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About some remarkable properties of generalized canonical analysis

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The method and the criterion proposed by J.D. Carroll for generalizing canonical analysis to more than two groups of variables, provide a very general model which unify various techniques of multi-dimensional data analysis.

Let X_i $i = 1, 2, \dots, p$ be p sets of m_i variables on n individuals ; each X_i will be considered as a $n \times m_i$ matrix. Various attempts (Horst (1961) ; Kettenring (1971) , Masson (1974)) has been made to generalize canonical analysis when $p \geq 3$ by finding directly p -uples of canonical variables $\underline{\xi}_i$ ($\underline{\xi}_i = X_i \underline{a}_i$) satisfying to some criteria of optimality. Most of them lead to tricky algorithms and the $\underline{\xi}_i$, generally, are no longer solutions of an eigenequation.

Carroll's method consists in finding an auxiliary variable, the most correlated with the p groups, in the following sense :

$$(1) \quad \sum_{i=1}^p R^2(\underline{z}, X_i) \text{ is maximal}$$

where R^2 is the square multiple correlation coefficient. This leads to the eigenequation :

$$(2) \quad \sum_{i=1}^p X_i (X_i' X_i)^{-1} X_i' \underline{z} = \lambda \underline{z} \quad \lambda \text{ max}$$

where $X_i (X_i' X_i)^{-1} X_i' = A_i$ is the orthogonal projector onto the subspace spanned by X_i .

Regressing \underline{z} onto the X_i gives then the canonical variates $\underline{\xi}_i$ and \underline{z} is both the mean and the first principal component of the $\underline{\xi}_i$. It may be noted (Ten BERGE (1977)) that this solution (equivalent to solutions proposed by HORST, Mc DONALD and others) is an improper generalization of canonical analysis because, if the $\underline{z}^{(k)}$ corresponding to successive eigenvalues λ_k of equation (2) are orthogonal the $\underline{\xi}_i^{(k)}$ are not orthogonal when $p > 2$.

Nevertheless, the main interest of Carroll's method stay in its criterion and in finding \underline{z} variates (more than the $\underline{\xi}_i$) which leads to various interpretations :

First it is a generalization of principal components analysis to groups of variables instead of single variables : in normalized p.c.a. the first principal component \underline{c} of $\underline{x}_1 ; \underline{x}_2 \dots \underline{x}_p$ provides a maximum for $\sum_{i=1}^p r^2 (\underline{c} ; \underline{x}_i)$

Thus Carroll's method comes down to a p.c.a. of the $\sum m_i$ variables with a special metric which takes into account the grouping of variables.

Then, applying Carroll's generalized canonical analysis to p nominal variables (each X_i is the array of the indicator variables of the categories of the i^{th} variable \underline{x}_i) leads exactly to the principal components of scales analysis of Guttman (1941), to the optimal quantification of Hayashi (1950), to the Benzecri-Lebart (1972) multiple correspondence analysis (See also Bouroche - Saporta - Tenenhaus (1975) and to the optimal scaling method Homals of J. De Leeuw (1978) which are all equivalent.

Since the multiple correlation coefficient with the indicator variables is nothing but a correlation ratio, all these methods may be presented in a fairly simple way : they consist in finding a numerical variable \underline{z} maximizing $\sum_{i=1}^p \eta^2 (\underline{z} ; \underline{x}_i)$

We have thus a general criterion for analyzing sets of nominal or numerical variables.

Neither J.D. Carroll nor others gave a simple equation for the $\underline{\xi}_i$: we prove here that the canonical variables can be obtained as the eigenvector of an $np \times np$ matrix whose blocks are the products of projectors $A_i A_j$; this is a generalization of a seemingly new property of ordinary canonical analysis. From this result, it can be easily shown that the $\underline{\xi}_i$ satisfy a mean orthogonality property :

$$\sum_{i=1}^p \text{cov} (\underline{\xi}_i^{(k)} , \underline{\xi}_i^{(l)}) = 0 \quad \text{if } k \neq l$$

.../...

which completes the weak orthogonality :

$$\text{cov} \left(\sum_{i=1}^p \xi_i^{(k)}, \sum_{i=1}^p \xi_i^{(l)} \right) = 0 \text{ if } k \neq l$$

of course writing the equation :

$$(3) \begin{pmatrix} A_1 & A_1 A_2 & \dots & A_1 A_p \\ A_2 A_1 & A_2 & \dots & \\ \vdots & \vdots & \ddots & \vdots \\ A_p A_1 & A_p A_2 & \dots & A_p \end{pmatrix} \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \vdots \\ \hat{\xi}_p \end{pmatrix} = \lambda \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_p \end{pmatrix}$$

has only a theoretical interest since it is of dimension np , the solution is in fact performed in dimension $\sum \xi_i$ by solving the equation (4)

$$(4) \begin{pmatrix} v_{11} & & & \\ & v_{22} & & \\ & & \ddots & \\ & & & v_{pp} \end{pmatrix}^{-1} \begin{pmatrix} v_{11} & v_{12} & \dots & v_{1p} \\ v_{21} & v_{22} & \dots & \\ \vdots & \vdots & \ddots & \vdots \\ & & & v_{pp} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix}$$

where $\xi_i = X_i \frac{a_i}{\lambda}$ and $v_{ij} = X_i' X_j$

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