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Alessandro Duca, Vahagn Nersesyan. Local exact controllability of the 1D nonlinear Schrödinger equation in the case of Dirichlet boundary conditions. 2022. hal-03579006

HAL Id: hal-03579006

<https://hal.science/hal-03579006>

Preprint submitted on 17 Feb 2022

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Local exact controllability of the 1D nonlinear Schrödinger equation in the case of Dirichlet boundary conditions

Alessandro Duca* Vahagn Nersesyan^{†‡}

February 17, 2022

Abstract

We consider the 1D nonlinear Schrödinger equation with bilinear control. In the case of Neumann boundary conditions, local exact controllability of this equation near the ground state has been proved by Beauchard and Laurent [BL10]. In this paper, we study the case of Dirichlet boundary conditions. To establish the controllability of the linearised equation, we use a bilinear control acting through four directions: three Fourier modes and one generic direction. The Fourier modes are appropriately chosen so that they satisfy a saturation property. These modes allow to control approximately the linearised Schrödinger equation. We show that the reachable set for the linearised equation is closed. This is achieved by representing the resolving operator as a sum of two linear continuous mappings: one is surjective (here the control in generic direction is used) and the other is compact. A mapping with dense and closed image is surjective, so the linearised Schrödinger equation is exactly controllable. Then local exact controllability of the nonlinear equation is derived using the inverse mapping theorem.

AMS subject classifications: 35Q55, 81Q93, 93B05

Keywords: Nonlinear Schrödinger equation, local exact controllability, linearisation, approximate controllability, saturation property

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0 Introduction

We study the controllability of the one-dimensional nonlinear Schrödinger (NLS) equation with bilinear control and Dirichlet boundary conditions. To simplify the presentation, we consider in this introduction the case of the cubic NLS equation

$$i\partial_t\psi = -\partial_{xx}^2\psi + \kappa|\psi|^2\psi + \langle u(t), Q(x)\rangle\psi, \quad x \in I = (0, 1), \quad (0.1)$$

$$\psi(t, 0) = \psi(t, 1) = 0, \quad (0.2)$$

where $Q : I \rightarrow \mathbb{R}^q$ is a given external field and κ is a real number. We fix any $T > 0$ and consider the amplitude $u : [0, T] \rightarrow \mathbb{R}^q$ as a control term and the solution at time T , i.e., $\psi(T)$, as a state.

To formulate the main result of this paper, let us introduce some notation. We consider $L^2(I; \mathbb{C})$ as a real Hilbert space endowed with the scalar product

$$\langle f, g \rangle_{L^2} = \operatorname{Re} \int_0^1 f(x)\overline{g(x)}dx$$

and the corresponding norm $\|\cdot\|_{L^2}$. Let A be the Dirichlet Laplacian operator

$$A = -\partial_{xx}^2, \quad \mathbb{D}(A) = H^2 \cap H_0^1(I; \mathbb{C}),$$

and let $\phi_k(x) = \sqrt{2}\sin(k\pi x)$, $k \geq 1$ be its eigenfunctions associated with the eigenvalues $\lambda_k = k^2\pi^2$. We use the spaces $H_{(0)}^s = \mathbb{D}(A^{\frac{s}{2}})$, $s \geq 0$ endowed with the scalar products $\langle f, g \rangle_{(s)} = \langle A^{\frac{s}{2}}f, A^{\frac{s}{2}}g \rangle_{L^2}$ and the corresponding norms $\|\cdot\|_{(s)}$. The system (0.1), (0.2) is supplemented with the initial condition

$$\psi(0, x) = \psi_0(x), \quad (0.3)$$

which is assumed to belong to the unit sphere \mathcal{S} in $L^2(I; \mathbb{C})$. The following is the main result of this paper.

Main Theorem. *Assume that $Q = (Q_1, \dots, Q_q)$ is a smooth field such that the vector space*

$$\mathbb{Q} = \text{span}_{\mathbb{R}}\{Q_k : k = 1, \dots, q\} \quad (0.4)$$

contains the functions $\mathbf{1}$, $\cos(\pi x)$, $\cos(2\pi x)$, and a function μ verifying the inequality

$$|\langle \mu \phi_1, \phi_k \rangle_{L^2}| \geq \frac{c}{k^3}, \quad k \geq 1 \quad (0.5)$$

for some number $c > 0$. Then there is an at most countable set $\mathcal{K} \subset (-\infty, 0)$ such that, for any $\kappa \in \mathbb{R} \setminus \mathcal{K}$, the NLS equation is locally exactly controllable near the ground state ϕ_1 . More precisely, for any $T > 0$, there is a number $\delta > 0$ such that, for any $\psi_0, \psi_1 \in H_{(0)}^3(I; \mathbb{C}) \cap \mathcal{S}$ with

$$\|\psi_j - \phi_1\|_{(3)} < \delta, \quad j = 0, 1,$$

there is a control $u \in L^2([0, T]; \mathbb{R}^q)$ and a solution $\psi \in C([0, T]; H_{(0)}^3(I; \mathbb{C}))$ of the problem (0.1)-(0.3) satisfying $\psi(T) = \psi_1$.

A more general version of this theorem is stated in Section 2 (see Theorem 2.2). In that version, the nonlinear term has the form $|\psi|^{2p}\psi$ with any integer $p \geq 1$, and the conditions on the field Q and the number κ are formulated in terms of a general saturation property.

The controllability of the Schrödinger equation (0.1) has been extensively studied in the literature in the case $\kappa = 0$. Note that in that case, even if the equation is linear in ψ , the associated control problem is still nonlinear. Ball, Marsden, and Slemrod [BMS82] proved that the reachable set for this equation from any initial condition in $H_{(0)}^2(I; \mathbb{C}) \cap \mathcal{S}$ with controls in L^2 has empty interior in $H_{(0)}^2(I; \mathbb{C}) \cap \mathcal{S}$. In particular, this means that the problem is not locally exactly controllable in that phase space. Beauchard [Bea05] obtained the first positive controllability result: in the case $Q(x) = x$, she proved local exact controllability in some $H_{(0)}^7$ -neighborhood of any eigenstate by using a Nash–Moser theorem. Beauchard and Coron [BC06] obtained exact controllability between neighborhoods of different eigenstates. Later, in the paper [BL10], Beauchard and Laurent found a way to use the classical inverse mapping theorem to prove local exact controllability of the Schrödinger equation; more precisely, they proved exact controllability in some $H_{(0)}^3$ -neighborhood of any eigenstate in the case when $Q = \mu$ satisfies condition (0.5). The methods of [BL10] have been further developed by Morancey and the authors of this paper in [Mor14, MN15, Duc20] to study simultaneous exact controllability of several Schrödinger equations.

All the above papers deal with the one-dimensional Schrödinger equation. In the multidimensional case, exact controllability remains an open problem. In that situation, approximate controllability property has been studied by many authors; for the first results we refer the reader to the works by Boscain et al. [CMSB09, BCCS12], Mirrahimi [Mir09], and the second author [Ner10].

In the case of the NLS equation (0.1) with $\kappa \neq 0$ and Neumann boundary conditions, local exact controllability is established by Beauchard and Laurent [BL10]. They use the inverse mapping theorem and exact controllability of the linearised equation. The latter is proved by using a convenient change of the unknown that reduces the original problem to another linear system with explicit spectrum. The controllability of the reduced system is proved by using a moment problem approach and the Ingham inequality. In the case of Dirichlet boundary conditions, it is not clear whether such reduction is possible, so we proceed in a different way.

We note that when $\mathbf{1}$ and $\cos(2\pi x)$ belong to the vector space \mathbb{Q} defined by (0.4), the ground state ϕ_1 is a stationary solution of the NLS equation (0.1) corresponding to some constant control u . The linearisation of the equation around the couple (ϕ_1, u) is given by

$$i\partial_t \xi = -\partial_{xx}^2 \xi - \pi^2 \xi + 2\kappa \phi_1^2 \operatorname{Re}(\xi) + \langle v(t), Q(x) \rangle \phi_1. \quad (0.6)$$

We prove exact controllability of this equation in two steps. First, we show that if the number κ is in the complement of some at most countable set \mathcal{K} , then the directions $\mathbf{1}$ and $\cos(\pi x)$ are saturating. As a consequence, we obtain approximate controllability of Eq. (0.6). The saturation argument employed here is inspired by the papers of Agrachev, Sarychev [AS05, AS06] and Shirikyan [Shi06], which study approximate controllability of the nonlinear Navier–Stokes and Euler systems.

Next we show that approximate controllability of Eq. (0.6) implies its exact controllability. To this end, we decompose the solution as follows $\xi = \xi_1 + \xi_2$, where ξ_1 and ξ_2 are solutions of equations

$$i\partial_t \xi_1 = -\partial_{xx}^2 \xi_1 - \pi^2 \xi_1 + \langle v(t), Q(x) \rangle \phi_1, \quad (0.7)$$

$$i\partial_t \xi_2 = -\partial_{xx}^2 \xi_2 - \pi^2 \xi_2 + 2\kappa \phi_1^2 \operatorname{Re}(\xi_1 + \xi_2). \quad (0.8)$$

Eq. (0.7) is exactly controllable. Indeed, as in [BL10], this can be seen by rewriting the control system as a moment problem and then by solving it with the help of the Ingham inequality and the assumption (0.5). On the other hand, we show that the resolving operator of Eq. (0.8) is compact. According to a functional analysis result, in a Banach space, the sum of compact and surjective linear continuous mappings has closed image. On the other hand, this image is dense, by approximate controllability of Eq. (0.6). An operator with closed and dense image is obviously surjective, so the linearised Schrödinger equation (0.6) is exactly controllable. Applying the inverse mapping theorem, we derive local exact controllability of the nonlinear equation.

In the case $\kappa \in \mathcal{K}$, the linearised control system may possibly miss one direction. However, we expect that using the nonlinear term one can prove that the result of the Main Theorem still remains true. It would be natural to study this case by applying a power series expansion in the spirit of the paper [CC04] by Coron and Crépeau (see also [Cor07, BC06]). This question will be considered elsewhere.

The use of saturation property to prove approximate controllability of the linearised Schrödinger equation and the argument allowing to derive exact controllability from approximate controllability are among the novelties of this paper. We believe these arguments can be employed in other useful situations.

Global controllability of the NLS equation (0.1) with $\kappa \neq 0$ is a challenging open problem. First results in this direction have been obtained recently by the authors [DN21] and by Coron et al. [CXZ21], who consider approximate controllability between some particular states (in a semiclassical sense in the second reference). From the Main Theorem and the time reversibility of the Schrödinger equation it follows that global exact controllability will be established if one shows approximate controllability to the ground state ϕ_1 in the $H_{(0)}^3$ -norm (see Theorem 3.2 in [Ner10] for the case $\kappa = 0$).

Acknowledgement

This paper was finalised when the second author was visiting the School of Mathematical Sciences of Shanghai Jiao Tong University. He thanks the institute for hospitality.

Notation

In this paper, we use the following notation.

$\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the Euclidian scalar product and norm in \mathbb{R}^q .

$\mathbf{1}$ is the function identically equal to 1 on the interval $I = (0, 1)$.

$L^2 = L^2(I; \mathbb{C})$ and $H^s = H^s(I; \mathbb{C})$, $s > 0$ are the usual Lebesgue and Sobolev spaces of functions $g : I \rightarrow \mathbb{C}$ with the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^s}$. In the case of the spaces of real-valued functions, we write $L^2(I; \mathbb{R})$ and $H^s(I; \mathbb{R})$.

For any $V \in H^3(I; \mathbb{R})$, we denote by A_V the Schrödinger operator

$$A_V = -\partial_{xx}^2 + V, \quad \mathbb{D}(A_V) = H^2 \cap H_0^1(I; \mathbb{C}). \quad (0.9)$$

$\{\phi_{k,V}\}$ is an orthonormal basis in $L^2(I; \mathbb{R})$ formed by eigenfunctions of A_V , and $\lambda_{1,V} < \dots < \lambda_{k,V} < \dots$ are the corresponding eigenvalues.

$H_{(V)}^s = \mathbb{D}(A_V^{\frac{s}{2}})$, $s \geq 0$. Note that

$$H_{(V)}^3 = H_{(0)}^3 = \{\psi \in H^3(I, \mathbb{C}) : \psi|_{x=0,1} = \psi''|_{x=0,1} = 0\}.$$

$$\ell^2 = \left\{ \{a_k\}_{k \geq 1} \in \mathbb{C}^{\mathbb{N}} : \sum_{k=1}^{+\infty} |a_k|^2 < +\infty \right\}, \quad \ell_r^2 = \left\{ \{a_k\}_{k \geq 1} \in \ell^2 : a_1 \in \mathbb{R} \right\}.$$

We write J_T instead of $[0, T]$.

Let X be a Banach space endowed with a norm $\|\cdot\|$.

$B_X(a, r)$ denotes the closed ball in X of radius $r > 0$ centred at $a \in X$.

$L^2(J_T; X)$ is the space of Borel-measurable functions $g : J_T \rightarrow X$ with the norm

$$\|g\|_{L^2(J_T; X)} = \left(\int_0^T \|g(t)\|_X^2 dt \right)^{\frac{1}{2}}.$$

$C(J_T; X)$ is the space of continuous functions $g : J_T \rightarrow X$ with the norm $\|g\|_{C(J_T; X)} = \max_{t \in J_T} \|g(t)\|_X$.

1 Controllability of the linearised equation

In this section, we study the controllability of the following linear Schrödinger equation (cf. Eq. (0.6)):

$$i\partial_t \xi = -\partial_{xx}^2 \xi + V(x)\xi - \lambda \xi + W(x) \operatorname{Re}(\xi) + \langle v(t), Q(x) \rangle \phi(x), \quad (1.1)$$

$$\xi(t, 0) = \xi(t, 1) = 0, \quad (1.2)$$

where $Q : I \rightarrow \mathbb{R}^q$ is a given field, $V, W : I \rightarrow \mathbb{R}$ are given potentials, $\phi = \phi_{1, V}$ is the ground state of the Schrödinger operator A_V (see (0.9)), and $\lambda = \lambda_{1, V}$ is the associated eigenvalue. The following well-posedness result is a consequence of Proposition 3.1.

Proposition 1.1. *For any $T > 0$, $Q \in H^3(I, \mathbb{R}^q)$, $V, W \in H^3(I; \mathbb{R})$, $v \in L^2(J_T; \mathbb{R}^q)$, and $\xi_0 \in H_{(0)}^3$, there is a unique solution $\xi \in C(J_T; H_{(0)}^3)$ of the problem (1.1), (1.2) satisfying $\xi(0) = \xi_0$. Moreover, there is a constant $C_T > 0$ such that*

$$\|\xi\|_{C(J_T; H_{(0)}^3)} \leq C_T (\|\xi_0\|_{(3)} + \|v\|_{L^2(J_T; \mathbb{R}^q)}).$$

Let

$$\mathcal{T}_\phi = \{\psi \in L^2(I, \mathbb{C}) : \operatorname{Re}\langle \psi, \phi \rangle_{L^2} = 0\}$$

be the tangent space to the unit sphere \mathcal{S} at ϕ . As the functions Q, V, W , and v are real-valued, we have $\xi(t) \in \mathcal{T}_\phi$ for any $t \in J_T$, provided that $\xi_0 \in \mathcal{T}_\phi$. Let

$$\mathcal{R} : H_{(0)}^3 \times L^2(J_T; \mathbb{R}^q) \rightarrow C(J_T; H_{(0)}^3), \quad (\xi_0, v) \mapsto \xi$$

be the resolving operator of the problem (1.1), (1.2), and let \mathcal{R}_T be its restriction at time T . In Section 1.1, we show that this problem is approximately controllable under some saturation condition. Then, in Section 1.2, assuming additionally that the vector space \mathbb{Q} spanned by the components of Q contains a function μ satisfying an inequality similar to (0.5), we prove exact controllability of the problem.

1.1 Approximate controllability

Assume that the field Q and the potentials V, W are smooth. Let us define finite-dimensional vector spaces by

$$\mathcal{H} = \operatorname{span}_{\mathbb{R}}\{Q_j \phi : j = 1, \dots, q\}$$

and

$$\mathcal{F}(\mathcal{H}) = \operatorname{span}_{\mathbb{R}}\{f + i(-\partial_{xx}^2 g + Vg - \lambda g + W \operatorname{Re}(g)) : f, g \in \mathcal{H}\}. \quad (1.3)$$

In the spirit of the papers [AS05, AS06, Shi06], we define a non-decreasing sequence of finite-dimensional spaces $\{\mathcal{H}_j\}$ in the following way

$$\mathcal{H}_0 = \mathcal{H}, \quad \mathcal{H}_j = \mathcal{F}(\mathcal{H}_{j-1}) \quad \text{for } j \geq 1, \quad \mathcal{H}_\infty = \bigcup_{j=0}^{\infty} \mathcal{H}_j. \quad (1.4)$$

Let P_1 be the orthogonal projection onto the closed subspace $H_{(0)}^3 \cap \mathcal{T}_\phi$ in $H_{(0)}^3$.

Definition 1.2. We say that a field Q is saturating for the problem (1.1), (1.2) if $\mathcal{H}_\infty \subset H_{(0)}^3$ and the projection $P_1 \mathcal{H}_\infty$ is dense in $H_{(0)}^3 \cap \mathcal{T}_\phi$.

Proposition 1.3. *Assume that Q is saturating. Then the problem (1.1), (1.2) is approximately controllable in the sense that the image of the linear mapping*

$$\mathcal{R}_T(0, \cdot) : L^2(J_T; \mathbb{R}^a) \rightarrow H_{(0)}^3 \cap \mathcal{T}_\phi, \quad v \mapsto \xi(T)$$

is dense in $H_{(0)}^3 \cap \mathcal{T}_\phi$ for $T > 0$.

Proof. Step 1. Reduction. For any $0 \leq \tau \leq t \leq T$, let

$$R(t, \tau) : H_{(0)}^3 \rightarrow H_{(0)}^3, \quad \xi_0 \mapsto \xi(T)$$

be the resolving operator of the problem

$$\begin{aligned} i\partial_t \xi &= -\partial_{xx}^2 \xi + V(x)\xi - \lambda\xi + W(x) \operatorname{Re}(\xi), \\ \xi(t, 0) &= \xi(t, 1) = 0, \\ \xi(\tau, x) &= \xi_0. \end{aligned}$$

Let the operator $\mathcal{A} : L^2(J_T; H_{(0)}^3) \rightarrow H_{(0)}^3$ be defined by

$$\mathcal{A}(v) = \int_0^T R(T, \tau)v(\tau) \, d\tau, \quad v \in L^2(J_T; H_{(0)}^3),$$

and let $P_{\mathcal{H}}$ be the orthogonal projection onto \mathcal{H} in $H_{(0)}^3$. The proposition will be proved if we show that the image of the operator

$$\mathcal{A}_1 : L^2(J_T, H_{(0)}^3) \rightarrow H_{(0)}^3 \cap \mathcal{T}_\phi, \quad \mathcal{A}_1 = \mathcal{A}P_{\mathcal{H}}$$

is dense in $H_{(0)}^3 \cap \mathcal{T}_\phi$. The latter will be achieved by showing that the kernel of the adjoint \mathcal{A}_1^* of \mathcal{A}_1 is trivial. Note that \mathcal{A}_1^* is given by

$$\mathcal{A}_1^* : H_{(0)}^3 \cap \mathcal{T}_\phi \rightarrow L^2(J_T, \mathcal{H}), \quad z \mapsto P_{\mathcal{H}}R(T, \cdot)^*z,$$

where $R(T, \tau)^* : H_{(0)}^3 \rightarrow H_{(0)}^3$ is the $H_{(0)}^3$ -adjoint of $R(T, \tau)$, $\tau \in J_T$.

Step 2. Triviality of the kernel of \mathcal{A}_1^ .* Let z be an arbitrary element of the kernel of \mathcal{A}_1^* . Our goal is to show that $z = 0$. To this end, we take any $g \in \mathcal{H}$ and note that

$$\langle g, R(T, \tau)^*z \rangle_{(3)} = 0 \quad \text{for almost any } \tau \in J_T.$$

By continuity in τ of $R(T, \tau)g$, this is equivalent to

$$\langle R(T, \tau)g, z \rangle_{(3)} = 0 \quad \text{for any } \tau \in J_T. \quad (1.5)$$

Taking $\tau = T$ in this equality, we see that z is orthogonal to \mathcal{H} in $H_{(0)}^3$. In what follows, we show that z is orthogonal to \mathcal{H}_j for any $j \geq 1$. This, together with the saturation assumption, will imply that $z = 0$.

Let us fix any $T_1 \in (0, T)$ and rewrite (1.5) as follows:

$$\langle R(T_1, \tau)g, R(T, T_1)^* z \rangle_{(3)} = 0 \quad \text{for any } \tau \in J_{T_1}. \quad (1.6)$$

Note that $\zeta(\tau) = R(T_1, \tau)g$ is the solution of the problem

$$\begin{aligned} i\partial_\tau \zeta &= -\partial_{xx}^2 \zeta + V(x)\zeta - \lambda\zeta + W(x) \operatorname{Re}(\zeta), \\ \zeta(\tau, 0) &= \zeta(\tau, 1) = 0, \\ \zeta(T_1, x) &= g(x). \end{aligned}$$

Taking the derivative of (1.6) in τ and choosing $\tau = T_1$, we get

$$\langle i(-\partial_{xx}^2 g + Vg - \lambda g + W \operatorname{Re}(g)), R(T, T_1)^* z \rangle_{(3)} = 0 \quad \text{for any } g \in \mathcal{H}.$$

Thus

$$\langle g, R(T, T_1)^* z \rangle_{(3)} = 0 \quad \text{for any } g \in \mathcal{H}_1.$$

As $T_1 \in (0, T)$ is arbitrary, we see that z is orthogonal to \mathcal{H}_1 in $H_{(0)}^3$. Iterating this argument, we derive orthogonality of z to \mathcal{H}_j for any $j \geq 1$. Since $\mathbb{P}_1 \mathcal{H}_\infty$ is dense in $H_{(0)}^3 \cap \mathcal{T}_\phi$, we conclude that $z = 0$. \square

Let us close this section with an example of saturating field Q . This example will be used in the proof of the Main Theorem formulated in the Introduction. Let us introduce the operator

$$A_\kappa g = -\partial_{xx}^2 g - \pi^2 g + 2\kappa\phi_1^2 g, \quad \mathbb{D}(A_\kappa) = H^2 \cap H_0^1(I; \mathbb{R}), \quad (1.7)$$

and let $\lambda_{1,\kappa} < \dots < \lambda_{k,\kappa} < \dots$ be the sequence of its eigenvalues. The following lemma is proved in Section 3.3.

Lemma 1.4. *There is an at most countable set $\mathbb{K} \subset (-\infty, 0]$ such that, for any $\kappa \in \mathbb{R} \setminus \mathbb{K}$ and any $k \geq 1$, we have $\lambda_{k,\kappa} \neq 0$.*

Recall that \mathbb{Q} denotes the vector space spanned by the components of Q (see (0.4)). The following proposition is proved in Section 3.4.

Proposition 1.5. *Let $V(x) = 0$ and $W(x) = 2\kappa\phi_1^2(x)$ for any $x \in I$, let $\kappa \in \mathbb{R} \setminus \mathbb{K}$, where \mathbb{K} is the set in Lemma 1.4, and let Q be a smooth field such that $\mathbf{1}, \cos(\pi x) \in \mathbb{Q}$. Then Q is saturating in the sense of Definition 1.2.*

1.2 Exact controllability

The aim of this section is to prove the following proposition.

Proposition 1.6. *Let V and W be smooth functions such that the following boundary conditions are verified:*

$$W(0) = W(1) = W'(0) = W'(1) = 0. \quad (1.8)$$

Moreover, assume that Q is saturating, and there is a function $\mu \in \mathbb{Q}$ satisfying the inequality

$$|\langle \mu\phi, \phi_{k,V} \rangle_{L^2}| \geq \frac{c}{k^3}, \quad k \geq 1 \quad (1.9)$$

for some $c > 0$. Then the problem (1.1), (1.2) is exactly controllable in the sense that the mapping $\mathcal{R}_T(0, \cdot) : L^2(J_T; \mathbb{R}^q) \rightarrow H_{(0)}^3 \cap \mathcal{T}_\phi$ is surjective for any $T > 0$.

Proof. To prove this proposition, we represent the solution ξ of the problem (1.1), (1.2) as follows $\xi = \xi_1 + \xi_2$, where ξ_1 and ξ_2 are the solutions of problems

$$\begin{aligned} i\partial_t \xi_1 &= -\partial_{xx}^2 \xi_1 + V(x)\xi_1 - \lambda\xi_1 + \langle v(t), Q(x) \rangle \phi, \\ \xi_1(t, 0) &= \xi_1(t, 1) = 0, \\ \xi_1(0, x) &= 0 \end{aligned}$$

and

$$\begin{aligned} i\partial_t \xi_2 &= -\partial_{xx}^2 \xi_2 + V(x)\xi_2 - \lambda\xi_2 + W(x) \operatorname{Re}(\xi_1 + \xi_2), \\ \xi_2(t, 0) &= \xi_2(t, 1) = 0 \\ \xi_2(0, x) &= 0. \end{aligned}$$

Let

$$\mathcal{R}_T^j(0, \cdot) : L^2(J_T; \mathbb{R}^q) \rightarrow H_{(0)}^3 \cap \mathcal{T}_\phi, \quad v \mapsto \xi_j(T), \quad j = 1, 2$$

be the resolving operators of these problems.

Lemma 1.7. *The mapping $\mathcal{R}_T^1(0, \cdot) : L^2(J_T; \mathbb{R}^q) \rightarrow H_{(0)}^3 \cap \mathcal{T}_\phi$ is surjective for any $T > 0$.*

Lemma 1.8. *The mapping $\mathcal{R}_T^2(0, \cdot) : L^2(J_T; \mathbb{R}^q) \rightarrow H_{(0)}^3 \cap \mathcal{T}_\phi$ is compact for any $T > 0$.*

Taking these lemmas for granted, let us complete the proof of the proposition. We have

$$\mathcal{R}_T(0, \cdot) = \mathcal{R}_T^1(0, \cdot) + \mathcal{R}_T^2(0, \cdot),$$

where the linear bounded operators $\mathcal{R}_T^1(0, \cdot)$ and $\mathcal{R}_T^2(0, \cdot)$ are, respectively, surjective and compact from $L^2(J_T; \mathbb{R}^q)$ to $H_{(0)}^3 \cap \mathcal{T}_\phi$. Lemma 3.5 implies that the image of the mapping $\mathcal{R}_T(0, \cdot)$ is closed in $H_{(0)}^3 \cap \mathcal{T}_\phi$. On the other hand, by Proposition 1.3, the image of $\mathcal{R}_T(0, \cdot)$ is dense in $H_{(0)}^3 \cap \mathcal{T}_\phi$. We conclude that $\mathcal{R}_T(0, \cdot)$ is surjective. \square

Proof of Lemma 1.7. Let us consider the problem

$$\begin{aligned} i\partial_t \xi_3 &= -\partial_{xx}^2 \xi_3 + V(x)\xi_3 - \lambda \xi_3 + v(t)\mu(x)\phi, \\ \xi_3(t, 0) &= \xi_3(t, 1) = 0, \\ \xi_3(0, x) &= 0, \end{aligned}$$

and denote by $\mathcal{R}_T^3(0, \cdot) : L^2(J_T; \mathbb{R}) \rightarrow H_{(0)}^3 \cap \mathcal{T}_\phi$ its resolving operator. From the assumption that $\mu \in \mathbb{Q}$ it follows that the image of $\mathcal{R}_T^3(0, \cdot)$ is contained in that of $\mathcal{R}_T^1(0, \cdot)$, so it suffices to prove the surjectivity of $\mathcal{R}_T^3(0, \cdot)$. The latter is proved by rewriting the system as a moment problem which is then solved using the Ingham inequality (see Proposition 4 in [BL10]). Indeed, ξ_3 satisfies the equality

$$\xi_3(t) = -i \int_0^t e^{-i(A_V - \lambda)(t-s)} (v(s)\mu(x)\phi) ds, \quad t \in J_T.$$

We write $\xi_3(T)$ in the form

$$\xi_3(T) = -i \sum_{k=1}^{+\infty} e^{-i(\lambda_{k,V} - \lambda)T} \langle \mu\phi, \phi_{k,V} \rangle_{L^2} \phi_{k,V} \int_0^T e^{i(\lambda_{k,V} - \lambda)s} v(s) ds,$$

which is equivalent to

$$\int_0^T e^{i(\lambda_{k,V} - \lambda)s} v(s) ds = \frac{ie^{i(\lambda_{k,V} - \lambda)T}}{\langle \mu\phi, \phi_{k,V} \rangle_{L^2}} \int_0^1 \xi_3(T, x) \phi_{k,V}(x) dx, \quad k \geq 1.$$

In view of assumption (1.9), for any $\tilde{\xi} \in H_{(0)}^3 \cap \mathcal{T}_\phi$, we have

$$\left\{ \frac{ie^{i(\lambda_{k,V} - \lambda)T}}{\langle \mu\phi, \phi_{k,V} \rangle_{L^2}} \int_0^1 \tilde{\xi}(x) \phi_{k,V}(x) dx \right\}_{k \geq 1} \in \ell_r^2.$$

By the asymptotic formula for the eigenvalues (e.g., see Theorem 4 in [PT87]),

$$\lambda_{k,V} = k^2 \pi^2 + \int_0^1 V(x) dx + r_k, \quad \text{where } \{r_k\}_{k \geq 1} \in \ell^2. \quad (1.10)$$

Hence, $\lambda_{k,V} - \lambda_{k-1,V} \rightarrow +\infty$ as $k \rightarrow +\infty$. Applying Corollary 1 in [BL10], we find that there is $v \in L^2(J_T; \mathbb{R})$ such that

$$\int_0^T e^{i\lambda_{k,V}s} v(s) ds = \frac{ie^{i\lambda_{k,V}T}}{\langle \mu\phi, \phi_{k,V} \rangle_{L^2}} \int_0^1 \tilde{\xi}(x) \phi_{k,V}(x) dx, \quad k \geq 1.$$

This shows that $\mathcal{R}_T^3(0, \cdot)$ is surjective and completes the proof of Lemma 1.7. \square

Proof of Lemma 1.8. The proof follows immediately from Corollary 3.3. Indeed, the operator $\mathcal{R}_T^2(0, \cdot)$ is compact since it can be represented as a composition of the compact operator Φ in Corollary 3.3 with linear continuous mappings. \square

2 Proof of the Main Theorem

Let us consider the NLS equation

$$i\partial_t\psi = -\partial_{xx}^2\psi + V(x)\psi + \kappa|\psi|^{2p}\psi + \langle u(t), Q(x) \rangle \psi, \quad x \in I, \quad (2.1)$$

$$\psi(t, 0) = \psi(t, 1) = 0, \quad (2.2)$$

where the field Q and the potential V are smooth, $p \geq 1$ is an integer, and κ is a real number. The following proposition establishes local well-posedness of this equation. It is proved in Section 3.2.

Proposition 2.1. *Assume that, for some $T > 0$, $\hat{\psi}_0 \in H_{(0)}^3$, and $\hat{u} \in L^2(J_T; \mathbb{R}^q)$, there is a solution $\hat{\psi} \in C(J_T; H_{(0)}^3)$ of the problem (2.1), (2.2) satisfying the initial condition $\hat{\psi}(0) = \psi_0$. Then there are positive numbers $\delta = \delta(T, \Lambda)$ and $C = C(T, \Lambda)$, where*

$$\Lambda = \|\hat{\psi}\|_{C(J_T; H_{(0)}^3)} + \|\hat{u}\|_{L^2(J_T; \mathbb{R}^q)}, \quad (2.3)$$

such that the following properties hold.

(i) *For any $\psi_0 \in H_{(0)}^3$ and $u \in L^2(J_T; \mathbb{R}^q)$ satisfying*

$$\|\psi_0 - \hat{\psi}_0\|_{(3)} + \|u - \hat{u}\|_{L^2(J_T; \mathbb{R}^q)} < \delta, \quad (2.4)$$

the problem (2.1), (2.2) has a unique solution $\psi \in C(J_T; H_{(0)}^3)$ satisfying the initial condition $\psi(0) = \psi_0$.

(ii) *Let Ψ_T be the mapping taking a couple (ψ_0, u) satisfying (2.4) to $\psi(T)$. Then Ψ_T is C^1 , and for any $v \in L^2(J_T; \mathbb{R}^q)$, we have $\partial_u \Psi_T(\hat{\psi}_0, \hat{u})v = \xi(T)$, where ξ is the solution of linearised system*

$$i\partial_t\xi = -\partial_{xx}^2\xi + V(x)\psi + (p+1)\kappa|\psi|^{2p}\xi + p\kappa\psi^2|\psi|^{2(p-1)}\bar{\xi} + \langle \hat{u}(t), Q(x) \rangle \xi + \langle v(t), Q(x) \rangle \hat{\psi}, \quad (2.5)$$

$$\xi(t, 0) = \xi(t, 1) = 0, \quad (2.6)$$

$$\xi(0, x) = 0. \quad (2.7)$$

The following theorem is a generalisation of the Main Theorem.

Theorem 2.2. *Assume that the field Q is saturating in the sense of Definition 1.2 with potentials $V(x)$ and $W(x) = 2p\kappa\phi^{2p}(x)$, and the vector space \mathbb{Q} contains the functions $\mathbf{1}$, ϕ^{2p} , and a function μ verifying inequality (1.9) for some $c > 0$. Then, for any $T > 0$, there is a number $\delta > 0$ such that, for any $\psi_0, \psi_1 \in H_{(0)}^3 \cap \mathcal{S}$ with*

$$\|\psi_j - \phi_1\|_{(3)} < \delta, \quad j = 0, 1, \quad (2.8)$$

there is a control $u \in L^2([0, T]; \mathbb{R}^q)$ and a solution $\psi \in C([0, T]; H_{(0)}^3)$ of the problem (2.1), (2.2) satisfying $\psi(0) = \psi_0$ and $\psi(T) = \psi_1$.

Proof. From the assumption that the functions $\mathbf{1}$ and ϕ^{2p} belong to \mathbb{Q} it follows that $\hat{\psi}(t) = \phi$ is a stationary solution of the problem (2.1), (2.2) corresponding to some constant control $\hat{u}(t)$ such that

$$\langle \hat{u}(t), Q(x) \rangle = -\kappa \phi(x)^{2p} - \lambda_{1,V}. \quad (2.9)$$

Proposition 2.1 implies that the operator $\Psi_T : (\psi_0, u) \mapsto \psi(T)$ is well-defined and C^1 -smooth in some neighbourhood of (ϕ, \hat{u}) . Furthermore, for any $v \in L^2(J_T; \mathbb{R}^q)$, we have $\partial_u \Psi_T(\phi, \hat{u})v = \xi(T)$, where, in view of (2.9), ξ is the solution of equation

$$i\partial_t \xi = -\partial_{xx}^2 \xi + V(x)\xi - \lambda_{1,V}\xi + 2p\kappa\phi^{2p} \operatorname{Re}(\xi) + \langle v(t), Q(x) \rangle \phi. \quad (2.10)$$

The conditions of Proposition 1.6 are satisfied for the linearised system (2.10), (2.6), (2.7), so it is exactly controllable. Applying the inverse mapping theorem to the mapping $u \mapsto \Psi_T(\phi, u)$, we find that there is a number $\delta > 0$ such that, for any $\psi_0, \psi_1 \in H_{(0)}^3 \cap \mathcal{S}$ verifying (2.8), there are controls $u_0, u_1 \in L^2([0, T]; \mathbb{R}^q)$ such that $\Psi_T(\phi_1, u_0) = \overline{\psi_0}$ and $\Psi_T(\phi_1, u) = \psi_1$. By the time-reversibility property of the Schrödinger equation, $\Psi_T(\psi_0, w) = \phi$ with $w(t) = u_0(T - t)$. Setting $u(t) = w(t)$ for $t \in [0, T]$ and $u(t) = u_1(t - T)$ for $t \in [T, 2T]$, we derive $\Psi_{2T}(\psi_0, u) = \psi_1$. As the time $T > 0$ is arbitrary, we complete the proof of Theorem 2.2. \square

Proof of the Main Theorem. Main Theorem is proved by applying Theorem 2.2 with $p = 1$ and $V = 0$. Indeed, the assumption that \mathbb{Q} contains the functions $\mathbf{1}$ and $\cos(2\pi x)$ implies that ϕ_1^2 is also in \mathbb{Q} . Proposition 1.5 implies the saturation assumption with $W(x) = 2\kappa\phi_1^2(x)$ and $\kappa \in \mathbb{R} \setminus \mathbb{K}$.

When¹ $\kappa = 0$, the linearised problem is still exactly controllable by Lemma 1.7, so the conclusions of the theorem hold in that case too. Thus we proved the Main Theorem with $\mathcal{K} = \mathbb{K} \setminus \{0\} \subset (-\infty, 0)$. \square

Remark 2.3. Let us emphasise that, under the conditions of the Main Theorem, the linearised problem (1.1), (1.2) with $V(x) = 0$, $W(x) = 2\kappa\phi_1^2(x)$, and $\kappa \in \mathcal{K}$ may still be exactly controllable. Indeed, in that situation we only know that the linearised system can miss at most one direction (see Corollary 3.4). As mentioned in the Introduction, even if the linearised system misses one direction, the nonlinear system can still be locally exactly controllable near the ground state.

Remark 2.4. It is not difficult to construct examples of saturating fields in the case of any integer $p \geq 1$. Indeed, the results of Lemma 1.4 and Proposition 1.5 generalise without difficulties to the case when $V(x) = 0$, $W(x) = 2p\kappa\phi^{2p}$, and $\mathbf{1}, \cos(\pi x), \dots, \cos(N_p\pi x) \in \mathbb{Q}$ with sufficiently large integer $N_p \geq 1$.

¹Note that zero belongs to the set \mathbb{K} in Lemma 1.4, since $\lambda_{1,0} = 0$.

3 Appendix

3.1 Well-posedness of the linear Schrödinger equation

Let us consider the following linear Schrödinger equation with a source term:

$$i\partial_t \xi = -\partial_{xx}^2 \xi + V(x)\xi + W(x) \operatorname{Re}(\xi) + f(t, x), \quad (3.1)$$

$$\xi(t, 0) = \xi(t, 1) = 0. \quad (3.2)$$

Well-posedness of this equation is essentially established in [BL10].

Proposition 3.1. *For any $T > 0$, $V, W \in H^3(I; \mathbb{R})$, $f \in L^2(J_T; H^3 \cap H_0^1(I; \mathbb{C}))$, and $\xi_0 \in H_{(0)}^3$, there is a unique solution $\xi \in C(J_T; H_{(0)}^3)$ of this problem with the initial condition $\xi(0) = \xi_0$ in the sense that*

$$\xi(t) = e^{-iA_V t} \xi_0 - i \int_0^t e^{-iA_V(t-s)} (W \operatorname{Re}(\xi(s)) + f(s)) ds, \quad t \in J_T.$$

Moreover, there is a constant $C_T > 0$ such that

$$\|\xi\|_{C(J_T; H_{(0)}^3)} \leq C_T \left(\|\xi_0\|_{(3)} + \|f\|_{L^2(J_T; H^3 \cap H_0^1(I; \mathbb{C}))} \right).$$

By Lemma 1 in [BL10], the function

$$G_f : t \mapsto \int_0^t e^{-iA_V(t-s)} f(s) ds \quad \text{belongs to } C(J_T; H_{(0)}^3) \quad (3.3)$$

and satisfies the inequality²

$$\|G_f\|_{C(J_T; H_{(0)}^3)} \leq C_T \|f\|_{L^2(J_T; H^3 \cap H_0^1(I; \mathbb{C}))}. \quad (3.4)$$

Proposition 3.1 is derived from this by applying a usual fixed point argument to the mapping $L : C(J_T; H_{(0)}^3) \rightarrow C(J_T; H_{(0)}^3)$ defined by

$$L(\xi)(t) = e^{-iA_V t} \xi_0 - i \int_0^t e^{-iA_V(t-s)} (W \operatorname{Re}(\xi(s)) + f(s)) ds, \\ \text{for } \xi \in C(J_T; H_{(0)}^3), \quad t \in J_T.$$

We shall not dwell on the details of the proof.

The next lemma follows from the Arzelà–Ascoli theorem in a standard way; again we skip the details.

Lemma 3.2. *Under the conditions of Proposition 3.1, the mapping*

$$G : L^2(J_T; H_{(0)}^3) \rightarrow C(J_T; H_{(0)}^3), \quad f \mapsto G_f$$

is compact.

²More precisely, in the paper [BL10], the case $V = 0$ is considered. The general case is proved in a similar way by using the asymptotics (1.10) for the eigenvalues of A_V .

Let us consider a particular case of the problem (3.1), (3.2) given by

$$i\partial_t \xi = -\partial_{xx}^2 \xi + V(x)\xi + W(x) \operatorname{Re}(\xi + \eta), \quad (3.5)$$

$$\xi(t, 0) = \xi(t, 1) = 0, \quad (3.6)$$

$$\xi(0, x) = 0, \quad (3.7)$$

and assume that W satisfies the boundary conditions (1.8). Then the mapping $\eta \mapsto W(x) \operatorname{Re}(\eta)$ is continuous from $L^2(J_T; H_{(0)}^3)$ to itself, and the following is an immediate consequence of Lemma 3.2.

Corollary 3.3. *Under the conditions of Proposition 3.1, the mapping*

$$\Phi : L^2(J_T; H_{(0)}^3) \rightarrow C(J_T; H_{(0)}^3), \quad \eta \mapsto \xi$$

is compact, where ξ is the solution of the problem (3.5)-(3.7).

3.2 Local well-posedness of the NLS equation

Here we prove Proposition 2.1.

Step 1. Uniqueness. Let $\psi_1, \psi_2 \in C(J_T; H_{(0)}^3)$ be solutions of (2.1), (2.2) with the same control $u \in L^2(J_T; \mathbb{R}^q)$ and the same initial condition $\psi_1(0) = \psi_2(0)$. Then the difference $\varphi = \psi_1 - \psi_2$ satisfies

$$\begin{aligned} i\partial_t \varphi &= -\partial_{xx}^2 \varphi + V(x)\varphi + \kappa|\psi_1|^{2p}\psi_1 - \kappa|\psi_2|^{2p}\psi_2 + \langle u(t), Q(x) \rangle \varphi, \\ \varphi(t, 0) &= \varphi(t, 1) = 0, \\ \varphi(0) &= 0. \end{aligned}$$

From (3.4) it follows that

$$\begin{aligned} \|\varphi(t)\|_{(3)}^2 &\leq C_T \left(\| |\psi_1|^{2p}\psi_1 - |\psi_2|^{2p}\psi_2 \|_{L^2(J_t; H^3 \cap H_0^1(I; \mathbb{C}))}^2 \right. \\ &\quad \left. + \| \langle u, Q \rangle \varphi \|_{L^2(J_t; H^3 \cap H_0^1(I; \mathbb{C}))}^2 \right) \\ &\leq C_1 \int_0^t (1 + |u(s)|^2) \|\varphi(s)\|_{(3)}^2 ds, \quad t \in J_T, \end{aligned}$$

where $C_1 > 0$ is a constant depending on $\|\psi_j\|_{C(J_T; H_{(0)}^3)}$, $j = 1, 2$. Applying the Gronwall inequality, we infer that $\varphi(t) = 0$ for any $t \in J_T$.

Step 2. Local-in-time existence. Let us take any $\psi_0 \in H_{(0)}^3$ and $u \in L^2(J_T, \mathbb{R}^q)$ satisfying (2.4), any $T_1 \in J_T$, any $\psi \in B_{C(J_{T_1}; H_{(0)}^3)}(\hat{\psi}, 1)$, and define

$$M(\psi)(t) = e^{-iAvt} \psi_0 - i \int_0^t e^{-iAv(t-s)} (\kappa|\psi(s)|^{2p}\psi(s) + \langle u(s), Q(x) \rangle \psi(s)) ds$$

for $t \in J_{T_1}$. Then (3.3) implies that $M(\psi) \in C(J_{T_1}; H_{(0)}^3)$. Moreover, using (3.4), we get

$$\begin{aligned} \|M(\psi)(t) - \hat{\psi}(t)\|_{(3)} &\leq C_2 \left(\|\psi_0 - \hat{\psi}_0\|_{(3)} + \|u - \hat{u}\|_{L^2(J_T; \mathbb{R}^q)} \right. \\ &\quad \left. + (\|\hat{u}\|_{L^2(J_{T_1}; \mathbb{R}^q)} + T_1^{\frac{1}{2}}) \|\psi - \hat{\psi}\|_{C(J_t; H_{(0)}^3)} \right), \quad t \in J_{T_1}, \end{aligned}$$

where³ $C_2(T, \Lambda) > 0$ (see (2.3)). This implies that M maps $B_{C(J_{T_1}; H_{(0)}^3)}(\hat{\psi}, 1)$ into itself for sufficiently small T_1 and δ . In a similar way, for any $\psi_1, \psi_2 \in B_{C(J_{T_1}; H_{(0)}^3)}(\hat{\psi}, 1)$ and $t \in J_{T_1}$, we have

$$\|M(\psi_1)(t) - M(\psi_2)(t)\|_{(3)} \leq C_3 \left(\|u\|_{L^2(J_{T_1}; \mathbb{R}^q)} + T_1^{\frac{1}{2}} \right) \|\psi_1 - \psi_2\|_{C(J_t; H_{(0)}^3)}.$$

Thus, M is a contraction in $B_{C(J_{T_1}; H_{(0)}^3)}(\hat{\psi}, 1)$ for sufficiently small T_1 and δ . Hence, there is $\psi \in B_{C(J_{T_1}; H_{(0)}^3)}(\hat{\psi}, 1)$ such that $M(\psi) = \psi$.

Step 3. Existence up to T . Let ψ be a maximal solution of (2.1), (2.2) corresponding to $\psi_0 \in H_{(0)}^3$ and $u \in L^2(J_T, \mathbb{R}^q)$ satisfying (2.4). Then there is a maximal time $T_* \in J_T$ such that ψ is defined on $[0, T_*)$ and

$$\|\psi(t)\|_{(3)} \rightarrow +\infty \quad \text{as } t \rightarrow T_*^- \text{ when } T_* < T.$$

The difference $\varphi = \psi - \hat{\psi}$ satisfies the inequality

$$\begin{aligned} \|\varphi(t)\|_{(3)} &\leq C_4 \delta + C_4 \int_0^t \left(\|\varphi(s)\|_{(3)} + \|\varphi(s)\|_{(3)}^{2p+1} \right) ds \\ &\quad + C_4 \left(\int_0^t |u(s)|^2 \|\varphi(s)\|_{(3)}^2 ds \right)^{\frac{1}{2}}, \quad t \in [0, T_*]. \end{aligned} \quad (3.8)$$

Let us denote

$$\tau = \inf\{t \in J_T : \|\varphi(t)\|_{(3)} = 1\},$$

where the infimum over an empty set is equal to T . Let us show that, for sufficiently small δ , we have $\tau = T$. Arguing by contradiction, let us assume that $\tau < T$. From (3.8) we derive

$$\|\varphi(t)\|_{(3)}^2 \leq C_5 \delta^2 + C_5 \int_0^t \|\varphi(s)\|_{(3)}^2 (1 + |u(s)|^2) ds, \quad t \in [0, \tau].$$

The Gronwall inequality implies

$$\|\varphi(t)\|_{(3)}^2 \leq C_5 \delta^2 \exp \left(C_5 \int_0^t (1 + |u(s)|^2) ds \right) < 1, \quad t \in [0, \tau)$$

for small δ and any $u \in L^2(J_T, \mathbb{R}^q)$ satisfying (2.4). This contradicts the definition of τ . Thus $\tau = T$ for small δ .

Step 4. Differentiability. The proof of C^1 regularity of the resolving operator is similar to the case considered in Sections 2.2 and 3.2 in [BL10], so we do not provide the details.

³All the constants C_i below depend on T and Λ .

3.3 Proof of Lemma 1.4

Let us fix any $k \geq 1$. Let $\phi_{k,\kappa}$ be the eigenfunction of the operator A_κ (see (1.7)) associated with the eigenvalue $\lambda_{k,\kappa}$. By Theorem 3 in Chapter 2 in [PT87], both $\phi_{k,\kappa}$ and $\lambda_{k,\kappa}$ are real-analytic functions in κ . By differentiating in κ the identity

$$(-\partial_{xx}^2 - \pi^2 + 2\kappa\phi_1^2 - \lambda_{k,\kappa}) \phi_{k,\kappa} = 0,$$

we obtain

$$(-\partial_{xx}^2 - \pi^2 + 2\kappa\phi_1^2 - \lambda_{k,\kappa}) \frac{d\phi_{k,\kappa}}{d\kappa} + \left(2\phi_1^2 - \frac{d\lambda_{k,\kappa}}{d\kappa}\right) \phi_{k,\kappa} = 0.$$

Taking the scalar product in $L^2(I; \mathbb{R})$ of this identity with $\phi_{k,\kappa}$, we get

$$\frac{d\lambda_{k,\kappa}}{d\kappa} = \langle 2\phi_1^2, \phi_{k,\kappa}^2 \rangle_{L^2} > 0$$

for any $\kappa \in \mathbb{R}$. We conclude that $\lambda_{k,\kappa}$ is strictly increasing in κ , so it can vanish for at most one value of $\kappa \in \mathbb{R}$. Moreover, as $\lambda_{1,0} = 0$, strict monotonicity of $\lambda_{k,\kappa}$ implies that $0 < \lambda_{1,\kappa} \leq \lambda_{k,\kappa}$ for any $k \geq 1$ and $\kappa > 0$. This completes the proof of the lemma.

3.4 Saturation property

This section is devoted to the proof of Proposition 1.5. It suffices to consider the case $q = 2$ and $Q(x) = (\mathbf{1}, \cos(\pi x))$. It is easy to see that

$$\mathcal{H} = \text{span}_{\mathbb{R}}\{\phi_1, \phi_2\}. \quad (3.9)$$

Let us denote

$$F(g) = i(-\partial_{xx}^2 g - \pi^2 g + 2\kappa\phi_1^2 \text{Re}(g)) \quad \text{for } g \in H^2(I; \mathbb{C}).$$

From (1.3) and (1.4) it follows that $F(g) \in \mathcal{H}_j$ for $g \in \mathcal{H}_{j-1}$ and $j \geq 1$. In particular, $F(g) \in \mathcal{H}_\infty$ for $g \in \mathcal{H}_\infty$. Furthermore, from (3.9) we derive

$$\mathcal{H}_\infty \subset \text{span}_{\mathbb{C}}\{\phi_k : k \geq 1\} \subset H_{(0)}^3.$$

The proposition will be established if we show that \mathcal{H}_∞ is dense in $H_{(0)}^3$. The proof of this is divided into three steps.

Step 1. First let us show that the eigenfunctions ϕ_{2k+1} , $k \geq 0$ belong to \mathcal{H}_∞ . We proceed by recurrence. By (3.9), we have $\phi_1 \in \mathcal{H} \subset \mathcal{H}_\infty$. Let us take any $N \geq 1$, assume that $\phi_{2k+1} \in \mathcal{H}_\infty$ for any $0 \leq k \leq N-1$, and prove that $\phi_{2N+1} \in \mathcal{H}_\infty$. We have

$$F(\phi_{2N-1}) = ic_{2N-1}\phi_{2N-1} + i2\kappa\phi_1^2\phi_{2N-1} \in \mathcal{H}_\infty,$$

where $c_{2N-1} = \pi^2((2N-1)^2 - 1)$. Combining this with the identity

$$\phi_1^2\phi_{2N-1} = \frac{1}{2}(2\phi_{2N-1} - \phi_{2N-3} - \phi_{2N+1}),$$

we obtain

$$F(F(\phi_{2N-1})) = (c_{2N-1}^2 - c_{2N-1}) \phi_{2N-1} + \frac{1}{2} (c_{2N-3} \phi_{2N-3} + c_{2N+1} \phi_{2N+1}) \in \mathcal{H}_\infty.$$

As $\phi_{2N-3}, \phi_{2N-1} \in \mathcal{H}_\infty$, we conclude that $\phi_{2N+1} \in \mathcal{H}_\infty$.

In a similar way, as $\phi_2 \in \mathcal{H} \subset \mathcal{H}_\infty$, one shows that $\phi_{2k} \in \mathcal{H}_\infty$ for any $k \geq 1$.

Step 2. Let us prove that the vector space

$$\mathcal{G} = \text{span}_{\mathbb{R}}\{iF(\phi_k) : k \geq 1\}$$

is dense in $H_{(0)}^3(I; \mathbb{R})$. Indeed, let A_κ be the operator defined by (1.7) with $\kappa \in \mathbb{R} \setminus \mathbb{K}$. Then $\lambda_{k,\kappa} \neq 0$ for any $k \geq 1$, so the image of A_κ is dense in $H_{(0)}^3(I; \mathbb{R})$. It remains to note that $A_\kappa g = -iF(g)$ for $g \in H^2$, so \mathcal{G} is dense in $H_{(0)}^3(I; \mathbb{R})$.

Step 3. Combining the results of steps 1 and 2, and using the fact that

$$\text{span}_{\mathbb{R}}\{\phi_k, F(\phi_k) : k \geq 1\} \subset \mathcal{H}_\infty,$$

we see that \mathcal{H}_∞ is dense in $H_{(0)}^3$. This completes the proof of the proposition.

Corollary 3.4. *Let $\mathbb{K} \subset (-\infty, 0]$ be the set in Lemma 1.4, $Q(x) = (\mathbf{1}, \cos(\pi x))$, $V(x) = 0$, and $W(x) = 2\kappa\phi_1^2(x)$. Then, for any $\kappa \in \mathbb{K}$, the codimension of \mathcal{H}_∞ in $H_{(0)}^3$ is one.*

Proof. As $\kappa \in \mathbb{K}$, there is a unique $k \geq 1$ such that $\lambda_{k,\kappa} = 0$. From the proof of Proposition 1.5 it follows that the vector space spanned by $i\phi_{k,\kappa}$ is the orthogonal complement of \mathcal{H}_∞ in $H_{(0)}^3$. \square

3.5 A closed image theorem

In this section, we formulate a simple functional analysis result used in the proof of exact controllability of the linear Schrödinger equation (see Section 1.2). Being unable to find a proper reference, we give a complete proof.

Let X and Y be Banach spaces, and let X^* and Y^* be their duals.

Lemma 3.5. *Assume that $A : X \rightarrow Y$ and $B : X \rightarrow Y$ are linear continuous operators such that A is surjective and B is compact. Then the image of $A + B$ is closed in Y .*

Proof. Let $A^* : Y^* \rightarrow X^*$ and $B^* : Y^* \rightarrow X^*$ be the adjoint operators of A and B . By Theorem 2.19 in [Bre11], the set $(A + B)(X)$ is closed in Y if and only if $(A^* + B^*)(Y^*)$ is closed in X^* . We will prove that $(A^* + B^*)(Y^*)$ is closed in X^* in three steps.

Step 1. Let us first show that the kernel of the operator $A^* + B^*$ is finite-dimensional. To this end, we prove that any bounded sequence $\{y_n\}$ in the kernel of $A^* + B^*$ has a convergent subsequence. Indeed, by Theorem 6.4 in [Bre11], B^* is compact. So there is a subsequence $\{y_{n_k}\}$ such that $\{B^*(y_{n_k})\}$ converges. The equality $A^*(y_{n_k}) + B^*(y_{n_k}) = 0$ implies that $\{A^*(y_{n_k})\}$ also converges.

As A^* is injective and $A^*(Y^*)$ is closed, the open mapping theorem implies that $\{y_{n_k}\}$ converges.

Thus the kernel of $A^* + B^*$ is finite-dimensional, so it is complemented. Without loss of generality, we can assume that $A^* + B^*$ is injective.

Step 2. Let $\{y_n\} \subset Y^*$ be a sequence such that $\{(A^* + B^*)(y_n)\}$ converges. Let us show that $\{y_n\}$ is bounded. Arguing by contradiction, let us assume that there is a subsequence such that $\|y_{n_k}\|_{Y^*} \rightarrow +\infty$. Then, for the sequence $\tilde{y}_n = y_{n_k}/\|y_{n_k}\|_{Y^*}$ we have $(A^* + B^*)(\tilde{y}_n) \rightarrow 0$ as $n \rightarrow +\infty$. By the fact that $\|\tilde{y}_n\|_{Y^*} = 1$ and the compactness of B^* , there is a subsequence $\{\tilde{y}_{\tilde{n}_k}\}$ such that $\{B^*(\tilde{y}_{\tilde{n}_k})\}$ converges. From the equality

$$A^*(\tilde{y}_{\tilde{n}_k}) = (A^* + B^*)(\tilde{y}_{\tilde{n}_k}) - B^*(\tilde{y}_{\tilde{n}_k})$$

it follows that $\{A^*(\tilde{y}_{\tilde{n}_k})\}$ converges too. As in step 1, this implies that $\{\tilde{y}_{\tilde{n}_k}\}$ converges to some limit \tilde{y} . Since $\|\tilde{y}_{\tilde{n}_k}\|_{Y^*} = 1$, we have $\|\tilde{y}\|_{Y^*} = 1$. By continuity,

$$(A^* + B^*)(\tilde{y}_{\tilde{n}_k}) \rightarrow (A^* + B^*)(\tilde{y}) \quad \text{as } k \rightarrow +\infty. \quad (3.10)$$

On the other hand, from the construction of $\{\tilde{y}_n\}$ it follows that

$$(A^* + B^*)(\tilde{y}_{\tilde{n}_k}) \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (3.11)$$

From (3.10) and (3.11) it follows that \tilde{y} is a non-zero element of the kernel of $A^* + B^*$. This contradicts the injectivity of $A^* + B^*$.

Step 3. Let $\{y_n\} \subset Y^*$ be a bounded sequence such that

$$(A^* + B^*)(y_n) \rightarrow x \quad \text{as } n \rightarrow +\infty. \quad (3.12)$$

By compactness of B^* , there is a subsequence $\{y_{n_k}\}$ such that $\{B^*(y_{n_k})\}$ converges. From (3.12) it follows that $\{A^*(y_{n_k})\}$ converges too. As above, this implies that $\{y_{n_k}\}$ converges to some limit y . By continuity and (3.12), we have $(A^* + B^*)(y) = x$. We conclude that the set $(A^* + B^*)(Y^*)$ is closed in X^* . \square

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