Absolutely summing weighted composition operators on Bloch spaces

Pascal Lefèvre, Tonie Fares

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Tonie Fares*, Pascal Lefèvre†

Abstract

We characterize p-summing composition operators $C_\varphi(f) = f \circ \varphi$ from a Bloch space $\mathcal{B}^\mu$ to another such space $\mathcal{B}^\beta$, where $\mu, \beta > 0$. The corresponding result on little Bloch-type spaces is also proved. In addition, we construct an example of a conformal mapping of the unit disk $\mathbb{D}$ into itself which has a contact point with the unit circle $\mathbb{T}$, and induces a compact composition operator, that fails to be p-summing for any $p \geq 1$. We also detail the case of lens maps. Moreover we explore the case of weighted composition operators $uC_\varphi(f) = u.(f \circ \varphi)$ and characterize such nuclear operators, and p-summing ones for a class of weights. We also show that compactness of a composition operator on $\mathcal{B}^\beta$ and $\mathcal{B}_0^\beta$ implies its compactness on Bergman spaces. Moreover when $\beta > 1$ the converse is also true for composition operators on $\mathcal{B}_0^\beta$.

1 Introduction and background

This paper is interested in composition operators, *i.e.* mapping $f \mapsto C_\varphi(f) = f \circ \varphi$, where $\varphi : \mathbb{D} \to \mathbb{D}$ (the symbol) is analytic, between Banach spaces of analytic functions over the unit disk $\mathbb{D}$. A major purpose of this field of investigation is to understand the link between the properties of the operator C_φ and the properties of the function φ. Of course it also depends on the spaces involved.

In this paper we focus on two features: the p-summing property for the operator C_φ (see below for the details and [7] or [22] for good references on this operator property); and, for the underlying space, the classical Bloch space (and its standard weighted generalization, as well as the little Bloch spaces). This problem was completely open except in the case $p = 1$: indeed in [10], we characterized nuclear composition operators C_φ on the classical Bloch spaces $\mathcal{B}$ and $\mathcal{B}_0$ and it turns out that nuclear composition operators on Bloch spaces are exactly 1-summing composition operators on these spaces.

It is then natural to wonder whether we can extend this in three natural directions: from 1 to p-summingness, adding (standard) weights to the spaces, and consider weighted composition operators. Nevertheless, according to us, the main point is to get a characterization of p-summing composition operators on the classical Bloch space. This problem is solved (among other things) in our paper.

More precisely, we obtain: the characterization of p-summing composition operators, for any $1 \leq p < +\infty$, on Bloch-type spaces; for weighted composition operators, we characterize nuclear

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ones on Bloch-type spaces. For the case $p > 1$, the computations are more delicate. We present a necessary condition and some sufficient ones that are unfortunately slightly different, except for a certain class of weights. Whereas the extension is fairly easy to get concerning the weight for the space, it turns out that managing the weight for the operator and extending the characterization from 1-summing to arbitrary p-summing is more difficult and requires more work. Actually, once our work was already completed, we discoreved that recently, using similar methods as in [10], Bonet & al. [3] extended the result on nuclearity to the case $\mathcal{B}^\beta$ with $\beta \geq 1$. However the results of the present paper are more general than [3] and [10] in many ways.

Throughout this paper, we will denote by $\mathcal{A}^p_\alpha = \mathcal{A}^p_\alpha(\mathbb{D}) = \mathcal{H}ol(\mathbb{D}) \cap L^p(\mathbb{D}, A_\alpha)$, $1 \leq p < \infty$ and $-1 < \alpha < +\infty$, the (weighted) Bergman space consisting of analytic functions f on the open unit disk $\mathbb{D}$ satisfying

$$\|f\|_{\mathcal{A}^p_\alpha} = \Big(\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \Big)^{1/p} < +\infty \,,$$

where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ and A denotes the normalized area measure on $\mathbb{D}$. When $\alpha = 0$, $\mathcal{A}^p_0$ is simply denoted $\mathcal{A}^p$.

For $1 \leq p < \infty$, $H^p = H^p(\mathbb{D})$ is the Hardy space consisting of analytic functions f on $\mathbb{D}$ satisfying

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} \Big(\int_{\mathbb{T}} |f(rz)|^p d\lambda \Big)^{1/p} < +\infty \,,$$

where λ denotes the Haar measure on $\mathbb{T}$.

The space $H^\infty = \mathcal{H}ol(\mathbb{D}) \cap L^\infty(\mathbb{D})$ is the Hardy space of the bounded analytic functions f on the open unit disk $\mathbb{D}$. It is equipped with its usual norm

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

The classical Bloch space $\mathcal{B}$ is defined as the space of analytic functions f on $\mathbb{D}$ which satisfy

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < +\infty.$$

See monographs [9], [13] for a classical background, [5] for a survey and [1] for more recent results on $\mathcal{B}$. One can also see [30] for more details on Bloch-type spaces.

In this paper we study composition operators on a class of generalized Bloch spaces. More precisely, for each $\beta > 0$, we let $\mathcal{B}^\beta$ denote the space of analytic functions f on $\mathbb{D}$ which satisfy

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |f'(z)| < +\infty.$$

When $\beta = 1$, we recover the classical Bloch space $\mathcal{B}^1 = \mathcal{B}$.
We will also be interested in the generalization of the little Bloch space $\mathcal{B}_0$ consisting of functions f in $\mathcal{B}$ such that

$$\lim_{|z| \to 1^-} (1 - |z|^2)|f'(z)| = 0.$$

Thus, for each $\beta > 0$, we let $\mathcal{B}_0^\beta$ denote the subspace of $\mathcal{B}^\beta$ consisting of functions f such that

$$\lim_{|z|\to 1^-} (1-|z|^2)^\beta |f'(z)| = 0.$$

It is well known that the Bloch-type space $\mathcal{B}^\beta$ is a Banach space when equipped with the norm

$$\|f\|_{\mathcal{B}^\beta} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |f'(z)|$$

and that $\mathcal{B}_0^\beta$ is a closed subspace of $\mathcal{B}^\beta$ (see [30] for instance).

An important tool in functional analysis is the duality, and it will play once again a key role below. Here the situation is well known and there is a natural description of the duality between Bloch-type spaces and the Bergman space $\mathcal{A}^1$. Actually, we have $(\mathcal{A}^1)^* \approx \mathcal{B}^\beta$. More precisely, given $h \in \mathcal{A}^1$ and $f \in \mathcal{B}^\beta$, we use the integral pairing

$$\langle h, f\rangle = \lim_{r\to 1^-} \int_{\mathbb{D}} h(rw)\overline{f(rw)}\, dA_{\beta-1}(w)$$

and, for some $c > 0$ and every $\Phi \in (\mathcal{A}^1)^*$, there exists $f \in \mathcal{B}^\beta$ satisfying $\Phi = \langle \cdot, f\rangle$ with $\|f\| \le c\|\Phi\|$ (see [30, Theorem 14]).
In the same spirit, we have $(\mathcal{B}_0^\beta)^* \approx \mathcal{A}^1$ under the same integral pairing

$$\langle h, f\rangle = \lim_{r\to 1^-} \int_{\mathbb{D}} h(rw)\overline{f(rw)}\, dA_{\beta-1}(w) \tag{1}$$

where $h \in \mathcal{B}_0^\beta$ and $f \in \mathcal{A}^1$ (see [30, Theorem 15]).

When $h \in H^\infty \cap \mathcal{B}_0^\beta$, we can simplify the preceding formula and write

$$\langle h, f\rangle = \int_{\mathbb{D}} h(w)\overline{f(w)}\, dA_{\beta-1}(w).$$

Note that, when $0 < \beta < 1$, the space $\mathcal{B}^\beta$ is contained in $H^\infty(\mathbb{D})$. More specifically f belongs to $\mathcal{B}^\beta$ if and only if it satisfies the following Lipschitz condition

$$\sup\left\{ \frac{|f(z)-f(w)|}{|z-w|^{1-\beta}};\ z \neq w \right\} < +\infty.$$

In particular, when $0 < \beta < 1$, the space $\mathcal{B}^\beta$ is contained in the disk algebra (see [30, Prop. 9]). When $\beta \ge 1$, it is easy to see by the Schwarz-Pick lemma that $H^\infty(\mathbb{D}) \subset \mathcal{B}^\beta$ (see [13, p.13]).

Another important tool is the point evaluation functional. Let $a \in \mathbb{D}$, the point evaluation functional δ_a is defined by $\delta_a(f) = f(a)$. This functional is bounded on $\mathcal{B}^\beta$ for all $\beta > 0$ and on $\mathcal{A}^1$ (see [9], [13] or [30]).

Given an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$, the composition operator C_φ is (formally) defined by

$$C_\varphi(f) = f \circ \varphi.$$

Its behavior on various spaces of analytic functions on $\mathbb{D}$ is already widely studied and still receives a lot of attention, even on classical spaces. Its weighted version $uC_\varphi : f \mapsto u(f\circ\varphi)$,

where u is an analytic function on $\mathbb{D}$, is a natural extension of C_φ, and generalizes the multiplier operators as well. The question of the boundedness and of the membership of classical operator ideals is natural when viewed on Bloch-type spaces.

Actually, for $0 < \beta < 1$, the boundedness of the composition operator C_φ on $\mathcal{B}^\beta$ was considered and solved first by Roan [23] and later by Madigan [16]. Then boundedness and compactness of C_φ on classical Bloch spaces were described by Madigan and Matheson in [17]. Contreras and Hernandez-Díaz [4] on one hand and Xiao [29] on the other hand generalized the work of Madigan and Matheson, characterizing bounded and compact composition operators on general Bloch-type spaces. The case of weighted composition operators on classical Bloch spaces was treated by Ohno and Zhao in [20]: they characterised boundedness and compactness. In 2003, Ohno, Stroethoff and Zhao [21] extended this work to Bloch-type spaces.

The study of absolutely summing composition operators was initiated by Shapiro-Taylor in the seminal paper [26]. Nevertheless except very specific cases, the problem was left open for classical spaces. On Bergman spaces, the problem was solved by Domenig in [8] and more recently it was solved on Hardy spaces H^p, when $p > 1$, by Rodríguez-Piazza and the second named author [15].

In this paper, we study p-summing weighted composition operators uC_φ on Bloch-type spaces. Nevertheless we start by some further results on compactness of composition operators on Bloch-type spaces. We discuss how compactness of C_φ on the Bloch-type space and the little Bloch-type space relates to its compactness on the Bergman space. In particular, we show that compactness of C_φ on $\mathcal{B}^\beta$ (with $\beta > 0$) or on $\mathcal{B}_0^\beta$ implies compactness of C_φ on the Bergman space $\mathcal{A}_\alpha^p$. Moreover when $\beta > 1$, we show that compactness of C_φ on $\mathcal{B}_0^\beta$ is equivalent to its compactness on $\mathcal{A}_\alpha^p$. In the third section, we first give a necessary condition for uC_φ to be p-summing for $p \geq 1$. Then we give some sufficient conditions. As a consequence, we obtain a characterization for p-summing composition operators on Bloch-type spaces, $p \geq 1$. We also obtain a characterization for nuclear weighted composition operators. For the case $p > 1$ we obtain a characterization only for a class of weights. This is the purpose of section 4. When $p = 1$ and $\beta = \mu = 1$ we recover the characterization of nuclear composition operators on classical Bloch spaces which was given in [10]. In section 5, we exhibit an example of a symbol φ with a contact point with the unit circle $\mathbb{T}$ that induces a compact composition operator on the classical Bloch space which is not p-summing for any $p \geq 1$. We also give an example of an inner symbol inducing a nuclear composition operator of the Bloch space $\mathcal{B}^\beta$ when $\beta \geq 1$. Finally we consider the example of the lens map. We give several basic properties of this symbol and we study the behaviour of its associated composition operator on $\mathcal{B}^\beta$ depending on $\beta > 0$.

Let us recall the definition of absolutely summing operators and nuclear operators.

Definition 1.1. *An operator $T : X \longrightarrow Y$ is p-summing, $1 \leq p < +\infty$, if there exists a constant C such that for all finite sequences $(x_j)_{j=1}^n \subset X$, we have*

$$\Big(\sum_{j=1}^n \|Tx_j\|_Y^p\Big)^{1/p} \leq C \sup_{\substack{x^* \in X^* \\ \|x^*\| \leq 1}} \sum_{j=1}^n \Big(|x^*(x_j)|^p\Big)^{1/p} = C \sup_{\substack{a \in \ell^{p'} \\ \|a\|_{p'} \leq 1}} \Big\| \sum_{j=1}^n a_j x_j \Big\|_X.$$

We define the p-summing norm of an operator T by the least admissible constant C and denote it by $\pi_p(T)$.

A generic example is the canonical identity from $C(K)$ to some $L^p(K,\mu)$ space (where μ is a Borel measure on a compact Hausdorff set K). See [7] for more informations on the subject, and [22] for the study of such operators viewed on Banach spaces of analytic functions. They are not compact in general (see the previous generic example). Nevertheless, and even if we do not use this fact, it is worth mentioning that on Bloch spaces, and little Bloch spaces, absolute summingness of the composition operators implies their compactness. It is due to the fact that Bloch spaces are isomorphic to ℓ^∞ and little Bloch spaces are isomorphic to c_0 (see [14]).

Definition 1.2. *An operator $T : X \longrightarrow Y$ is said to be nuclear if there exist a sequence $(x_n^*) \subset X^*$ and a sequence $(y_n) \subset Y$ such that $\sum_n \|x_n^*\|\|y_n\| < \infty$ and*

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n \,,$$

where $x_n^* \otimes y_n : X \to Y$ is defined to be the mapping $x \mapsto x_n^*(x)y_n$.

In other words, nuclear operators are absolutely convergent series of rank one operators. Therefore they are compact.

As usual, the notation $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$. In the same way, the notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

We also use the conjugate exponent of $p \in [1,\infty[$, denoted by p', which is defined by the relation $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. When $p = 1$, $p' = \infty$ and $\frac{1}{p'} = 0$.

2 Further results on compactness of composition operators

In this section we give a property of monotony concerning the compactness of composition operators on Bloch-type spaces with respect to the parameter $\beta > 0$. Surprisingly it does not appear in the literature. We also show that compactness of composition operators on the Bloch spaces induces compactness on Bergman spaces. Moreover when $\beta > 1$ the compactness on the little Bloch spaces is equivalent to the one on Bergman spaces. We start by recalling the characterization of compact composition operators on Bloch-type spaces. Denote

$$F(\varphi, z, \mu, \beta) = \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^\mu} \,.$$

Theorem 2.1. *([29, Theorem 3.1])*
Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic, and $\mu, \beta \in (0,\infty)$. Then

(i) $C_\varphi : \mathcal{B}^\mu$ or $\mathcal{B}_0^\mu \longrightarrow \mathcal{B}^\beta$ is compact $\Longleftrightarrow \lim\limits_{|\varphi(z)|\to 1} F(\varphi, z, \mu, \beta) = 0.$

(ii) $C_\varphi : \mathcal{B}^\mu$ or $\mathcal{B}_0^\mu \longrightarrow \mathcal{B}_0^\beta$ is compact $\Longleftrightarrow \lim\limits_{|z|\to 1} F(\varphi, z, \mu, \beta) = 0.$

Curiously the following fact does not appear in the literature:

Corollary 2.2. *If $C_\varphi : \mathcal{B}^\beta \longrightarrow \mathcal{B}^\beta$ is compact and $\gamma > \beta$ then $C_\varphi : \mathcal{B}^\gamma \longrightarrow \mathcal{B}^\gamma$ is compact.*

We point out that this is also true for little Bloch-type spaces.

Proof. In the first part of the proof we assume that $\varphi(0) = 0$.
Since C_φ is compact on $\mathcal{B}^\beta$, we have $\lim\limits_{|\varphi(z)|\to 1} F(\varphi, z, \beta, \beta) = 0$.
We want to prove that $\lim\limits_{|\varphi(z)|\to 1} F(\varphi, z, \gamma, \gamma) = 0$.

But,

$$F(\varphi, z, \gamma, \gamma) = \frac{(1-|z|^2)^\gamma |\varphi'(z)|}{(1-|\varphi(z)|^2)^\gamma} = \frac{(1-|z|^2)^\beta |\varphi'(z)| (1-|z|^2)^{\gamma-\beta}}{(1-|\varphi(z)|^2)^\beta (1-|\varphi(z)|^2)^{\gamma-\beta}}.$$

Then thanks to Schwarz's lemma (since $\varphi(0) = 0$) we have $|\varphi(z)| \leq |z|$, for every $z \in \mathbb{D}$.

So

$$\Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^{\gamma-\beta} \lesssim \Big(\frac{1-|z|}{1-|\varphi(z)|}\Big)^{\gamma-\beta} \leq 1.$$

We get

$$F(\varphi, z, \gamma, \gamma) = \frac{(1-|z|^2)^\gamma |\varphi'(z)|}{(1-|\varphi(z)|^2)^\gamma} \lesssim F(\varphi, z, \beta, \beta) \underset{|\varphi(z)|\to 1}{\longrightarrow} 0.$$

To prove that C_φ is compact on $\mathcal{B}^\gamma$ even when φ does not fix the origin, we use a standard strategy, but we give the details for the sake of completeness. We use the Möbius map $\varphi_a(z) = \dfrac{a-z}{1-\overline{a}z}$, for $a = \varphi(0)$. Then the holomorphic function $\psi = \varphi_a \circ \varphi$ takes $\mathbb{D}$ into itself and fixes the origin. Hence C_ψ satisfies the first case so it is compact on $\mathcal{B}^\gamma$ as soon as it is compact on $\mathcal{B}^\beta$ and $\gamma > \beta$. Now by the self-inverse property of φ_a we have $\varphi = \varphi_a \circ \psi$ and this translates into the operator equation $C_\varphi = C_\psi C_{\varphi_a}$. It is easy to check that

$$\sup_{z\in\mathbb{D}} F(\varphi_a, z, \gamma, \gamma) \leq \Big(\frac{1+|a|}{1-|a|}\Big)^{|1-\gamma|},$$

which implies that C_{φ_a} is bounded on $\mathcal{B}^\gamma$ ([29, Theorem 2.2]). By the ideal property of compact operators (see for example [7, P. 37, 2.4]) C_φ is a compact composition operator on $\mathcal{B}^\gamma$, and this ends the proof. □

In [17], Madigan and Matheson have noticed using the Julia-Caratheodory theorem, that if φ has a finite angular derivative at some point of $\mathbb{T}$, then C_φ cannot be compact on $\mathcal{B}$. Actually we extend this remark to Bloch-type spaces and get the following result, which does not appear (as far as we saw) in the literature.

Proposition 2.3. *For $1 \leq p < \infty$, $\alpha > -1$ and $\beta > 0$ we have:*

$$C_\varphi : \mathcal{B}_0^\beta \longrightarrow \mathcal{B}_0^\beta \text{ compact} \Longrightarrow C_\varphi : \mathcal{A}_\alpha^p \longrightarrow \mathcal{A}_\alpha^p \text{ compact.}$$

The converse is not true in general. We show in section 5 that for the lens map Λ_a, the composition operator C_{Λ_a} is not compact on $\mathcal{B}_0$. But it is well known that the lens map induces a compact (even nuclear) composition operator on the Bergman space.
Note that this theorem is no longer true if we replace the Bergman space $\mathcal{A}_\alpha^p$ by the Hardy space H^p. Smith in [27] constructs an inner function φ that induces a compact composition operator on $\mathcal{B}_0$. But it is well known (see [25, Theorem 2.6]) that if C_φ is compact on H^p then $|\varphi^*| < 1$, where $\varphi^*(e^{it}) = \lim\limits_{r\to 1^-} \varphi(re^{it})$ for almost every t.

Proof. We give here a selfcontained argument. Assume that C_φ is not compact on $\mathcal{A}^p_\alpha$, then (see [19, Theorem 3.5])

$$\liminf_{|z|\to 1^-} \frac{1-|\varphi(z)|}{1-|z|} < \infty.$$

Which implies that there exists $w \in \mathbb{T}$ such that,

$$0 < \liminf_{z\to w} \frac{1-|\varphi(z)|}{1-|z|} = \delta < \infty.$$

By Julia-Caratheodory's theorem ([24, p.57]), we get that for some $\eta \in \mathbb{T}$, the angular derivative of φ at w exists and is equal to $\varphi'(w) = \angle \lim_{z\to w} \dfrac{\eta - \varphi(z)}{w-z} = w\overline{\eta}\delta$.
Moreover, $\angle \lim_{z\to w} \varphi'(z) = w\overline{\eta}\delta$. Hence,

$$\angle \lim_{z\to w} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta} = \frac{1}{\delta^{\beta-1}} \neq 0\,,$$

which implies (by Theorem 2.1) that C_φ is not compact on the little Bloch-type space. □

In fact, when $\beta > 1$, we have an equivalence.

Theorem 2.4. *For $1 \le p < \infty$, $\alpha > -1$ and $\beta > 1$ we have:*

$$C_\varphi : \mathcal{B}^\beta_0 \longrightarrow \mathcal{B}^\beta_0 \text{ compact} \iff C_\varphi : \mathcal{A}^p_\alpha \longrightarrow \mathcal{A}^p_\alpha \text{ compact.}$$

Proof. By the preceding proposition, we only need to prove the sufficiency. Assume that C_φ is compact on $\mathcal{A}^p_\alpha$, then

$$\lim_{|z|\to 1} \frac{1-|z|}{1-|\varphi(z)|} = 0.$$

By Schwarz-Pick lemma we have

$$\frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^\beta} \lesssim \Big(\frac{1-|z|}{1-|\varphi(z)|}\Big)^{\beta-1} \underset{|z|\to 1}{\longrightarrow} 0\,,$$

which ends the proof of the theorem. □

The compactness of a composition operator on the Bloch-type space also implies compactness on the Bergman space. The proof of this result is more technical than the corresponding result for little Bloch-type spaces (see Prop.2.3).

Theorem 2.5. *For $1 \le p < \infty$, $\alpha > -1$ and $\beta > 0$ we have:*

$$C_\varphi : \mathcal{B}^\beta \longrightarrow \mathcal{B}^\beta \text{ compact} \Longrightarrow C_\varphi : \mathcal{A}^p_\alpha \longrightarrow \mathcal{A}^p_\alpha \text{ compact.}$$

We need the following lemma for its proof.

Lemma 2.6. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic, $\beta > \frac{1}{2}$ and $\alpha > \max(2\beta - 3, -1)$. Let $\gamma = 2 + \alpha - 2\beta$. The operator*

$$\begin{array}{rccc} T & : & L^2(\mathbb{D}, dA_\alpha) & \longrightarrow & L^2(\mathbb{D}, dA_{\gamma,\varphi}) \\ & & f & \longmapsto & Tf(z) = \displaystyle\int_{\mathbb{D}} \frac{(1-|z|^2)^\beta \overline{w}}{(1-z\overline{w})^{3+\alpha}} f(w) dA_\alpha(w) \end{array}$$

is bounded.

Here $A_{\gamma,\varphi}$ stands for the pull back measure of A_γ relative to φ: $A_{\gamma,\varphi}(E) = A_\gamma\big(\varphi^{-1}(E)\big)$ for $E \subset \mathbb{D}$ measurable.

We point out too that $\alpha > -1$ and $\gamma > -1$.

Proof. It suffices to show that the operator

$$\begin{array}{rcccl} S & : & L^2(\mathbb{D}, dA_\alpha) & \longrightarrow & L^2(\mathbb{D}, dA_{\gamma,\varphi}) \\ & & f & \longmapsto & Sf(z) = \displaystyle\int_{\mathbb{D}} \frac{(1-|z|^2)^\beta |w|}{|1-z\overline{w}|^{3+\alpha}} f(w) dA_\alpha(w) \end{array}$$

is bounded.
We are going to use the Schur's test. Let $h^2(z) = \frac{1}{(1-|z|^2)^{2\beta-1}}$ and $g^2(z) = \frac{1}{(1-|z|^2)^\beta}$, we show that

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^\beta |w|}{|1-z\overline{w}|^{3+\alpha}} g^2(z) dA_{\gamma,\varphi}(z) \lesssim h^2(w), \quad w \in \mathbb{D}. \tag{2}$$

and

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^\beta |w|}{|1-z\overline{w}|^{3+\alpha}} h^2(w) dA_\alpha(w) \lesssim g^2(z), \quad z \in \mathbb{D}, \tag{3}$$

Indeed,

$$\begin{aligned} \int_{\mathbb{D}} \frac{(1-|z|^2)^\beta |w|}{|1-z\overline{w}|^{3+\alpha}} g^2(z) dA_{\gamma,\varphi}(z) &= |w| \int_{\mathbb{D}} \frac{1}{|1-z\overline{w}|^{3+\alpha}} dA_{\gamma,\varphi}(z) \\ &\leq \int_{\mathbb{D}} \frac{1}{|1-\varphi(z)\overline{w}|^{3+\alpha}} dA_\gamma(z)\ . \end{aligned}$$

But we know that C_φ is bounded on $\mathcal{A}_\gamma$ so we get

$$\begin{aligned} \int_{\mathbb{D}} \frac{(1-|z|^2)^\beta |w|}{|1-z\overline{w}|^{3+\alpha}} g^2(z) dA_{\gamma,\varphi}(z) &\lesssim \int_{\mathbb{D}} \frac{(1-|z|^2)^\gamma}{|1-z\overline{w}|^{3+\alpha}} dA(z) \\ &\lesssim \frac{1}{(1-|w|^2)^{2\beta-1}} = h^2(w) \end{aligned}$$

thanks to Theorem 1.7 of [13] for $\beta > \dfrac{1}{2}$.
In the same way we have,

$$\begin{aligned} \int_{\mathbb{D}} \frac{(1-|z|^2)^\beta |w|}{|1-z\overline{w}|^{3+\alpha}} h^2(w) dA_\alpha(w) &\lesssim (1-|z|^2)^\beta \int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha-2\beta+1}}{|1-z\overline{w}|^{3+\alpha}} dA(w) \\ &\lesssim \frac{1}{(1-|z|^2)^\beta} = g^2(z). \end{aligned}$$

As a conclusion we have that T is a bounded operator:

$$\|Tf\|_{L^2(\mathbb{D}, dA_{\gamma,\varphi})} \leq C \|f\|_{L^2(\mathbb{D}, dA_\alpha)}.$$

□

Proof of Theorem 2.5. Assume that the operator C_φ is compact on $\mathcal{B}^\beta$, $\beta > 0$ and choose $\alpha > \max(0, 2\beta - 3)$ so that $\gamma = 2 + \alpha - 2\beta > -1$. We want to prove that C_φ is compact on $\mathcal{A}^2_\alpha$. It suffices to show that if (f_n) is a bounded sequence in $\mathcal{A}^2_\alpha$ that converges to 0 uniformly on every compact subset of $\mathbb{D}$, then $\|C_\varphi(f_n)\|_{\mathcal{A}^2_\alpha} \to 0$.

By the Littlewood-Paley formula we have:

$$\|C_\varphi(f_n)\|^2_{\mathcal{A}^2_\alpha} \lesssim |(f_n \circ \varphi)(0)|^2 + \int_{\mathbb{D}} (1 - |z|^2)^{\alpha+2} |(f_n \circ \varphi)'(z)|^2 dA(z)$$

for every $n \geq 1$. Since $f_n(\varphi(0)) \underset{n\to\infty}{\longrightarrow} 0$, it remains to show that the integral tends to 0.

Since C_φ is compact on $\mathcal{B}^\beta$, for every $\varepsilon > 0$, there exists some $0 < \delta < 1$ such that $\dfrac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\beta} \leq \varepsilon$, for $\delta < |\varphi(z)| < 1$. This implies that

$$\begin{aligned}
\int_{\mathbb{D}} (1 - |z|^2)^{\alpha+2} |(f_n \circ \varphi)'(z)|^2 dA(z) &\leq \int_{\{|\varphi(z)| \leq \delta\}} (1 - |z|^2)^{\alpha+2} |\varphi'(z)|^2 |(f_n' \circ \varphi)(z)|^2 dA(z) \\
&\quad + \varepsilon^2 \int_{\{\delta < |\varphi(z)| < 1\}} (1 - |z|^2)^{2+\alpha-2\beta} (1 - |\varphi(z)|^2)^{2\beta} |(f_n' \circ \varphi)(z)|^2 dA(z) \\
&\lesssim \int_{\{|\varphi(z)| \leq \delta\}} (1 - |z|^2)^{\alpha+2} |\varphi'(z)|^2 |(f_n' \circ \varphi)(z)|^2 dA(z) \\
&\quad + \varepsilon^2 \int_{\varphi(\{\delta < |\varphi(z)| < 1\})} (1 - |z|^2)^{2\beta} |f_n'(z)|^2 dA_{\gamma,\varphi}(z)
\end{aligned}$$

Therefore, using the fact that $\alpha + 2 \geq 2$ and that $\|\varphi\|_{\mathcal{B}} \leq 1$ in the first integral, we get

$$\int_{\mathbb{D}} (1-|z|^2)^{\alpha+2} |(f_n \circ \varphi)'(z)|^2 dA(z) \lesssim \int_{\{|\varphi(z)| \leq \delta\}} |(f_n' \circ \varphi)(z)|^2 dA(z) + \varepsilon^2 \int_{\mathbb{D}} (1-|z|^2)^{2\beta} |f_n'(z)|^2 dA_{\gamma,\varphi}(z).$$

Denote by I_n the first integral and by J_n the second one. Since f_n' tends to 0 uniformly on the closed disk $\overline{D}(0, \delta)$, I_n tends to 0. It remains to show that the sequence (J_n) is bounded. But for $f_n \in \mathcal{A}^2_\alpha$ by the reproducing kernel formula we have,

$$\forall z \in \mathbb{D}, \quad f_n(z) = (\alpha + 1) \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha}{(1 - z\overline{w})^{(2+\alpha)}} f_n(w) dA(w).$$

Differentiating under the integral sign and multiplying by $(1 - |z|^2)^\beta$, we obtain

$$Tf_n(z) = (1 - |z|^2)^\beta f_n'(z) = C \int_{\mathbb{D}} \frac{(1 - |z|^2)^\beta \overline{w}}{(1 - z\overline{w})^{3+\alpha}} f_n(w) dA_\alpha(w).$$

By the preceding lemma we have, $\|Tf_n\|_{L^2(\mathbb{D}, dA_{\gamma,\varphi})} \leq C \|f_n\|_{L^2(\mathbb{D}, dA_\alpha)}$.
As a conclusion we have, $J_n \leq C\|f_n\|^2_{\mathcal{A}^2_\alpha}$. Since (f_n) is bounded in $\mathcal{A}^2_\alpha$, then $\|C_\varphi(f_n)\|_{\mathcal{A}^2_\alpha} \longrightarrow 0$ when $n \to \infty$, and C_φ is compact on $\mathcal{A}^2_\alpha$ where $\beta > \frac{1}{2}\cdot$ Hence C_φ is compact on $\mathcal{A}^p_\alpha$ in the case where $\beta > \frac{1}{2}\cdot$

Now for $0 < \beta \leq \frac{1}{2}$, if the composition operator is compact on $\mathcal{B}^\beta$ then by corollary 2.2, C_φ is also compact on $\mathcal{B}^\beta$ for $\beta > \frac{1}{2}\cdot$

But we can conclude that C_φ is compact on $\mathcal{A}^2_\alpha$ for any $\alpha > -1$ since this property actually does not depend on the parameter α as soon as $\alpha > -1$. This ends the proof of the theorem.

□

3 Absolutely summing weighted composition operators

We start this section by some preliminary results which we shall need for the proofs of our theorems.

Lemma 3.1. *Let $f \in \mathcal{A}^p_\alpha$ where $\alpha > -1$, and $p \geq 1$. We have*

$$\forall z \in \mathbb{D}, \quad |f(z)|^p \lesssim \int_{\mathbb{D}} \frac{|f(w)|^p (1-|w|^2)^\alpha}{|1-\overline{w}z|^{\alpha+2}}\, dA(w).$$

Remark 1. *Actually when p is an integer and $f \in \mathcal{A}^p_\alpha$, thanks to the reproducing kernel formula for the Bergman space (see for example [13, Cor. 1.5]), we clearly have the equality*

$$\forall z \in \mathbb{D}, \quad f^p(z) = (\alpha+1)\int_{\mathbb{D}} \frac{f^p(w)(1-|w|^2)^\alpha}{(1-\overline{w}z)^{\alpha+2}} dA(w).$$

Proof. Let $f \in \mathcal{A}^p_\alpha$. The rotation invariance of dA_α gives

$$f(0) = \int_{\mathbb{D}} f(w) dA_\alpha(w).$$

Then, since $p \geq 1$ and dA_α is a probability measure, we have

$$|f(0)|^p \leq \int_{\mathbb{D}} |f(w)|^p \, dA_\alpha(w).$$

Replacing f by $f \circ \varphi_z$, where φ_z is a Möbius map, making a change of variables and using the properties of φ_z, we obtain

$$\begin{aligned}
|f(z)|^p &\lesssim \int_{\mathbb{D}} |f\circ\varphi_z(w)|^p (1-|w|^2)^\alpha dA(w) \\
&\lesssim \int_{\mathbb{D}} |f(w)|^p (1-|\varphi_z(w)|^2)^\alpha |\varphi_z'(w)|^2 dA(w) \\
&\approx \int_{\mathbb{D}} |f(w)|^p \frac{(1-|z|^2)^{\alpha+2}(1-|w|^2)^\alpha}{|1-z\overline{w}|^{4+2\alpha}} dA(w) \\
&\lesssim \int_{\mathbb{D}} \frac{|f(w)|^p (1-|w|^2)^\alpha}{|1-z\overline{w}|^{\alpha+2}} dA(w).
\end{aligned}$$

□

We state the following result as a lemma, but is already known (see [30, Cor.4]).

Lemma 3.2. *Let $\gamma > 0$ and $g \in \mathcal{B}^\gamma$, we have*

$$\forall z \in \mathbb{D}, \quad g(z) = g(0) + \int_{\mathbb{D}} \frac{g'(w)(1-|w|^2)^\gamma}{\overline{w}(1-\overline{w}z)^{\gamma+1}} dA(w).$$

Lemma 3.3. *Let $G : \mathbb{D} \to \mathbb{C}$ analytic, $p \geq 1$, $\alpha > -1$ and $q \geq 0$. We have*

$$\int_{\mathbb{D}} |G(w)|^p \, dA_\alpha(w) \approx \int_{\mathbb{D}} |G(w)|^p |w|^q \, dA_\alpha(w)\,,$$

where the underlying constants depend only on p, q and α (but not on G).

Proof. Of course, we have to prove something when G belongs to $\mathcal{A}^p_\alpha$.

Moreover $\displaystyle\int_{\mathbb{D}} |G(w)|^p |w|^q \, dA_\alpha(w) \leq \int_{\mathbb{D}} |G(w)|^p \, dA_\alpha(w)$.

On the other hand, we have

$$\int_{\mathbb{D}} |G(w)|^p \, dA_\alpha(w) = \int_{D(0,1/2)} |G(w)|^p \, dA_\alpha(w) + \int_{\mathbb{D}\setminus D(0,1/2)} |G(w)|^p \, dA_\alpha(w).$$

The second integral is easy to handle:

$$\int_{\mathbb{D}\setminus D(0,1/2)} |G(w)|^p \, dA_\alpha(w) \leq 2^q \int_{\mathbb{D}\setminus D(0,1/2)} |w|^q |G(w)|^p \, dA_\alpha(w) \leq 2^q \int_{\mathbb{D}} |w|^q |G(w)|^p \, dA_\alpha(w).$$

Taking n as the integer just larger than q/p and using that the norm of point evaluation in a on $\mathcal{A}^p_\alpha$ is $(1-|a|^2)^{-(2+\alpha)/p}$, the first integral is lower than

$$2^{np} \sup_{|w|=1/2} |w^n.G(w)|^p \leq 2^{np} \Big(\frac{4}{3}\Big)^{(2+\alpha)} \int_{\mathbb{D}} |w|^{np} |G(w)|^p \, dA_\alpha(w).$$

Hence

$$\int_{D(0,1/2)} |G(w)|^p \, dA_\alpha(w) \lesssim \int_{\mathbb{D}} |G(w)|^p |w|^q \, dA_\alpha(w).$$

□

We now state our necessary condition which is valid for all $p \geq 1$ and any $\mu, \beta > 0$.

3.1 Necessary condition

Theorem 3.4. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$, $u : \mathbb{D} \longrightarrow \mathbb{C}$ be analytic, and μ, $\beta > 0$. If the weighted composition operator $uC_\varphi : \mathcal{B}^\mu \longrightarrow \mathcal{B}^\beta$ is p-summing, $p \geq 1$, Then*

(C1) $\displaystyle J_1 = \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)}(1-|z|^2)^{\beta p}|\varphi'(z)|^p}{|1-\overline{w}\varphi(z)|^{(2+\mu)p}} |u(z)|^p dA(w) < +\infty.$

and

(C2) $\displaystyle J_2 = \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)}}{|1-\overline{w}\varphi(z)|^{(1+\mu)p}} (1-|z|^2)^{\beta p}|u'(z)|^p dA(w) < +\infty.$

Proof. Assume that $uC_\varphi : \mathcal{B}^\mu \longrightarrow \mathcal{B}^\beta$ is p-summing, then $uC_\varphi : \mathcal{B}^\mu_0 \longrightarrow \mathcal{B}^\beta$ is also p-summing.

Thanks to Pietsch's theorem (see [7, Th. 2.12]) there exists a Borel probability measure ν on $\big(B_{(\mathcal{B}^\mu_0)^*}, \sigma((\mathcal{B}^\mu_0)^*, \mathcal{B}^\mu_0)\big)$ (here B_Z denotes the unit ball of a Banach space Z), such that

$$\|u.(f\circ\varphi)\|_{\mathcal{B}^\beta} \leq \pi_p(uC_\varphi) \Big(\int_{B_{(\mathcal{B}^\mu_0)^*}} |\xi(f)|^p \, d\nu(\xi) \Big)^{1/p}, \tag{4}$$

for every $f \in \mathcal{B}^\mu_0$.

Now, for every w in $\mathbb{D}$, we consider $f_w(z) = \dfrac{(1-|w|^2)^{2/p'}}{(1-\overline{w}z)^{1+\mu}}$ which lies in $\mathcal{B}_0^\mu \cap H^\infty$.

From (4), we get

(5)

$$\sup_{z\in\mathbb{D}}(1-|w|^2)^{2(p-1)}(1-|z|^2)^{\beta p}\Big|\frac{(1+\mu)\overline{w}u(z)\varphi'(z)}{(1-\overline{w}\varphi(z))^{2+\mu}}+\frac{u'(z)}{(1-\overline{w}\varphi(z))^{1+\mu}}\Big|^p \le K^p\int_{B_{(\mathcal{B}_0^\mu)^*}}|\xi(f_w)|^p d\nu(\xi)$$

where K stands for the π_p constant of the operator uC_φ.

But thanks to the duality between $\mathcal{B}_0^\mu$ and $\mathcal{A}^1$, there exists some numerical constant $\gamma \ge 1$ satisfying: for every $\xi \in B_{(\mathcal{B}_0^\mu)^*}$, there exists $h \in \gamma B_{\mathcal{A}^1}$ such that $\xi(f) = \langle h, f\rangle$ for every $f \in \mathcal{B}_0^\mu$. In particular, thanks to the reproducing kernel formula for the Bergman space,

$$\begin{aligned}
\xi(f_w) &= \langle h, f_w\rangle = \int_{\mathbb{D}} \overline{h(z)} f_w(z)(1-|z|^2)^{\mu-1}dA(z)\\
&= (1-|w|^2)^{2/p'}\int_{\mathbb{D}}\frac{\overline{h(z)}}{(1-\overline{w}z)^{1+\mu}}(1-|z|^2)^{\mu-1}dA(z)\\
&= (1-|w|^2)^{2/p'}\frac{1}{\mu}\int_{\mathbb{D}}\frac{\overline{h(z)}}{(1-\overline{w}z)^{2+\mu-1}}dA_{\mu-1}(z)\\
&= \frac{1}{\mu}(1-|w|^2)^{2/p'}\overline{h(w)}\,.
\end{aligned}$$

We finally use the fact that the point-evaluation δ_z is a bounded linear functional on $\mathcal{A}^1$, with norm equal to $\dfrac{1}{(1-|z|^2)^2}$ (see for example [9, Theorem 1]). Therefore

$$|\xi(f_w)|^p \le \frac{1}{\mu^p}\|h\|_{\mathcal{A}^1}^{p-1}|h(w)| \le \frac{\gamma^{p-1}}{\mu^p}\,|h(w)|\,. \tag{6}$$

Integrating over $\mathbb{D}$ inequality (5), we get

$$\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}(1-|w|^2)^{2(p-1)}(1-|z|^2)^{\beta p}\Big|\frac{(1+\mu)\overline{w}u(z)\varphi'(z)}{(1-\overline{w}\varphi(z))^{2+\mu}}+\frac{u'(z)}{(1-\overline{w}\varphi(z))^{1+\mu}}\Big|^p dA(w) < \infty\,. \tag{7}$$

• Now, we focus on the case $p > 1$:
for every w, z_0 in $\mathbb{D}$, we consider

$$F(z) = \frac{a}{(1-\overline{w}z)^{1+\mu}} + \frac{b}{(1-\overline{w}z)^{\mu}}\,,$$

where $a, b \in \mathbb{C}$ are chosen so that $F(\varphi(z_0)) = \dfrac{(1-|w|^2)^{2/p'}}{(1-\overline{w}\varphi(z_0))^{1+\mu}}$ and $F'(\varphi(z_0)) = 0$. Clearly $F \in \mathcal{B}_0^\mu \cap H^\infty$. Actually the values of a and b are easy to compute:

$$a = -\mu(1-|w|^2)^{2/p'} \qquad \text{and} \qquad b = (\mu+1)\frac{(1-|w|^2)^{2/p'}}{1-\overline{w}\varphi(z_0)}\,.$$

From (4) applied to F, we get

$$(8)\qquad (1-|z_0|^2)^\beta |u'(z_0).F(\varphi(z_0))| \le \|u(F\circ\varphi)\|_{\mathcal{B}^\beta} \le \pi_p(u.C_\varphi)\Big(\int_{B_{(\mathcal{B}_0^\mu)^*}} |\xi(F)|^p\, d\nu(\xi)\Big)^{1/p},$$

We have now to estimate $|\xi(F)|$ for any ξ in the unit ball of $(\mathcal{B}_0^\mu)^*$. We adapt the previous computation (recall (6)) with the same notations and we have

$$|\xi(F)| \lesssim |\xi(f_w)| + |\xi(C_w)|, \quad \text{where } C_w(z) = \frac{b}{(1-\overline{w}z)^\mu}\cdot$$

$$\overline{\xi(C_w)} = \overline{\langle h, C_w\rangle} = (\mu+1)\frac{(1-|w|^2)^{2/p'}}{1-w\overline{\varphi(z_0)}}\int_{\mathbb{D}} \frac{h(z)(1-|z|^2)^{\mu-1}}{(1-\overline{z}w)^\mu} dA(z)$$

Let $G(w) = \displaystyle\int_{\mathbb{D}} \frac{h(z)(1-|z|^2)^{\mu-1}}{(1-\overline{z}w)^\mu} dA(z)$.

Then, using Fourier convolution between functions on the torus $\mathbb{T}$, we can write

$$w^2G''(w) = \mu(\mu+1)\int_{\mathbb{D}} \frac{w^2\overline{z}^2 h(z)(1-|z|^2)^{\mu-1}}{(1-\overline{z}w)^{\mu+2}} dA(z) = \mu(\mu+1) h_{\sqrt{r}} * Q_{\sqrt{r}\rho}(u)\,,$$

where $Q_{\sqrt{r}\rho}(u) = \displaystyle\int_0^1 q_{\sqrt{r}\rho}(u)(1-\rho)^{\mu-1}\, d\rho$, and $q(t) = t^2(1-t)^{-(\mu+2)}$, for $t\in\mathbb{D}$, with $w = ru$ (i.e $r = |w|$ and $u\in\mathbb{T}$).

Therefore, for all ξ in the unit ball of $\mathcal{B}_0^\mu$, since $p-2 > -1$, (by [13, Proposition 1.11] and Lemma 3.3), we have

$$\begin{aligned}
\int_{\mathbb{D}} |\xi(C_w)|^p dA(w) &\lesssim \int_{\mathbb{D}} (1-|w|^2)^{p-2}|G(w)|^p dA(w)\\
&= \|G\|^p_{\mathcal{A}^p_{p-2}} \approx \|(1-|w|)^2 G''\|^p_{L^p(\mathbb{D},dA_{p-2})}\\
&\approx \|w^2G''\|^p_{L^p(\mathbb{D},dA_{3p-2})}\\
&\approx \int_{\mathbb{D}} (1-|w|)^{3p-2}|w^2G''(w)|^p dA(w)\\
&= \int_0^1 2r^{1+2p}(1-r)^{3p-2}\|G''_r\|^p_{H^p} dr\,,
\end{aligned}$$

where H^p is the classical Hardy space. But,

$$r^{2p}\|G''_r\|^p_{H^p} \lesssim \|h_{\sqrt{r}}\|^p_{H^p}\|Q_{\sqrt{r}\rho}\|^p_{H^1} \lesssim \|h_{\sqrt{r}}\|^p_{H^p}\Big(\int_{\mathbb{D}} |q(\sqrt{r}z)|(1-|z|)^{\mu-1}dA(z)\Big)^p,$$

and

$$\int_{\mathbb{D}} |q(\sqrt{r}z)|(1-|z|)^{\mu-1}dA(z) \approx \int_{\mathbb{D}} \frac{(1-|z|)^{\mu-1}}{|1-\sqrt{r}z|^{\mu+2}} dA(z) \approx \frac{1}{1-r},$$

where we used theorem 1.7 of [13] for the last estimate.
We get,

$$\int_{\mathbb{D}} |\xi(C_w)|^p dA(w) \lesssim \int_0^1 2r(1-r)^{2p-2}\|h_{\sqrt{r}}\|^p_{H^p} dr.$$

Again, using the norm of the evaluation functional on $\mathcal{A}^1$ which is equal to $\dfrac{1}{(1-|z|^2)^2}$, we have

$$\|h_{\sqrt{r}}\|_{H^p}^p = \int_{\mathbb{T}} \left|h(\sqrt{r}z)\right|^p \, d\lambda(z) \leq \frac{\|h\|_{\mathcal{A}^1}^{p-1}}{(1-r)^{2(p-1)}} \|h_{\sqrt{r}}\|_{H^1}.$$

Therefore,

$$\int_{\mathbb{D}} |\xi(C_w)|^p dA(w) \lesssim \gamma^{p-1} \int_0^1 2r\|h_{\sqrt{r}}\|_{H^1} dr = \gamma^{p-1} \int_0^1 4s^3 \|h_s\|_{H^1} ds \leq 2\gamma^{p-1}\|h\|_{\mathcal{A}^1} \leq 2\gamma^p.$$

Taking the supremum over z_0 and integrating inequality (8) over $\mathbb{D}$, we get,

$$\int_{\mathbb{D}} \sup_{z_0 \in \mathbb{D}} (1-|w|^2)^{2(p-1)} (1-|z_0|^2)^{\beta p} \Big| \frac{u'(z_0)}{(1-\overline{w}\varphi(z_0))^{1+\mu}} \Big|^p \, dA(w) < \infty \tag{9}$$

Hence we get condition $(C2)$. Then, combining (7) and (9), we obtain $(C1)$.

Remark: let us mention that the proof can be simplified when $p\mu > 2p-1$. Adapting the preceding proof we have

$$\begin{aligned} \overline{\xi(C_w)} &= (\mu+1)\frac{(1-|w|^2)^{2/p'}}{1-w\overline{\varphi(z_0)}} \int_{\mathbb{D}} \frac{h(z)(1-|z|^2)^{\mu-1}}{(1-\overline{z}w)^{\mu}} dA(z) \\ &= (\mu+1)\frac{(1-|w|^2)^{2/p'}}{1-w\overline{\varphi(z_0)}} G(w). \end{aligned}$$

Here we do not use the convolution. Actually we have

$$\begin{aligned} \int_{\mathbb{D}} |\xi(C_w)|^p dA(w) &\lesssim \int_{\mathbb{D}} (1-|w|^2)^{p-2} |G(w)|^p dA(w) \\ &= \|G\|_{\mathcal{A}^p_{p-2}}^p \approx \|(1-|w|^2)G'\|_{L^p(\mathbb{D},A_{p-2})}^p \\ &= \int_{\mathbb{D}} (1-|w|^2)^{2(p-1)} \Big| \int_{\mathbb{D}} \frac{\mu \overline{z} h(z)(1-|z|^2)^{\mu-1}}{(1-\overline{z}w)^{\mu+1}} dA(z) \Big|^p dA(w) \end{aligned}$$

Now by theorem 1.9 of [13], we have

$$\int_{\mathbb{D}} |\xi(C_w)|^p dA(w) \lesssim \int_{\mathbb{D}} (1-|w|^2)^{2(p-1)} |h(w)|^p dA(w) \lesssim \gamma^p.$$

Therefore, from (8), we get

$$\int_{\mathbb{D}} \sup_{z_0 \in \mathbb{D}} (1-|w|^2)^{2(p-1)} (1-|z_0|^2)^{\beta p} \Big| \frac{u'(z_0)}{(1-\overline{w}\varphi(z_0))^{\mu+1}} \Big|^p \, dA(w) < \infty.$$

• Now, we focus on the case $p = 1$:
We adapt the preceding method and consider for $\delta \in (0,1)$, for every w, z_0 in $\mathbb{D}$, the functions

$$F(z) = \frac{a}{(1-\overline{w}z)^{\mu+1}} + \frac{b}{(1-\overline{w}z)^{\mu+\delta}}$$

where $a, b \in \mathbb{C}$ still chosen so that $F(\varphi(z_0)) = \dfrac{1}{(1-\overline{w}\varphi(z_0))^{\mu+1}}$ and $F'(\varphi(z_0)) = 0$.

We have $a = -\dfrac{\mu+\delta}{1-\delta}$ and $b = \dfrac{\mu+1}{(1-\delta)(1-\overline{w}\varphi(z_0))^{1-\delta}}$.

We apply (4) to F in the same way and we have again to estimate $|\xi(F)|$ for any ξ in the unit ball of $(\mathcal{B}_0^\mu)^*$. The term $|\xi(f_w)|$ was already estimated from above. Let us choose $\delta = 1/2$ to simplify the expressions (but any value of $\delta \in (0,1)$ would work) and write $D_w(z) = \dfrac{b}{(1-\overline{w}z)^{\mu+\frac{1}{2}}}$.

$$\overline{\xi(D_w)} = \bar{b}\int_{\mathbb{D}} \frac{h(z)(1-|z|^2)^{\mu-1}}{(1-w\overline{z})^{\mu+\frac{1}{2}}} dA(z) = \bar{b}G(w).$$

Then,

$$wG'(w) = (\mu+\frac{1}{2})\int_{\mathbb{D}} \frac{w\overline{z}h(z)(1-|z|^2)^{\mu-1}}{(1-\overline{z}w)^{\mu+\frac{3}{2}}} dA(z) = (\mu+\frac{1}{2})\int_0^1 h_{\sqrt{r}} * Q_{\sqrt{r}\rho}(u)(1-\rho)^{\mu-1}\, d\rho$$

where $Q(t) = t(1-t)^{-(\mu+\frac{3}{2})}$, for $t \in \mathbb{D}$, and $w = ru$ (i.e $r = |w|$ and $u \in \mathbb{T}$).
We have for every ξ in the unit ball of $(\mathcal{B}_0^\mu)^*$:

$$\begin{aligned}
\int_{\mathbb{D}} |\xi(D_w)| dA(w) &\lesssim \|G\|_{\mathcal{A}^1_{-\frac{1}{2}}} \approx \|(1-|w|)G'\|_{L^1(\mathbb{D}, dA_{-\frac{1}{2}})} \\
&\approx \|wG'\|^p_{L^1(\mathbb{D}, dA_{\frac{1}{2}})} \\
&\approx \int_{\mathbb{D}} (1-|w|)^{\frac{1}{2}} |wG'(w)| dA(w) \\
&= \int_0^1 2r^2(1-r)^{\frac{1}{2}} \|G'_r\|_{H^1} dr.
\end{aligned}$$

But,

$$r\|G'_r\|_{H^1} \lesssim \int_0^1 \|h_{\sqrt{r}}\|_{H^1} \|Q_{\sqrt{r}\rho}\|_{H^1} (1-\rho)^{\mu-1}\, d\rho \approx \|h_{\sqrt{r}}\|_{H^1} \int_{\mathbb{D}} |Q(\sqrt{r}z)|(1-|z|)^{\mu-1} dA(z),$$

and ([13, Theorem 1.7]),

$$\int_{\mathbb{D}} |Q(\sqrt{r}z)|(1-|z|)^{\mu-1} dA(z) \approx \frac{1}{(1-r)^{\frac{1}{2}}}.$$

We get,

$$\int_{\mathbb{D}} |\xi(D_w)| dA(w) \lesssim \int_0^1 2r \|h_{\sqrt{r}}\|_{H^1} dr \leq 2\gamma.$$

The proof now ends like in the case $p > 1$.

Remark: once again the proof can be simplified when $\mu > 1$: we have,

$$\xi(D_w) = \frac{2(\mu+1)}{(1-\overline{w}\varphi(z_0))^{\frac{1}{2}}} \int_{\mathbb{D}} \frac{\overline{h(z)}(1-|z|^2)^{\mu-1}}{(1-z\overline{w})^{\mu+\frac{1}{2}}} dA(z).$$

Then,

$$\begin{aligned}\int_{\mathbb{D}}|\xi(D_w)|dA(w) &= \int_{\mathbb{D}}\Big|\frac{2(\mu+1)}{(1-\overline{w}\varphi(z_0))^{\frac{1}{2}}}\int_{\mathbb{D}}\frac{h(z)(1-|z|^2)^{\mu-1}}{(1-\overline{z}w)^{\mu+\frac{1}{2}}}dA(z)\Big|dA(w)\\ &\lesssim \int_{\mathbb{D}}\Big(\int_{\mathbb{D}}\frac{(1-|w|)^{-\frac{1}{2}}}{|1-\overline{z}w|^{\mu+\frac{1}{2}}}dA(w)\Big)|h(z)|(1-|z|^2)^{\mu-1}dA(z)\\ &\lesssim \int_{\mathbb{D}}\frac{|h(z)|(1-|z|^2)^{\mu-1}}{(1-|z|^2)^{\mu-1}}dA(z)\leq\gamma.\end{aligned}$$

We used again Theorem 1.7 of [13] for $\mu>1$. □

3.2 Sufficient conditions

We are going to present several sufficient conditions. The first one is valid for classical Bloch spaces, the second one is valid for Bloch-type spaces but we have to separate cases according to the value of μ relative to 1. The last one is true only for $p=1$ on Bloch-type spaces.

Theorem 3.5. *Let $\varphi:\mathbb{D}\to\mathbb{D}$ be analytic, $u\in\mathcal{B}$, and $p\geq 1$. Assume that the following conditions are satisfied*

(C'1) $J_1'=\displaystyle\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|w|^2)^{2(p-1)}(1-|z|^2)^p|\varphi'(z)|^p}{|1-\overline{w}\varphi(z)|^{3p}}|u(z)|^pdA(w)<+\infty\,.$

(C'2) $J_2'=\displaystyle\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|w|^2)^{2(p-1)}}{|w|.|1-\overline{w}\varphi(z)|^{2p}}(1-|z|^2)|u'(z)|dA(w)<+\infty\,.$

Then the weighted composition operator $uC_\varphi:\mathcal{B}\longrightarrow\mathcal{B}$ is p-summing.

Remark 2. *This theorem has to be compared on one hand to Theorem 3.4: $J_1'=J_1$ when $\mu=\beta=1$ and $J_2'\approx J_2$ when $p=1$. In other words we have an equivalence in this case of classical Bloch spaces for $p=1$. In that case it coincides with Theorem 3.7.*

The only difference between J_2' and J_2 concerns the power of $(1-|z|^2)|u'(z)|$ in the integral for $\mu=\beta=1$: that is why we miss the full characterization for classical Bloch spaces when $p>1$.

On the other hand, the second condition has to be compared also to the second one in Theorem 3.6 (see also Theorem 3.7 as mentioned).

Proof. We want to prove that uC_φ is p-summing. We consider $f_1,\dots,f_N\in\mathcal{B}$ satisfying

$$\sup_{a\in B_{\ell^{p'}}}\Big\|\sum_{j=1}^N a_jf_j\Big\|_{\mathcal{B}}=1.$$

Equivalently we have

$$\sup_{w\in\mathbb{D}}\sum_{j=1}^N|f_j'(w)|^p(1-|w|^2)^p=\sup_{a\in B_{\ell^{p'}}}\sup_{w\in\mathbb{D}}\Big|\sum_{j=1}^N a_jf_j'(w)(1-|w|^2)\Big|^p=1. \tag{10}$$

We wish to get an upper estimate for

$$\Big(\sum_{j=1}^{N}\|u(f_j\circ\varphi)\|_{\mathcal{B}}^p\Big)^{1/p}$$

which is less than

$$\Big(\sum_{j=1}^{N}\Big|u(0).f_j(\varphi(0))\Big|^p\Big)^{1/p}+\Big(\sum_{1\le j\le N}\sup_{z\in\mathbb{D}}\Big((1-|z|^2)|u(z)||\varphi'(z)||f_j'(\varphi(z))|\Big)^p\Big)^{1/p}$$
$$+\Big(\sum_{1\le j\le N}\sup_{z\in\mathbb{D}}\Big((1-|z|^2)|u'(z)||f_j(\varphi(z))|\Big)^p\Big)^{1/p}$$

by the triangular inequality on ℓ^p.

The term $\Big(\sum_{j=1}^{N}\Big|u(0).f_j(\varphi(0))\Big|^p\Big)^{1/p}$ is clearly bounded by $|u(0)|.\|\delta_{\varphi(0)}\|_{\mathcal{B}^*}$ using linearization like in (10).

Now we focus on the second term $S_2=\Big(\sum_{j=1}^{N}\sup_{z\in\mathbb{D}}\Big((1-|z|^2)|u(z)||\varphi'(z)||f_j'(\varphi(z))|\Big)^p\Big)^{1/p}$. For each $1\le j\le N$, there exists some $z_j\in\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}}\Big((1-|z|^2)|u(z)||\varphi'(z)||f_j'(\varphi(z))|\Big)^p\le(1-|z_j|^2)^p|u(z_j)|^p|\varphi'(z_j)|^p|f_j'(\varphi(z_j))|^p+2^{-j}$$

so that we need to control

$$\sum_{j=1}^{N}(1-|z_j|^2)^p|u(z_j)|^p|\varphi'(z_j)|^p|f_j'(\varphi(z_j))|^p.$$

Using Lemma 3.1 with $\alpha=3p-2>0$, this is less than

$$\sum_{j=1}^{N}(1-|z_j|^2)^p|u(z_j)|^p|\varphi'(z_j)|^p\int_{\mathbb{D}}\frac{|f_j'(w)|^p(1-|w|^2)^{3p-2}}{|1-\overline{w}\varphi(z_j))|^{3p}}\,dA(w)$$

itself being less than

$$\int_{\mathbb{D}}\Big(\sum_{j=1}^{N}|f_j'(w)|^p(1-|w|^2)^p\Big)\sup_{1\le j\le N}\frac{(1-|z_j|^2)^p(1-|w|^2)^{2(p-1)}}{|1-\overline{w}\varphi(z_j))|^{3p}}|u(z_j)|^p|\varphi'(z_j)|^p\,dA(w).$$

On one hand $\Big(\sum_{j=1}^{N}|f_j'(w)|^p(1-|w|^2)^p\Big)\le 1$ by (10).

On the other hand, we have for every $w\in\mathbb{D}$ and every $j\in\{1\dots,N\}$,

$$\frac{(1-|z_j|^2)^p(1-|w|^2)^{2(p-1)}}{|1-\overline{w}\varphi(z_j))|^{3p}}|u(z_j)|^p|\varphi'(z_j)|^p\le\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^p(1-|w|^2)^{2(p-1)}}{|1-\overline{w}\varphi(z))|^{3p}}|u(z)|^p|\varphi'(z)|^p.$$

Therefore our second quantity S_2 is indeed uniformly bounded by $(1+J_1')^{\frac{1}{p}}$ thanks to condition $(C'1)$.

Now we have to get a uniform estimate from above for the third term

$$S_3 = \Big(\sum_{j=1}^{N} \sup_{z\in\mathbb{D}} \Big((1-|z|^2)|u'(z)||f_j(\varphi(z))|\Big)^p\Big)^{1/p}.$$

By linearization and as before, we can find $\big(a_j\big)_{1\le j\le N}$ in the unit ball of $\ell^{p'}$ and, for each $1\le j\le N$, some $z_j\in\mathbb{D}$ such that

$$S_3 \le 1+\sum_{j=1}^{N} a_j(1-|z_j|^2)|u'(z_j)|\,|f_j(\varphi(z_j))| \le 1+\|u\|_{\mathcal{B}}+\sum_{j=1}^{N} a_j(1-|z_j|^2)|u'(z_j)|\,|\widetilde{f}_j(\varphi(z_j))|,$$

where $\widetilde{f}_j = f_j - f_j(0)$, since

$$\sum_{j=1}^{N} a_j(1-|z_j|^2)|u'(z_j)|\,|f_j(0)| \le \|u\|_{\mathcal{B}}\Big|\sum_{j=1}^{N} a_j f_j(0)\Big| \le \|u\|_{\mathcal{B}}\Big\|\sum_{j=1}^{N} a_j f_j\Big\|_{\mathcal{B}} \le \|u\|_{\mathcal{B}}.$$

Now we can use Lemma 3.2 for each $1\le j\le N$ to the function $\widetilde{f}_j$ at point $\varphi(z_j)$, with $\gamma = 2p-1>0$, since $\widetilde{f}_j\in\mathcal{B}\subset\mathcal{B}^\gamma$ (because $p\ge 1$). We get

$$S_3 \le 1+\|u\|_{\mathcal{B}}+\sum_{j=1}^{N} a_j(1-|z_j|^2)|u'(z_j)|\int_{\mathbb{D}}\frac{\widetilde{f}_j'(w)(1-|w|^2)^{2p-1}}{\overline{w}(1-\overline{w}\varphi(z_j)))^{2p}}\,dA(w).$$

Rearranging, we obtain

$$S_3 \le 1+\|u\|_{\mathcal{B}}+\int_{\mathbb{D}}\Big|\sum_{j=1}^{N} a_j f_j'(w)(1-|w|^2)\Big|\sup_{z\in\mathbb{D}}|u'(z)|\frac{(1-|z|^2)(1-|w|^2)^{2(p-1)}}{|w||1-\overline{w}\varphi(z))|^{2p}}\,dA(w).$$

At last, $\Big|\sum_{j=1}^{N} a_j f_j'(w)(1-|w|^2)\Big| \le 1$ by (10) and we conclude that $S_3\le 1+\|u\|_{\mathcal{B}}+J_2'$. □

A second result giving a sufficient condition for weighted composition operators to be p-summing on Bloch-type spaces is the following:

Theorem 3.6. *Let $\varphi:\mathbb{D}\longrightarrow\mathbb{D}$, $u:\mathbb{D}\longrightarrow\mathbb{C}$ be analytic, $p\ge 1$ and μ, $\beta>0$. Assume that $uC_\varphi:\mathcal{B}^\mu\longrightarrow\mathcal{B}^\beta$ is bounded.*

- *For $\mu>1$ if*

$$(C"0)\quad J_1=\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta p}(1-|w|^2)^{2(p-1)}|u(z)|^p|\varphi'(z)|^p}{|1-\overline{w}\varphi(z)|^{(\mu+2)p}}dA(w)<\infty,$$

and

$$(C"1) \quad J_2 = \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta p}(1-|w|^2)^{2(p-1)}|u'(z)|^p}{|1-\overline{w}\varphi(z)|^{(\mu+1)p}} dA(w) < \infty,$$

then $uC_\varphi : \mathcal{B}^\mu \longrightarrow \mathcal{B}^\beta$ *is p-summing.*

- *For* $\mu = 1$ *if*

$$(C"2) \quad J"_1 = \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta p}(1-|w|^2)^{2(p-1)}|u(z)|^p|\varphi'(z)|^p}{|1-\overline{w}\varphi(z)|^{3p}} dA(w) < \infty,$$

and

$$(C"3) \quad K_1 = \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta p}(1-|w|^2)^{2(p-1)}|u'(z)|^p \log\left(\frac{1}{1-|w|^2}\right)^p}{|1-\overline{w}\varphi(z)|^{2p}} dA(w) < \infty,$$

then $uC_\varphi : \mathcal{B} \longrightarrow \mathcal{B}^\beta$ *is p-summing.*

- *For* $0 < \mu < 1$ *if*

$$(C"4) \quad J_1 = \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta p}(1-|w|^2)^{2(p-1)}|u(z)|^p|\varphi'(z)|^p}{|1-\overline{w}\varphi(z)|^{(\mu+2)p}} dA(w) < \infty,$$

and

$$(C"5) \quad K_2 = \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta p}(1-|w|^2)^{(\mu+1)p-2}|u'(z)|^p}{|1-\overline{w}\varphi(z)|^{(1+\mu)p}} dA(w) < \infty,$$

then $uC_\varphi : \mathcal{B}^\mu \longrightarrow \mathcal{B}^\beta$ *is p-summing.*

Proof. The proof of this theorem uses several similar ideas as in the proof of Theorem 3.5. Nevertheless, there are some technical differences and it leads to different conditions.

We consider $f_1, \ldots, f_N \in \mathcal{B}^\mu$ satisfying

$$\sup_{a\in B_{\ell^{p'}}} \Big\| \sum_{j=1}^N a_j f_j \Big\|_{\mathcal{B}^\mu} = 1.$$

Equivalently

$$\sup_{w\in\mathbb{D}} \sum_{j=1}^N |f_j'(w)|^p (1-|w|^2)^{\mu p} = 1. \tag{11}$$

We wish to get an upper estimate for

$$\Big(\sum_{j=1}^N \|u(f_j\circ\varphi)\|^p_{\mathcal{B}^\beta} \Big)^{1/p}.$$

The term $\Big(\sum_{j=1}^{N}\Big|u(0).f_j(\varphi(0))\Big|^p\Big)^{1/p}$ is bounded by $|u(0)|.\|\delta_{\varphi(0)}\|_{(\mathcal{B}^\mu)^*}$.

Now we focus on the second term

$$S_2=\Big(\sum_{j=1}^{N}\sup_{z\in\mathbb{D}}\Big((1-|z|^2)^\beta|u(z)||\varphi'(z)||f_j'(\varphi(z))|\Big)^p\Big)^{1/p}.$$

As before, we need to control

$$\sum_{j=1}^{N}(1-|z_j|^2)^{\beta p}|u(z_j)|^p|\varphi'(z_j)|^p|f_j'(\varphi(z_j))|^p,\quad \text{for any arbitrary } z_j\in\mathbb{D}.$$

We use once again Lemma 3.1 with $\alpha=(2+\mu)p-2>0$, then the preceding quantity is less than

$$\sum_{j=1}^{N}(1-|z_j|^2)^{\beta p}|u(z_j)|^p|\varphi'(z_j)|^p\int_{\mathbb{D}}\frac{|f_j'(w)|^p(1-|w|^2)^{(2+\mu)p-2}}{|1-\overline{w}\varphi(z_j)|^{(2+\mu)p}}dA(w)\,.$$

Rearranging, this is less than

$$\Big(\sup_w\sum_{j=1}^{N}(1-|w|^2)^{\mu p}|f_j'(w)|^p\Big)\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta p}(1-|w|^2)^{2(p-1)}|u(z)|^p|\varphi'(z)|^p}{|1-\overline{w}\varphi(z)|^{(2+\mu)p}}\,dA(w)\,.$$

Since $\Big(\sum_{j=1}^{N}|f_j'(w)|^p(1-|w|^2)^{\mu p}\Big)\le 1$ (by (11)), we conclude that S_2 is uniformly bounded by $(1+J_1)^{\frac{1}{p}}$ thanks to condition (C"0).

We still need to get a uniform estimate from above for the third term

$$S_3=\Big(\sum_{j=1}^{N}\sup_{z\in\mathbb{D}}\Big((1-|z|^2)^\beta|u'(z)||f_j(\varphi(z))|\Big)^p\Big)^{1/p}.$$

In other words, we want to upper estimate

$$\sum_{j=1}^{N}(1-|z_j|^2)^{\beta p}|u'(z_j)|^p|f_j(\varphi(z_j))|^p,\quad \text{for any } z_j\in\mathbb{D}.$$

We use again Lemma 3.1, this time with $\alpha=(1+\mu)p-2>-1$. Then the preceding quantity is less than

$$\sum_{j=1}^{N}(1-|z_j|^2)^{\beta p}|u'(z_j)|^p\int_{\mathbb{D}}\frac{(1-|w|^2)^{(1+\mu)p-2}|f_j(w)|^p}{|1-\varphi(z_j)\overline{w}|^{(1+\mu)p}}dA(w)$$

itself being less than

$$\Big(\sup_{w\in\mathbb{D}}\sum_{j=1}^{N}(1-|w|^2)^{p(\mu-1)}|f_j(w)|^p\Big)\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta p}(1-|w|^2)^{2(p-1)}|u'(z)|^p}{|1-\varphi(z)\overline{w}|^{(1+\mu)p}}dA(w).$$

Thanks to (C"1) this is less than

$$J_2\Big(\sup_{w\in\mathbb{D}}\sum_{j=1}^{N}(1-|w|^2)^{p(\mu-1)}|f_j(w)|^p\Big).$$

Now we use the norm of point evaluation function δ_w on $\mathcal{B}^\mu$.

When $\mu>1$, we have $\|\delta_w\|_{(\mathcal{B}^\mu)^*}\approx\dfrac{1}{(1-|w|^2)^{(\mu-1)}}$, then

$$\begin{aligned}\sup_{w\in\mathbb{D}}\sum_{j=1}^{N}(1-|w|^2)^{p(\mu-1)}|f_j(w)|^p &\approx \sup_{w\in\mathbb{D}}\sup_{a\in B_{\ell^{p'}}}\frac{1}{\|\delta_w\|^p_{(\mathcal{B}^\mu)^*}}\Big|\sum_{j=1}^{N}a_jf_j(w)\Big|^p\\ &\lesssim \sup_{a\in B_{\ell^{p'}}}\Big\|\sum_{j=1}^{N}a_jf_j\Big\|^p_{\mathcal{B}^\mu}=1.\end{aligned}$$

Therefore, S_3 is uniformly bounded by $(1+J_2)^{1/p}$, and $uC_\varphi:\mathcal{B}^\mu\longrightarrow\mathcal{B}^\beta$ is p-summing.

When $\mu=1$, we have $\|\delta_w\|_{\mathcal{B}^*}\approx\log\dfrac{1}{1-|w|^2}$. We need to control

$$\int_{\mathbb{D}}\Big(\sum_{j=1}^{N}|f_j(w)|^p\Big)\sup_j\frac{(1-|z_j|^2)^{\beta p}(1-|w|^2)^{2(p-1)}|u'(z_j)|^p}{|1-\overline{w}\varphi(z_j)|^{2p}}dA(w).$$

We multiply and divide by $\Big(\log\dfrac{1}{1-|w|^2}\Big)^p$, getting

$$\Big(\sup_{w\in\mathbb{D}}\sum_{j=1}^{N}|f_j(w)|^p\Big(\log\frac{1}{1-|w|^2}\Big)^{-p}\Big)\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta p}(1-|w|^2)^{2(p-1)}|u'(z)|^p\big(\log\frac{1}{1-|w|^2}\big)^p}{|1-\overline{w}\varphi(z)|^{2p}}dA(w).$$

Thanks to (C"3), this is less than

$$K_1\Big(\sup_{w\in\mathbb{D}}\sum_{j=1}^{N}|f_j(w)|^p\Big(\log\frac{1}{1-|w|^2}\Big)^{-p}\Big).$$

Since $\|\delta_w\|_{\mathcal{B}^*}\approx\log\dfrac{1}{1-|w|^2}$, we conclude as before that

$$\sup_{w\in\mathbb{D}}\sum_{j=1}^{N}|f_j(w)|^p\Big(\log\frac{1}{1-|w|^2}\Big)^{-p}\lesssim\sup_{a\in B_{\ell^{p'}}}\Big\|\sum_{j=1}^{N}a_jf_j\Big\|^p_{\mathcal{B}}=1.$$

Therefore, S_3 is uniformly bounded by $(1+K_1)^{1/p}$, and $uC_\varphi:\mathcal{B}\longrightarrow\mathcal{B}^\beta$ is p-summing.

Finally when $\mu<1$, we have $\|\delta_w\|_{(\mathcal{B}^\mu)^*}\approx 1$. We need to control

$$\Big(\sup_{w\in\mathbb{D}}\sum_{j=1}^{N}|f_j(w)|^p\Big)\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta p}(1-|w|^2)^{(1+\mu)p-2}|u'(z)|^p}{|1-\varphi(z)\overline{w}|^{(1+\mu)p}}dA(w).$$

Thanks to (C"5), this is less than

$$K_2\Big(\sup_{w\in\mathbb{D}}\sum_{j=1}^{N}|f_j(w)|^p\Big).$$

Since $\|\delta_w\|_{(\mathcal{B}^\mu)^*}\approx 1$, we conclude that

$$\sup_{w\in\mathbb{D}}\sum_{j=1}^{N}|f_j(w)|^p\lesssim\sup_{a\in B_{\ell^{p'}}}\Big\|\sum_{j=1}^{N}a_jf_j\Big\|_{\mathcal{B}^\mu}^p=1.$$

Therefore, S_3 is uniformly bounded by $(1+K_2)^{1/p}$, and $uC_\varphi:\mathcal{B}^\mu\longrightarrow\mathcal{B}^\beta$ is p-summing. This ends the proof of the theorem. □

In the special case $p=1$, we succeed in obtaining the characterization for 1-summing weighted composition operators on Bloch-type spaces. We prove now that necessary conditions in Theorem 3.4 are sufficient.

Theorem 3.7. *Let $\varphi:\mathbb{D}\to\mathbb{D}$ be analytic, $u\in\mathcal{B}^\beta$, and $\mu,\beta>0$. If the following conditions are satisfied*

(C1) $J_1=\displaystyle\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|\varphi'(z)|}{|1-\overline{w}\varphi(z)|^{\mu+2}}|u(z)|dA(w)<+\infty.$

(C2) $J_2=\displaystyle\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{|1-\overline{w}\varphi(z)|^{\mu+1}}|u'(z)|dA(w)<+\infty.$

Then the weighted composition operator $uC_\varphi:\mathcal{B}^\mu\longrightarrow\mathcal{B}^\beta$ is 1-summing.

Proof. Suppose that $(C1)$ and $(C2)$ are satisfied, we want to prove that uC_φ is 1-summing. Using the same technique as before, we consider $f_1,...,f_N$ elements in $\mathcal{B}^\mu$ satisfying

$$\sup_{|\varepsilon_j|=1}\Big\|\sum_{j=1}^{N}\varepsilon_jf_j\Big\|_{\mathcal{B}^\mu}=1.$$

Therefore,

$$\sup_{w\in\mathbb{D}}\sum_{j=1}^{N}|f_j'(w)|(1-|w|^2)^\mu=1. \tag{12}$$

We wish to get an upper estimate for

$$\sum_{j=1}^{N}\|u(f_j\circ\varphi)\|_{\mathcal{B}^\beta}.$$

The term $\displaystyle\sum_{j=1}^{N}|u(0)f_j(\varphi(0))|$ is bounded by $|u(0)|.\|\delta_{\varphi(0)}\|_{(\mathcal{B}^\mu)^*}$.
For the estimate of the second term we apply once again Lemma 3.1 (or the reproducing kernel

formula) to the function f_j' at point $\varphi(z_j)$ with $\alpha = \mu$. As in the preceding proofs we obtain by $(C1)$ that S_2 is bounded by $1 + J_1$.
The difference here is given in the estimate of the last term.
We need to control

$$\sum_{j=1}^{N}(1-|z_j|^2)^{\beta}|u'(z_j)|f_j(\varphi(z_j))|\,.$$

which is less than

$$\|u\|_{\mathcal{B}^\beta} + \sum_{j=1}^{N}(1-|z_j|^2)^{\beta}|u'(z_j)|\,|\widetilde{f}_j(\varphi(z_j))|,$$

with $\widetilde{f}_j = f_j - f_j(0)$, since

$$\sum_{j=1}^{N}(1-|z_j|^2)^{\beta}|u'(z_j)|\,|f_j(0)| \leq \|u\|_{\mathcal{B}^\beta}\Big|\sum_{j=1}^{N}\varepsilon_j f_j(0)\Big| \leq \|u\|_{\mathcal{B}^\beta}\Big\|\sum_{j=1}^{N}\varepsilon_j f_j\Big\|_{\mathcal{B}^\mu} \leq \|u\|_{\mathcal{B}^\beta},$$

where $|\varepsilon_j| = 1$.
In this case, we apply lemma 3.2 to the function $\widetilde{f}_j$ at $\varphi(z_j)$ with $\gamma = \mu$. We obtain that

$$\sum_{j=1}^{N}(1-|z_j|^2)^{\beta}|u'(z_j)|\widetilde{f}_j(\varphi(z_j))|$$

is less than

$$\sum_{j=1}^{N}(1-|z_j|^2)^{\beta}|u'(z_j)|\int_{\mathbb{D}}\frac{(1-|w|^2)^{\mu}|\widetilde{f}_j'(w)|}{|\overline{w}||1-\varphi(z_j)\overline{w}|^{(1+\mu)}}dA(w)$$

itself being less than

$$\Big(\sup_w\sum_{j=1}^{N}(1-|w|^2)^{\mu}|f_j'(w)|\Big)\int_{\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta}|u'(z)|}{|\overline{w}||1-\varphi(z)\overline{w}|^{(1+\mu)}}dA(w)\,.$$

Now since $\Big(\sum_{j=1}^{N}|f_j'(w)|(1-|w|^2)^{\mu}\Big) \leq 1$, we conclude that S_3 is bounded by $1 + \|u\|_{\mathcal{B}^\beta} + J_2$ thanks to $(C2)$.
Hence, $uC_\varphi : \mathcal{B}^\mu \longrightarrow \mathcal{B}^\beta$ is 1-summing, and this ends the proof of the theorem. □

4 Consequences

4.1 Absolutely summing composition operators

As an immediate corollary of theorems 3.4 and 3.6, we mention the important case of composition operators.

Theorem 4.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic, $\mu, \beta > 0$ and $p \geq 1$. The following assertions are equivalent:*

(a) The composition operator $C_\varphi : \mathcal{B}^\mu \longrightarrow \mathcal{B}^\beta$ is p-summing.

(b) The composition operator $C_\varphi : \mathcal{B}^\mu_0 \longrightarrow \mathcal{B}^\beta$ is p-summing.

(c) $Q_{\mu,\beta}(p) = \displaystyle\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)}(1-|z|^2)^{\beta p}|\varphi'(z)|^p}{|1-\overline{w}\varphi(z)|^{(2+\mu)p}} dA(w) < +\infty.$

Moreover, when $\varphi \in \mathcal{B}^\beta_0$, the preceding assertions are also equivalent to

(d) The composition operator $C_\varphi : \mathcal{B}^\mu_0 \longrightarrow \mathcal{B}^\beta_0$ is p-summing.

Remark: using the fact that $\|.\|_{\mathcal{B}^\beta} \lesssim \|.\|_\infty$ when $\beta \geq 1$ and applying this to the function $z \mapsto \dfrac{1}{(1-\overline{w}\varphi(z))^{(1+\mu)}}$, we point out that (having Lemma 3.3 in mind)

$$Q_{\mu,\beta}(p) \lesssim \int_{\mathbb{D}} \sup_{Z\in\varphi(\mathbb{D})} \frac{(1-|w|^2)^{2(p-1)}}{|1-\overline{w}Z|^{(1+\mu)p}} dA(w)$$

which depends only on μ and the geometry of $\varphi(\mathbb{D})$.

In the same spirit, when $\beta < 1$:

$$\int_{\mathbb{D}} \sup_{Z\in\varphi(\mathbb{D})} \frac{(1-|w|^2)^{2(p-1)}}{|1-\overline{w}Z|^{(1+\mu)p}} dA(w) \lesssim Q_{\mu,\beta}(p)\,.$$

This remark may be useful to get some sufficient or necessary condition (according to the value of μ): see an application in Prop.4.6. Similar ideas are used in Cor. 4.5.

Combining again theorem 3.4 and 3.6 we obtain a characterization of p-summing weighted composition operators, $p \geq 1$, for $\mu > 1$.

Theorem 4.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic, $u \in \mathcal{B}^\beta$, $p \geq 1$, $\beta > 0$ and $\mu > 1$. The following assertions are equivalent:*

(a) The weighted composition operator $uC_\varphi : \mathcal{B}^\mu \longrightarrow \mathcal{B}^\beta$ is p-summing.

(b) The weighted composition operator $uC_\varphi : \mathcal{B}^\mu_0 \longrightarrow \mathcal{B}^\beta$ is p-summing.

(c) The following conditions are satisfied

(c1) $J_1 = \displaystyle\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)}(1-|z|^2)^{\beta p}|\varphi'(z)|^p}{|1-\overline{w}\varphi(z)|^{(\mu+2)p}} |u(z)|^p dA(w) < +\infty.$

(c2) $J_2 = \displaystyle\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)}}{|1-\overline{w}\varphi(z)|^{(\mu+1)p}} (1-|z|^2)^{\beta p}|u'(z)|^p dA(w) < +\infty.$

Moreover, when $\varphi \in \mathcal{B}^\beta_0$, the preceding assertions are also equivalent to

(d) The weighted composition operator $uC_\varphi : \mathcal{B}^\mu_0 \longrightarrow \mathcal{B}^\beta_0$ is p-summing.

4.2 Nuclear weighted composition operators

Now combining theorem 3.4 and 3.7, and since 1-summing operators on Bloch-type spaces are nuclear (see [10, Lemma 2.1] and [14, Theorem 1.1]), we obtain a characterization of nuclear weighted composition operators on Bloch-type spaces.

Theorem 4.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic, $u \in \mathcal{B}^\beta$ and μ, $\beta > 0$. The following assertions are equivalent:*

(a) The weighted composition operator $uC_\varphi : \mathcal{B}^\mu \longrightarrow \mathcal{B}^\beta$ is nuclear.

(b) The weighted composition operator $uC_\varphi : \mathcal{B}_0^\mu \longrightarrow \mathcal{B}^\beta$ is nuclear.

(c) The following conditions are satisfied

(c1) $J_1 = \displaystyle\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{|1-\overline{w}\varphi(z)|^{\mu+2}} |u(z)| dA(w) < +\infty.$

(c2) $J_2 = \displaystyle\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{|1-\overline{w}\varphi(z)|^{\mu+1}} |u'(z)| dA(w) < +\infty.$

Moreover, when $\varphi \in \mathcal{B}_0^\beta$, the preceding assertions are also equivalent to

(d) The weighted composition operator $uC_\varphi : \mathcal{B}_0^\mu \longrightarrow \mathcal{B}_0^\beta$ is nuclear.

4.3 Other consequences

We focus on composition operators on Bloch-type spaces. We deduce from Theorem 4.1 a property of monotony with respect to the parameter β.

Corollary 4.4. *If $C_\varphi : \mathcal{B}^\beta \longrightarrow \mathcal{B}^\beta$ is p-summing and $\gamma > \beta$ then $C_\varphi : \mathcal{B}^\gamma \longrightarrow \mathcal{B}^\gamma$ is p-summing.*

Proof. As in the proof of Corollary 2.2 we first assume that $\varphi(0) = 0$.
Since C_φ is p-summing on $\mathcal{B}^\beta$ we have $Q_{\beta,\beta}(p) < \infty$. We want to prove that $Q_{\gamma,\gamma}(p)$ is finite. But

$$\begin{aligned} Q_{\gamma,\gamma}(p) &= \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)}(1-|z|^2)^{\gamma p}|\varphi'(z)|^p}{|1-\overline{w}\varphi(z)|^{(2+\gamma)p}} dA(w) \\ &= \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)}(1-|z|^2)^{\beta p}|\varphi'(z)|^p(1-|z|^2)^{(\gamma-\beta)p}}{|1-\overline{w}\varphi(z)|^{(2+\beta)p}|1-\overline{w}\varphi(z)|^{(\gamma-\beta)p}} dA(w)\ . \end{aligned}$$

Thanks to Schwarz's lemma (since $\varphi(0) = 0$) we have $|\varphi(z)| \le |z|$, for every $z \in \mathbb{D}$, therefore

$$\Big(\frac{1-|z|^2}{|1-\overline{w}\varphi(z)|}\Big)^{(\gamma-\beta)p} \lesssim \Big(\frac{1-|z|}{1-|\varphi(z)|}\Big)^{(\gamma-\beta)p} \le 1.$$

We get

$$Q_{\gamma,\gamma}(p) = \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)}(1-|z|^2)^{\gamma p}|\varphi'(z)|^p}{|1-\overline{w}\varphi(z)|^{(2+\gamma)p}} dA(w) \lesssim Q_{\beta,\beta}(p) < \infty.$$

To prove that C_φ is p-summing on $\mathcal{B}^\gamma$ even when φ does not fix the origin, we proceed as in the proof of Corollary 2.2. □

The following is a strengthening of the fact that any bounded composition operator $C_\varphi : \mathcal{B}^\mu \longrightarrow \mathcal{B}^\beta$ is automatically compact when $\beta > \mu$. It can be viewed as a quantified version of this fact.

Corollary 4.5. *Let $\beta \geq 1$, $\mu > 0$ and $p \geq 1$ satisfying $\beta > \mu + \frac{1}{p}\cdot$ Then $C_\varphi : \mathcal{B}^\mu \longrightarrow \mathcal{B}^\beta$ is p-summing.*

In particular, when $\beta > \mu + 1$, then $C_\varphi : \mathcal{B}^\mu \longrightarrow \mathcal{B}^\beta$ is r-summing, for every $r \geq 1$.

In other words, when $\beta > \mu$ and $\beta \geq 1$, any bounded composition operator $C_\varphi : \mathcal{B}^\mu \longrightarrow \mathcal{B}^\beta$ is more than compact: it is p-summing for every $p > \dfrac{1}{\beta-\mu}\cdot$

Proof. Under the asumption $\beta > \mu + \dfrac{1}{p}$, we are going to prove that $Q_{\mu,\beta}(p) < +\infty$ (see Theorem 4.1).

In a standard way, in order to simplify the computations, we assume that $\varphi(0) = 0$. We use similar ideas as in the remark after Th.4.1.

Since $\beta p > \mu p + 1$, we can choose a suitable $\delta \in (0,1)$ such that $\beta p - p > \mu p - p + 1 + \delta$. Thanks to Schwarz's lemma (and since $\beta \geq 1$), we have

$$\frac{(1-|z|)^{(\beta-1)p}}{(1-|\varphi(z)|)^{\mu p - p + 1 + \delta}} \leq 1.$$

We get

$$\begin{aligned} Q_{\mu,\beta}(p) &= \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)}(1-|z|^2)^{\beta p}|\varphi'(z)|^p}{|1-\overline{w}\varphi(z)|^{(2+\mu)p}}\, dA(w) \\ &\lesssim \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} (1-|w|^2)^{2(p-1)} \left(\frac{(1-|z|^2)|\overline{w}\varphi'(z)|}{|1-\overline{w}\varphi(z)|^{3-\frac{(1+\delta)}{p}}} \right)^p dA(w) \end{aligned}$$

where we used again Lemma 3.3.

Now recall that $\|.\|_{\mathcal{B}} \lesssim \|.\|_\infty$ and apply this to the function $z \mapsto \dfrac{1}{(1-\overline{w}\varphi(z))^{2-\frac{(1+\delta)}{p}}}\cdot$

We get

$$Q_{\mu,\beta}(p) \lesssim \int_{\mathbb{D}} \frac{1}{(1-|w|)^{(1-\delta)}} dA(w) < +\infty.$$

□

Proposition 4.6. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be a symbol such that $C_\varphi : \mathcal{B}^\mu \to \mathcal{B}^\beta$ is p-summing for some $p \geq 1$, where $\beta \in (0,1)$ and $\mu > 0$. Then*

$$\int_{\varphi(\mathbb{D})} \frac{1}{(1-|w|)^{2+p(\mu-1)}} dA(w) < \infty.$$

In particular,

(i) If $A(\mathbb{D} \setminus \varphi(\mathbb{D})) = 0$, then we must have $\mu < 1/p'$.

(ii) When $\mu > 1$, we must have

$$\int_{\varphi(\mathbb{D})} \frac{1}{(1-|w|)^{N}} dA(w) < \infty$$

for every $N \geq 1$.

(iii) If $\mu \geq \dfrac{1}{p'}$, we cannot have $A(\mathbb{D} \setminus \varphi(\mathbb{D})) = 0$ and φ cannot be inner.

(iv) Assume that φ is inner (or merely $A(\mathbb{D} \setminus \varphi(\mathbb{D})) = 0$). Then $C_\varphi : \mathcal{B}^\beta \to \mathcal{B}^\beta$ cannot be p-summing for any $p \geq 1$.

We shall see below that the situation is very different as soon as $\beta \geq 1$ (see Th.5.3).

Proof. We already pointed out in a remark after Theorem 4.1 that

$$\int_{\mathbb{D}} \sup_{Z \in \varphi(\mathbb{D})} \frac{(1-|w|^2)^{2(p-1)}}{|1-\overline{w}Z|^{(1+\mu)p}} dA(w) \lesssim Q_{\mu,\beta}(p)$$

which is finite with our assumptions. In particular

$$\int_{\varphi(\mathbb{D})} \frac{1}{(1-|w|^2)^{2+p(\mu-1)}} dA(w) \leq \int_{\varphi(\mathbb{D})} \sup_{Z \in \varphi(\mathbb{D})} \frac{(1-|w|^2)^{2(p-1)}}{|1-\overline{w}Z|^{(1+\mu)p}} dA(w) < +\infty.$$

In particular, when $A(\mathbb{D} \setminus \varphi(\mathbb{D})) = 0$, we must have

$$\int_{\mathbb{D}} \frac{1}{(1-|w|^2)^{2+p(\mu-1)}} dA(w) < \infty$$

hence $2 + p(\mu - 1) < 1$ which means that $\mu < 1/p'$. This proves (i).

Assume that $\beta \in (0,1)$ and $\mu > 1$. Since C_φ is p-summing it is q-summing for every $q \geq p$. Once N is fixed, we can choose q large enough to satisfy $2 + q(\mu - 1) \geq N$. This proves (ii).

When φ is inner then $\varphi(\mathbb{D}) = \mathbb{D}$ a.e ([11, Cor.2.3, p.118]) and the conclusion for (iii) follows.

At last if $C_\varphi : \mathcal{B}^\beta \to \mathcal{B}^\beta$ were p-summing for some $p \geq 1$ then, thanks to Cor.4.4, it would be p-summing on $\mathcal{B}^{\beta'}$ for every $\beta' > \beta$. But we can choose such a β' in $(1/p', 1)$. And (iii) leads to a contradiction. □

5 Examples

In this section we are going to give an example of a cusp map that induces a compact composition operator on the Bloch space which is not p-summing for any $p \geq 1$. On the other hand, we give an example of an inner symbol φ such that C_φ is nuclear (hence p-summing for every $p \geq 1$) on $\mathcal{B}^\beta$ for every $\beta \geq 1$. We also consider the example of the lens map which is often considered when studying composition operators on Banach spaces of analytic functions on $\mathbb{D}$. We give several basic properties of this symbol and we study the behaviour of its associated composition operator on $\mathcal{B}^\beta$ depending on $\beta > 0$.
We recall the definition of a cusp map.

Definition 5.1. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be a univalent analytic self map.*
Assume that $\overline{\varphi(\mathbb{D})} \cap \mathbb{T} = \{1\}$, the region $\varphi(\mathbb{D})$ is said to have a nontangential cusp at 1 if

$$dist(w, \partial\varphi(\mathbb{D})) = o(|1-w|) \quad as\ w \longrightarrow 1\ in\ \varphi(\mathbb{D}).$$

Also $\varphi(\mathbb{D})$ lies inside a Stolz angle if there exist $r, M > 0$ such that

$$|1-w| \leq M(1-|w|), \quad if\ |1-w| < r,\ and\ w \in \varphi(\mathbb{D}).$$

Madigan and Matheson showed (in [17, Theorem 5]) that if φ is univalent and if $\varphi(\mathbb{D})$ has a nontangential cusp at 1 and touches the unit circle at no other point, then C_φ is a compact operator on $\mathcal{B}$. Actually the same is true on Bloch-type spaces. In [10] we gave an example of a cusp more flattened than the one of Madigan and Matheson that induces a p-summing composition operator on $\mathcal{B}$. Again the same remains true on $\mathcal{B}^\beta$. Our next example will satisfy the properties of definition 5.1 and hence induces a compact composition operator on the Bloch space but fails to satisfy the condition $Q_{1,1}(p) < +\infty$ for any finite $p \geq 1$.

Example 1. We consider in the sequel a map Φ such that its domain $\Phi(\mathbb{D})$ is bounded by some convex curves of type $\gamma_1(t) = (1-t, \frac{t}{\theta(t)})$ and $\gamma_2(t) = (1-t, -\frac{t}{\theta(t)})$, for t in a neighborhood of 0 and $\theta : (0,1) \longrightarrow (0, +\infty)$, such that $\theta(t)$ tends to infinity when t tends to 0:

Theorem 5.2. *Let $p \geq 1$ and Φ be the map defined in Example 1 associated to some θ satisfying*

$$\int_0^{1/2} \frac{1}{s\theta(s)^{p+1}} ds = \infty.$$

Then, $C_\Phi : \mathcal{B} \longrightarrow \mathcal{B}$ is a compact composition operator which fails to be p-summing.
In particular there exists a compact composition operator on $\mathcal{B}$ which is not p-summing for any $p \geq 1$.

Proof. Let us define $\gamma(x) = \dfrac{1-x}{\theta(1-x)}$.

We consider now the symbol Φ associated to θ like in Example 1. It is clear that $\Phi(\mathbb{D})$ has a nontangential cusp at 1:

for a point $z \in \mathbb{D}$ such that $\Phi(z) = x + iy$ and $\Im m\Phi(z) \geq 0$, we have

$$dist(\Phi(z), \gamma) \leq \gamma(x) = o(1-x) = o(|1-\Phi(z)|).$$

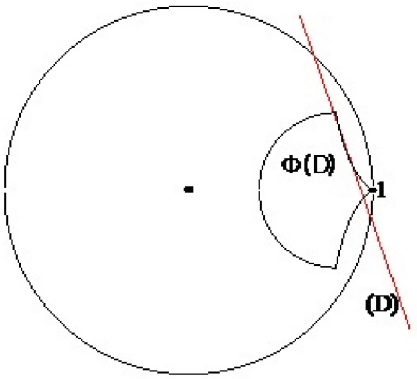

Figure 1: Domain of $\Phi(\mathbb{D})$

By symmetry, the same happens when $\Im m\Phi(z) \leq 0$, and then

$$dist(\Phi(z), \partial\Phi(\mathbb{D})) = o(|1 - \Phi(z)|).$$

Thus, C_Φ is a compact composition operator on $\mathcal{B}$.
Now we are going to prove that $Q_{1,1}(p) = +\infty$ for every $p \geq 1$. For this we use the Koebe distortion theorem [12, p.13]:

$$\frac{1}{4}(1 - |z|^2)|\Phi'(z)| \leq dist(\Phi(z), \partial\Phi(\mathbb{D})) \leq (1 - |z|^2)|\Phi'(z)|$$

and the following fact: consider $z \in \mathbb{D}$ such that $\Im m\Phi(z) \geq 0$ and write $\Phi(z) = x + iy$. The tangent (D) at point $(x, \gamma(x))$ is totally under the curve γ (by convexity of γ). So that the distance from $\Phi(z)$ to γ is greater than the distance from $\Phi(z)$ to (D):

$$dist(\Phi(z), \gamma) \geq dist(\Phi(z), (D)).$$

By symmetry, the same happens when $\Im m\Phi(z) \leq 0$, and we have

$$dist(\Phi(z), \partial\Phi(\mathbb{D})) \geq dist(\Phi(z), (D)). \tag{13}$$

Since (D) is the tangent at the point $(x, \gamma(x))$, it has the following equation

$$(D): \quad Y = \gamma(x) + \gamma'(x)(X - x).$$

Therefore the distance from $\Phi(z) = x + iy$ to the tangent (D) is the following

$$dist(\Phi(z), (D)) = \frac{|y - \gamma(x)|}{\sqrt{\gamma'^2(x) + 1}}. \tag{14}$$

Now we compute the integral $Q_{1,1}(p)$. By the Koebe distortion theorem we have:

$$\begin{aligned} Q_{1,1}(p) &= \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)}(1-|z|^2)^p|\Phi'(z)|^p}{|1-\overline{w}\Phi(z)|^{3p}} dA(w) \\ &\geq \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)} dist(\Phi(z),\partial\Phi(\mathbb{D}))^p}{|1-\overline{w}\Phi(z)|^{3p}} dA(w) \\ &\geq \int_{\Phi(\mathbb{D})} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)} dist(\Phi(z),\partial\Phi(\mathbb{D}))^p}{|1-\overline{w}\Phi(z)|^{3p}} dA(w)\,. \end{aligned}$$

Now for every $w \in \Phi(\mathbb{D})$, we can write w as $\Phi(z)$ where $z \in \mathbb{D}$ and we have

$$\begin{aligned} Q_{1,1}(p) &\geq \int_{\Phi(\mathbb{D})} \frac{(1-|w|^2)^{2(p-1)} dist(w,\partial\Phi(\mathbb{D}))^p}{(1-|w|^2)^{3p}} dA(w) \\ &\geq \int_{\Phi(\mathbb{D})} \frac{(dist(w,\partial\Phi(\mathbb{D}))^p}{(1-|w|^2)^{p+2}} dA(w)\,. \end{aligned}$$

From (13) and (14), integrating on $\Phi(\mathbb{D}) \cap \{\Im m z > 0\} \cap \{\Re e z \in (\frac{1}{2}, 1)\}$ we get,

$$\begin{aligned} Q_{1,1}(p) &\geq \int_{1/2}^1 \int_0^{\gamma(x)} \frac{|y-\gamma(x)|^p}{(\gamma'^2(x)+1)^{p/2}(1-|x^2+y^2|)^{p+2}} dydx \\ &= \int_{1/2}^1 \int_0^{\gamma(x)} \frac{(\gamma(x)-y)^p}{(\gamma'^2(x)+1)^{p/2}((1-x^2)-y^2)^{p+2}} dydx\,. \end{aligned}$$

Now since $0 \leq y \leq \gamma(x) = o(1-x)$ in the integral, we have

$$(1-x^2)-y^2 \sim 1-x^2 \approx 1-x \quad \text{when } x \longrightarrow 1.$$

It is easy to check that $\gamma'(x) \longrightarrow 0$ when $x \longrightarrow 1$. Finally we have,

$$\begin{aligned} Q_{1,1}(p) &\gtrsim \int_{1/2}^1 \int_0^{\gamma(x)} \frac{(\gamma(x)-y)^p}{(1-x)^{p+2}} dydx \\ &\gtrsim \int_{1/2}^1 \frac{\gamma^{p+1}(x)}{(1-x)^{p+2}} dx \\ &\gtrsim \int_{1/2}^1 \frac{1}{(1-x)(\theta(1-x))^{p+1}} dx \end{aligned}$$

and this last integral diverges.

For a concrete example, just choose a convex function θ such that $\theta(t) = \ln(\ln(\frac{1}{t}))$, for $t < e^{-1}$. □

Example 2. As we already mentioned, we gave in [10] an example of a cusp map (with a contact point on the torus) inducing a nuclear composition operator on the Bloch space $\mathcal{B}$. We are going to present a more striking example of an inner symbol also inducing a nuclear operator on $\mathcal{B}^\beta$. This shows that the situation in Prop.4.6(iv), where $\beta < 1$, is very different compared to the case $\beta \geq 1$.

Theorem 5.3. *There exists a Blaschke product ϕ such that $C_\phi : \mathcal{B}^\beta \longrightarrow \mathcal{B}^\beta$ is nuclear for every $\beta \geq 1$. In particular, C_ϕ is p-summing on $\mathcal{B}^\beta$ for every $\beta \geq 1$ and every $p \geq 1$.*

Proof. From [2, Th.2], we know that there exists a Blaschke product ϕ such that

$$\forall z \in \mathbb{D}\,, \quad (1-|z|)|\phi'(z)| \leq (1-|\phi(z)|)^3\,.$$

Thanks to the Schwarz-Pick inequality, we know that $1-|z| \lesssim 1-|\phi(z)|$ for $z \in \mathbb{D}$.
Since $\beta \geq 1$, this implies that

$$\forall z \in \mathbb{D}\,, \quad (1-|z|)^\beta|\phi'(z)| \lesssim (1-|\phi(z)|)^{2+\beta}\,.$$

We can use Th.4.3 with $\beta = \mu$ (we see that J_1 is finite and that $J_2 = 0$ since $u = 1$ here), or use directly the main result of [10] when $\beta = 1$. We conclude that $C_\phi : \mathcal{B}^\beta \longrightarrow \mathcal{B}^\beta$ is nuclear. □

Example 3. *(The lens map)*
First let us recall that the Cayley transform is an automorphism from the unit disk $\mathbb{D}$ onto the right half plane $\mathbb{C}_0 = \{z \in \mathbb{C} \mid Re(z) > 0\}$ and is defined by

$$\chi(z) = \frac{1+z}{1-z}\,.$$

Its inverse is $\chi^{-1}(z) = \dfrac{z-1}{z+1}\,.$
The lens map is then defined in the following way:
Given $a \in (0,1)$, we define the symbol $\Lambda_a : \mathbb{D} \longrightarrow \mathbb{D}$ by

$$\Lambda_a(z) = \frac{(\chi(z))^a - 1}{(\chi(z))^a + 1} \qquad \text{for } z \in \mathbb{D}.$$

Actually we have $z \in \mathbb{D} \xrightarrow{\chi} \chi(z) \xrightarrow{Z^a} (\chi(z))^a \xrightarrow{\chi^{-1}} \Lambda_a(z)$ and the geometric representation of $\Lambda_a(\mathbb{D})$ is the following:

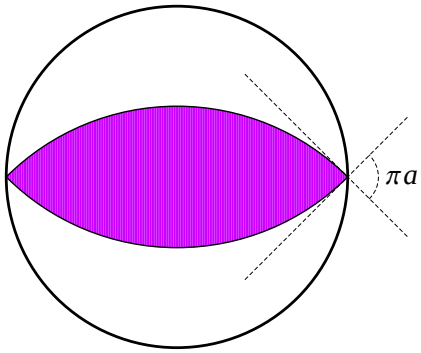

Figure 2: Domain of $\Lambda_a(\mathbb{D})$

Let us give several basic properties of this symbol.

Proposition 5.4. *Let $a \in (0,1)$. We have*

(i) $\overline{\Lambda_a(\mathbb{D})} \cap \mathbb{T} = \{-1, 1\}$. *Actually Λ_a can be extended to $\overline{\mathbb{D}}$ and the contact points correspond to $z = -1$ and $z = 1$.*

(ii) $\Lambda_a(\mathbb{D})$ *is included in an angular sector.*

(iii) For every $z \in \mathbb{D}$, $\Lambda_a'(z) = \dfrac{4a}{(1-z)^2}(\chi(z))^{a-1}\dfrac{1}{\big((\chi(z))^a+1\big)^2}\cdot$

(iv) In particular, when $z \to 1$, we have $|\Lambda_a'(z)| \approx |1-z|^{a-1}$.

Proof. (i). The only critical points are -1 and 1 but $\chi(1) = \infty$ and $\chi^{-1}(\infty) = 1$. In the same way $\chi(-1) = 0$ and $\chi^{-1}(0) = -1$.

(ii). We are first interested in the neighborhood of 1 for the lens. We only have to justify that $\chi^{-1}(z)$ belongs to a fixed Stolz domain when $z \in \mathbb{C}_0$ with $|arg(z)| \leq \pi a/2$ and $|z| \to +\infty$.

But for such z, we have

$$|1 - \chi^{-1}(z)| = \frac{2}{|z+1|} \sim \frac{2}{|z|}$$

whereas

$$1 - |\chi^{-1}(z)| \sim \frac{1}{2}(1 - |\chi^{-1}(z)|^2) \sim \frac{2Re(z)}{|z|^2}\cdot$$

The inequality $Re(z) \geq \cos(\pi a/2)|z|$, satisfied by all such z, settles the justification.

The same occurs around point -1.

(iii) We can write $\Lambda_a(z) = \chi^{-1}\Big((\chi(z))^a\Big)$.

We have $\chi'(z) = \dfrac{2}{(1-z)^2}$ and $\big(\chi^{-1}\big)'(z) = \dfrac{2}{(z+1)^2}\cdot$

Therefore, by the chain rule formula,

$$\Lambda_a'(z) = a\chi'(z)(\chi(z))^{a-1}\big(\chi^{-1}\big)'\Big((\chi(z))^a\Big) = \frac{4a}{(1-z)^2}(\chi(z))^{a-1}\frac{1}{\big((\chi(z))^a+1\big)^2}\cdot$$

(iv) When $z \to 1$, we have $|\chi(z)| \approx \dfrac{1}{|1-z|}$ and we get

$$|\Lambda_a'(z)| \approx |1-z|^{-(a+1)}.|1-z|^{2a}$$

which is the result. □

From (iv) of the preceding proposition we get

Proposition 5.5. *We have*

- *When $\beta < 1 - a$, the symbol Λ_a does not even belong to $\mathcal{B}^\beta$.*
- *The symbol Λ_a belongs to $\mathcal{B}^{1-a}$.*

- *When $\beta > 1 - a$, the symbol Λ_a belongs to $\mathcal{B}_0^\beta$.*

Concerning the behavior of the associated composition operator on the Bloch space , we have the following result:

Theorem 5.6. *Let $\beta > 0$. Then*

- *For $\beta < 1$, the composition operator C_{Λ_a} is not bounded on $\mathcal{B}^\beta$.*
- *For $\beta = 1$, the composition operator C_{Λ_a} is bounded but not compact on $\mathcal{B}$.*
- *For $\beta > 1$, the composition operator C_{Λ_a} is compact on $\mathcal{B}_0^\beta$.*

Proof. We start by the case $\beta \leq 1$: For $n \geq 1$, define $f_n(z) = n^{\beta-1} z^n$ for $z \in \mathbb{D}$, which belongs to $\mathcal{B}^\beta$ with bounded norm (not depending on n). This sequence (f_n) converges uniformly to 0 on every compact subset of $\mathbb{D}$.

Nevertheless we shall see that $\|f_n \circ \Lambda_a\|_{\mathcal{B}^\beta} \gtrsim 1$. Indeed

$$\|f_n \circ \Lambda_a\|_{\mathcal{B}^\beta} = \sup_{z \in \mathbb{D}} n^\beta (1 - |z|^2)^\beta |\Lambda_a'(z)| . |\Lambda_a(z)|^{n-1}.$$

Take $\gamma > 0$ and choose $z = z_n = 1 - n^{-\gamma} \to 1$:

$$|\Lambda_a'(z_n)| \approx n^{\gamma(1-a)} \qquad \text{and} \qquad \Lambda_a(z_n) = 1 - \frac{2}{(\chi(z_n))^a + 1} = 1 - 2^{1-a} n^{-a\gamma} + o\Big(n^{-a\gamma}\Big).$$

The choice $\gamma = 1/a > 1$ leads to

$$|\Lambda_a(z_n)|^{n-1} = \Big(1 - 2^{1-a} n^{-a\gamma} + o\big(n^{-a\gamma}\big)\Big)^{n-1} \approx 1 \cdot$$

We get

$$\|f_n \circ \Lambda_a\|_{\mathcal{B}^\beta} \gtrsim n^{\beta(1-\gamma)} n^{\gamma(1-a)}.$$

Since $\beta(1-\gamma) + \gamma(1-a) = \beta - \beta\gamma + \gamma - 1 = (\gamma - 1)(1 - \beta)$, we get the conclusion.

Now when $\beta > 1$: it suffices to show that Λ_a satisfies

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)^\beta |\Lambda_a'(z)|}{(1 - |\Lambda_a(z)|^2)^\beta} = 0.$$

From the estimates of proposition 5.4 we have, when $z \to 1$

$$\frac{(1 - |z|^2)^\beta |\Lambda_a'(z)|}{(1 - |\Lambda_a(z)|^2)^\beta} \approx \frac{(1 - |z|^2)^\beta |1 - z|^{a-1}}{|1 - z|^{\beta a}} \lesssim (1 - |z|^2)^{(\beta-1)(1-a)} \longrightarrow 0.$$

The same occurs when $z \to -1$ and these are the only two points we have to focus on. □

Theorem 5.7. *Let $p \geq 1$ and $\beta > 1$, if $p > \dfrac{a}{(\beta - 1)(1 - a)}$ then the composition operator $C_{\Lambda_a} : \mathcal{B}^\beta \longrightarrow \mathcal{B}^\beta$ is p-summing.*

In particular, if $a < 1 - \dfrac{1}{\beta}$, then C_{Λ_a} is nuclear on $\mathcal{B}^\beta$.

Proof. Suppose that $p > \dfrac{a}{(\beta-1)(1-a)}$, we want to prove that C_{Λ_a} is p-summing. By theorem 4.1 it suffices to show that

$$Q_\beta(p) = \int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)}(1-|z|^2)^{\beta p}|\Lambda_a'(z)|^p}{|1-\overline{w}\Lambda_a(z)|^{(2+\beta)p}} dA(w) < +\infty.$$

Let $\delta \in (0,1)$, such that

$$\beta p + p(a-1) - a(\beta p + 1 + \delta) \geq 0.$$

Then for every $w, z \in \mathbb{D}$ we have,

$$\sup_{z\in\mathbb{D}} \frac{(1-|w|^2)^{2(p-1)}(1-|z|^2)^{\beta p}|\Lambda_a'(z)|^p}{|1-\overline{w}\Lambda_a(z)|^{(2+\beta)p}} \lesssim \frac{1}{(1-|w|^2)^{(1-\delta)}}$$

since $|1-\overline{w}\Lambda_a(z)|^{(2+\beta)p} \geq (1-|w|^2)^{2p-1-\delta}(1-|\Lambda_a(z)|)^{\beta p+1+\delta}$, $1-|\Lambda_a(z)| \approx |1-z|^a$ and $|\Lambda_a'(z)| \approx |1-z|^{(a-1)}$.
The function $w \in \mathbb{D} \mapsto (1-|w|^2)^{-(1-\delta)} \in L^1(\mathbb{D}, dA)$ so we get the conclusion in this case. □

Acknowledgements. The authors are partially supported by the grant ANR-17-CE40-0021 of the French National Research Agency ANR (project Front).

We wish to thank warmly the referee for her/his very careful reading of this paper and her/his suggestions to improve the quality of the redaction.

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