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Interior estimates of first order for initial value problems in the octonion algebra

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Abstract

In this paper we use a Cauchy-Type integral formula for analytic octonionic-valued functions, to give an interior estimate of first order, in the supremum norm, for such functions. These estimates are used to solve an initial value problem with a \mathbb{O} -analytic function as initial data, via the Banach fixed point theorem.

Keywords: Interior estimates; Octonions; \mathbb{O} -analytic functions; initial value problem.

Resumen

En este trabajo usamos una fórmula integral de tipo Cauchy para funciones analíticas con valores en los octoniones, para dar un estimado interior de primer orden, en la norma del supremo, para tales funciones. Estos estimados se utilizan para

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resolver un problema de valor inicial con una función \mathbb{O} –analítica como dato inicial, a través del teorema de punto fijo de Banach.

Palabras claves: Estimados interiores; Octoniones; funciones \mathbb{O} –analíticas; problema de valor inicial.

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^8 and

$$\partial_x = \partial_0 + e_1\partial_1 + \cdots + e_7\partial_7,$$

where $x = (x_0, x_1, \dots, x_7) \in \mathbb{R}^8$ and $\partial_j = \frac{\partial}{\partial x_j}$, for $j = 0, 1, \dots, 7$. Then a function $u \in C^1(\mathbb{R}^8; \mathbb{O})$ is said to be \mathbb{O} –**analytic** in Ω , if $\partial_x u = 0$ in Ω .

Initial value problems of type

$$\partial_t u = \mathcal{F}(t, x, u, \partial_0 u, \partial_1 u, \dots, \partial_7 u), \quad (1)$$

$$u(0, x) = \varphi(x), \quad (2)$$

where t is the time and u is a function depending on t and x , can be solved by the contraction principle in case the initial function φ belongs to an associated space whose elements satisfy an interior estimate of first order.

Solutions of (1)–(2) are fixed points of the associated integro-differential operator:

$$Tu(t, x) = \varphi(x) + \int_0^t \mathcal{F}(\tau, x, u, \partial_0 u, \partial_1 u, \dots, \partial_7 u) d\tau \quad (3)$$

and conversely.

In order to apply the Banach fixed-point theorem, (3) must be a contractive operator. Hence, we need to estimate it in a suitable function space whose elements depend only on the space variable x . This can be done by the so-called interior estimates for the elements of the associated space \mathcal{X} . Interior estimates of first order describe the behaviour of the derivatives of a function near the boundary of a bounded domain. The space \mathcal{X} is a space of functions such that, given a linear differential operator \mathcal{G} depending only on the space variables, any element u of \mathcal{X} is such that $\mathcal{G}u = 0$ implies $\mathcal{G}(\mathcal{F}u) = 0$. That is, \mathcal{X} is given by the solutions of the equation $\mathcal{G}u = 0$,

satisfying $\mathcal{F}(\mathcal{X}) \subseteq \mathcal{X}$. In this paper we give an interior estimate of first order for analytic octonion-valued functions and exhibit how to apply the method of interior estimates and associated spaces to solve some kind of initial value problems of the form (1)–(2).

Ariza et al. (see [1]) gave interior estimates in the supremum norm for a class of generalized monogenic functions with integral representations in a Clifford-type algebra. In [2] an interior estimate in the L_p -norm for functions in Clifford-type algebras was given. The problem of interior estimates for the derivatives of generalized q -holomorphic vectors was studied in [3], and Tutschke (see [4]) gave interior estimates in the sup-norm for generalized monogenic functions satisfying a differential equation with an anti-monogenic right-hand side. Initial value problems of type (1)–(2) were studied in [5–7] in the context of quaternions and Clifford-type algebras. For more details on the associated and interior estimates method for solving initial value problems, we refer the reader to [8–11].

The paper is organized as follows: in section 2 we give some definitions and known results as background for the follows sections. In section 3 we give some estimates for integrals of \mathbb{O} -analytic functions; in section 4 we obtain an interior estimate of first order for \mathbb{O} -analytic functions using the Cauchy integral formula for such functions. Applications to initial value problems are presented in section 5 and concluding remarks in section 6.

2 Preliminaries

In this section we recall some concepts, notations and results to be used in the next sections.

2.1 Associated Spaces

Definition 1. *Given a differential equation*

$$\mathcal{G}u = 0,$$

where \mathcal{G} is a given partial differential operator with respect to the space-like variables x_i and coefficients not depending on time t . A second differential operator \mathcal{F} is called **associated** to \mathcal{G} if \mathcal{F} transforms the set of all solutions to the differential equation $\mathcal{G}u = 0$ into itself, i.e., if

$$\mathcal{G}u = 0 \Rightarrow \mathcal{G}(\mathcal{F}u) = 0$$

The function space containing all solutions to the differential equation $\mathcal{G}u = 0$ is called an **associated space** to \mathcal{F} .

For example, in the complex plane, consider $\mathcal{G} = \partial_z$ and $\mathcal{F} = \frac{d}{dz}$. Since the complex derivative of a holomorphic function is again holomorphic, we have that \mathcal{F} is associated to \mathcal{G} . This means that the space of holomorphic functions is associated to the complex differentiation.

2.2 Interior Estimates of First Order

An interior estimate (see [5, 12]) is an estimate of the derivatives of a function that is true in a sub-domain (of the given domain of definition), which has a positive distance to the boundary of the domain of definition. Interior estimates describe the behaviour of the derivatives of a function near the boundary of a bounded domain. Formally, we have the following definition.

Definition 2. Let Ω be a bounded domain of \mathbb{R}^n , for some $n \in \mathbb{N}$, and $\mathcal{B}(\Omega)$ be a Banach space of functions defined in Ω . Suppose Ω' is a sub-domain of Ω such that $\text{dist}(\Omega', \partial\Omega) > 0$. If u is given function in Ω , then, an estimate of the form

$$\|\partial_{x_j} u\|_{\mathcal{B}(\Omega')} \leq \frac{C}{\text{dist}(\Omega', \partial\Omega)} \|u\|_{\mathcal{B}(\Omega)},$$

is called an **interior estimate of first order**, where C is a constant neither depending on u nor on Ω' . See [13].

2.3 Octonions

Octonions \mathbb{O} , or Cayley numbers [14], are considered the non-associative extension of quaternions \mathbb{H} . An arbitrary octonion $x \in \mathbb{O}$ is of the form:

$$x = x_0 + \underline{x},$$

or, equivalently,

$$x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7,$$

where $x_0 \in \mathbb{R}$ is the **real part** of the octonion and $\underline{x} = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7$, is the **vector part** of it, with $x_1, \dots, x_7 \in \mathbb{R}$. The **conjugate** of octonion x is:

$$\bar{x} = x_0 - \underline{x}.$$

In composition algebras the following relation is established $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$, it is known that the real \mathbb{R} and the complex \mathbb{C} have the commutative and associative properties, the quaternions \mathbb{H} has only the associative property, meanwhile the octonions \mathbb{O} , does not have the associative and commutative properties [15–17]. With the Cayley-Dickson construction, the multiplication of pairs of quaternion in the space $\mathbb{H} \oplus \mathbb{H}e_4$ is

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}), \text{ where } a, b, c, d \in \mathbb{H}.$$

Multiplying the pairs of quaternions $\mathbb{H} \oplus \mathbb{H}e_4$ we obtain an 8-dimensional algebra generated by the elements:

$$\begin{aligned} e_0 &= (1, 0), & e_1 &= (i, 0), & e_2 &= (j, 0), & e_3 &= (k, 0), \\ e_4 &= (0, 1), & e_5 &= (0, i), & e_6 &= (0, j), & e_7 &= (0, k). \end{aligned}$$

If $e_0 = 1$, using the product definition you get the following table :

Tab. 1: Octonion multiplication table.

\cdot	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

From Table 1 we can see that

- (i) e_1, \dots, e_7 are square roots of -1 .
- (ii) e_i and e_j anti-commute when $i \neq j$ and $i, j = 1, \dots, 7$, that is

$$e_i e_j = -e_j e_i.$$

- (iii) $e_0 = 1$ is the unit element of the algebra.

(iv) The index cycling identity holds:

$$e_i e_j = e_k \implies e_{i+1} e_{j+1} = e_{k+1},$$

where we think the indices as living in \mathbb{Z}_7 , and

(v) The index doubling identity holds:

$$e_i e_j = e_k \implies e_{2i} e_{2j} = e_{2k}.$$

Using the table, it is an easy task to see that this algebra is not associative. In fact, we have

$$(e_1 e_2) e_5 = e_3 e_5 = -e_6, \quad \text{but} \quad e_1 (e_2 e_5) = e_1 e_7 = e_6.$$

We also see that the elements $1, e_1, e_2, e_3$ generate the quaternion algebra, i.e., \mathbb{H} can be seen as a subalgebra of \mathbb{O} .

Next, we show some results about the algebraic structure of octonions. The proof of its can be checked in [18].

Proposition 1. *If $x, y \in \mathbb{O}$, with $x, y \neq 0$ then:*

$$x(xy) = x^2 y, \quad (xy)y = xy^2, \quad (xy)x = x(yx).$$

Proposition 2. *For all $x, y \in \mathbb{O}$ $y r \in \mathbb{R}$, then:*

$$x(ry) = (xr)y.$$

Proposition 3. [*Moufang identities*] *For each $x, y, z \in \mathbb{O}$:*

$$(xy)(zx) = (x(yz))x = x((yz)x).$$

Given a non-zero octonion x , its inverse element is represented as:

$$x^{-1} = \frac{\bar{x}}{|x|^2}.$$

where the **norm** of x is defined by $|x| = \sqrt{x\bar{x}} = \sqrt{\bar{x}x}$.

Proposition 4. *For all $x, y \in \mathbb{O}$, the norm of \mathbb{O} satisfies the composition law*

$$|xy| = |x||y|.$$

Proposition 5. *If $x, y \in \mathbb{O}$, then:*

$$\overline{\bar{x}} = x \quad \wedge \quad \overline{xy} = \bar{y} \bar{x}.$$

2.3.1 Cauchy-Riemann Operator and \mathbb{O} -analytic functions

Let Ω be a domain in \mathbb{R}^8 . We consider functions $f : \Omega \rightarrow \mathbb{O}$, i.e., f has the form

$$f(x) = \sum_0^7 e_k f_k(x), \quad \text{with } f_k : \Omega \rightarrow \mathbb{R}, \quad k = 0, 1, \dots, 7.$$

Definition 3. Given $x = (x_0, x_1, \dots, x_7) \in \mathbb{R}^8$, the **Cauchy-Riemann operator** is given by

$$\partial_x = \partial_{x_0} + e_1 \partial_{x_1} + \dots + e_7 \partial_{x_7}, \quad (4)$$

where $\partial_{x_k} = \frac{\partial}{\partial x_k}$, for $k = 0, 1, \dots, 7$. Its vector part,

$$\partial_{\underline{x}} = e_1 \partial_{x_1} + \dots + e_7 \partial_{x_7},$$

is known as the **Dirac operator**. Their conjugates are

$$\overline{\partial_x} = \partial_{x_0} - \partial_{\underline{x}} \quad \text{and} \quad -\partial_{\underline{x}},$$

respectively.

Definition 4. Let $\Omega \subset \mathbb{O}$ be a bounded domain in \mathbb{R}^8 and $f \in C^1(\mathbb{R}^8; \mathbb{O})$, if $\partial_x f = 0$ (resp., $f \partial_x = 0$) in Ω , then f is called **left (resp., right) \mathbb{O} -analytic** in Ω .

Remark 1. Note that

$$\partial_x f = \sum_{i=0}^7 e_i \partial_{x_i} f = \sum_{i,j=0}^7 e_i e_j \partial_{x_i} f_j,$$

while

$$f \partial_x = \sum_{i=0}^7 f e_i \partial_{x_i} = \sum_{i,j=0}^7 e_j e_i \partial_{x_i} f_j.$$

Since

$$\overline{\partial_x} \partial_x = \partial_x \overline{\partial_x} = \Delta = \partial_{x_0}^2 + \partial_{x_1}^2 + \dots + \partial_{x_7}^2,$$

any left (or right) \mathbb{O} -analytic function in Ω is harmonic in Ω . Moreover, if ϕ is any real-valued harmonic function in Ω , then $f = \overline{\partial_x} \phi$ is both left and right \mathbb{O} -analytic function. Particularly, if

$$\Gamma_8(x) = \frac{-1}{6\omega_8|x|^6},$$

then

$$\Phi(x) = \overline{\partial_x} \Gamma_8(x) = \frac{\overline{x}}{\omega_8 |x|^8}$$

is both left and right \mathbb{O} -analytic everywhere away from the origin. Here, ω_8 is the surface area of the unit sphere in \mathbb{R}^8 .

Definition 5. *The Cauchy kernel in \mathbb{O} is given by*

$$\Phi(x - \zeta) = \frac{\overline{x} - \overline{\zeta}}{\omega_8 |x - \zeta|^8} =: \sum_{s=0}^7 \Phi_s e_s, \quad (5)$$

which is a left and right \mathbb{O} -analytic function everywhere away from $x = \zeta$.

2.3.2 Cauchy Integral Formula for Analytic Functions in \mathbb{O}

Let M be an 8-dimensional, compact, oriented C^∞ -manifold with boundary ∂M contained in some domain Ω of \mathbb{R}^8 . For $j = 0, 1, \dots, 7$, let

$$d\widehat{x}_j = dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_7$$

and

$$d\sigma(x) = \sum_0^7 (-1)^j e_j d\widehat{x}_j.$$

For each $x \in \partial M$, let $n(x) = \sum_0^7 n_j e_j$ be the outer unit normal to ∂M

at x . Then $(-1)^j d\widehat{x}_j = n_j(x) dS(x)$, where $dS(x)$ is the scalar element of surface area on ∂M . Consequently, on ∂M ,

$$d\sigma = ndS.$$

The well-known Cauchy integral formula has a version in the octonion algebra, see [19, 20]. This formula reads

Theorem 1. *(Cauchy Integral formula)*

Let M and Ω be as above. If f is an \mathbb{O} -analytic function in Ω , then

$$\int_{\partial M} \Phi(x - \zeta) (d\sigma(x) f(x)) = \begin{cases} f(\zeta), & \text{if } \zeta \in \overset{\circ}{M}, \\ 0, & \text{if } \zeta \in \Omega \setminus M. \end{cases} \quad (6)$$

Remark 2. *Note the importance of the parenthesis in the left hand side of (6), in view of non-associativity of octonions.*

3 Some Important Estimates

Let Ω be a bounded domain in \mathbb{R}^8 , and $f : \Omega \rightarrow \mathbb{O}$ given by

$$f(x) = \sum_{i=0}^7 f_i(x) e_i,$$

with $f_i : \Omega \rightarrow \mathbb{R}$, for $i = 0, 1, \dots, 7$. Then, by definition of the norm,

$$|f(x)|^2 = \sum_{i=0}^7 |f_i(x)|^2 \Rightarrow |f_i(x)| \leq |f(x)|.$$

On the other hand, since

$$\int_{\Omega} f(x) \, dx = \sum_{i=0}^7 \left(\int_{\Omega} f_i(x) \, dx \right) e_i,$$

and

$$\left| \int_{\Omega} f_i(x) \, dx \right| \leq \int_{\Omega} |f_i(x)| \, dx \leq \int_{\Omega} |f(x)| \, dx,$$

we have that

$$\begin{aligned} \left| \int_{\Omega} f(x) \, dx \right|^2 &= \sum_{i=0}^7 \left| \int_{\Omega} f_i(x) \, dx \right|^2 \\ &\leq \sum_{i=0}^7 \left(\int_{\Omega} |f(x)| \, dx \right)^2 \\ &= 8 \left(\int_{\Omega} |f(x)| \, dx \right)^2. \end{aligned}$$

Thus,

$$\left| \int_{\Omega} f(x) \, dx \right| \leq 2\sqrt{2} \int_{\Omega} |f(x)| \, dx. \quad (7)$$

Consider now Γ , the boundary of the domain Ω , and an integral of the form

$$\int_{\Gamma} f(x) \, dS(x).$$

If f is a real-valued function, it's clear that

$$\left| \int_{\Gamma} f(x) dS(x) \right| \leq \int_{\Gamma} |f(x)| dS(x),$$

then if f is an \mathbb{O} -valued function, a similar calculation to that given above leads to the inequality

$$\left| \int_{\Gamma} f(x) dS(x) \right| \leq 2\sqrt{2} \int_{\Gamma} |f(x)| dS(x).$$

Finally, if f is of the form $f(x) = g(x)(n(x)h(x))$, where g and h are \mathbb{O} -valued functions, and n the octonionic value outer unit normal to ∂M at x , then

$$|f(x)| = |g(x)(n(x)h(x))| = |g(x)||h(x)|,$$

and, using the last inequality, we obtain

$$\left| \int_{\Gamma} g(x) (d\sigma(x)h(x)) \right| \leq 2\sqrt{2} \int_{\Gamma} |g(x)| |h(x)| dS(x). \quad (8)$$

4 Interior Estimates in \mathbb{O}

In this section we will give an interior estimate of first order for \mathbb{O} -analytic functions. In order to do that, let f be an \mathbb{O} -analytic function in a bounded domain Ω and M be as in 2.3.2. For $\zeta \in \overset{\circ}{M}$, the Cauchy integral formula implies that

$$f(\zeta) = \int_{\partial M} \Phi(x - \zeta) (d\sigma(x)f(x)), \quad (9)$$

then

$$\frac{\partial}{\partial \zeta_i} f(\zeta) = \int_{\partial M} \frac{\partial}{\partial \zeta_i} (\Phi(x - \zeta)) (d\sigma(x)f(x)). \quad (10)$$

From (8), we see that

$$\left| \frac{\partial}{\partial \zeta_i} f(\zeta) \right| \leq 2\sqrt{2} \int_{\partial M} \left| \frac{\partial}{\partial \zeta_i} (\Phi(x - \zeta)) \right| |f(x)| dS(x). \quad (11)$$

Since for $i = 0, 1, \dots, 7$,

$$\frac{\partial}{\partial \zeta_i} \Phi(x - \zeta) = \frac{1}{\omega_8} \cdot \left[\frac{1}{|x - \zeta|^8} e_i + 8 \frac{(\bar{x} - \bar{\zeta})(x_i - \zeta_i)}{|x - \zeta|^{10}} \right],$$

the triangular inequality implies that

$$\begin{aligned} \left| \frac{\partial}{\partial \zeta_i} (\Phi(x - \zeta)) \right| &\leq \frac{1}{\omega_8} \left[\frac{1}{|x - \zeta|^8} |e_i| + 8 \left| \frac{(\bar{x} - \bar{\zeta})(x_i - \zeta_i)}{|x - \zeta|^{10}} \right| \right] \\ &\leq \frac{1}{\omega_8} \left[\frac{1}{|x - \zeta|^8} + 8 \frac{|x - \zeta| |x - \zeta|}{|x - \zeta|^{10}} \right] \\ &= \frac{9}{\omega_8} \frac{1}{|x - \zeta|^8}. \end{aligned}$$

Using this inequality and (11) we have

$$\left| \frac{\partial}{\partial \zeta_i} f(\zeta) \right| \leq \frac{18\sqrt{2}}{\omega_8} \int_{\partial M} \frac{|f(x)|}{|x - \zeta|^8} dS(x). \quad (12)$$

Consider now the set $\{x \in \mathbb{R}^8 : |x - \zeta| = \delta\}$ instead of ∂M , $\zeta \in \overset{\circ}{M}$ and $\delta < \text{dist}(\zeta, \partial M)$. Then, from (12),

$$\begin{aligned} \left| \frac{\partial}{\partial \zeta_i} f(\zeta) \right| &\leq \frac{18\sqrt{2}}{\omega_8} \int_{|x-\zeta|=\delta} \frac{|f(x)|}{\delta^8} dS(x) \\ &\leq \frac{18\sqrt{2}}{\omega_8 \delta^8} \sup_{|x-\zeta|=\delta} |f(x)| \cdot \int_{|x-\zeta|=\delta} dS(x) \\ &\leq \frac{18\sqrt{2}}{\delta} \sup_{x \in M} |f(x)| \\ &\leq \frac{18\sqrt{2}}{\delta} \|f(x)\|_M. \end{aligned}$$

If $\delta \rightarrow \text{dist}(\zeta, \partial M)$, we get

$$\left| \frac{\partial}{\partial \zeta_i} f(\zeta) \right| \leq \frac{18\sqrt{2}}{\text{dist}(\zeta, \partial M)} \|f(x)\|_M. \quad (13)$$

Choosing $M' \subset M$, a subdomain of M , we have, for $\zeta \in M'$,

$$\text{dist}(M', \partial M) \leq \text{dist}(\zeta, \partial M). \quad (14)$$

From (13) and (14) follows that

$$\left| \frac{\partial}{\partial \zeta_i} f(\zeta) \right| \leq \frac{18\sqrt{2}}{\text{dist}(M', \partial M)} \|f(x)\|_M. \quad (15)$$

The last inequality implies, finally, that

$$\left\| \frac{\partial f}{\partial \zeta_i} \right\|_{M'} \leq \frac{C}{\text{dist}(M', \partial M)} \|f\|_M,$$

with $C = 18\sqrt{2}$. Thus, we have proved the following theorem.

Theorem 2. *Let f be an \mathbb{O} -analytic function in Ω . If M , M' and Ω are as above, then an interior estimate of first order is given by*

$$\left\| \frac{\partial f}{\partial \zeta_i} \right\|_{M'} \leq \frac{C}{\text{dist}(M', \partial M)} \|f\|_M, \quad (16)$$

where $C = 18\sqrt{2}$, holds.

Remark 3. *It should be noted that estimates of the type (16) for \mathbb{O} -analytic functions can be obtained in some Clifford algebras. Is the case, for example, of Cl_7 , which is an associative algebra with zero divisors. The idea in both cases is to obtain an interior estimate by using the corresponding Cauchy Integral formula, and the properties of the norm in the underlying algebra.*

5 Application to Initial Value Problems in \mathbb{O}

In this section we show how to apply the method of associated spaces and interior estimates to solve initial value problems of the form (1)–(2). First, we give the following definition.

Definition 6. *Let Ω be a domain in \mathbb{R}^n . An **exhaustion of Ω** is a sequence of compact subsets K_1, K_2, \dots of Ω , satisfying i) $K_s \subset \Omega$, ii) $K_s \subset K_{s+1}$, for each s and iii) Each compact subset of Ω is contained in K_s provided s is sufficiently small.*

Next we consider an exhaustion of Ω as follows (see [13]):

- $K_s = \overline{\Omega_s}$, where Ω_s are sub-domains of Ω , $0 < s < s_0$ and s_0 is some suitable positive number.

- Each $x \in \Omega$ (with the only exception of some fixed point $x_0 \in \Omega$) belongs to the boundary of a uniquely determined domain $\Omega_{s(x)}$ of the exhaustion.
- The distance of a smaller sub-domain from the boundary of a larger sub-domain can be estimated uniformly from below by the difference of the indices characterizing the two sub-domains under consideration. That is, there exists a constant c_0 such that

$$\text{dist}(\Omega_{s'}, \partial\Omega_s) \geq c_0(s - s') \quad (17)$$

for any pair s, s' with $0 < s' < s < s_0$.

Note that, since the initial function given in (2) is defined only in Ω , it may have singularities at $\partial\Omega$, so, the nearer a point x is of $\partial\Omega$, the smaller the t -interval in which the solution is expected to exist. This means that the solution will be given in conical domains in the x, t -space. Note that $s_0 - s(x)$ can be used as a measure for the distance of $x \in \Omega$ from the boundary $\partial\Omega$. Thus, this conical domain can be defined as

$$M = \{(t, x) : x \in \Omega, 0 \leq t < \eta(s_0 - s(x))\}.$$

The base of M is Ω , while its height depends on η , which will be fixed later. The intersection of M with a plane $t = \tilde{t}$, where $0 < \tilde{t} < \eta s_0$, has as projection in the x -space, the sub-domain $\Omega_{\tilde{s}}$, where \tilde{s} is defined by

$$\tilde{t} = \eta(s_0 - \tilde{s}). \quad (18)$$

Define, now, the pseudo-distance $d(t, x) = s_0 - s(x) - \frac{t}{\eta}$, measuring the distance of $(t, x) \in M$ from the lateral surface of M . Let B_s be the space of all continuous functions u in Ω_s , equipped with the norm

$$\|u\|_s := \max_{i=0}^7 \left(\sup_{\Omega_s} |u_i(x)| \right).$$

It turns out that B_s equipped with this norm is a Banach space. Let $B_*(M)$ be the set of all function $f = f(t, x)$ which are continuous in M . Note that $f(\tilde{t}, x)$ belongs to $B_{s(x)}$ for fixed \tilde{t} if only $s(x) < \tilde{s}$, where \tilde{s} is given by (18). Define in $B_*(M)$ the $*$ -norm

$$\|f\|_* = \sup_{(t,x) \in M} \|f(t, \cdot)\|_{s(x)} d(t, x) < +\infty. \quad (19)$$

This definition implies that

$$\|f(t, \cdot)\|_{s(x)} \leq \frac{\|f\|_*}{d(t, x)}, \quad (20)$$

for any point $(t, x) \in M$.

It is not difficult to prove that $B_*(M)$ with the norm defined by (19) is a Banach space.

Proposition 6. *If $0 < s' < s'' < s_0$ and u is \mathbb{O} -analytic in Ω , then*

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{\Omega_{s'}} \leq \frac{C}{\text{dist}(\Omega_{s'}, \partial\Omega_{s''})} \|u\|_{\Omega_{s''}}. \quad (21)$$

Proof. It's enough to use the interior estimate (16) with $M' = \Omega_{s'}$ and $M = \Omega_{s''}$. \square

Finally, we prove the main theorem of this section.

Theorem 3. *Let \mathcal{F} be the operator defined by the right-hand side of equation (1) satisfying a Lipschitz condition with respect to u and its first order derivatives, i.e.,*

$$|\mathcal{F}u| \leq C_1|u| + \sum_{j=0}^7 C_{\mathcal{F}j} |\partial_{\xi_j} u|, \quad (22)$$

where C_1 and $C_{\mathcal{F}j}$, for $j = 0, 1, \dots, 7$, are constants in $(0, 1)$. If \mathcal{F} is associated to the Cauchy–Riemann operator ∂_x , then the initial value problem (1)–(2) is solvable provided the initial function is \mathbb{O} -analytic (i.e., $\partial_x \varphi(x) = 0$).

Proof. First, note that solutions of the initial value problem (1)–(2) are fixed points of the operator

$$Tu(t, x) = \varphi(x) + \int_0^t \mathcal{F}(\tau, x, u, \partial_0 u, \partial_1 u, \dots, \partial_7 u) d\tau, \quad (23)$$

and conversely. Let

$$B_*^{\partial_x}(M) = \{w \in B_*(M) : \partial_x w(t, \cdot) = 0, \text{ for each fixed } t\}.$$

By the Weierstrass approximation theorem, we see that $B_*^{\partial_x}(M)$ is a closed subset of $B_*(M)$. So, $B_*^{\partial_x}(M)$ is also a Banach space. Since \mathcal{F} is associated to the Cauchy–Riemann operator ∂_x and φ is \mathbb{O} -analytic, we have that

$$\partial_x(Tu(t, x)) = \partial_x \left(\varphi(x) + \int_0^t \mathcal{F}(\tau, x, u, \partial_0 u, \partial_1 u, \dots, \partial_7 u) d\tau \right)$$

that is

$$\partial_x(Tu(t, x)) = \partial_x \varphi(x) + \int_0^t \partial_x (\mathcal{F}(\tau, x, u, \partial_0 u, \partial_1 u, \dots, \partial_7 u)) d\tau = 0.$$

Thus, $T : B_*^{\partial_x}(M) \rightarrow B_*^{\partial_x}(M)$. Now let $(t, x) \in M$, so that, $d(t, x) = s_0 - s(x) - \frac{t}{\eta} > 0$. Define

$$r = \frac{1}{2}d(t, x) \quad \text{and} \quad \tilde{s} = s(x) + r.$$

Since $0 < d(t, x) \leq s_0 - s(x)$, one has $s(x) < s_0$ and

$$\tilde{s} = s(x) + \frac{1}{2}d(t, x) \leq s(x) + \frac{1}{2}(s_0 - s(x)) = \frac{1}{2}(s(x) + s_0) \leq s_0.$$

Let \tilde{x} be an arbitrary point with $s(\tilde{x}) = \tilde{s}$, i.e., $\tilde{x} \in \partial\Omega_{\tilde{s}}$. Then,

$$d(t, \tilde{x}) = s_0 - s(\tilde{x}) - \frac{t}{\eta} = s_0 - (s(x) + r) - \frac{t}{\eta} = d(t, x) - r = r.$$

Applying estimate (20) at the point (t, \tilde{x}) , one has

$$\|u(t, \cdot)\|_{\tilde{s}} \leq \frac{\|u\|_*}{d(t, \tilde{x})} = \frac{\|u\|_*}{r}.$$

From Proposition 6, with $s' = s(x)$ and $s'' = \tilde{s}$, and using (17) we have

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{s(x)} \leq \frac{C}{\text{dist}(\Omega_{s(x)}, \partial\Omega_{\tilde{s}})} \|u\|_{\tilde{s}} \leq \frac{C}{c_0} \cdot \frac{1}{\tilde{s} - s(x)} \cdot \|u\|_{\tilde{s}} \leq \frac{C}{c_0} \frac{\|u\|_*}{r^2} = \frac{4C}{c_0} \cdot \frac{\|u\|_*}{d^2(t, x)}.$$

On the other hand, since $\tilde{s} < s_0$, then $1 < \frac{s_0}{\tilde{s} - s(x)}$ and

$$\|u(t, \cdot)\|_{s(x)} \leq \frac{\|u\|_*}{d(t, x)} \leq \frac{s_0}{\tilde{s} - s(x)} \cdot \frac{\|u\|_*}{d(t, x)} = 2s_0 \cdot \frac{\|u\|_*}{d^2(t, x)}.$$

From (22), we have

$$\|\mathcal{F}u(t, \cdot)\|_{s(x)} \leq C_1 \|u\|_{s(x)} + \sum_{j=1}^7 C_{\mathcal{F}_j} \|\partial_{x_j} u(t, \cdot)\|_{s(x)} \leq \frac{K}{d^2(t, x)} \cdot \|u\|_*,$$

where

$$K = 2C_1s_0 + \frac{28C}{c_0} \cdot C_{\mathcal{F}} \quad \text{and} \quad C_{\mathcal{F}} = \max_{j=1}^7 \{C_{\mathcal{F}_j}\}.$$

Finally, since

$$\left\| \int_0^t \mathcal{F}(\tau, x, u, \partial_{x_i} u) d\tau \right\|_{s(x)} \leq K \int_0^t \frac{d\tau}{d^2(\tau, x)} \cdot \|u\|_* \leq \frac{K \cdot \eta}{d(t, x)} \|u\|_*,$$

then

$$\left\| \int_0^t \mathcal{F}(\tau, x, u, \partial_{x_i} u) \right\|_* \leq K \cdot \eta \|u\|_*.$$

Hence, if η is small enough (for example, $\eta < 1/K$), $T : B_*^{\partial_x}(M) \rightarrow B_*^{\partial_x}(M)$ is a contraction. By the Banach fixed point theorem, T has a unique fixed point, which is the unique solution of the initial value problem (1)–(2). \square

Remark 4. *Note that the proofs given in this section are valid in other algebras, such as other Clifford-type algebras and bi-complex numbers, because the structure of the octonions does not play an important role in them.*

6 Concluding Remarks

We obtained an interior estimate on the sup-norm for analytic functions in the octonion algebra. This estimate can be used to solve initial value problems of type Cauchy- Kovaleskaya which has an \mathbb{O} -analytic function as initial data satisfying the equation $\mathcal{G}u = 0$.

A proof of the initial value problem using the method of associated spaces and interior estimates is given. Interior estimates for an \mathbb{O} -analytic functions in other norms can be investigated. Likewise, initial value problems for another class of initial functions can be considered.

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