examples 5.7 and 8.8, respectively).

**Corollary 7.6.** *Let  $\mathbf{v}$  be a vocabulary. For any class  $\mathcal{C}$  of  $\mathbf{v}$ -structures, the following are equivalent:*

- (i)  $\mathcal{C}$  is  $\text{PC}(\mathcal{L}_{\infty\infty})$ ;
- (ii)  $\mathcal{C}$  is  $\text{RPC}(\mathcal{L}_{\infty\infty})$ ;
- (iii) *there are an accessible category  $\mathcal{S}$  and a continuous functor  $\Phi: \mathcal{S} \rightarrow \mathbf{Str}(\mathbf{v})$  such that  $\mathcal{C} = \text{im } \Phi$ .*

*Proof.* By the Uniformization Theorem for accessible categories and functors (Adámek and Rosický [1, 2.19]; see also Proposition 2.3.5 and Theorem 2.3.10 in Makkai and Paré [22]), for any accessible category  $\mathcal{S}$  and any continuous functor  $\Phi$  from  $\mathcal{S}$  to some accessible category, there are arbitrarily large cardinals  $\kappa$  such that  $\mathcal{S}$  is a  $\kappa^+$ -accessible category and  $\Phi$  is a  $\kappa^+$ -accessible functor. Apply Theorem 7.5. □

## 8. FROM RPC TO PC IN THE SINGULAR CASE

Due to the regularity requirement on the infinite cardinal  $\lambda$  in Lemma 6.7, the argument of the proof of Theorem 6.8 does not extend to the case where  $\lambda$  is singular. The latter will require a completely different approach, resting on the following elementary observation of cardinal arithmetic. This will lead to Theorem 8.6, which, together with Theorem 6.8, shows that  $\text{PC}(\mathcal{L}_{\infty\lambda}) = \text{RPC}(\mathcal{L}_{\infty\lambda})$  for any infinite cardinal  $\lambda$  (regardless of whether  $\lambda$  is regular).

**Lemma 8.1.** *The following statements hold, for every infinite cardinal  $\lambda$ :*

- (1) *If  $\lambda$  is regular, then there are arbitrarily large cardinals  $\kappa$  such that  $\kappa^\lambda < \kappa^\lambda$ .*
- (2) *If  $\lambda$  is singular, then for every cardinal  $\kappa \geq 2^\lambda$  there exists  $\alpha < \lambda$  such that  $\kappa^\lambda = \kappa^\alpha$ ; hence  $\kappa^\lambda = \kappa^\lambda$ .*

*Proof.* *Ad (1).* Let  $\alpha$  be an ordinal and set  $\kappa \stackrel{\text{def}}{=} \beth_{\alpha+\lambda}$  (we are using the standard notation for Beth numbers). Then  $\kappa^\lambda = \kappa$  has cofinality  $\lambda$ . On the other hand (cf. Jech [14, Corollary 5.13]),  $\text{cf}(\kappa^\lambda) > \lambda$ ; whence  $\kappa^\lambda < \kappa^\lambda$ .

*Ad (2).* The conclusion is obvious for  $\kappa = 2^\lambda$  (take  $\alpha = 1$ ) so we may assume that  $\kappa > 2^\lambda$ . The least cardinal  $\mu$  such that  $\kappa^\lambda = \mu^\lambda$  satisfies  $\mu > 2^\lambda$  as well. Note

that  $\mu > \lambda$ , and  $\bar{\mu}^\lambda < \mu$  whenever  $\bar{\mu} < \mu$  (otherwise  $\mu \leq \bar{\mu}^\lambda$  thus  $\kappa^\lambda = \mu^\lambda \leq \bar{\mu}^\lambda$  and thus  $\kappa^\lambda = \bar{\mu}^\lambda$ , a contradiction).

If  $\mu > \lambda$ , then, applying Jech [14, Theorem 5.20(iii,a)],  $\kappa^\lambda = \mu^\lambda = \mu \leq \kappa$  thus  $\kappa^\lambda = \kappa$  and we are done.

Suppose that  $\text{cf } \mu \leq \lambda$ . Since  $\lambda$  is singular,  $\alpha \stackrel{\text{def}}{=} \text{cf } \mu < \lambda$ . Moreover, applying Jech [14, Theorem 5.20(iii,b)],  $\kappa^\lambda = \mu^\lambda = \mu^\alpha \leq \kappa^\alpha$ ; whence  $\kappa^\lambda = \kappa^\alpha$ .  $\square$

**Definition 8.2.** A *multiterm*, in a vocabulary  $\mathbf{v}$ , is a family  $\mathbf{t} = (\mathbf{t}_j \mid j \in J)$  of terms of  $\mathbf{v}$ . If each  $\mathbf{t}_j$  has all its variables in a set  $I$ , we will write  $\mathbf{t}: {}^I\mathbf{M} \rightarrow {}^J\mathbf{M}$ . Moreover, for any  $\mathbf{v}$ -structure  $\mathbf{M}$ , we will denote  $\mathbf{t}^{\mathbf{M}}: {}^I\mathbf{M} \rightarrow {}^J\mathbf{M}$ ,  $x \mapsto (\mathbf{t}_j^{\mathbf{M}}(x) \mid j \in J)$ , and call  $\mathbf{t}^{\mathbf{M}}$  a *multiterm function* on  $\mathbf{M}$ . Multiterms  $\mathbf{s}: {}^I\mathbf{M} \rightarrow {}^J\mathbf{M}$  and  $\mathbf{t}: {}^J\mathbf{M} \rightarrow {}^K\mathbf{M}$  may be “composed” to the multiterm  $\mathbf{t} \circ \mathbf{s} \stackrel{\text{def}}{=} (\mathbf{t}_k(\mathbf{s}_j \mid j \in J) \mid k \in K): {}^I\mathbf{M} \rightarrow {}^K\mathbf{M}$ .

**Lemma 8.3.** For all cardinals  $\alpha$  and  $\lambda$ , with  $\lambda$  singular and  $0 < \alpha < \lambda$ , there are a  $\lambda$ -ary vocabulary  $\mathbf{b}$ , consisting of operations only, and multiterms  $\mathbf{g}: {}^\alpha\mathbf{M} \rightarrow {}^\lambda\mathbf{M}$  and  $\mathbf{h}: {}^\lambda\mathbf{M} \rightarrow {}^\alpha\mathbf{M}$  of  $\mathbf{b}$ , such that every set  $M$  with  $(\text{card } M)^\alpha = (\text{card } M)^\lambda$  expands to a  $\mathbf{b}$ -structure  $\mathbf{M}$  satisfying both statements  $(\forall \bar{x})_{(\alpha)}((\mathbf{h} \circ \mathbf{g})(\bar{x}) = \bar{x})$  and  $(\forall \bar{y})_{(\lambda)}((\mathbf{g} \circ \mathbf{h})(\bar{y}) = \bar{y})$  (that we shall from now on abbreviate by  $\mathbf{h} \circ \mathbf{g} = \text{id}$  and  $\mathbf{g} \circ \mathbf{h} = \text{id}$ , respectively).

*Note.* Setting  $\kappa \stackrel{\text{def}}{=} \text{card } M$  and by virtue of the well known identity  $\kappa^\lambda = (\kappa^\lambda)^{\text{cf } \lambda}$ ,  $\kappa^\alpha = \kappa^\lambda$  is equivalent to  $\kappa^\alpha = \kappa^\lambda$ , that is,  $\kappa^\alpha = \kappa^\beta$  whenever  $\alpha \leq \beta < \lambda$ . Also, without the restriction that  $\mathbf{b}$  be  $\lambda$ -ary, Lemma 8.3 would be trivial.

*Proof.* We begin with the case where  $\text{cf } \lambda \leq \alpha \leq \lambda$ . Let  $\lambda = \sum_{i < \text{cf } \lambda} \lambda_i$  where each  $\alpha \leq \lambda_i < \lambda$ , and set  $\lambda_{<i} \stackrel{\text{def}}{=} \sum_{i' < i} \lambda_{i'}$  (ordinal sum) whenever  $i \leq \text{cf } \lambda$ . Let  $\mathbf{b}$  consist of operation symbols  $f_{i,j}$  and  $f'_{i,k}$ , for  $i < \text{cf } \lambda$ ,  $j < \lambda_i$ , and  $k < \alpha$ , where each  $\text{ar}(f_{i,j}) = \alpha$  and  $\text{ar}(f'_{i,k}) = \lambda_i$ . Moreover, pick a bijection  $(\text{cf } \lambda) \times \alpha \rightarrow \alpha$ ,  $(i, k) \mapsto \langle i, k \rangle$ , with inverse  $\alpha \rightarrow (\text{cf } \lambda) \times \alpha$ ,  $k \mapsto (p(k), q(k))$ . We let

$$\mathbf{g}_j(\bar{x})_{(\alpha)} \stackrel{\text{def}}{=} f_{i,\bar{j}}(x_{\langle i,k \rangle} \mid k < \alpha), \text{ whenever } j = \lambda_{<i} + \bar{j} \text{ with } \bar{j} < \lambda_i;$$

$$\mathbf{g}(\bar{x})_{(\alpha)} \stackrel{\text{def}}{=} (\mathbf{g}_j(\bar{x}) \mid j < \lambda);$$

$$\mathbf{h}_k(\bar{y})_{(\lambda)} \stackrel{\text{def}}{=} f'_{p(k),q(k)}(y_{\lambda_{<p(k)} + \bar{j}} \mid \bar{j} < \lambda_{p(k)}), \text{ whenever } k < \alpha;$$

$$\mathbf{h}(\bar{y})_{(\lambda)} \stackrel{\text{def}}{=} (\mathbf{h}_k(\bar{y}) \mid k < \alpha).$$

Let us define interpretations  $f_{i,j}$  and  $f'_{i,k}$ , of the symbols  $f_{i,j}$  and  $f'_{i,k}$ , on any set  $M$ , with cardinality  $\kappa$ , such that  $\kappa^\alpha = \kappa^\lambda$ . For each  $i < \text{cf } \lambda$ , it follows from the equation  $\kappa^\alpha = \kappa^{\lambda_i}$  that there are mutually inverse bijections  $f_i: {}^\alpha M \rightarrow {}^{\lambda_i} M$ ,  $x \mapsto (f_{i,j}(x) \mid j < \lambda_i)$  and  $f'_i: {}^{\lambda_i} M \rightarrow {}^\alpha M$ ,  $y \mapsto (f'_{i,k}(y) \mid k < \alpha)$ . For each  $x \in {}^\alpha M$ , we define  $g(x)$  as the concatenation of all  $f_i(x_{\langle i,k \rangle}) \mid k < \alpha$  for  $i < \text{cf } \lambda$ : that is, setting  $x_{\langle i,- \rangle} \stackrel{\text{def}}{=} (x_{\langle i,k \rangle} \mid k < \alpha)$  whenever  $x \in {}^\alpha M$  and  $i < \text{cf } \lambda$ ,

$$g(x) = f_0(x_{\langle 0,- \rangle}) \frown f_1(x_{\langle 1,- \rangle}) \frown \cdots \frown f_i(x_{\langle i,- \rangle}) \frown \cdots$$

Hence  $g$  is a bijection from  ${}^\alpha M$  onto  ${}^\lambda M$ , and  $g(x) = (g_j(x) \mid j < \lambda)$  for every  $x \in {}^\alpha M$ , where  $g_j(x) = f_{i,\bar{j}}(x_{\langle i,- \rangle})$  whenever  $j = \lambda_{<i} + \bar{j}$  with  $\bar{j} < \lambda_i$ .

Set  $h = g^{-1} = (h_k \mid k < \alpha)$ . Let  $y \in {}^\lambda M$  and  $x = h(y)$ . Then for each  $i < \text{cf } \lambda$ ,  $f_i(x_{\langle i, - \rangle}) = (y_{\lambda_{<i+\bar{j}} \mid \bar{j} < \lambda_i})$ , thus  $x_{\langle i, - \rangle} = f'_i(y_{\lambda_{<i+\bar{j}} \mid \bar{j} < \lambda_i})$ . It follows that each  $h_k(y) = x_k = x_{\langle p(k), - \rangle} = f'_{\langle p(k), q(k) \rangle}(y_{\lambda_{<p(k)+\bar{j}} \mid \bar{j} < \lambda_{p(k)}})$ .

If  $0 < \alpha < \text{cf } \lambda$ , then  $\kappa^\alpha = \kappa^\lambda$  entails  $\kappa^{\text{cf } \lambda} = \kappa^\alpha = \kappa^\lambda$ , thus it suffices to compose the multiterms associated to the latter equation, *via* the argument above, to a pair of multiterms giving rise to mutually inverse bijections between  ${}^\alpha \kappa$  and  ${}^{\text{cf } \lambda} \kappa$ .  $\square$

Let  $\alpha$  and  $\lambda$  be cardinals, with  $\lambda$  singular and  $0 < \alpha < \lambda$ , and let  $\mathbf{v}$  and  $\mathbf{w}$  be vocabularies with  $\mathbf{v} \subseteq \mathbf{w}$  and  $\mathbf{w} \setminus \mathbf{v}$  consisting only of relations with arities ordinals smaller than or equal to  $\lambda$ . Let  $\mathbf{b}$  be a  $\lambda$ -ary vocabulary, which we may assume to be disjoint from  $\mathbf{w}$ , with terms  $\mathbf{g}$  and  $\mathbf{f}$  satisfying the conclusion of Lemma 8.3.

Denote<sup>8</sup> by  $\mathbf{w}|\alpha$  the vocabulary with the same symbols and arities as  $\mathbf{w}$ , except for the relation symbols  $R \in \mathbf{w} \setminus \mathbf{v}$  with  $\text{ar}(R) = \lambda$ , each of which is replaced by a new relation symbol  $R_\alpha$  with arity  $\alpha$ . Note, in particular, that  $\mathbf{v} \subseteq \mathbf{w}|\alpha$ .

To any formula  $E$  in  $\mathcal{L}_{\infty \lambda^+}(\mathbf{w} \cup \mathbf{b})$  we associate the collection of its  $(\mathbf{g}, \mathbf{h})$ -contractions (with respect to the pair  $(\mathbf{g}, \mathbf{h})$ ), which are *preformulas* of  $\mathcal{L}_{\infty \lambda}(\mathbf{w}|\alpha \cup \mathbf{b})$  with the same free variables as  $E$ , by the following inductive process:

- If  $E$  is atomic, then it is its only  $(\mathbf{g}, \mathbf{h})$ -contraction, except in case  $E = R(\vec{t})$  for a relation symbol  $R \in \mathbf{w} \setminus \mathbf{v}$  with arity  $\lambda$ , in which case the only  $(\mathbf{g}, \mathbf{h})$ -contraction of  $E$  is  $R_\alpha(\mathbf{h}(\vec{t}))$ .
- If  $E$  is  $\neg F$ , then its  $(\mathbf{g}, \mathbf{h})$ -contractions are the  $\neg F'$  for  $(\mathbf{g}, \mathbf{h})$ -contractions  $F'$  of  $F$ .
- If  $E$  is  $\bigvee_{i \in I} E_i$ , then its  $(\mathbf{g}, \mathbf{h})$ -contractions are the  $\bigvee_{i \in I} E'_i$  for  $(\mathbf{g}, \mathbf{h})$ -contractions  $E'_i$  of  $E_i$ .
- If  $E$  is  $(\exists Y)F$ , where  $Y \subseteq \text{fvar}(F)$  has less than  $\lambda$  elements, then the  $(\mathbf{g}, \mathbf{h})$ -contractions of  $E$  are the  $(\exists Y)F'$  for  $(\mathbf{g}, \mathbf{h})$ -contractions  $F'$  of  $F$ .
- If  $E$  is  $(\exists Y)F$ , where  $Y \subseteq \text{fvar}(F)$  has  $\lambda$  elements, then the  $(\mathbf{g}, \mathbf{h})$ -contractions of  $E$  are all formulas of the form  $(\exists \vec{z} \underset{(\alpha)}{)} F'(y_j \mapsto \mathbf{g}_j(\vec{z}) \mid j < \lambda)$  (with free variables of  $E$  being assigned to themselves), for one-to-one enumerations  $(y_j \mid j < \lambda)$  of  $Y$ .

Although a formula usually has more than one  $(\mathbf{g}, \mathbf{h})$ -contraction, all its  $(\mathbf{g}, \mathbf{h})$ -contractions are logically equivalent.

Every  $\mathbf{w}$ -structure  $M$  such that  $(\text{card } M)^\alpha = (\text{card } M)^\lambda$  has a  $(\mathbf{w} \cup \mathbf{b})$ -expansion, which we shall also denote by  $M$ , such that the interpretations  $g \stackrel{\text{def}}{=} \mathbf{g}^M$  and  $h \stackrel{\text{def}}{=} \mathbf{h}^M$  are mutually inverse. The  $(\mathbf{g}, \mathbf{h})$ -contraction of  $M$  is the  $((\mathbf{w}|\alpha) \cup \mathbf{b})$ -structure on  $M$ , where all symbols in  $\mathbf{w} \cup \mathbf{b}$  which are not relation symbols in  $\mathbf{w} \setminus \mathbf{v}$  with arity  $\lambda$  have the same interpretations in  $M$  and  $M'$ , and for every relation symbol  $R \in \mathbf{w} \setminus \mathbf{v}$  with arity  $\lambda$ ,

$$R_\alpha^{M'} \stackrel{\text{def}}{=} \{\vec{x} \in {}^\alpha M \mid M \models \text{Rg}(\vec{x})\} = h[R^M]. \quad (8.1)$$

**Lemma 8.4.** *Every  $((\mathbf{w}|\alpha) \cup \mathbf{b})$ -structure  $M'$ , satisfying both statements  $\mathbf{h} \circ \mathbf{g} = \text{id}$  and  $\mathbf{g} \circ \mathbf{h} = \text{id}$ , is the  $(\mathbf{g}, \mathbf{h})$ -contraction of a unique  $(\mathbf{w} \cup \mathbf{b})$ -structure  $M$  satisfying those statements.*

*Proof.* For any relation symbol  $R \in \mathbf{w} \setminus \mathbf{v}$  with arity  $\lambda$ ,  $R^M$  can be recovered from  $R_\alpha^{M'}$  *via* (8.1): namely,  $R^M = g[R_\alpha^{M'}]$ .  $\square$

<sup>8</sup>This notation is slightly overloaded: indeed,  $\mathbf{w}|\alpha$  also depends of  $\mathbf{v}$ .

**Lemma 8.5.** *Let  $M$  be a  $(\mathbf{w} \cup \mathbf{b})$ -structure satisfying both statements  $\mathbf{h} \circ \mathbf{g} = \text{id}$  and  $\mathbf{g} \circ \mathbf{h} = \text{id}$ , and let  $M'$  be the  $(\mathbf{g}, \mathbf{h})$ -contraction of  $M$ . Then for every  $E \in \mathcal{L}_{\infty\lambda+}(\mathbf{w} \cup \mathbf{b})$ , every  $(\mathbf{g}, \mathbf{h})$ -contraction  $E'$  of  $E$ , and every  $\vec{a} \in \text{var}(E)M$ ,  $M \models E(\vec{a})$  iff  $M' \models E'(\vec{a})$ .*

*Proof.* By induction on the formula  $E$ . If  $E$  is atomic, then the result is trivial unless  $E = R(\vec{t})$  where  $R \in \mathbf{w} \setminus \mathbf{v}$  with arity  $\lambda$  and  $\vec{t} = (t_\xi \mid \xi < \lambda)$  for terms  $t_\xi$  of  $\mathbf{w} \cup \mathbf{b}$ . Then  $M' \models E'(\vec{a})$  iff  $M' \models R_\alpha(\mathbf{h}(\vec{t}(\vec{a})))$ , iff  $M \models R(\mathbf{gh}(\vec{t}(\vec{a})))$  (because each  $t_\xi^M(\vec{a}) = t_\xi^{M'}(\vec{a})$ ), iff  $M \models R(\vec{t}(\vec{a}))$ , that is,  $M \models E(\vec{a})$ . The induction steps corresponding to disjunction and negation are trivial.

The only nontrivial remaining step is thus the one where  $E$  is  $(\exists Y)F$  where  $Y$  is a set of free variables of  $F$  with cardinality  $\lambda$ . Any  $(\mathbf{g}, \mathbf{h})$ -contraction  $E'$  of  $E$  has the form  $(\exists \vec{z})_{(\alpha)} F'(y_j \mapsto g_j(\vec{z}) \mid j < \lambda)$ , for a  $(\mathbf{g}, \mathbf{h})$ -contraction  $F'$  of  $F$  and a one-to-one enumeration  $(y_j \mid j < \lambda)$  of  $Y$ . Hence  $M' \models E'(\vec{a})$  iff there exists  $\vec{c} \in {}^\alpha M$  such that  $M' \models F'(y_j \mapsto g_j(\vec{c}) \mid j < \lambda)$ . Since  $g$  is surjective,  $(g_j(\vec{c}) \mid j < \lambda)$  can take any value in  ${}^\lambda M$ , so  $M' \models E'(\vec{a})$  iff  $M' \models F'(\vec{a}, \vec{b})$  for some  $\vec{b} \in {}^Y M$ . By the induction hypothesis, this is equivalent to saying that  $M \models F(\vec{a}, \vec{b})$  for some  $\vec{b} \in {}^Y M$ ; that is,  $M \models E(\vec{a})$ .  $\square$

**Theorem 8.6.** *Let  $\lambda$  be a singular cardinal. Then  $\text{PC}(\mathcal{L}_{\infty\lambda}) = \text{PC}(\mathcal{L}_{\infty\lambda+}) = \text{RPC}(\mathcal{L}_{\infty\lambda+}) = \text{RPC}(\mathcal{L}_{\infty\lambda})$ .*

*Proof.* We know from Theorem 6.8 that  $\text{PC}(\mathcal{L}_{\infty\lambda+}) = \text{RPC}(\mathcal{L}_{\infty\lambda+})$ . Hence, due to the trivial containments  $\text{PC}(\mathcal{L}_{\infty\lambda}) \subseteq \text{RPC}(\mathcal{L}_{\infty\lambda}) \subseteq \text{RPC}(\mathcal{L}_{\infty\lambda+})$ , it suffices to prove the containment  $\text{PC}(\mathcal{L}_{\infty\lambda+}) \subseteq \text{PC}(\mathcal{L}_{\infty\lambda})$ .

Let  $\mathcal{C}$  be a  $\text{PC}(\mathcal{L}_{\infty\lambda+})$ -class over a vocabulary  $\mathbf{v}$ . By Theorem 4.11, there are a vocabulary  $\mathbf{w}$  containing  $\mathbf{v}$ , with each symbol in  $\mathbf{w} \setminus \mathbf{v}$  of arity an ordinal smaller than or equal to  $\lambda$ , and a sentence  $E \in \mathcal{L}_{\infty\lambda+}(\mathbf{w})$  such that  $\mathcal{C} = \{N \upharpoonright_{\mathbf{v}} \mid N \in \text{Mod}_{\mathbf{w}}(E)\}$ . Replacing every operation in  $\mathbf{w} \setminus \mathbf{v}$  by a new relation symbol, viewed as its graph, and modifying  $E$  accordingly, we may assume that  $\mathbf{w} \setminus \mathbf{v}$  consists of relation symbols only.

For every nonzero cardinal  $\alpha < \lambda$ , let  $\mathbf{b}_\alpha$  be a  $\lambda$ -ary vocabulary, with terms  $\mathbf{g}_\alpha$  and  $\mathbf{h}_\alpha$ , satisfying the conclusion of Lemma 8.3. We may assume that  $\mathbf{w}$  and the  $\mathbf{b}_\alpha$  are mutually disjoint, and that the additional relation symbols  $R_\alpha$  are all pairwise distinct and do not belong to  $\mathbf{w} \cup \bigcup_{\xi \neq \alpha} \mathbf{b}_\xi$ . Any  $(\mathbf{g}_\alpha, \mathbf{h}_\alpha)$ -contraction  $E_\alpha$  of  $E$  is a preformula in  $\mathcal{L}_{\infty\lambda}((\mathbf{w} \upharpoonright_\alpha) \cup \mathbf{b}_\alpha)$  with the same free variables as  $E$  (there are none); thus  $E_\alpha$  is a sentence in  $\mathcal{L}_{\infty\lambda}((\mathbf{w} \upharpoonright_\alpha) \cup \mathbf{b}_\alpha)$ . The disjunction  $E^*$ , over all nonzero  $\alpha < \lambda$ , of all statements  $(\mathbf{g}_\alpha \circ \mathbf{h}_\alpha = \text{id}) \wedge (\mathbf{h}_\alpha \circ \mathbf{g}_\alpha = \text{id}) \wedge E_\alpha$ , is an  $\mathcal{L}_{\infty\lambda}$ -statement over  $\mathbf{w}^* \stackrel{\text{def}}{=} \bigcup_{0 < \alpha < \lambda} ((\mathbf{w} \upharpoonright_\alpha) \cup \mathbf{b}_\alpha)$ . Hence,  $\mathcal{C}^* \stackrel{\text{def}}{=} \{N \upharpoonright_{\mathbf{v}} \mid N \in \text{Mod}_{\mathbf{w}^*}(E^*)\}$  is a  $\text{PC}(\mathcal{L}_{\infty\lambda})$ -definable class. Therefore, in order to conclude the proof, it suffices, by Lemma 3.2, to verify that  $\mathcal{C}^* \subseteq \mathcal{C}$  and that  $\mathcal{C} \setminus \mathcal{C}^*$  is small-abstract.

Let  $M \in \mathcal{C}^*$ . By definition, there are a nonzero cardinal  $\alpha < \lambda$  and a  $\mathbf{w}^*$ -structure  $N^*$ , satisfying  $(\mathbf{g}_\alpha \circ \mathbf{h}_\alpha = \text{id}) \wedge (\mathbf{h}_\alpha \circ \mathbf{g}_\alpha = \text{id}) \wedge E_\alpha$ , such that  $M = N^* \upharpoonright_{\mathbf{v}}$ . By Lemma 8.4,  $N^* \upharpoonright_{(\mathbf{w} \upharpoonright_\alpha) \cup \mathbf{b}_\alpha}$  is the  $(\mathbf{g}_\alpha, \mathbf{h}_\alpha)$ -contraction of a unique  $(\mathbf{w} \cup \mathbf{b}_\alpha)$ -structure  $N_\alpha$ . Since  $N^* \models E_\alpha$ , it follows from Lemma 8.5 that  $N_\alpha \models E$ . Therefore,  $M = N^* \upharpoonright_{\mathbf{v}} = N_\alpha \upharpoonright_{\mathbf{v}}$  belongs to  $\mathcal{C}$ . This completes the proof that  $\mathcal{C}^* \subseteq \mathcal{C}$ .

Let  $M \in \mathcal{C}$  with cardinality  $\kappa \geq 2^\lambda$ . Then  $M = N \upharpoonright_{\mathbf{v}}$  for some  $N \in \text{Mod}_{\mathbf{w}}(E)$ . By Lemma 8.1, there exists a nonzero cardinal  $\alpha < \lambda$  such that  $\kappa^\alpha = \kappa^\lambda$ . By Lemma 8.3,  $N$  expands to a  $(\mathbf{w} \cup \mathbf{b}_\alpha)$ -structure  $N_\alpha$  satisfying both statements

$h_\alpha \circ g_\alpha = \text{id}$  and  $g_\alpha \circ h_\alpha = \text{id}$ . The  $(g_\alpha, h_\alpha)$ -contraction  $N'_\alpha$  of  $N_\alpha$  is a  $((\mathbf{w}|\alpha) \cup \mathbf{b}_\alpha)$ -structure, thus it expands in turn to a  $\mathbf{w}^*$ -structure  $N^*$ . From  $N \models E$  it follows that  $N_\alpha \models E$ , thus, by Lemma 8.5,  $N'_\alpha \models E_\alpha$ ; that is,  $N^* \models E_\alpha$ . Therefore,  $N^* \models E^*$ , so  $M = N^* \upharpoonright_{\mathbf{v}} \in \mathcal{C}^*$ . This proves that  $\mathcal{C} \setminus \mathcal{C}^*$  is contained in the small-abstract class  $\{M \in \mathbf{Str}(\mathbf{v}) \mid \text{card } M < 2^\lambda\}$ .  $\square$

Bringing together Theorems 6.8 and 8.6, we obtain:

**Corollary 8.7.**  $\text{PC}(\mathcal{L}_{\infty\lambda}) = \text{RPC}(\mathcal{L}_{\infty\lambda})$ , for any infinite cardinal  $\lambda$ .

The influence of the regularity / singularity of  $\lambda$  on the behavior of  $\text{PC}(\mathcal{L}_{\infty\lambda})$  (thus also  $\text{RPC}(\mathcal{L}_{\infty\lambda})$ ) can be seen on the following example. That example also shows that the condition that  $\mathbf{v}$  be  $\lambda$ -ary cannot be dispensed with in the statement of Theorem 7.5.

**Example 8.8** (The Empty). *Let  $\lambda$  be an infinite cardinal and let the vocabulary  $\mathbf{v}_\lambda$  consist of the single relation symbol  $R$  with  $\text{ar}(R) = \lambda$ . Set  $\mathcal{E}_\lambda \stackrel{\text{def}}{=} \mathbf{Mod}_{\mathbf{v}_\lambda}(E_\lambda)$  where  $E_\lambda$  is the  $\mathcal{L}_{\infty\lambda^+}(\mathbf{v}_\lambda)$ -sentence  $(\forall \vec{x}) \neg R(\vec{x})$ . Then  $\mathcal{E}_\lambda$  is the range of a finitely  $(\lambda)$  continuous functor on  $\mathbf{Set}_{\neq \emptyset}$ . Moreover,  $\mathcal{E}_\lambda$  is  $\text{PC}(\mathcal{L}_{\infty\lambda})$  iff  $\lambda$  is singular.*

*Proof.* Let  $\Phi: \mathbf{Set}_{\neq \emptyset} \rightarrow \mathcal{E}_\lambda$ ,  $M \mapsto (M, \emptyset)$ , naturally extended at morphism level. Then  $\mathcal{E}_\lambda$  is the range of  $\Phi$ . Moreover,  $\mathcal{E}_\lambda$  is  $\mathcal{L}_{\infty\lambda^+}$ -definable, thus  $\text{PC}(\mathcal{L}_{\infty\lambda^+})$ . By Theorem 8.6, if  $\lambda$  is singular, then  $\mathcal{E}_\lambda$  is also  $\text{PC}(\mathcal{L}_{\infty\lambda})$ .

Finally, let  $\lambda$  be regular and suppose that  $\mathcal{E}_\lambda = \{M \upharpoonright_{\mathbf{v}_\lambda} \mid M \in \mathbf{Mod}_{\mathbf{w}}(E)\}$  for some vocabulary  $\mathbf{w}$  extending  $\mathbf{v}_\lambda$  and some statement  $E \in \mathcal{L}_{\infty\lambda}(\mathbf{w})$ . Set  $\theta \stackrel{\text{def}}{=} (\text{card } \mathbb{T}_\lambda(\mathbf{w}))^\lambda$  (cf. Notation 4.1). By Lemma 8.1, there exists an infinite cardinal  $\kappa \geq \theta$  such that  $\kappa^\lambda < \kappa^\lambda$ . Since  $(\kappa, \emptyset) \in \mathcal{E}_\lambda$ , there exists  $M \in \mathbf{Mod}_{\mathbf{w}}(E)$  such that  $(\kappa, \emptyset) = (M, R^M)$ .

Let  $(v_\xi \mid \xi < \lambda)$  be a  $\lambda$ -sequence of distinct variables, set  $X_\alpha \stackrel{\text{def}}{=} \{v_\xi \mid \xi < \alpha\}$  for each  $\alpha < \lambda$ , and  $D = \bigcup_{\alpha < \lambda} D_\alpha$  where, for  $\alpha < \lambda$ ,

$$D_\alpha \stackrel{\text{def}}{=} \left\{ (s_\xi^M(\vec{a}) \mid \xi < \lambda) \mid \text{all } s_\xi \in \mathbb{T}_\lambda(\mathbf{w}), \bigcup_{\xi < \lambda} \text{var}(s_\xi) \subseteq X_\alpha, \text{ and } \vec{a} \in X_\alpha^\kappa \right\}.$$

Then  $\text{card } D \leq \theta \cdot \kappa^\lambda = \kappa^\lambda$ . Since  $D \subseteq {}^\lambda \kappa$  and  $\kappa^\lambda < \kappa^\lambda$ , there exists  $\vec{a} \in {}^\lambda \kappa \setminus D$ . Let  $M'$  be the model obtained from  $M$  by changing  $R^M$  (which is empty) to  $\{\vec{a}\}$ . Since every formula in  $\mathcal{L}_{\infty\lambda}(\mathbf{w})$  has less than  $\lambda$  free variables, and since  $\{\vec{a}\} \cap D = \emptyset \cap D = \emptyset$ ,  $M$  and  $M'$  satisfy the same atomic  $\mathcal{L}_{\infty\lambda}$ -sentences with parameters from  $\kappa$ , hence (as  $|M| = |M'| = \kappa$ ) the same  $\mathcal{L}_{\infty\lambda}$ -sentences. In particular,  $M' \models E$ , so  $(\kappa, \{\vec{a}\}) = M' \upharpoonright_{\mathbf{v}_\lambda}$  belongs to  $\mathcal{E}_\lambda$ ; a contradiction.  $\square$

## 9. BACK-AND-FORTH SYSTEMS, ANTI-ELEMENTARITY, AND TUURI'S INTERPOLATION THEOREM

**9.1. Karttunen's back-and-forth systems.** There are a number of works establishing *non-representability*, of a given class  $\mathcal{C}$  of structures over a given vocabulary, as the class of models of any *class* of  $\mathcal{L}_{\infty\lambda}$ -sentences, for any regular cardinal  $\lambda$ . Those negative results are achieved by establishing, in one form or another, that  $\mathcal{C}$  is not closed under a suitable notion of back-and-forth equivalence. It turns out that the following instance of the latter, introduced in Karttunen [19] as “partial isomorphism with weak  $\lambda$ -extension property”, denoted there as  $M \cong_w^{\lambda_e} N$  and

that we will denote here in the form  $\mathbf{M} \triangleleft_{\lambda} \mathbf{N}$ , is sufficiently general to encompass all those situations.

**Definition 9.1.** Let  $\lambda$  be a regular cardinal and let  $\mathbf{M}$  and  $\mathbf{N}$  be models over the same vocabulary. A  $\lambda$ -back-and-forth system, in short  $\lambda$ -BFS, between  $\mathbf{M}$  and  $\mathbf{N}$ , is a poset  $(\mathcal{F}, \triangleleft)$ , together with a function  $f \mapsto \bar{f}$  with domain  $\mathcal{F}$ , such that each  $\bar{f}$  is an isomorphism from a submodel  $\mathbf{d}(f)$  of  $\mathbf{M}$  onto a submodel  $\mathbf{r}(f)$  of  $\mathbf{N}$ , and the following statements hold:

- (1) For all  $f, g \in \mathcal{F}$ ,  $f \triangleleft g$  implies that  $\bar{f} \subseteq \bar{g}$  (i.e.,  $\bar{g}$  extends  $\bar{f}$ ).
- (2) The poset  $(\mathcal{F}, \triangleleft)$  is  $\lambda$ -inductive, that is, every  $\lambda$ -small subchain of  $(\mathcal{F}, \triangleleft)$  has an upper bound in  $\mathcal{F}$ .
- (3) For every  $f \in \mathcal{F}$  and every  $x \in M$ , there exists  $g \in \mathcal{F}$  such that  $f \triangleleft g$  and  $x \in \mathbf{d}(g)$ .
- (4) For every  $f \in \mathcal{F}$  and every  $y \in N$ , there exists  $g \in \mathcal{F}$  such that  $f \triangleleft g$  and  $y \in \mathbf{r}(g)$ .

We will sum this up by writing  $\mathcal{F}: \mathbf{M} \triangleleft_{\lambda} \mathbf{N}$ . Let  $\mathbf{M} \triangleleft_{\lambda} \mathbf{N}$  hold if there exists a  $\lambda$ -BFS between  $\mathbf{M}$  and  $\mathbf{N}$ .

The condition, added by Karttunen in [19, Definition 3.2], that  $(\mathcal{F}, \triangleleft)$  be a tree, is redundant, in the sense that any  $\mathcal{F}: \mathbf{M} \triangleleft_{\lambda} \mathbf{N}$  gives rise to some  $\mathcal{F}': \mathbf{M} \triangleleft_{\lambda} \mathbf{N}$  where  $\mathcal{F}'$  is a tree: indeed, let  $\mathcal{F}'$  consist of all  $\triangleleft$ -ascending sequences  $(f_{\xi} \mid \xi \leq \alpha)$  from  $\mathcal{F}$ , with  $\alpha < \lambda$  and  $f_0 = e$  is a fixed element of  $\mathcal{F}$ , partially ordered under extension, and precompose the assignment  $f \mapsto \bar{f}$  with the “projection”  $\mathcal{F}' \rightarrow \mathcal{F}$ ,  $(f_{\xi} \mid \xi \leq \alpha) \mapsto f_{\alpha}$ .

The case where  $\triangleleft$  is function extension,  $f = \bar{f}$ ,  $\mathbf{d}(f) = \text{dom } f$ , and  $\mathbf{r}(f) = \text{rng } f$  whenever  $f \in \mathcal{F}$ , is Dickmann’s original  $\simeq_{\lambda}^{p,e}$  from [6, Definition 4.2.3].

As many aspects of the present paper are category-theoretical, we should point the reader to Beke and Rosický’s categorical take on back-and-forth presented in [3].

Many instances of  $\lambda$ -BFSs arise from the functors considered in the author’s paper [39], as stated in the forthcoming Proposition 9.3.

**Notation 9.2.** For any set  $\Omega$ , denote<sup>9</sup> by  $[\Omega]^{\text{inj}}$  the subcategory of **Set** whose objects are all the nonempty subsets of  $\Omega$  and whose arrows are the one-to-one maps between those.

**Proposition 9.3.** *Let  $\lambda$  be a regular cardinal, let  $\mathbf{v}$  be a  $\lambda$ -ary vocabulary, let  $\Omega$  be a set, let  $X, Y \subseteq \Omega$ , and let  $\Gamma: [\Omega]^{\text{inj}} \rightarrow \mathbf{Str}(\mathbf{v})$  be a  $\lambda$ -continuous functor. If either  $\text{card } X = \text{card } Y$  or  $\lambda \leq \min\{\text{card } X, \text{card } Y\}$ , then  $\Gamma(X) \triangleleft_{\lambda} \Gamma(Y)$ .*

*Note.* We are not assuming, in the statement of Proposition 9.3, that  $\Gamma$  turns embeddings to embeddings (that assumption would in fact trivialize the result). Nonetheless, by [39, Proposition 4.5], this is automatically the case for embeddings with domain of cardinality at least  $\lambda$ .

*Proof.* The conclusion is trivial if  $\text{card } X = \text{card } Y$ : let  $\mathcal{F}$  be a singleton, acting on  $\Gamma(X)$  via  $\Gamma(f)$  where  $f$  is any bijection from  $X$  onto  $Y$ .

Suppose now that  $\lambda \leq \min\{\text{card } X, \text{card } Y\}$ . We define  $\mathcal{F}$  as the set of bijections  $f: U \rightarrow V$  where  $U \in [X]^{<\lambda}$  and  $V \in [Y]^{<\lambda}$ , ordered by function extension (i.e.,  $f \triangleleft g$  if  $g$  extends  $f$ ). Further, denoting by  $\text{id}_S^T$  the inclusion map from  $S$  into  $T$ ,

<sup>9</sup>In [39],  $[\Omega]^{\text{inj}}$  is denoted by  $\mathfrak{P}_{\text{inj}}(\Omega)$ .

and setting  $e_S^T \stackrel{\text{def}}{=} \Gamma(\text{id}_S^T)$ , whenever  $S \subseteq T \subseteq \Omega$ , we set  $\mathbf{d}(f) = \mathbf{M}_U \stackrel{\text{def}}{=} e_U^X[\Gamma(U)]$  and  $\mathbf{r}(f) = \mathbf{N}_V \stackrel{\text{def}}{=} e_V^Y[\Gamma(V)]$ . Trivially,  $(\mathcal{F}, \trianglelefteq)$  is a  $\lambda$ -inductive poset.

**Claim.** *There exists a unique  $\mathbf{v}$ -isomorphism  $\bar{f}: \mathbf{M}_U \rightarrow \mathbf{N}_V$  such that  $\bar{f} \circ e_U^X = e_V^Y \circ \Gamma(f)$ .*

*Proof of Claim.* Let  $e_U^X(x_1) = e_U^X(x_2)$  where  $x_1, x_2 \in \Gamma(U)$ . Since  $\Gamma$  is a  $\lambda$ -continuous functor and  $X$  is the colimit of its  $\lambda$ -small subsets, we get

$$(\Gamma(X), e_S^X \mid S \in [X]^{<\lambda}) = \varinjlim (\Gamma(S), e_S^T \mid S \subseteq T \text{ in } [X]^{<\lambda}). \quad (9.1)$$

Hence, using Lemma 2.1, there exists  $X' \in [X]^{<\lambda}$  containing  $U$  such that

$$e_U^{X'}(x_1) = e_U^{X'}(x_2). \quad (9.2)$$

Since  $\text{card } X' < \lambda \leq \text{card } Y$ , the bijection  $f: U \rightarrow V$  extends to a one-to-one map  $g: X' \rightarrow Y$ . From  $g \circ \text{id}_{X'}^X = \text{id}_Y^Y \circ f$  it follows that  $\Gamma(g) \circ e_U^{X'} = e_V^Y \circ \Gamma(f)$ . From (9.2) it thus follows that  $e_V^Y \Gamma(f)(x_1) = e_V^Y \Gamma(f)(x_2)$ .

This completes the proof of the existence of a unique map  $\bar{f}: \mathbf{M}_U \rightarrow \mathbf{N}_V$  such that  $\bar{f} \circ e_U^X = e_V^Y \circ \Gamma(f)$ . A similar argument shows that  $\bar{f}$  is a  $\mathbf{v}$ -homomorphism.

By applying the above to the inverse map  $f^{-1}: V \rightarrow U$ , we obtain a unique  $\mathbf{v}$ -homomorphism  $\bar{g}: \mathbf{N}_V \rightarrow \mathbf{M}_U$  such that  $\bar{g} \circ e_V^Y = e_U^X \circ \Gamma(f^{-1})$ . It follows that  $\bar{g}\bar{f}e_U^X = \bar{g}e_V^Y \Gamma(f) = e_U^X \Gamma(f^{-1}) \Gamma(f) = e_U^X$ , whence  $\bar{g}\bar{f}$  is the identity on  $\mathbf{M}_U$ . Similarly,  $\bar{f}\bar{g}$  is the identity on  $\mathbf{N}_V$ . Therefore,  $\bar{f}$  and  $\bar{g}$  are mutually inverse isomorphisms.  $\square$  Claim.

It is then straightforward to verify that  $f \trianglelefteq g$  implies  $\bar{f} \subseteq \bar{g}$ , for all  $f, g \in \mathcal{F}$ . Let us verify Item (3) of Definition 9.1. Let  $f: U \rightarrow V$  in  $\mathcal{F}$  and let  $x \in \Gamma(X)$ . We need to enlarge  $f$  to some  $g \in \mathcal{F}$  such that  $x \in \mathbf{d}(g)$ . By (9.1) and Lemma 2.1, there exists a  $\lambda$ -small subset  $X'$  of  $X$ , containing  $U$ , such that  $x \in M_{X'}$ . Since  $\text{card } X' < \lambda \leq \text{card } Y$ ,  $f$  extends to a bijection  $g: X' \rightarrow Y'$  for some  $Y' \in [Y]^{<\lambda}$ . Now  $f \trianglelefteq g$  and  $x \in M_{X'} = \mathbf{d}(g)$ , so  $g$  is as required. The verification of Item (4) of Definition 9.1 is similar.  $\square$

**9.2. The infinitely deep languages  $\mathcal{M}_{\kappa\lambda}$ .** The infinitely deep language  $\mathcal{M}_{\kappa\lambda}$  (here  $\lambda \leq \kappa$  are either infinite cardinals or  $\infty$ ) extends the infinitary language  $\mathcal{L}_{\kappa\lambda}$  in the sense that infinite alternations of quantifiers are allowed; satisfaction is then defined in terms of existence of a winning strategy for the ‘‘semantic game’’. For example, the satisfaction of a ‘‘formula’’ of the form  $\forall x_0 \exists x_1 \forall x_2 \cdots \mathbf{E}(x_0, x_1, \dots)$  is expressed by the existence of a winning strategy for player  $\exists$  in a standard Gale-Stewart game. While the latter has ‘‘clock’’  $\omega$ , the games underlying the formulas from  $\mathcal{M}_{\kappa\lambda}$  may be clocked by *trees*. Since a full formal description of the logics  $\mathcal{M}_{\kappa\lambda}$  will not be needed here, we will omit it, and refer the reader to Hyttinen [13], Karttunen [19], Rantala [30], Tuuri [34] for more details. The main result of Tuuri [34], extending a result from Hyttinen [13], states as follows.

**Proposition 9.4.** *Let  $\kappa$  be a regular cardinal, set  $\lambda \stackrel{\text{def}}{=} \kappa^{\kappa}$ , let  $\mathbf{v}$  be a  $\kappa$ -ary vocabulary, and let  $\mathbf{E}$  and  $\mathbf{F}$  be sentences of  $\mathcal{L}_{\kappa+\kappa}(\mathbf{v})$  such that the conjunction  $\mathbf{E} \wedge \mathbf{F}$  has no  $\mathbf{v}$ -model. Then there exists a sentence  $\mathbf{G}$  of  $\mathcal{M}_{\lambda+\lambda}$ , with vocabulary the intersection of the vocabularies of  $\mathbf{E}$  and  $\mathbf{F}$ , such that every  $\mathbf{v}$ -structure satisfies both sentences  $\mathbf{E} \Rightarrow \mathbf{G}$  and  $\mathbf{F} \Rightarrow \sim \mathbf{G}$ .*

Here,  $\sim G$  denotes the sentence obtained by interchanging  $\bigvee$  and  $\bigwedge$ ,  $\exists$  and  $\forall$ ,  $A$  and  $\neg A$  in the expression of  $G$  by a tree-clocked game; it implies the usual negation  $\neg G$  (which, however, is no longer an  $\mathcal{M}_{\lambda+\lambda}$ -sentence). By a counterexample due to Malitz [23, Theorem 4.2],  $\mathcal{M}_{\lambda+\lambda}$  cannot be replaced by  $\mathcal{L}_{\infty\infty}$  in the statement of Proposition 9.4. Karttunen states in [19, Theorem 3.5] that whenever  $M$  and  $N$  are  $\mathbf{v}$ -structures,  $M \equiv_{\lambda} N$  implies<sup>10</sup> that  $M$  and  $N$  satisfy the same sentences in a logic, denoted there by  $\mathcal{N}_{\infty\lambda}$ , extending  $\mathcal{M}_{\infty\lambda}$ , with uniquely defined syntax but allowing for several (quite unwieldy) nonequivalent definitions of satisfaction. Since we will need that result only for the far better behaved  $\mathcal{M}_{\infty\lambda}$ , we record here the corresponding statement.

**Proposition 9.5.** *Let  $M$  and  $N$  be  $\mathbf{v}$ -structures. If  $M \equiv_{\lambda} N$ , then  $M$  and  $N$  satisfy the same  $\mathcal{M}_{\infty\lambda}(\mathbf{v})$ -sentences.*

## 10. APPLICATION TO INTRACTABILITY RESULTS

All intractability results stated in the author's paper [39] establish a property, called there *anti-elementarity*, for pairs of classes  $\mathcal{C}_0, \mathcal{C}_1$  of  $\mathbf{v}$ -structures with  $\mathcal{C}_0 \subseteq \mathcal{C}_1$ . By definition,  $(\mathcal{C}_0, \mathcal{C}_1)$  is anti-elementary if there are arbitrarily large infinite cardinals  $\lambda < \kappa$ , with  $\lambda$  regular and a  $\lambda$ -continuous functor  $\Gamma: [\kappa]^{\text{inj}} \rightarrow \mathbf{Str}(\mathbf{v})$ , such that  $\Gamma(\lambda) \in \mathcal{C}_0$  and  $\Gamma(\kappa) \notin \mathcal{C}_1$ . Then every class  $\mathcal{C}$  with  $\mathcal{C}_0 \subseteq \mathcal{C} \subseteq \mathcal{C}_1$  is anti-elementary, in the sense that  $(\mathcal{C}, \mathcal{C})$  is anti-elementary. Since  $\Gamma(\lambda)$  and  $\Gamma(\kappa)$  are  $\mathcal{L}_{\infty\lambda}$ -elementarily equivalent, it follows that  $\mathcal{C}$  is not closed under  $\mathcal{L}_{\infty\lambda}$ -elementary equivalence. In fact, by Proposition 9.3,  $\Gamma(\lambda) \equiv_{\lambda} \Gamma(\kappa)$ , thus it follows from Proposition 9.5 that  $\mathcal{C}$  is not closed under  $\mathcal{M}_{\infty\lambda}$ -elementary equivalence. In particular, by Proposition 9.4,  $\mathcal{C}$  cannot be simultaneously PC( $\mathcal{L}_{\infty\infty}$ ) and co-PC( $\mathcal{L}_{\infty\infty}$ ).

We shall illustrate that line of argument with a sample of extensions of various intractability results mostly arising from [39].

**10.1. Posets of convex  $\ell$ -subgroups in  $\ell$ -groups.** Denote by  $\text{Cs}_c G$  the (distributive) lattice with zero consisting of all finitely generated convex  $\ell$ -subgroups of an  $\ell$ -group  $G$ . As we verified in [38], this lattice is *Cevian*, that is, it has a binary operation  $\searrow$  such that all inequalities  $x \leq y \vee (x \searrow y)$ ,  $(x \searrow y) \wedge (y \searrow x) = 0$ , and  $x \searrow z \leq (x \searrow y) \vee (y \searrow z)$  hold on that lattice. The class  $\mathbf{Cev}$  of all Cevian lattices is obviously PC( $\mathcal{L}_{\omega\omega}$ )-definable.

Our next result involves the (non-commutative) diagram  $\vec{A}$ , of Archimedean  $\ell$ -groups and  $\ell$ -embeddings, introduced in [38, § 4] (also described in [39, § 12]).

**Theorem 10.1.** *For any accessible category  $\mathcal{G}$  of  $\ell$ -groups and  $\ell$ -homomorphisms, containing all objects and arrows of  $\vec{A}$  and closed under products and  $\lambda$ -directed colimits of  $\ell$ -groups, for some regular cardinal  $\lambda$ , there is no co-PC( $\mathcal{L}_{\infty\infty}$ ) class  $\mathcal{C}$  of Cevian lattices containing  $\text{Cs}_c \mathcal{G}$ . In particular,  $\text{Cs}_c \mathcal{G}$  is not co-PC( $\mathcal{L}_{\infty\infty}$ ).*

*Proof.* By assumption,  $\mathcal{G}$  contains the category denoted by  $\mathcal{A}(\theta, \vec{A})$  in [39, Notation 12.2], for some regular cardinal  $\theta$ . By [39, Theorem 12.3], there are arbitrarily large regular cardinals  $\lambda < \kappa$  with  $\lambda$ -continuous functors  $\Gamma: [\kappa]^{\text{inj}} \rightarrow \mathbf{Str}(\vee, \wedge, 0)$  such that  $\Gamma(\lambda) \in \text{Cs}_c \mathcal{G}$  and  $\Gamma(\kappa) \notin \mathbf{Cev}$ . Since  $\lambda$  is arbitrarily large, we may assume

<sup>10</sup>Strictly speaking, the results of [19] are stated there for *finitary* vocabularies. Nonetheless, it is not hard to verify that the part concerning Proposition 9.5 extends to  $\lambda$ -ary vocabularies. It is plausible that this would extend to  $\mathcal{N}_{\infty\lambda}$  as well, but we did not verify this.

that  $\mathcal{G}$  is closed under  $\lambda$ -directed colimits. Since the functor  $\text{Cs}_c$  is finitely continuous on all  $\ell$ -groups, its restriction  $\Phi$  to  $\mathcal{G}$  is  $\lambda$ -continuous, thus, by Theorem 7.5, the class  $\text{Cs}_c \mathcal{G} = \Phi(\mathcal{G})$  is  $\text{PC}(\mathcal{L}_{\infty\lambda})$ . Since  $\mathcal{C}$  is  $\text{co-PC}(\mathcal{L}_{\infty\infty})$  and by Proposition 9.4, there exists an  $\mathcal{M}_{\infty\infty}(\vee, \wedge, 0)$ -sentence  $\mathbf{E}$  such that  $\text{Cs}_c \mathcal{G} \subseteq \mathbf{Mod}_{(\vee, \wedge, 0)}(\mathbf{E}) \subseteq \mathcal{C}$ . In particular,  $\Gamma(\lambda) \models \mathbf{E}$  and  $\Gamma(\kappa) \not\models \mathbf{E}$ .

On the other hand, it follows from Proposition 9.3 that  $\Gamma(\lambda) \simeq_{\lambda} \Gamma(\kappa)$ . By Proposition 9.5,  $\Gamma(\lambda)$  and  $\Gamma(\kappa)$  satisfy the same  $\mathcal{M}_{\infty\lambda}(\vee, \wedge, 0)$ -sentences; a contradiction.  $\square$

Theorem 10.1 applies, in particular, to  $\mathcal{G}$  defined as any variety of  $\ell$ -groups containing all Archimedean  $\ell$ -groups. The particular case, where  $\mathcal{G}$  is the category of all Abelian  $\ell$ -groups, can thus be viewed as another negative solution to Mundici's *MV-spectrum Problem* [25, Problem 2] handled in [39]: it was proved there that the  $\text{PC}(\mathcal{L}_{\infty\omega})$  class  $\text{Cs}_c \mathcal{G}$  is not closed under any  $\mathcal{L}_{\infty\lambda}$ -elementary equivalence; now we know that it is not  $\text{co-PC}(\mathcal{L}_{\infty\infty})$  either, and that it is in fact not contained in any  $\text{co-PC}(\mathcal{L}_{\infty\infty})$  class of Cevian lattices.

Our next result aims at extending Theorem 10.1 to the functor that to every  $\ell$ -group  $G$  associates its  $(\vee, 0)$ -semilattice  $\text{Id}_c G$  of finitely generated  $\ell$ -ideals (which is isomorphic to the  $(\vee, 0)$ -semilattice of all finitely generated congruences of  $G$ ). This result will involve the forgetful functor  $\Psi$  introduced in Gillibert and Wehrung [10, § 5.1] (see also [39, § 13]), that to every “semilattice-metric space”  $(A, \delta, \tilde{A})$  of type 1, with  $\tilde{A}$  generated by the range of  $\delta$ , associates its underlying  $(\vee, 0)$ -semilattice  $\tilde{A}$ . Recalling the full definition of  $\Psi$  will not be needed here. However, the reader might find it helpful to record that  $\text{im } \Psi$  contains all  $(\vee, 0)$ -semilattices either of the form  $\text{Id}_c R \stackrel{\text{def}}{=} \text{semilattice of all finitely generated two-sided ideals of a ring } R$ , or  $\text{Id}_c G$  for an  $\ell$ -group  $G$ , or  $\text{Sub}_c M \stackrel{\text{def}}{=} \text{semilattice of all finitely generated submodules of a module } M$  (cf. [39, Proposition 13.1]). Our statement will also involve the (non-commutative) diagrams  $\vec{C}_{a,g}$ , indexed by non-representable<sup>11</sup>  $\ell$ -groups  $G$  and elements  $a, g \in C$  such that  $a > 0$  and  $a \wedge (g + a - g) = 0$ , introduced in [39, § 13].

**Theorem 10.2.** *For any accessible category  $\mathcal{G}$  of  $\ell$ -groups and  $\ell$ -homomorphisms, containing all objects and arrows of some  $\vec{C}_{a,g}$  for some non-representable  $\ell$ -group  $C$ , and closed under products and  $\lambda$ -directed colimits for some regular cardinal  $\lambda$ , there is no  $\text{co-PC}(\mathcal{L}_{\infty\infty})$  class  $\mathcal{C}$  such that  $\text{Id}_c \mathcal{G} \subseteq \mathcal{C} \subseteq \text{im } \Psi$ . In particular,  $\text{Id}_c \mathcal{G}$  is not  $\text{co-PC}(\mathcal{L}_{\infty\infty})$ .*

*Note.* The case of categories of representable  $\ell$ -groups is covered by Theorem 10.1: indeed, whenever  $G$  is representable,  $\text{Id}_c G$  is a lattice homomorphic image of  $\text{Cs}_c G$ ; hence it is Cevian (cf. [38, Proposition 5.10]).

*Proof.* By assumption,  $\mathcal{G}$  contains some  $\mathcal{A}(\theta, \vec{C}_{a,g})$  (cf. [39, Notation 12.2]) for some regular cardinal  $\theta$ . By [39, Theorem 13.8], there are arbitrarily large regular cardinals  $\lambda < \kappa$  with  $\lambda$ -continuous functors  $\Gamma: [\kappa]^{\text{inj}} \rightarrow \mathbf{Str}(\vee, \wedge, 0)$  such that  $\Gamma(\lambda) \in \text{Id}_c \mathcal{G}$  and  $\Gamma(\kappa) \notin \text{im } \Psi$ . Since  $\lambda$  is arbitrarily large, we may assume that  $\mathcal{G}$  is closed under  $\lambda$ -directed colimits. Since the functor  $\text{Id}_c$  is finitely continuous on all  $\ell$ -groups, its restriction  $\Phi$  to  $\mathcal{G}$  is  $\lambda$ -continuous, thus, by Theorem 7.5, the class  $\text{Id}_c \mathcal{G} = \Phi(\mathcal{G})$  is  $\text{PC}(\mathcal{L}_{\infty\lambda})$ . Since  $\mathcal{C}$  is  $\text{co-PC}(\mathcal{L}_{\infty\infty})$  and by Proposition 9.4, there

<sup>11</sup>Recall that an  $\ell$ -group is *representable* if it embeds into a product of totally ordered groups.

exists an  $\mathcal{M}_{\infty}(V, 0)$ -sentence  $\mathbf{E}$  such that  $\text{Cs}_c \mathcal{G} \subseteq \mathbf{Mod}_{(V, 0)}(\mathbf{E}) \subseteq \mathcal{C}$ . The desired conclusion then follows as in the last part of the argument of the proof of Theorem 10.2.  $\square$

For a further extension of the method of Subsection 10.1 to an *a priori* non-co-PC class, specific to the case of Abelian  $\ell$ -groups, see Section 11.

**10.2. Posets of ideals in rings.** Our next result is an analogue of Theorem 10.2 for (two-sided) ideal lattices of rings. It involves the (non-commutative) diagrams  $\vec{R}_{\mathbb{k}}$  of matrix algebras over fields  $\mathbb{k}$ , introduced in [39, § 13]. Its proof is, *mutatis mutandis*, identical to the one of Theorem 10.2, using [39, Theorem 13.15] in place of [39, Theorem 13.8], and we omit it.

**Theorem 10.3.** *For any accessible category  $\mathcal{R}$  of unital rings and unital ring homomorphisms, containing all objects and arrows of some  $\vec{R}_{\mathbb{k}}$ , and closed under products and  $\lambda$ -directed colimits for some regular cardinal  $\lambda$ , there is no co-PC( $\mathcal{L}_{\infty}$ ) class  $\mathcal{C}$  such that  $\text{Id}_c \mathcal{R} \subseteq \mathcal{C} \subseteq \text{im } \Psi$ . In particular,  $\text{Id}_c \mathcal{R}$  is not co-PC( $\mathcal{L}_{\infty}$ ).*

Theorem 10.3 applies, in particular, to the category of all unital von Neumann regular rings, also to the one of all unital rings.

**10.3. Nonstable  $\mathbf{K}_0$ -theory of rings.** We denote by  $V(R)$  the (commutative) monoid of isomorphism classes  $[X]$  of all finitely generated projective right  $R$ -modules  $X$ , for any unital ring  $R$ . A ring  $R$  is *V-semiprimitive* if for all idempotent matrices  $a$  and  $b$  of the same finite order over  $R$ , if  $aR$  and  $bR$  have no nontrivial isomorphic direct summands, then  $ab = 0$  (this can be expressed as a left-right symmetric statement). Our next statement involves the class  $\mathcal{C}$  of all monoids that can be represented as images of some  $V(R)$ , for a  $V$ -semiprimitive ring  $R$ , by some “pre- $V$ -homomorphism” as defined in [39, § 14] (this class of monoid homomorphisms includes all isomorphisms). Again, a full formal definition of  $\mathcal{C}$  will not be needed here. The proof of Theorem 10.4 is, *mutatis mutandis*, identical to the one of Theorem 10.2, using [39, Theorem 14.6] in place of [39, Theorem 13.8], and we omit it.

**Theorem 10.4.** *For any accessible category  $\mathcal{R}$  of  $V$ -semiprimitive unital rings and unital ring homomorphisms, containing all objects and arrows of some  $\vec{R}_{\mathbb{k}}$ , and closed under products and  $\lambda$ -directed colimits for some regular cardinal  $\lambda$ , there is no co-PC( $\mathcal{L}_{\infty}$ ) class  $\mathcal{C}'$  of monoids such that  $V(\mathcal{R}) \subseteq \mathcal{C}' \subseteq \mathcal{C}$ . In particular,  $V(\mathcal{R})$  is not co-PC( $\mathcal{L}_{\infty}$ ).*

Theorem 10.4 applies, in particular, to the category  $\mathbf{vNRing}$  of all unital von Neumann regular rings with unital ring homomorphisms, and also to the category  $\mathbf{RR}_0$  of all unital  $C^*$ -algebras of real rank zero and unit-preserving  $C^*$ -homomorphisms (take  $\mathbb{k} \stackrel{\text{def}}{=} \mathbb{C}$  for the latter). Note that in the latter case, directed colimits are not preserved (the  $C^*$ -colimit is the topological closure of the algebraic colimit); nevertheless,  $\lambda$ -directed colimits are preserved whenever  $\lambda$  is uncountable.

**10.4. Coordinatizability of sectionally complemented modular lattices.** A lattice  $L$  is *coordinatizable* if it is isomorphic to the lattice  $\mathbb{L}(R)$  of all principal right ideals of any (not necessarily unital) von Neumann regular ring  $R$ . This implies that  $L$  is sectionally complemented and modular. Conversely, if a sectionally

complemented modular lattice  $L$  has a countable cofinal sequence, then the coordinatizability of  $L$  follows from an algebraic condition with geometric inspiration, introduced in Jónsson [17], called *existence of a large (partial) 4-frame*. The category  $\mathcal{A}$ , introduced in [39, § 15], is a category of von Neumann regular rings, with additional structure, arising from matrix units, ensuring that every member of  $\mathbb{L}(\mathcal{A})$  has a large 4-frame. It is  $\omega_1$ -accessible, and the restriction  $\Phi$  of the functor  $\mathbb{L}$  to  $\mathcal{A}$  is finitely continuous. Hence, using the  $(\kappa, < \omega, \lambda) \rightarrow \rho$  notation from Erdős *et al.* [8], together with the same token as the one used in the proof of Theorem 10.1 above, and applying [39, Theorem 5.9], we obtain the following.

**Theorem 10.5.** *Assume that for every cardinal  $\lambda$  there exists a cardinal  $\kappa$  such that  $(\kappa, < \omega, \lambda) \rightarrow \aleph_1$  (this holds if there is a proper class of Erdős cardinals). Then the class of all coordinatizable lattices with a large 4-frame is not  $\text{co-PC}(\mathcal{L}_{\infty\infty})$ .*

Let us now move to intractability results obtained from earlier sources than [39], not obtained *via* anti-elementarity.

Removing the large 4-frame condition but adding a unit element, we move to studying the class  $\mathcal{CL}$  of all coordinatizable, complemented, modular lattices. We verified in [35] that  $\mathcal{CL}$  is not the class of all models of any  $\mathcal{L}_{\infty\infty}$ -sentence — in fact, the argument of the proof of [35, Theorem 9.4] shows that  $\mathcal{CL}$  is not closed under  $\mathcal{L}_{\infty\lambda}$ -elementary equivalence, for any infinite cardinal  $\lambda$  (thus solving a problem from Jónsson [16]). Consider the class  $\mathcal{CL}_2$  of all 2-distributive<sup>12</sup> coordinatizable complemented modular lattices.

**Theorem 10.6.** *There is no  $\text{co-PC}(\mathcal{L}_{\infty\infty})$  class  $\mathcal{C}'$  of lattices with  $\mathcal{CL}_2 \subseteq \mathcal{C}' \subseteq \mathcal{CL}$ . In particular, neither  $\mathcal{CL}_2$  nor  $\mathcal{CL}$  is  $\text{co-PC}(\mathcal{L}_{\infty\infty})$ .*

*Outline of proof.* For any set  $X$  with more than one element, denote by  $M_X$  the (2-distributive, complemented, modular) lattice  $X \sqcup \{0, 1\}$  with the only nontrivial relations  $0 < x < 1$  for  $x \in X$ . Let  $\lambda$  be a regular cardinal, set  $\kappa \stackrel{\text{def}}{=} \lambda^+$ . Due to the need for introducing the  $\lambda$ -BFS  $\mathcal{F}_\lambda$  below, we need to modify the definitions of the lattices  $\bar{L}_\lambda$  and  $L_\lambda$  introduced in [35, § 9], by changing “finite” to “ $\lambda$ -small”. Our new lattices, for which we keep the same notation as in [35, § 9], are thus the following:

$$\begin{aligned} \bar{L}_\lambda &\stackrel{\text{def}}{=} \left\{ x \in {}^\kappa M_{\lambda+1} \mid (\exists X \in [\kappa]^{<\lambda})(x \upharpoonright_{\kappa \setminus X} \text{ is constant}) \right\}, \\ L_\lambda &\stackrel{\text{def}}{=} \{ x \in \bar{L}_\lambda \mid x_\infty \in M_\lambda \}. \end{aligned}$$

where  $x_\infty$  denotes the constant value of  $x_\xi$  for  $\xi$  outside a large enough  $\lambda$ -small subset of  $\kappa$ . It is straightforward to verify that the lattices  $\bar{L}_\lambda$  and  $L_\lambda$  are both complemented, modular, and 2-distributive.

**Claim.** *The lattice  $\bar{L}_\lambda$  is coordinatizable whereas  $L_\lambda$  is not.*

*Proof of Claim.* The proof that  $\bar{L}_\lambda$  is coordinatizable follows the argument presented at the beginning of the proof of [35, Theorem 9.4]: if  $D$  denotes any division ring with  $\lambda$  elements, then  $\bar{L}_\lambda$  is coordinatized by the von Neumann regular ring

$$\left\{ x \in {}^\kappa M_2(D) \mid (\exists X \in [\kappa]^{<\lambda})(x \upharpoonright_{\kappa \setminus X} \text{ is constant}) \right\}.$$

<sup>12</sup>A lattice  $L$  is 2-distributive if  $x \wedge (y_0 \vee y_1 \vee y_2) = \bigvee_{i < j} (x \wedge (y_i \vee y_j))$  for all  $x, y_0, y_1, y_2 \in L$ .

Although the proof that  $L_\lambda$  is not coordinatizable is much harder, it works along the same lines as the one of [35, Theorem 9.3], with the following modifications. Change  $\kappa$  and  $\kappa^+$  (from [35, Theorem 9.3]) to  $\lambda$  and  $\kappa$ , respectively; change “finite” to “ $\lambda$ -small”. Moreover, redefine  $J_X$ , for  $X \in [\kappa]^{<\lambda}$ , as the ideal  $\{x \in R \mid \varepsilon(xR) \upharpoonright_{\kappa \setminus X} = 0\}$ , and denote by  $J$  the union of all  $J_X$ . We keep the ideal  $I = \{\varepsilon(xR) \mid x \in J\} = \{x \in L_\lambda \mid x_\infty = 0\}$ . The remainder of the argument is, *mutatis mutandis*, identical to the one of the proof of [35, Theorem 9.3].  $\square$  Claim.

Since  $\mathbb{L}$  defines a finitely continuous functor on the finitely accessible category of all unital von Neumann regular rings, it follows from Theorem 7.5 that the class  $\mathcal{C}\mathcal{L}$  is  $\text{PC}(\mathcal{L}_{\infty\omega})$ ; thus so is also the class  $\mathcal{C}\mathcal{L}_2$  of all 2-distributive members of  $\mathcal{C}\mathcal{L}$ .

We claim that  $\bar{L}_\lambda \rightleftharpoons_\lambda L_\lambda$ . To that end, denote by  $\mathcal{F}_\lambda$  the set of all  $\kappa$ -sequences  $f = (f_\xi \mid \xi < \kappa)$ , constant on the complement of a  $\lambda$ -small set, where each  $f_\xi: \mathbf{d}(f_\xi) \rightarrow \mathbf{r}(f_\xi)$  is a bijection between  $\lambda$ -small subsets  $\mathbf{d}(f_\xi)$  and  $\mathbf{r}(f_\xi)$  of  $\lambda + 1$ , and such that  $\mathbf{r}(f_\infty) \subseteq \lambda$ . For  $f, g \in \mathcal{F}_\lambda$ , let  $f \leq g$  hold if each  $g_\xi$  extends  $f_\xi$ . To each  $f \in \mathcal{F}_\lambda$  we assign the map

$$\bar{f}: \bar{L}_\lambda \cap \prod_{\xi < \kappa} M_{\mathbf{d}(f_\xi)} \rightarrow L_\lambda \cap \prod_{\xi < \kappa} M_{\mathbf{r}(f_\xi)}, \quad (x_\xi \mid \xi < \kappa) \mapsto (\bar{f}_\xi(x_\xi) \mid \xi < \kappa)$$

where each  $\bar{f}_\xi: M_{\mathbf{d}(f_\xi)} \rightarrow M_{\mathbf{r}(f_\xi)}$  is the unique 0, 1-preserving extension of  $f_\xi$ . This defines a  $\lambda$ -BFS  $\mathcal{F}_\lambda: \bar{L}_\lambda \rightleftharpoons_\lambda L_\lambda$ .

Since  $\mathcal{C}\mathcal{L}_2 \subseteq \mathcal{C}'$  and both  $\mathcal{C}\mathcal{L}_2$  and the complement of  $\mathcal{C}'$  are  $\text{PC}(\mathcal{L}_{\infty\infty})$ , there exists by Proposition 9.4 an  $\mathcal{M}_{\infty\infty}(\vee, \wedge)$ -sentence  $\mathbf{E}$  with  $\mathcal{C}\mathcal{L}_2 \subseteq \mathbf{Mod}_{(\vee, \wedge)}(\mathbf{E}) \subseteq \mathcal{C}'$ . Since  $\bar{L}_\lambda \in \mathcal{C}\mathcal{L}_2$  and  $L_\lambda \notin \mathcal{C}'$ , it follows that  $\bar{L}_\lambda \models \mathbf{E}$  and  $L_\lambda \not\models \mathbf{E}$ ; a contradiction since  $\bar{L}_\lambda \rightleftharpoons_\lambda L_\lambda$  and by Proposition 9.5.  $\square$

**10.5. Real spectra of commutative unital rings.** A subset  $P$  in a commutative, unital ring  $A$  is a *cone* if it is both an additive and multiplicative submonoid of  $A$  such that  $x^2 \in P$  whenever  $x \in A$ . The cone  $P$  is *prime* if  $P \cup (-P) = A$  and  $P \cap (-P)$  is a prime ideal of  $A$ . The *real spectrum* of  $A$  is the set  $\text{Spec}_r A$  of all prime cones of  $A$ , endowed with the topology generated by the sets  $\{P \in \text{Spec}_r A \mid a \notin P\}$  for  $a \in A$ . The topological space  $\text{Spec}_r A$  is *spectral*, which is equivalent to saying that it is the Stone dual of a bounded distributive lattice (cf. Stone [31], Johnstone [15, § II.3], Grätzer [12, § II.5]). Mellor and Tressl proved in [24, Theorem 5.1] that the class  $\mathcal{S}$  of Stone duals of real spectra is not closed under (strong)  $\lambda$ -back-and-forth equivalence, for any infinite  $\lambda$ ; hence it is not the class of models of any class of  $\mathcal{L}_{\infty\lambda}$ -sentences, for any infinite cardinal  $\lambda$ .

**Theorem 10.7.** *The class  $\mathcal{S}$ , of Stone duals of real spectra of commutative unital rings, is not  $\text{co-PC}(\mathcal{L}_{\infty\infty})$ .*

*Proof.* We first argue that  $\mathcal{S}$  is the image of a finitely continuous functor on a finitely accessible category. This can be done in more than one way. For example, the argument sketched at the end of the Introduction in [39], translated from spectra of Abelian  $\ell$ -groups to *Brumfiel spectra* (known to be equivalent to real spectra), shows that  $\mathcal{S}$  is  $\text{PC}(\mathcal{L}_{\omega_1\omega})$ . A more direct way to do this follows from Johnstone [15, p. 218], where the Stone dual of  $\text{Spec}_r A$ , which we will denote by  $\Phi(A)$ , is expressed as the bounded distributive lattice defined by generators  $D_r(a)$ , with  $a \in A$ , subjected to specific relations (e.g.,  $D_r(a) \wedge D_r(b) \leq D_r(a + b) \leq D_r(a) \vee D_r(b)$ ,

etc.). This entails that  $\Phi$  is a finitely continuous functor from the (finitely accessible) category of all commutative unital rings to the (finitely accessible) category of all bounded distributive lattices. By Theorem 7.5, the class  $\mathcal{S} = \text{im } \Phi$  is  $\text{PC}(\mathcal{L}_{\infty\omega})$ -definable.

If  $\mathcal{S}$  were also  $\text{co-PC}(\mathcal{L}_{\infty\infty})$ , then, by Proposition 9.4, it would be the class of models of some  $\mathcal{M}_{\infty\lambda}(\vee, 0)$ -sentence  $\mathbf{E}$ , for a regular cardinal  $\lambda$ . In particular,  $\mathcal{S}$  is closed under  $\hookrightarrow_{\lambda}$ , in contradiction with Mellor and Tressl [24, Theorem 5.1].  $\square$

## 11. BEYOND $\text{co-PC}$ : PLOŠČICA'S CONDITION AND $\ell$ -SPECTRA

Denote by  $\mathcal{A}$  the class of all Abelian  $\ell$ -groups. By Theorem 10.1,  $\text{Id}_c \mathcal{A}$  is not a  $\text{co-PC}(\mathcal{L}_{\infty\infty})$  class of lattices. On the other hand,  $\text{Id}_c \mathcal{A}$  is closed under isomorphic copies, thus, using Lemma 3.2, it is the intersection of all  $\text{co-PC}(\mathcal{L}_{\infty\omega})$  classes containing it. We shall introduce a condition, originating in Ploščica [28], expressible as a conjunction, ranging over all infinite cardinals, of  $\text{co-PC}(\mathcal{L}_{\infty\infty})$ -sentences, satisfied by all members of  $\text{Id}_c \mathcal{A}$ . We shall also prove (cf. Theorem 11.6) that under the GCH, this condition, together with any  $\text{co-PC}(\mathcal{L}_{\infty\infty})$ -sentence satisfied by all members of  $\text{Id}_c \mathcal{A}$ , is still not sufficient to characterize  $\text{Id}_c \mathcal{A}$ .

**Definition 11.1.** A lattice  $S$  has *countably based differences* (in abbreviation CBD; see Wehrung [37, § 10]) if every pair  $(x, y)$  has a *difference basis*, that is, a descending sequence  $(z_n \mid n < \omega)$  of elements of  $S$  such that for every  $z \in S$ ,  $x \leq y \vee z$  iff  $z_n \leq z$  for some  $n < \omega$ .

In particular, for every Abelian  $\ell$ -group  $G$ , the lattice  $\text{Id}_c G$  satisfies CBD.

**Definition 11.2.** For any cardinal number  $\lambda$ , define *Ploščica's Condition at  $\lambda$* , denoted by  $\text{Plo}_{\lambda}$ , the statement of lattice theory that for every element  $a$  and every  $\lambda$ -sequence  $(\mathfrak{m}_{\xi} \mid \xi < \lambda)$  of maximal ideals of  $\downarrow a$ , the quotient  $\downarrow a / \bigcap_{\xi < \lambda} \mathfrak{m}_{\xi}$  has at most  $2^{\lambda}$  elements (cf. Subsection 2.1 for the notation  $D/I$ ).

We leave the verification of the following lemma as an exercise.

**Lemma 11.3.** CBD is equivalent to an  $\mathcal{L}_{\omega_1\omega_1}$ -statement of lattice theory, and each  $\text{Plo}_{\lambda}$  is equivalent to a  $\text{co-PC}(\mathcal{L}_{\theta\theta})$ -statement of lattice theory where  $\theta \stackrel{\text{def}}{=} (2^{\lambda})^+$ .

**Lemma 11.4.** The following statements hold, for any distributive lattice  $D$  with zero:

- (1)  $D$  satisfies  $\text{Plo}_n$  whenever  $n < \omega$ .
- (2) If  $D$  satisfies  $\text{Plo}_{\kappa}$  for every  $\kappa \leq \text{card } D$ , then it satisfies all  $\text{Plo}_{\kappa}$ .
- (3) Suppose that  $D$  has CBD. For every cardinal  $\lambda$  with uncountable cofinality, if  $D \models \text{Plo}_{\alpha}$  whenever  $\alpha < \lambda$ , then  $D \models \text{Plo}_{\lambda}$ .
- (4) Suppose that  $D$  has CBD. For every infinite cardinal  $\lambda$ ,  $D \models \text{Plo}_{\lambda}$  implies  $D \models \text{Plo}_{\lambda^+}$ .

*Proof.* Ad (1). For every  $a \in D$  and every finite sequence  $(\mathfrak{m}_i \mid i < n)$  of maximal ideals of  $\downarrow a$ , it follows from the distributivity of  $D$  that the diagonal map embeds  $\downarrow a / \bigcap_{i < n} \mathfrak{m}_i$  into  $\prod_{i < n} (\downarrow a / \mathfrak{m}_i)$ . Since each  $\mathfrak{m}_i$  is a maximal ideal of  $\downarrow a$ ,  $\downarrow a / \mathfrak{m}_i$  is the two-element lattice.

Ad (2). Every intersection of a collection of subsets of  $D$  (here maximal ideals of some  $D \downarrow a$ ) is the intersection of a subcollection of size at most  $\text{card } D$ .

Ad (3). Setting  $\mathfrak{b}_{\alpha} \stackrel{\text{def}}{=} \bigcap_{\xi < \alpha} \mathfrak{m}_{\xi}$  for  $\alpha \leq \lambda$ ,  $\text{card}(\downarrow a / \mathfrak{b}_{\alpha}) \leq 2^{\text{card } \alpha} \leq 2^{\lambda}$  whenever  $\alpha < \lambda$ , thus it suffices to prove that the diagonal map  $\varepsilon: \downarrow a / \mathfrak{b}_{\lambda} \rightarrow \prod_{\alpha < \lambda} (\downarrow a / \mathfrak{b}_{\alpha})$  is

an order-embedding. Let  $x, y \in \downarrow a$  and let  $(z_n \mid n < \omega)$  be a difference basis of  $(x, y)$ . Then  $\varepsilon(x/\mathfrak{b}_\lambda) \leq \varepsilon(y/\mathfrak{b}_\lambda)$  means that for every  $\alpha < \lambda$  there exists  $n < \omega$  such that  $z_n \in \mathfrak{b}_\alpha$ . Denote by  $n_\alpha$  the least such  $n$ . Since the  $\lambda$ -sequence  $(\mathfrak{b}_\alpha \mid \alpha < \lambda)$  is descending, the  $\lambda$ -sequence  $(n_\alpha \mid \alpha < \lambda)$  is ascending. Since  $\text{cf } \lambda > \omega$ ,  $\{n_\alpha \mid \alpha < \lambda\}$  is bounded above, say by an integer  $m$ . Then  $z_m \in \mathfrak{b}_\lambda$ , so  $x \leq y \pmod{\mathfrak{b}_\lambda}$ .

*Ad* (4) follows immediately from (3).  $\square$

Denote by  $\text{Plo}_{\leq \kappa}$  (resp.,  $\text{Plo}_{< \infty}$ ) the conjunction of all  $\text{Plo}_\alpha$  over  $\alpha \leq \kappa$  (resp., over all  $\alpha$ ). A mild extension of the argument of the proof of Plošćica [28, Theorem 2.1] yields the following observation, of which we include a proof for convenience.

**Proposition 11.5.** *For every Abelian  $\ell$ -group  $G$ , the lattice  $\text{Id}_c G$  satisfies  $\text{Plo}_{< \infty}$ .*

*Proof.* By Lemma 11.4(1), we may assume that  $\lambda$  is infinite. The elements of  $\text{Id}_c G$  are the  $\langle u \rangle \stackrel{\text{def}}{=} \{x \in G \mid (\exists n < \omega)(|x| \leq nu)\}$  where  $u \in G^+$ , and then, for that element,  $(\text{Id}_c G) \downarrow \langle u \rangle = \text{Id}_c \langle u \rangle$ . Setting  $H \stackrel{\text{def}}{=} \langle u \rangle$  (an  $\ell$ -ideal of  $G$ ), we thus need to verify that for every  $\lambda$ -sequence  $(\mathfrak{m}_\xi \mid \xi < \lambda)$  of maximal ideals of  $\text{Id}_c H$ , with intersection  $\mathfrak{a} \stackrel{\text{def}}{=} \bigcap_{\xi < \lambda} \mathfrak{m}_\xi$ , the quotient  $(\text{Id}_c H)/\mathfrak{a}$  has at most  $2^\lambda$  elements. Every  $M_\xi \stackrel{\text{def}}{=} \{x \in H \mid \langle x \rangle \in \mathfrak{m}_\xi\}$  is a maximal  $\ell$ -ideal of  $H$ , thus  $H/M_\xi$  can be embedded into the reals (cf. Anderson and Feil [2, Theorem 2.3]). Further, setting  $A \stackrel{\text{def}}{=} \bigcap_{\xi < \lambda} M_\xi$ , the diagonal map embeds  $H/A$  into  $\prod_{\xi < \lambda} (H/M_\xi)$ ; it follows that  $\text{card}(H/A) \leq (2^{\aleph_0})^\lambda = 2^\lambda$ . Therefore  $(\text{Id}_c H)/\mathfrak{a}$ , which is isomorphic to  $\text{Id}_c(H/A)$  (cf. Wehrung [37, Lemma 2.5]), has at most  $2^\lambda$  elements.  $\square$

By Theorem 10.1, for any co-PC( $\mathcal{L}_{\infty\infty}$ ) sentence  $E$  of lattice theory, there is no cardinal  $\kappa$  such that  $E \wedge \text{Plo}_{\leq \kappa}$  characterizes  $\text{Id}_c \mathcal{A}$ . We shall now verify that under the GCH, the same can be said about  $\text{Plo}_{< \infty}$ .

**Theorem 11.6.** *Suppose that the GCH holds. Then for every co-PC( $\mathcal{L}_{\infty\infty}$ ) class  $\mathcal{C}$  of lattices containing the class  $\text{Id}_c \mathcal{A}$ , there exists a bounded distributive lattice in  $\mathcal{C} \setminus \text{Id}_c \mathcal{A}$  that satisfies  $\text{Plo}_{< \infty}$ .*

*Proof.* Since  $\mathcal{C}$  is co-PC( $\mathcal{L}_{\infty\infty}$ ), it is co-PC( $\mathcal{L}_{\kappa^{3+\kappa^{2+}}$ ) for some regular cardinal  $\kappa$  (where  $\kappa^{n+}$  denotes the  $n$ -th cardinal successor of  $\kappa$ ). Setting  $\lambda \stackrel{\text{def}}{=} \kappa^{2+}$ , it follows from [39, Theorem 12.3] that there exists a  $\lambda$ -continuous functor  $\Delta: [\lambda^{2+}]^{\text{inj}} \rightarrow \mathbf{Str}(\vee, \wedge, 0)$  such that  $\Delta(\lambda) \in \text{Id}_c \mathcal{A}$ ,  $\Delta(\lambda^{2+})$  is not Cevian (thus it does not belong to  $\text{Id}_c \mathcal{A}$ ), and (due to the GCH)  $\text{card } \Delta(X) \leq \max\{\lambda, \text{card } X\}$  for every  $X \subseteq \lambda^{2+}$ . Since  $\text{Id}_c \mathcal{A}$  and the complement of  $\mathcal{C} \cap \mathbf{Mod}_{(\vee, \wedge, 0)}(\text{Plo}_{\leq \kappa})$  are both PC( $\mathcal{L}_{\lambda+\lambda}$ ) (cf. Lemma 11.3), with  $\text{Id}_c \mathcal{A} \subseteq \mathcal{C} \cap \mathbf{Mod}_{(\vee, \wedge, 0)}(\text{Plo}_{\leq \kappa})$ , it follows from Tuuri's Interpolation Theorem (Proposition 9.4) that there exists an  $\mathcal{M}_{\lambda+\lambda}(\vee, \wedge, 0)$ -sentence  $E$  such that  $\text{Id}_c \mathcal{A} \subseteq \mathbf{Mod}_{(\vee, \wedge, 0)}(E) \subseteq \mathcal{C} \cap \mathbf{Mod}_{(\vee, \wedge, 0)}(\text{Plo}_{\leq \kappa})$ . In particular,  $\Delta(\lambda)$  is a bounded distributive lattice satisfying  $\text{CBD} \wedge E \wedge \text{Plo}_{\leq \kappa}$ . Since  $\Delta(\lambda) \hookrightarrow_\lambda \Delta(\lambda^{2+})$  (cf. Proposition 9.3) and by Proposition 9.5, it follows that  $\Delta(\lambda^{2+})$  is a bounded distributive lattice satisfying  $E$ ; whence  $\Delta(\lambda^{2+}) \in \mathcal{C}$ . By Lemma 11.4(4),  $\Delta(\lambda^{2+})$  satisfies  $\text{Plo}_{\leq \kappa^{n+}}$  for every  $n < \omega$ . Taking  $n = 2$ , we get  $\Delta(\lambda^{2+}) \models \text{Plo}_{\leq \lambda^{2+}}$ . By Lemma 11.4(2), it follows that  $\Delta(\lambda^{2+})$  satisfies  $\text{Plo}_{< \infty}$ , and thus is as required.  $\square$

In the proof above,  $\Delta(\lambda^{2+}) = \Delta(\kappa^{4+})$  has cardinality  $\kappa^{4+}$ . In particular, applying Theorem 11.6 to the class  $\mathcal{C}$  of all distributive 0-lattices with CBD, we can take

$\kappa = \aleph_0$ , so we get the bounded distributive lattice  $\Delta(\omega_4)$ . Due to the GCH, this lattice has cardinality  $\aleph_4$ , is completely normal, satisfies CBD together with  $\text{Plo}_{<\infty}$ , but does not belong to  $\text{Id}_c \mathcal{A}$  (it is not Cevian). We do not know whether the conclusion of Theorem 11.6 holds in full generality (e.g., without the GCH). We find it plausible that this might follow from a closer look at the condensate construction from [39], within bounded distributive lattices.

## 12. A FINITE BOUNDED POSET FOR WHICH CLL FAILS

The poset of the title is represented in the left hand side of Figure 12.2 and denoted there by  $P_0$ . The ‘‘Condensate Lifting Lemma’’ CLL is stated in Gillibert and Wehrung [10], Wehrung [39]. This section provides a counterexample to [10, Problem 1].

Recall that a  $(\vee, 0)$ -semilattice is an idempotent commutative monoid; in order to avoid any notational confusion between the semilattice operation and logical disjunction, we will in this section denote such structures additively. Letting  $x \leq y$  stand for  $x + y = y$ , a  $(\vee, 0)$ -semilattice  $S$  is *distributive* (cf. Grätzer [12]) if for all  $a, b, c \in S$  such that  $c \leq a + b$  there are  $x \leq a$  and  $y \leq b$  in  $S$  such that  $c = x + y$ .

Consider the category  $\mathbf{DSLat}^0$  of all distributive  $(\vee, 0)$ -semilattices with monoid homomorphisms. By the Ershov-Pudlák Lemma (cf. Ershov [9], Pudlák [29]; see Wehrung [36, Theorem 7-4.2] for a discussion), every distributive  $(\vee, 0)$ -semilattice is the directed union of its finite distributive 0-subsemilattices. Hence,  $\mathbf{DSLat}^0$  is a finitely accessible category; it has all products.

The main result of this section (i.e., Theorem 12.4) implies that  $\mathbf{DSLat}^0$  does not have all pushouts or, for that matter, pullbacks; hence it is not locally presentable. To that end, consider the Boolean semilattice  $\mathbf{2} \stackrel{\text{def}}{=} \{0, 1\}$  and define  $(\vee, 0)$ -semilattice embeddings  $\iota: \mathbf{2} \hookrightarrow {}^2\mathbf{2}$ ,  $\alpha, \beta: {}^2\mathbf{2} \hookrightarrow {}^3\mathbf{2}$ , and  $\gamma_3, \gamma_4: {}^3\mathbf{2} \hookrightarrow {}^4\mathbf{2}$  by

$$\begin{aligned} \iota(x) &= (x, x); & \alpha(x, y) &\stackrel{\text{def}}{=} (x, x, y); & \beta(x, y) &\stackrel{\text{def}}{=} (y, x, x); \\ \gamma_3(x, y, z) &= (x, y, z, x + z); & \gamma_4(x, y, z) &= (x, y, z, x + y + z), \end{aligned}$$

for all  $x, y, z \in \mathbf{2}$ . It is straightforward to verify that the diagram  $\vec{D}$  represented in Figure 12.1 is commutative.

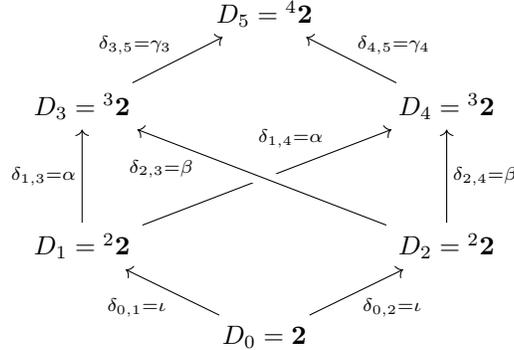


FIGURE 12.1. The commutative diagram  $\vec{D}$

This diagram is indexed by the poset  $P_0$ , with universe  $\{0, 1, 2, 3, 4, 5\}$ , represented on the left hand side of Figure 12.2. The intersection  $L \stackrel{\text{def}}{=} \gamma_3[{}^3\mathbf{2}] \cap \gamma_4[{}^3\mathbf{2}] = \{0000, 1001, 0011, 1101, 0111, 1011, 1111\}$  is a 0, 1-subsemilattice of  ${}^4\mathbf{2}$ , represented on the right hand side of Figure 12.2.

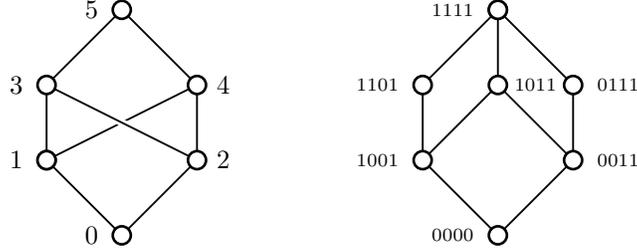


FIGURE 12.2. The posets  $P_0$  and  $L \stackrel{\text{def}}{=} \gamma_3[{}^3\mathbf{2}] \cap \gamma_4[{}^3\mathbf{2}]$

Our next task will be representing  $\mathbf{DSLat}^0$  as the image of an accessible functor (by Theorem 5.5 this is certainly possible). Moreover, the domain of that functor will be constructed as a (finitary, congruence-distributive) *variety of algebras*. This will be performed by a simple Skolemization argument. Consider the extension  $\mathbf{v}^* \stackrel{\text{def}}{=} (+, 0, \mathbf{f})$  of the vocabulary  $\mathbf{v} \stackrel{\text{def}}{=} (+, 0)$ , where the operation symbols  $+$ ,  $0$ , and  $\mathbf{f}$  have respective arities 2, 0, and 3. Denote by  $\mathcal{V}$  the variety of  $\mathbf{v}^*$ -algebras defined by the identities

$$x + x = x; \quad x + y = y + x; \quad x + (y + z) = (x + y) + z; \quad x + 0 = x; \quad (12.1)$$

$$\mathbf{f}(x + y, x, x) = x; \quad (12.2)$$

$$\mathbf{f}(x, y, z) + x = x; \quad (12.3)$$

$$\mathbf{f}(x + y, z, z) = \mathbf{f}(x, y, \mathbf{f}(x + y, z, z)) + \mathbf{f}(y, x, \mathbf{f}(x + y, z, z)). \quad (12.4)$$

In particular, the identities (12.1) define the variety of all  $(\vee, 0)$ -semilattices.

**Proposition 12.1.** *A  $(\vee, 0)$ -semilattice  $S$  is distributive iff it expands to a member of  $\mathcal{V}$ .*

*Proof.* Every member  $\mathbf{S}$  of  $\mathcal{V}$  satisfies both implications  $z \leq x + y \Rightarrow z = \mathbf{f}(x + y, z, z)$  and  $z = \mathbf{f}(x + y, z, z) \Rightarrow z = \mathbf{f}(x, y, z) + \mathbf{f}(y, x, z)$  (apply (12.2) and (12.4), respectively), thus also the implication

$$z \leq x + y \implies z = \mathbf{f}(x, y, z) + \mathbf{f}(y, x, z). \quad (12.5)$$

Owing to (12.3),  $\mathbf{S}$  has thus distributive underlying  $(\vee, 0)$ -semilattice.

Let, conversely,  $(S, +, 0)$  be a distributive  $(\vee, 0)$ -semilattice. For all  $a, b, c \in S$  with  $c \leq a + b$ , there are  $f_0(a, b, c) \leq a$  and  $f_1(a, b, c) \leq b$  in  $S$  such that  $c = f_0(a, b, c) + f_1(a, b, c)$ . We may further assume that  $a \leq b$  implies  $f_0(b, a, a) = f_1(a, b, a) = a$ . The map  $f: {}^3S \rightarrow S$  defined by the rule

$$f(a, b, c) \stackrel{\text{def}}{=} \begin{cases} f_0(a, b, c) + f_1(b, a, c), & \text{if } c \leq a + b, \\ a, & \text{otherwise} \end{cases}, \quad \text{for all } a, b, c \in S,$$

obviously satisfies the identities (12.2) and (12.3). Let  $a, b, c \in S$  and set  $c' \stackrel{\text{def}}{=} f(a + b, c, c)$ . Then  $c' \leq a + b$  by (12.3), thus

$$f(a, b, c') + f(b, a, c') = f_0(a, b, c') + f_1(a, b, c') + f_0(b, a, c') + f_1(b, a, c') = c',$$

so  $f$  satisfies (12.4).  $\square$

Although some of its defining equations (12.1)–(12.4) may seem a bit unnatural, the variety  $\mathcal{V}$  turns out to be surprisingly well-behaved.

**Proposition 12.2.** *The variety  $\mathcal{V}$  is congruence-distributive (i.e., every member of  $\mathcal{V}$  has distributive congruence lattice).*

*Proof.* By applying (12.5) to the inequality  $x \leq x + y$ , we obtain that the equation  $x = f(x, y, x) + f(y, x, x)$  is valid in  $\mathcal{V}$ . Using (12.3), it follows that for the term  $m(x, y, z) \stackrel{\text{def}}{=} f(x, y, z) + f(y, z, x) + f(z, x, y)$ , all equations  $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$  are valid in  $\mathcal{V}$  (i.e.,  $m$  is a *majority term* for  $\mathcal{V}$ ). Now apply Jónsson [18, Theorem 2.1] (for  $n = 2$ ).  $\square$

By Proposition 12.1, the forgetful functor  $\Phi: \mathcal{V} \rightarrow \mathbf{Str}(\mathbf{v})$ ,  $(S, +, 0, f) \mapsto (S, +, 0)$  has range  $\mathbf{DSLat}^0$ . This functor is faithful, finitely accessible, and it preserves all products.

We shall now argue that the essential surjectivity of  $\Phi$  does not extend to diagrams, in a strong sense tailoring Problem 1 in Gillibert and Wehrung [10]. As in [10], say that a homomorphism  $\varphi: S \rightarrow T$  of  $(\vee, 0)$ -semilattices is *weakly distributive* if for all  $s \in S$  and  $t_1, t_2 \in T$  such that  $\varphi(s) \leq t_1 + t_2$  there are  $s_1, s_2 \in S$  such that  $s \leq s_1 + s_2$  and each  $\varphi(s_i) \leq t_i$ . Since the functor  $\Phi$  preserves all directed colimits, and since any canonical projection  $E \times F \rightarrow E$  is weakly distributive, Condition  $\text{PROJ}(\Phi, \mathcal{S}^{\Rightarrow})$  of [10, Definition 3.4.1] is satisfied.

Let  $P_1 \stackrel{\text{def}}{=} P_0 \cup \{\mu\}$  where the additional “middle element”  $\mu$  is greater than 1 and 2 but smaller than 3 and 4. Say that a  $P_0$ -indexed commutative diagram is *mid-extendible* if it extends to a  $P_1$ -indexed diagram in  $\mathbf{DSLat}^0$ .

**Proposition 12.3.** *Every diagram in  $\mathbf{DSLat}^0$  of the form  $\Phi(\vec{E})$ , where  $\vec{E}$  is a  $P_0$ -indexed commutative diagram in  $\mathcal{V}$ , is mid-extendible.*

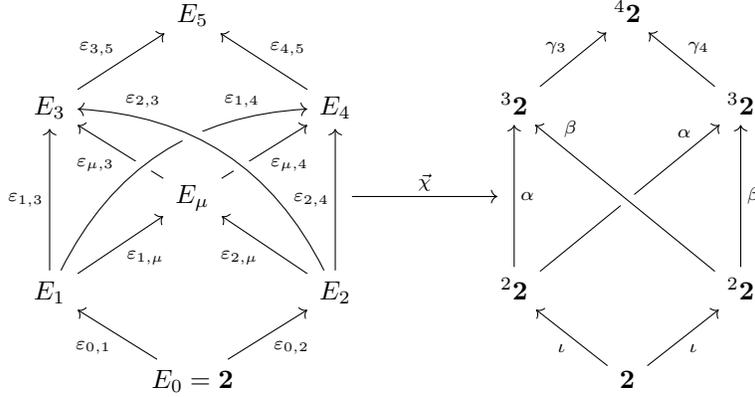
*Proof.* Write  $\vec{E} = (E_p, \varepsilon_{p,q} \mid p \leq q \text{ in } P_0)$  and set

$$E_\mu \stackrel{\text{def}}{=} \{(x_3, x_4) \in E_3 \times E_4 \mid \varepsilon_{3,5}(x_3) = \varepsilon_{4,5}(x_4)\},$$

the pullback of  $(\varepsilon_{j,5}: E_j \rightarrow E_5 \mid j \in \{3, 4\})$ . The missing maps are provided by  $\varepsilon_{i,\mu} \stackrel{\text{def}}{=} (\varepsilon_{i,3}, \varepsilon_{i,4})$  ( $i \in \{1, 2\}$ ) and  $\varepsilon_{\mu,j} \stackrel{\text{def}}{=} \text{projection } E_\mu \rightarrow E_j$  ( $j \in \{3, 4\}$ ).  $\square$

**Theorem 12.4.** *There are no mid-extendible  $P_0$ -indexed commutative diagram  $\vec{E}$  in  $\mathbf{DSLat}^0$  and no natural transformation  $\vec{\chi}: \vec{E} \rightarrow \vec{D}$  such that  $\chi_0$  is surjective and  $\chi_1$  and  $\chi_2$  are both weakly distributive. In particular,  $\vec{D}$  is not mid-extendible; thus it is not isomorphic to  $\Phi(\vec{E})$  for any diagram  $\vec{E}$  in  $\mathcal{V}$ .*

*Proof.* Denote by  $\vec{E}' \stackrel{\text{def}}{=} (E_p, \varepsilon_{p,q} \mid p \leq q \text{ in } P_0 \cup \{\mu\})$  the given extension of  $\vec{E}$  to a  $P_1$ -indexed commutative diagram in  $\mathbf{DSLat}^0$ . We represent it, together with the natural transformation  $\vec{\chi}$ , on the left hand side of Figure 12.3. Note that the “mid component”  $\chi_\mu$  is undefined at that point.

FIGURE 12.3. The natural transformation  $\vec{\chi}$ 

Now we set  $\chi_\mu \stackrel{\text{def}}{=} \chi_5 \varepsilon_{\mu,5}$ . For each  $j \in \{3,4\}$ ,  $\chi_\mu = \chi_5 \varepsilon_{j,5} \varepsilon_{\mu,j} = \gamma_j \chi_j \varepsilon_{\mu,j}$  has range contained in  $\gamma_j[\mathbf{3}\mathbf{2}]$ ; whence  $\chi_\mu: E_\mu \rightarrow L$ . Moreover,  $\chi_\mu \varepsilon_{i,\mu} = \chi_5 \varepsilon_{i,5}$  whenever  $i \in \{1,2\}$ .

Since  $\chi_0$  is surjective,  $1 = \chi_0(e)$  for some  $e \in E_0$ . Since

$$\chi_1 \varepsilon_{0,1}(e) = \iota \chi_0(e) = \iota(1) = (1,1) = (1,0) + (0,1),$$

it follows from the weak distributivity of the map  $\chi_1$  and the distributivity of the semilattice  $E_1$  that there are  $a_1, b_1 \in E_1$  such that  $\varepsilon_{0,1}(e) = a_1 + b_1$ ,  $\chi_1(a_1) = (1,0)$ , and  $\chi_1(b_1) = (0,1)$ . Likewise, there are  $a_2, b_2 \in E_2$  such that  $\varepsilon_{0,2}(e) = a_2 + b_2$ ,  $\chi_2(a_2) = (1,0)$ , and  $\chi_2(b_2) = (0,1)$ . We compute  $\chi_5 \varepsilon_{1,5}(a_1) = \delta_{1,5} \chi_1(a_1) = \delta_{1,5}(1,0) = (1,1,0,1)$ . Similarly,  $\chi_5 \varepsilon_{1,5}(b_1) = (0,0,1,1)$ ,  $\chi_5 \varepsilon_{2,5}(a_2) = (0,1,1,1)$ , and  $\chi_5 \varepsilon_{2,5}(b_2) = (1,0,0,1)$ . Since

$$\varepsilon_{1,\mu}(a_1) \leq \varepsilon_{1,\mu}(a_1 + b_1) = \varepsilon_{1,\mu} \varepsilon_{0,1}(e) = \varepsilon_{0,\mu}(e) = \varepsilon_{2,\mu}(a_2) + \varepsilon_{2,\mu}(b_2)$$

and  $E_\mu$  is distributive, there are  $x, y \in E_\mu$  such that  $\varepsilon_{1,\mu}(a_1) = x + y$  whereas  $x \leq \varepsilon_{2,\mu}(a_2)$  and  $y \leq \varepsilon_{2,\mu}(b_2)$ . By applying the homomorphism  $\chi_\mu$  to both sides of the equation  $\varepsilon_{1,\mu}(a_1) = x + y$ , we obtain that  $(1,1,0,1) = \chi_\mu(x) + \chi_\mu(y)$ , with elements  $\chi_\mu(x), \chi_\mu(y) \in L$  such that  $\chi_\mu(x) \leq (0,1,1,1)$  and  $\chi_\mu(y) \leq (1,0,0,1)$ . Since  $\chi_\mu(x)$  belongs to  $L$ , it follows (cf. Figure 12.2) that  $\chi_\mu(x) = (0,0,0,0)$ , whence  $(1,1,0,1) \leq (1,0,0,1)$ , a contradiction.  $\square$

Theorem 12.4 provides a negative solution for [10, Problem 1]: although the functor  $\Phi$  is both surjective on objects (cf. Proposition 12.1) and finitely continuous with domain a variety, it is not essentially surjective on  $P_0$ -indexed diagrams of  $(\vee, 0)$ -embeddings, even up to weakly distributive natural transformations.

By using CLL (viz. [10, Theorem 3.4.2]), it can be verified that the counterexample underlying Theorem 12.4 has no analogue for diagrams indexed by finite lattices, or, more generally, finite *almost join-semilattices with zero* as defined in [10]. This means that for every finite almost join-semilattice  $P$  with zero and every  $P$ -indexed commutative diagram  $\vec{D}$  in  $\mathbf{DSLAt}^0$ , there are a  $P$ -indexed commutative diagram  $\vec{E}$  in  $\mathcal{V}$  and a natural transformation  $\vec{\chi}: \Phi(\vec{E}) \rightarrow \vec{D}$  whose components are all *ideal-induced* in the sense of [10]; thus, *a fortiori*, both surjective and weakly distributive.

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