

**REPORT ON THE PAPER “ $\mathrm{SL}_2(\mathbb{R})$ -DYNAMICS ON THE MODULI
SPACE OF ONE-HOLED DILATION TORI”
BY ADRIEN BOULANGER AND SELIM GHAZOUANI**

1. OVERVIEW

The paper under review extends results on the so-called Teichmüller geodesic flow to moduli spaces of dilation surfaces, which arise naturally when studying affine interval exchange transformations. The study of moduli spaces of translation surfaces is an important part of the study of the geometry and the dynamical properties of the translation surface itself (see for instance the results by Eskin, Masur and Veech on the ergodicity of this flow and on the Siegel–Veech constants). One important feature is that moduli spaces admit an $\mathrm{SL}_2(\mathbb{R})$ –action that is intimately related to the directional flow on the surface.

The authors, A. Boulanger and S. Ghazouani, proved a dichotomy for this action and deduce some properties for the flow on dilation tori. For translation surfaces, the corresponding results were proved by Masur in a celebrated paper in 1992 (it is a classical theorem of Weyl for linear foliations of the torus). This was later sharpened and extended by Cheung (2004), Cheung–Eskin (2006) and Cheung–Masur (2006). Note that there exists a combinatorial version of Masur’s criterion established by Boshernitzan (1985) for interval exchange transformations. More recently Treviño extended the criterion to infinite translation surfaces and Bratteli diagrams (2016).

There is no doubt that the results are original and interesting, and deserve to be published on *Israel Journal of Mathematics*. The authors use the following strategy:

- (1) Dilation structures can be obtained in gluing sides of pentagons in the plane.
- (2) An explicit parametrisation of the space of pentagons allows them to get a nice parametrisation of the moduli space with log-dilation parameters, where the $\mathrm{SL}_2(\mathbb{R})$ –action is easy to understand.
- (3) A classical trick permits to shift the question of density of $\mathrm{SL}_2(\mathbb{R})$ –orbit on moduli spaces to $\mathrm{SL}_2(\mathbb{Z})$ –action on \mathbb{R}_2^+ : this leads to a proof of the dichotomy.
- (4) For the orbits of the Teichmüller flow, they appeal to the coding given by the Rauzy–Veech induction. This step is subtle since sometimes the algorithm stops in finite time.

2. OPINION

This paper is an interesting addition to the expanding literature on flows on moduli spaces, and, more generally, on dilation surfaces. One has to say that the theory of interval exchange transformations is well-developed, but in the case of affine interval exchange transformations (or dilation surfaces), many basic questions remain unanswered. One difficulty is that the associated moduli spaces are not finite volume.

The paper is generally clear and it is mathematically correct (I did followed all particular cases and all the technical details, and it seems to me completely convincing). Our main criticism is about the exposition, besides the (large number

of) misprints listed below. The scheme of the proof should appear in the introduction, which will be very helpful in understanding all the strategy. Also it would be helpful to explain what are the difficulties for surfaces of genus $g > 1$, and for non polygonable dilation surfaces.

In conclusion, despite the expository shortcomings outlined above, the paper represents an interesting contribution to the study of dilation surfaces. The authors bring in the technique of Rauzy–Veech induction into this topic, and continue its study (introduced by Fougeron and the authors). For this reason **we strongly recommend acceptance** on *Israel Journal of Mathematics* after the minor issues below are addressed.

3. REMARKS

Here is a list of major remarks and misprints I have noticed during the readings, page by page. I used the following convention: If a line or a paragraph number is negative it should be counted from the bottom of the page (I do not count blank lines).

- page 2, Theorem 1.1: Linear holonomy is not defined. Also replace “does not generates a discrete subgroup” by “linear holonomy group is dense”.
- page 2, line after Theorem 1.1: What is the difference between “We also prove that all trajectories of the Teichmüller flow escape to infinity” and the last statement of the theorem?
- page 3, line 9: Moreover, any direction, “in” is missing.
- page 3, line 11: The foliation \mathcal{F}_θ is *not* well-defined. Indeed you must define what is the horizontal direction on the surface (you only defined a dilation structure).
- page 3, line 14: Why the complex structure is well-defined at S ?
- page 3, line 17: Usually “trivial” means identity map. Could you be more precise?
- page 3, line 26: can be characterized.
- page 3, line -12: equal to the set the singularities, “of” is missing.
- page 4, -4: these data.
- page 5, line 4: What is a dilation automorphism isotopic to the identity map?
- page 5, line 23: $\mathcal{D}(g, n, \partial)$: “; ρ ” is missing in the (). Same remark line 33.
- page 5, line 26: this is confusing: we define the action of $\mathrm{SL}_2(\mathbb{R})$ and then use the action of $\mathrm{GL}_2^+(\mathbb{R})$ to conclude that there is a well-defined action of $\mathrm{SL}_2(\mathbb{R})$
- page 5, line 29: of the set (remove s).
- page 5, line -4: please quote Veech’s work on zippered rectangle’s construction.
- page 6: in the caption of Figure 2: up to a some dilation: remove “some”. Also remove the “.” after one another.

- page 6, line 1: $\mathcal{D}(1, 1, 1)$: please use the correct definitions: Either $\mathcal{D}(1, 1, 1; *)$, or \mathcal{D} . Same remark few lines below.
- page 6, Lemma 3.1: Is it true that every dilation structure can be represented by a *convex* polygon in the plane? Or is this specific to the tori? Could the authors be more precise and explain what are the different properties in genus 1 and higher genera?
- page 6, line -3: by definition \mathfrak{D} . Remove \mathfrak{D} .
- page 7, line 7: “one it then left”?
- page 7, line 10: one of sides. Add “its” before sides.
- page 7, line 18: up to dilation: do you mean up to dilation and rotation?
- page 8, line 5: I do not understand why you require that not both $\nu_1 < 1$ and $\nu_2 < 1$ hold simultaneously. For instance in Figure 5, if you rotate the figure by π then you get a room with $\nu'_i = \nu_i^{-1} > 1$. Probably the misunderstanding is in the definition of room and the boundary component. Could the authors be more precise on that point?
- page 8, line 4: “pentagonal” replace to “pentagon”.
- page 8, line 13 (Global): Sometimes you write Figure, figure, figure 5, Figure 5, etc. Please be consistent! Usually one uses “Figure 5” and “figure”. The same applies for lemmas, theorems...
- page 8, line -14: what is a “natural” map? Is this different from an “equivariant” map? If yes, please give a definition. If no, please remove this terminology.
- page 8, line -2: typo $(\mathfrak{B}, (\mu_1, \mu_2))$. One line below: $\tilde{p}(A \cdot (\mathfrak{B}, (\mu_1, \mu_2)))$
- page 9: Figure 6: Could you please add the vectors e_i and e'_i ?
- page 9, line -17: I guess Figure 7 shows how to get the *last* formula (third and fourth). Could you check?
- page 10: The authors spent some efforts to define log-dilation parameters on \mathcal{Q} and eventually they restrict the discussion to \mathbb{R}_2^+ . Could the authors define log-dilation parameters on \mathbb{R}_2^+ directly? (since the map \mathcal{P} is defined with \mathbb{R}_2^+).
- page 11, line -18: replace “place” by “plane”.
- page 11, line -10: typo: Gauss’s algorithm (same remark few lines later).
- page 12, line 20: The “door” is not defined! Why the set of direction pointing inwards is $] - \pi/2, \pi/2[$? (and not $[-\pi/2, \pi/2]$).
- page 12, line -14, a curve cannot be transverse to an angle.
- page 13, line 3: values in $[0, 1]$. You allow 0 and 1? This seems to contradict Definition 4.1.
- page 13, line 4: “reducing the problem to ...”. Which problem? You should rename this subsection e.g. “from directional foliations to (ρ_A, ρ_B) -maps. Also this paragraph is not very clear. Could the authors do a better job in clarifying this?

- page 13, line 11: “of countably many families”. Why countably and not finite, since \mathbb{S}^1 is compact?
- page 13, line -7: “not strictly speaking the standard Rauzy induction”. Why it is not the same induction? Be more precise please.
- page 13, Figure 11 (caption): type $\mathcal{R}(A')$ replace by $\mathcal{R}(T)(A')$.
- page 15, lines 3&4: A is winner two times.
- page 16, line 1: typo “set in made”.
- page 16, line 3: typo “therein”.
- page 16, line 6: typo “Then main difference”.
- page 16, line -7: typo “we will some”.
- page 16: I propose to make this section more readable. You first formulate a general statement including Proposition 4.4 and Theorem 4.2. You then give a proof when $\rho_A > 1$ and $\rho_B < 1$. Eventually you quote the work [1].
- page 17, Section 4.4: there is one missing reference.
- page 17, Theorem 5.1: You changed the notation from page 6 (where elements of \mathcal{D} were denoted by \mathfrak{D}). Please be consistent. Also could you recall the definition of “diverges”?
- page 17, proof of the theorem. (T_n) is a subsequence of which sequence? (same question in the statement of Proposition 5.3). Also the function $\Theta(\cdot)$ is not defined.
- page 17, Proposition 5.3. $\rho(C_n)$ is not defined.
- page 18, line 6: I guess you want to write $\mathcal{F}_\theta(T) = \mathcal{F}_{A\theta}(A \cdot T)$?
- page 18, line 10: typo remove one “)”.
- page 18, Theorem 5.5: Teichmuller, replace u by ü
- page 18, line -6: typo ρ_A, ρ_B .
- page 19, typos (caption of Figure 12), and line 2.
- page 20, Proposition 5.6: The proposition claims that $\rho_A^n \rightarrow 1$ but $\rho_A > 1$. Could you explain?
- page 20, line -19: typo Replace θ_2 by θ_2 .
- page 20, line after the title of Section 6: Remove “problems and”.
- page 21, Conjecture 1: implicitly you are saying that \mathcal{T} is a connected subset of the ambient stratum. Is this clear? Could you explain?
- page 21, Genus 2 surfaces. I don’t understand the definition of \mathcal{R} . I guess it corresponds exactly to the moduli space of dilation surfaces with 6π conical angle. Why is this different? Probably I misunderstand.
- page 21, typo line -23 (componentS) line -20 (corollarIES) line -17 (I don’t understand the sentence - there is no verb).