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## About some remarkable properties of generalized canonical analysis

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The method and the criterion proposed by J.D. Carroll for generalizing canonical analysis to more than two groups of variables, provide a very general model which unify various techniques of multi-dimensional data analysis.

Let  $X_i$   $i = 1, 2, \dots, p$  be  $p$  sets of  $m_i$  variables on  $n$  individuals ; each  $X_i$  will be considered as a  $n \times m_i$  matrix. Various attempts (Horst (1961) ; Kettenring (1971) , Masson (1974)) has been made to generalize canonical analysis when  $p \geq 3$  by finding directly  $p$ -uples of canonical variables  $\underline{\xi}_i$  ( $\underline{\xi}_i = X_i \underline{a}_i$ ) satisfying to some criteria of optimality. Most of them lead to tricky algorithms and the  $\underline{\xi}_i$  , generally, are no longer solutions of an eigenequation.

Carroll's method consists in finding an auxiliary variable, the most correlated with the  $p$  groups, in the following sense :

$$(1) \quad \sum_{i=1}^p R^2(\underline{z}, X_i) \text{ is maximal}$$

where  $R^2$  is the square multiple correlation coefficient. This leads to the eigenequation :

$$(2) \quad \sum_{i=1}^p X_i (X_i' X_i)^{-1} X_i' \underline{z} = \lambda \underline{z} \quad \lambda \text{ max}$$

where  $X_i (X_i' X_i)^{-1} X_i' = A_i$  is the orthogonal projector onto the subspace spanned by  $X_i$  .

Regressing  $\underline{z}$  onto the  $X_i$  gives then the canonical variates  $\underline{\xi}_i$  and  $\underline{z}$  is both the mean and the first principal component of the  $\underline{\xi}_i$  . It may be noted (Ten BERGE (1977)) that this solution (equivalent to solutions proposed by HORST, Mc DONALD and others) is an improper generalization of canonical analysis because, if the  $\underline{z}^{(k)}$  corresponding to successive eigenvalues  $\lambda_k$  of equation (2) are orthogonal the  $\underline{\xi}_i^{(k)}$  are not orthogonal when  $p > 2$ .

Nevertheless, the main interest of Carroll's method stay in its criterion and in finding  $\underline{z}$  variates (more than the  $\underline{\xi}_i$ ) which leads to various interpretations :

First it is a generalization of principal components analysis to groups of variables instead of single variables : in normalized p.c.a. the first principal component  $\underline{c}$  of  $\underline{x}_1 ; \underline{x}_2 \dots \underline{x}_p$  provides a maximum for  $\sum_{i=1}^p r^2 (\underline{c} ; \underline{x}_i)$

Thus Carroll's method comes down to a p.c.a. of the  $\sum m_i$  variables with a special metric which takes into account the grouping of variables.

Then, applying Carroll's generalized canonical analysis to  $p$  nominal variables (each  $X_i$  is the array of the indicator variables of the categories of the  $i^{\text{th}}$  variable  $\underline{x}_i$ ) leads exactly to the principal components of scales analysis of Guttman (1941), to the optimal quantification of Hayashi (1950), to the Benzecri-Lebart (1972) multiple correspondence analysis (See also Bouroche - Saporta - Tenenhaus (1975) and to the optimal scaling method Homals of J. De Leeuw (1978) which are all equivalent.

Since the multiple correlation coefficient with the indicator variables is nothing but a correlation ratio, all these methods may be presented in a fairly simple way : they consist in finding a numerical variable  $\underline{z}$  maximizing  $\sum_{i=1}^p \eta^2 (\underline{z} ; \underline{x}_i)$

We have thus a general criterion for analyzing sets of nominal or numerical variables.

Neither J.D. Carroll nor others gave a simple equation for the  $\underline{\xi}_i$  : we prove here that the canonical variables can be obtained as the eigenvector of an  $np \times np$  matrix whose blocks are the products of projectors  $A_i A_j$  ; this is a generalization of a seemingly new property of ordinary canonical analysis. From this result, it can be easily shown that the  $\underline{\xi}_i$  satisfy a mean orthogonality property :

$$\sum_{i=1}^p \text{cov} ( \underline{\xi}_i^{(k)} , \underline{\xi}_i^{(l)} ) = 0 \quad \text{if } k \neq l$$

.../...

which completes the weak orthogonality :

$$\text{cov} \left( \sum_{i=1}^p \xi_i^{(k)}, \sum_{i=1}^p \xi_i^{(l)} \right) = 0 \text{ if } k \neq l$$

of course writing the equation :

$$(3) \begin{pmatrix} A_1 & A_1 A_2 & \dots & A_1 A_p \\ A_2 A_1 & A_2 & \dots & \\ \vdots & \vdots & \ddots & \vdots \\ A_p A_1 & A_p A_2 & \dots & A_p \end{pmatrix} \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \vdots \\ \hat{\xi}_p \end{pmatrix} = \lambda \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_p \end{pmatrix}$$

has only a theoretical interest since it is of dimension  $np$ , the solution is in fact performed in dimension  $\sum \xi_i$  by solving the equation (4)

$$(4) \begin{pmatrix} v_{11} & & & \\ & v_{22} & & \\ & & \ddots & \\ & & & v_{pp} \end{pmatrix}^{-1} \begin{pmatrix} v_{11} & v_{12} & \dots & v_{1p} \\ v_{21} & v_{22} & \dots & \\ \vdots & \vdots & \ddots & \vdots \\ v_{p1} & v_{p2} & \dots & v_{pp} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix}$$

where  $\xi_i = X_i a_i$  and  $v_{ij} = X_i' X_j$

REFERENCES

- J.M.F. Ten BERGE (1977) Optimizing factorial invariance. Thesis. University of Groningen.
- J.P. BENZECRI (1972) - Sur l'analyse des tableaux binaires associés à une correspondance multiple Lab. Stat. Math. Université Paris VI.
- J.M. BOUROCHE - G. SAPORTA - M. TENENHAUS (1975) - Generalized canonical analysis of qualitative data. US Japan seminar on multidimensional scaling. San Diego.
- J.D. CARROLL (1968) - Generalization of canonical correlation analysis to three or more sets of variables. Proceedings 76<sup>th</sup> convention American Psychological Association 227-228
- Y. ESCOUFIER - F. CAILLIEZ - J.P. PAGES (1978) Geometrie et techniques particulieres en analyse factorielle. European Meeting of the Psychometric Society. Uppsala and Revue de Statistique appliquée XXVII n° 1 5-28 (1979)
- L.L. GUTTMAN (1941) The quantification of a class of attributes. A theory and method of scale construction in P. Horst ed. "the prediction of personal adjustment" Social Science Research Council.
- C. HAYASHI (1950) On the quantification of qualitative data from the mathematico - statistical point of view. Annals Inst-Stat. Math 2 n° 1.
- P. HORST (1961) Relation among m sets of measures. Psychometrika 26 129-149.
- R.J. KETTENRING (1971) Canonical analysis of several sets of variables. Biometrika 58 433-451
- L. LEBART - A. MORINEAU - N. TABARD (1977) Techniques de la description statistique Dunod (Paris).
- J. De LEEUW (1978) Homals. Dept of Data Theory. University Of Leyden
- R.P. MAC DONALD (1968) A unified treatment of the weighting problem Psychometrika 33 351-381.
- M. MASSON (1974) Processus linéaires et analyse de données non linéaire. Thèse d'état. Université Paris VI
- G. SAPORTA (1975) Liaison entre plusieurs ensembles de variables et codages de données qualitatives. Thèse 3e cycle Université Paris V.

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