# Hilbert-Samuel formula over adelic curves

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# HILBERT-SAMUEL FORMULA OVER ADELIC CURVES

*by*

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***Abstract.*** — We establish, in the setting of Arakelov geometry adelic curves, an arithmetic Hilbert-Samuel theorem describing the asymptotic behaviour of the metrized graded linear series of an adelic line bundle in terms of its arithmetic intersection number.

## 1. Introduction

In algebraic geometry, Hilbert function measures the growth of graded linear series of a line bundle on a projective variety. Let $k$ be a field, $X$ be an integral projective scheme of degree $d \in \mathbb{N}$ over $\operatorname{Spec} k$, and $L$ be an invertible $\mathcal{O}_X$-module. The Hilbert function of $L$ is defined as

$$H_L : \mathbb{N} \longrightarrow \mathbb{N}, \quad H_L(n) := \dim_k(H^0(X, L^{\otimes n})).$$

If $L$ is ample, then the following asymptotic estimates holds:

$$H_L(n) = \frac{(L^d)}{d!} n^d + o(n^d). \tag{1.1}$$

This formula, which relates the asymptotic behaviour of the Hilbert function and the auto-intersection number of $L$, is for example a consequence of Hirzebruch-Riemann-Roch theorem and Serre's vanishing theorem. It turns out that the construction and the asymptotic estimate of Hilbert function have analogue in various context, such as graded algebra, local multiplicity, relative volume of two metrics, etc.

In Arakelov geometry, an arithmetic analogue of Hilbert function has been introduced by Gillet and Soulé **[18]** and an analogue of the asymptotic formula (1.1) has been deduced from their arithmetic Riemann-Roch theorem. This result is called an arithmetic Hilbert-Samuel theorem. Let $\mathscr{X}$ be a regular integral projective scheme of dimension $d+1$ over $\operatorname{Spec}\mathbb{Z}$, and $\overline{\mathscr{L}} = (\mathscr{L}, \varphi)$ be a Hermitian line bundle on $\mathscr{X}$, namely an invertible $\mathcal{O}_{\mathscr{X}}$-module $\mathscr{L}$ equipped with a smooth metric $\varphi$ on $\mathscr{L}(\mathbb{C})$. For any integer $n \in \mathbb{N}$, we let $\|\cdot\|_{n\varphi}$ be the norm on the real vector space $H^0(\mathscr{X}, \mathscr{L}) \otimes_{\mathbb{Z}} \mathbb{R}$ defined as follows

$$\forall\, s \in H^0(\mathscr{X}, \mathscr{L}) \otimes_{\mathbb{Z}} \mathbb{R} \subseteq H^0(\mathscr{X}_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}}^{\otimes n}), \quad \|s\|_{n\varphi} = \sup_{x \in \mathscr{X}(\mathbb{C})} |s|_{n\varphi}(x).$$

Then the couple $(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi})$ forms a lattice in a normed vector space. Recall that its arithmetic Euler-Poincaré characteristic is

$$\chi(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi}) = \ln \frac{\operatorname{vol}(\{s \in H^0(\mathscr{X}, \mathscr{L}^{\otimes n}) \otimes_{\mathbb{Z}} \mathbb{R} \,:\, \|s\|_{n\varphi} \leqslant 1\})}{\operatorname{covol}(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi})}$$

where $\operatorname{vol}(\cdot)$ denotes a Haar measure on the real vector space $H^0(\mathscr{X}, \mathscr{L}) \otimes_{\mathbb{Z}} \mathbb{R}$, and $\operatorname{covol}(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi})$ denotes the covolume of the lattice $H^0(\mathscr{X}, \mathscr{L}^{\otimes n})$ with respect to the Haar measure $\operatorname{vol}(\cdot)$, that is, the volume of any fundamental domain of this lattice. In this setting the arithmetic Hilbert-Samuel theorem shows that, in the case where $\mathscr{L}$ is relatively ample and the metric $\varphi$ is positive, the sequence

$$\frac{\chi(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi})}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N},\ n \geqslant 1$$

converges to the arithmetic intersection number $(\overline{\mathscr{L}}^{d+1})$. In the case where $\overline{\mathscr{L}}$ is ample, the arithmetic Hilbert-Samuel theorem also permits to relate the asymptotic behaviour (when $n \to +\infty$) of

$$\operatorname{card}(\{s \in H^0(\mathscr{X}, \mathscr{L}^{\otimes n}) \,:\, \|s\|_{n\varphi} \leqslant 1\})$$

to the arithmetic intersection number of $\overline{\mathscr{L}}$. These results have various applications in arithmetic geometry, such as Vojta's proof of Mordell conjecture, equidistribution problem and Bogomolov conjecture, etc. The arithmetic Hilbert-Samuel theorem has then been reproved in various setting and also been generalized in works such as **[1, 16, 21]**.

Recently, a new framework of Arakelov geometry has been proposed in **[13]**, which allows to consider arithmetic geometry over any countable field. Let $K$ be a field. A structure of proper adelic curve with underlying field $K$ is given by a family of absolute values $(|\cdot|_\omega)_{\omega\in\Omega}$ on $K$ parametrized by a measure space $(\Omega, \mathcal{A}, \nu)$, which satisfies a product formula of the form

$$\forall\, a \in K^\times, \quad \int_\Omega \ln|a|_\omega \,\nu(\mathrm{d}\omega) = 0.$$

We assume that, either $K$ is countable, or the $\sigma$-algebra $\mathcal{A}$ is discrete. The geometry of numbers and the arithmetic intersection theory in the setting of adelic curves have been developed respectively in the works **[13]** and **[15]**. Note that in general it is not possible to consider global integral models of an adelic curve. Several classic notions and constructions, such as integral lattice and its covolume, do not have adequate analogue over adelic curves. It turns out that a modified and generalized form of normed lattice — adelic vector bundle — has a natural avatar in the setting of adelic curves. An adelic vector bundle consists of a finite-dimensional vector space $V$ over $K$ equipped with a family of norms $(\|\cdot\|_\omega)_{\omega\in\Omega}$ on vector spaces $V_\omega = V\otimes_K K_\omega$ (where $K_\omega$ denotes the completion of $K$ with respect to the absolute value $|\cdot|_\omega$), which satisfy dominancy and measurability conditions. The Arakelov degree of the adelic vector bundle

$$\overline{V} = (V, (\|\cdot\|_\omega)_{\omega\in\Omega})$$

is then defined as

$$\widehat{\deg}(\overline{V}) := -\int_\Omega \ln\|s_1\wedge\cdots\wedge s_r\|_\omega \,\nu(\mathrm{d}\omega),$$

where $(s_i)_{i=1}^r$ is an arbitrary basis of $E$ over $K$. This notion is a good candidate to replace the Euler-Poincaré characteristic.

Let $\pi : X \to \operatorname{Spec} K$ be a projective scheme over $\operatorname{Spec} K$. For any $\omega\in\Omega$, let $X_\omega = X\times_{\operatorname{Spec} K}\operatorname{Spec} K_\omega$ and let $X_\omega^{\mathrm{an}}$ be the analytic variety associated with $X_\omega$ (in the sense of Berkovich if $|\cdot|_\omega$ is non-Archimedean). If $E$ is a vector bundle on $X$, namely a locally free $\mathcal{O}_X$-module of finite rank, we denote by $E_\omega$ the pull-back of $E$ on $X_\omega$. As *adelic vector bundle* on $X$, we refer to the data $\overline{E} = (E, (\psi_\omega)_{\omega\in\Omega})$ consisting of a vector bundle $E$ on $X$ and a family $(\psi_\omega)_{\omega\in\Omega}$ of continuous metrics on $E_\omega$, $\omega\in\Omega$, which satisfy dominancy and measurability conditions. It turns out that, if $X$ is geometrically reduced, then the vector space of global sections $H^0(X,E)$ equipped with supremum norms $(\|\cdot\|_{\psi_\omega})_{\omega\in\Omega}$ forms an adelic vector bundle $\pi_*(\overline{E})$ on the base adelic curve.

Let $\pi : X\to\operatorname{Spec} K$ be a geometrically integral projective scheme of dimension $d\geqslant 0$ over $\operatorname{Spec} K$ and $\overline{L} = (L,\varphi)$ be an adelic line bundle on $X$, that is, an adelic vector bundle of rank 1 on $X$. Assume that the line bundle $L$ is ample. We introduce

the notion of $\chi$-volume as

$$\widehat{\mathrm{vol}}_\chi(\overline{L}) = \limsup_{n\to+\infty} \frac{\widehat{\deg}(\pi_*(\overline{L}^{\otimes n}))}{n^{d+1}/(d+1)!}.$$

In view of the similarity between Arakelov degree and Euler-Poincaré characteristic of Euclidean lattices, the notion of $\chi$-volume is analogous to that of sectional capacity introduced in [**22**]. Moreover, similarly to the number field case, we show in Theorem-Definition 5.5 that the above superior limit defining the $\chi$-volume is actually a limite. However, from the methodological view, we do not follow the classic approaches, which are difficultly implantable in the adelic curve setting. Our strategy consists in casting the Arakelov geometry over an adelic curve to that in the particular case where the adelic curve contains a single copy of the trivial absolute value on $K$, that is, the absolute value $|\cdot|_0$ such that $|a|_0 = 1$ for any $a \in K \setminus \{0\}$. More precisely, to each adelic vector bundle $\overline{V} = (V, (\|\cdot\|_\omega)_{\omega\in\Omega})$, we associate an ultrametric norm $\|\cdot\|_0$ on $V$ (where we consider the trivial absolute value $|\cdot|_0$) via Harder-Narasimhan theory in the form of $\mathbb{R}$-filtrations, such that

$$\big|\widehat{\deg}(V, (\|\cdot\|_\omega)_{\omega\in\Omega}) - \widehat{\deg}(V, \|\cdot\|_0)\big| \leqslant \frac{1}{2}\nu(\Omega_\infty)\dim_K(V)\ln(\dim_K(V)),$$

where $\Omega_\infty$ denotes the set of $\omega \in \Omega$ such that $|\cdot|_\omega$ is Archimedean. Then the convergence of the suite defining $\widehat{\mathrm{vol}}_\chi(\overline{L})$ follows from a limite theorem of normed graded linear series as follows (see Theorem 4.4 and Corollary 4.5 for this result in a more general form and for more details):

***Theorem 1.1.*** — *Assume that the graded $K$-algebra $\bigoplus_{n\in\mathbb{N}} H^0(X, L^{\otimes n})$ is of finite type. For any integer $n \geqslant 1$, let $\|\cdot\|_n$ be a norm on $H^0(X, L^{\otimes n})$ (where we consider the trivial absolute value on $K$). Assume that*

(a) $\inf_{s\in V_n\setminus\{0\}} \ln\|s\|_n = O(n)$ *when* $n \to +\infty$,
(b) *for any* $(n, m) \in \mathbb{N}^2_{\geqslant 1}$ *and any* $(s_n, s_m) \in V_n \times V_m$, *one has*

$$\|s_n \cdot s_m\|_{n+m} \leqslant \|s_n\|_n \cdot \|s_m\|_m.$$

*Then the sequence*

$$\frac{\widehat{\deg}(V_n, \|\cdot\|_n)}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N},\ n \geqslant 1$$

*converges in* $\mathbb{R}$.

In view of the classic Hilbert-Samuel theorems in algebraic geometry and in Arakelov geometry, it is natural to compare the $\chi$-volume to the arithmetic intersection number of adelic line bundles that we have introduced in [**15**] (see also the work [**19**] on heights of varieties over $M$-fields). Let $\pi : X \to \operatorname{Spec} K$ be a projective scheme of dimension $d \geqslant 0$ over $K$ and $\overline{L} = (L, \varphi)$ be an adelic line bundle on $X$

such that $L$ is ample and the metrics in the family $\varphi$ are semi-positive. Then the arithmetic self-intersection number of $\overline{L}$ is written in a recursive way as

$$(\overline{L}^{d+1}) = \frac{1}{N}\left[(\overline{L}|^d_{\mathrm{div}(s)}) - \int_\Omega \int_{X_\omega^{\mathrm{an}}} \ln|s|_{\varphi_\omega}(x)\, c_1(L_\omega, \varphi_\omega)^d(\mathrm{d}x)\, \nu(\mathrm{d}\omega)\right], \tag{1.2}$$

where $N$ is a positive integer, and $s$ is a global section of $L^{\otimes N}$ which intersect properly with all irreducible components of the projective scheme $X$. The main result of the article is then the following theorem (see Theorem 10.1):

***Theorem 1.2.*** — *Let $X$ be a geometrically integral projective scheme over* $\operatorname{Spec} K$ *and $\overline{L} = (L, \varphi)$ be an adelic line bundle on $X$. Assume that $L$ is ample and all metrics in the family $\varphi$ are semi-positive, then the following equality holds:*

$$\widehat{\mathrm{vol}}_\chi(\overline{L}) = (\overline{L}^{d+1}).$$

Note that in the literature there exists a local version of the Hilbert-Samuel theorem which establishes an equality between the relative volume of two metrics and the relative Monge-Ampère energy between them. We refer the readers to **[3]** for the Archimedean case and to **[8, 6]** for the non-Archimedean case (see also **[7]**). These results show that, for a fixed ample line bundle $L$ on $X$, the difference between $\widehat{\mathrm{vol}}_\chi(\overline{L})$ and $(\overline{L}^{d+1})$ does not depend on the choice of the metric family on $L$ (see Proposition 7.4 and Remark 7.6). Moreover, by an argument of projection to a projective space (on which the arithmetic Hilbert-Samuel theorem can be proved by explicit computation, see Proposition 7.11), one can show that the inequality $\widehat{\mathrm{vol}}_\chi(\overline{L}) \geqslant (\overline{L}^{d+1})$ holds (see Step 2 of the proof of Theorem 10.1).

In view of the recursive formula (1.2) defining the self-intersection number, a natural idea to prove the above theorem could be an argument of induction, following the approach of **[1]** by using an adaptation to non-Archimedean setting of some technics in complex analytic geometry developed in **[6, 17]**. However, it seems that a refinement in the form of an asymptotic development of the function defining the local relative volume is needed to realize this strategy. Unfortunately such refinement is not yet available. Our approach consists in casting the arithmetic data of $\overline{L}$ to a series of metrics over a trivially valued field. This could be considered as a higher-dimensional generalization of the approach of Harder-Narasimhan $\mathbb{R}$-filtration mentioned above. What is particular in the trivial valuation case is that the local geometry becomes automatically global, thanks to the trivial "product formula". In this case, the arithmetic Hilbert-Samuel theorem follows from the equality between the relative volume and the relative Monge-Ampère energy with respect to the trivial metric (see Theorem 8.2). Note that this result also shows that, in the case of a projective curve over a trivially valued field, the arithmetic intersection number defined in **[15]** coincides with that constructed in a combinatoric way in **[14]** (see Remark 8.3). The comparison of divers invariants of $\overline{L}$ with respect to its casting to the trivial valuation case provides the opposite inequality $\widehat{\mathrm{vol}}_\chi(\overline{L}) \leqslant (\overline{L}^{d+1})$.

As a sequel to the above arguments in terms of trivially valued fields, our way towards the arithmetic Hilbert-Samuel theorem over a adelic curve gives a new approach even for the classical case.

As an application, we prove the following higher dimensional generalization of Hodge index theorem (see Corollaries 10.3 and 10.4).

***Theorem 1.3.*** — *Let $X$ be a geometrically integral projective scheme of dimension $d \geqslant 0$ over $\operatorname{Spec} K$ and $\overline{L} = (L, \varphi)$ be an adelic line bundle on $X$. Assume that $L$ is nef and all metrics in the family $\varphi$ are semi-positive, then the inequality $\widehat{\operatorname{vol}}(\overline{L}) \geqslant (\overline{L}^{d+1})$ holds. In particular, if $(\overline{L}^{d+1}) > 0$, then the line bundle $L$ is big.*

Theorem 1.2 naturally leads to the following refinement of the arithmetic Hilbert-Samuel theorem, in introducing a tensor product by an adelic vector bundle on $X$ (see Corollary 10.5).

***Theorem 1.4.*** — *Let $X$ be a geometrically integral projective scheme over $\operatorname{Spec} K$, $d$ be the dimension of $X$, $\overline{L} = (L, \varphi)$ be an adelic line bundle on $X$ and $\overline{E} = (E, \psi)$ be an adelic vector bundle on $X$. Assume that $L$ is ample and the metrics in $\varphi$ are semi-positive. Moreover we suppose that either $\operatorname{rk}(E) = 1$ or $X$ is normal. Then one has*

$$\lim_{n \to +\infty} \frac{\widehat{\deg}(H^0(X, L^{\otimes n} \otimes E), (\|\cdot\|_{n\varphi_\omega + \psi_\omega})_{\omega \in \Omega})}{n^{d+1}/(d+1)!} = \operatorname{rk}(E)(\overline{L}^{d+1}).$$

The rest of the article is organized as follows. In the second section we introduce the notation that we use all through the article. In the third second, we consider metric families on vector bundles and discuss their dominancy and measurability. In the fourth section, we study normed graded linear series over a trivially valued field and prove the limite theorem of their volumes. Then in the fifth section we deduce the limite theorem for graded algebra of adelic vector bundles over a general adelic curve, which proves in particular that the sequence defining the arithmetic volume function actually converges. in the sixth section we show that the arithmetic Hilbert-Samuel theorem in the original form implies the generalized form with tensor product by an adelic vector bundle. In the seventh section, we prove that the difference of the $\chi$-volume and the arithmetic intersection product does not depend on the choice of the metric family. In the eighth section, we prove the main theorem in the particular case where the adelic curve contains a single copy of the trivial absolute value. In the ninth section, we explain the method of casting to the trivial valuation case. Finally, in the tenth and last section, we prove the main theorem and deduce the generalized Hodge index theorem.

## 2. Notation and preliminaries

**2.1.** — Throughout the article, we fix a proper adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$, where $K$ is a commutative field, $(\Omega, \mathcal{A}, \nu)$ is a measure space and $\phi = (|\cdot|_\omega)_{\omega\in\Omega}$ is a family of absolute values on $K$ parametrized by $\Omega$, such that, for any $a \in K^\times$, $(\omega \in \Omega) \mapsto \ln|a|_\omega$ is integrable on $(\Omega, \mathcal{A}, \nu)$, and the following "product formula" holds:

$$\forall\, a \in K, \quad \int_\Omega \ln|a|_\omega \,\nu(\mathrm{d}\omega) = 0.$$

For any $\omega \in \Omega$, we denote by $K_\omega$ the completion of $K$ with respect to the absolute value $|\cdot|_\omega$.

We assume that, either the $\sigma$-algebra $\mathcal{A}$ is discrete, or the field $K$ is countable. Moreover, we denote by $\Omega_\infty$ the set of $\omega \in \Omega$ such that $|\cdot|_\omega$ is Archimedean. Note that $\nu(\Omega_\infty) < +\infty$.

Moreover, for $\omega \in \Omega_\infty$, we always assume that $|a|_\omega = a$ for any $a \in \mathbb{Q}_{\geqslant 0}$.

**2.2.** — Let $V$ be a finite-dimensional vector space over $K$. As *norm family* on $V$, we refer to a family $(\|\cdot\|_\omega)_{\omega\in\Omega}$, where $\|\cdot\|_\omega$ is a norm on $V_\omega := V \otimes_K K_\omega$.

Let $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ and $\xi' = (\|\cdot\|'_\omega)_{\omega\in\Omega}$ be norm families on $V$. For any $\omega \in \Omega$, we denote by $d_\omega(\xi, \xi')$ the following number

$$\sup_{s\in V\setminus\{0\}} \Big|\ln\|s\|_\omega - \ln\|s\|'_\omega\Big|.$$

In the case where $V = \mathbf{0}$, by convention $d_\omega(\xi, \xi') = 0$.

**2.3.** — As *adelic vector bundle* on $S$, we refer to the data $\overline{V} = (V, \xi)$ which consists of a finite-dimensional vector space $V$ over $K$ and a family of norms $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ on $V_\omega := V \otimes_K K_\omega$, satisfying the following conditions:

(1) the norm family $\xi$ is strongly dominated, that is, there exist an integrable function $C : \Omega \to \mathbb{R}_{\geqslant 0}$ and a basis $(e_i)_{i=1}^r$ of $V$ over $K$, such that, for any $\omega \in \Omega$ and any $(\lambda_1, \ldots, \lambda_r) \in K_\omega^r \setminus \{(0, \ldots, 0)\}$,

$$\Big|\ln\|\lambda_1 e_1 + \cdots + \lambda_r e_r\|_\omega - \ln \max_{i\in\{1,\ldots,r\}} |\lambda_i|_\omega\Big| \leqslant C(\omega).$$

(2) the norm family $\xi$ is *measurable*, that is, for any $s \in V$, the function $(\omega \in \Omega) \mapsto \|s\|_\omega$ is $\mathcal{A}$-measurable.

In the article, we only consider adelic vector bundles which are ultrametric over non-Archimedean places, namely we assume that the norm $\|\cdot\|_\omega$ is ultrametric once the absolute value $|\cdot|_\omega$ is non-Archimedean. If in addition the norm $\|\cdot\|_\omega$ is induced by an inner product whenever $|\cdot|_\omega$ is Archimedean, we say that $\overline{V}$ is *Hermitian*. If $\dim_K(V) = 1$, we say that $\overline{V}$ is an *adelic line bundle* (note that an adelic line bundle is necessarily Hermitian).

If $\overline{V}$ is an adelic vector bundle on $S$, any vector subspace (resp. quotient vector space) of $V$ together with the family of restricted norms (resp. quotient norms) forms also an adelic vector bundle on $S$, which is called an *adelic vector subbundle* (resp.

*quotient adelic vector bundle*) of $\overline{V}$. Note that if $\overline{V}$ is Hermitian, then all its adelic vector subbundles and quotient adelic vector bundles are Hermitian.

**2.4.** — Let $\overline{V} = (V, (\|\cdot\|_\omega)_{\omega\in\Omega})$ be an adelic vector bundle on $S$, we define the *Arakelov degree* of $\overline{V}$ as

$$\widehat{\deg}(\overline{V}) := -\int_\Omega \ln \|e_1 \wedge \cdots \wedge e_r\|_{\omega,\det}\, \nu(\mathrm{d}\omega),$$

where $(e_i)_{i=1}^r$ is a basis of $V$ over $K$, and $\|\cdot\|_{\omega,\det}$ denotes the determinant norm of $\|\cdot\|_\omega$, which is defined as (where $r = \dim_K(V)$)

$$\forall\, \eta \in \det(V) = \Lambda^r(V), \quad \|\eta\|_{\omega,\det} = \inf_{\eta = s_1\wedge\cdots\wedge s_r} \|s_1\| \cdots \|s_r\|.$$

Let $\widehat{\deg}_+(\overline{V})$ be the *positive degree* of $\overline{V}$, which is defined as

$$\widehat{\deg}_+(\overline{V}) = \sup_{W\subseteq V} \widehat{\deg}(\overline{W}),$$

where $W$ runs over the set of vector subspaces of $V$, and in the adelic vector bundle structure of $\overline{W}$ we consider the restricted norms. In the case where $V$ is non-zero, we denote by $\widehat{\mu}(\overline{V})$ the quotient $\widehat{\deg}(\overline{V})/\dim_K(V)$, called the *slope* of $V$. We define the *minimal slope* of $\overline{V}$ as

$$\widehat{\mu}_{\min}(\overline{V}) := \inf_{V\twoheadrightarrow W\neq\{0\}} \widehat{\mu}(\overline{W}),$$

where $\overline{W}$ runs over the set of all non-zero quotient adelic vector bundles of $\overline{V}$.

**2.5.** — Let $\overline{V}$ be a non-zero adelic vector bundle on $S$. For any $t \in \mathbb{R}$, we let

$$\mathcal{F}^t(\overline{V}) = \sum_{\substack{\{0\}\neq W\subseteq V\\ \widehat{\mu}_{\min}(\overline{W})\geqslant t}} W,$$

where $W$ runs over the set of all non-zero vector subspaces of $V$ such that the minimal slope of $W$ equipped with the family of restricted norms is $\geqslant t$. We call $(\mathcal{F}^t(\overline{V}))_{t\in\mathbb{R}}$ the *Harder-Narasimhan $\mathbb{R}$-filtration* of $\overline{V}$. In the case where $\overline{V}$ is Hermitian, the following equality holds (see [**13**, Theorem 4.3.44]):

$$\widehat{\deg}(\overline{V}) = -\int_{\mathbb{R}} t\, \mathrm{d}(\dim_K(\mathcal{F}^t(\overline{V}))),$$

$$\widehat{\deg}_+(\overline{V}) = -\int_0^{+\infty} t\, \mathrm{d}(\dim_K(\mathcal{F}^t(\overline{V}))) = \int_0^{+\infty} \dim_K(\mathcal{F}^t(\overline{V}))\, \mathrm{d}t.$$

In general one has (see [**13**, Propositions 4.3.50 and 4.3.51, and Corollary 4.3.52])

$$0 \leqslant \widehat{\deg}(\overline{V}) + \int_{\mathbb{R}} t\, \mathrm{d}(\dim_K(\mathcal{F}^t(\overline{V}))) \leqslant \frac{1}{2}\nu(\Omega_\infty)\dim_K(V)\ln(\dim_K(V)),$$

$$0 \leqslant \widehat{\deg}_+(\overline{V}) - \int_0^{+\infty} \dim_K(\mathcal{F}^t(\overline{V}))\, \mathrm{d}t \leqslant \frac{1}{2}\nu(\Omega_\infty)\dim_K(V)\ln(\dim_K(V)).$$

**2.6.** — Let $\overline{V}=(V,(\|\cdot\|_{V,\omega})_{\omega\in\Omega})$ and $\overline{W}=(W,(\|\cdot\|_{W,\omega})_{\omega\in\Omega})$ be adelic vector bundles on $S$. For any $\omega\in\Omega$ such that $|\cdot|_\omega$ is non-Archimedean, let $\|\cdot\|_\omega$ be the $\varepsilon$-tensor product on $V_\omega\otimes_{K_\omega}W_\omega$, of the norms $\|\cdot\|_{V,\omega}$ and $\|\cdot\|_{W,\omega}$. Note that, for any $T\in V_\omega\otimes_{K_\omega}W_\omega$, one has

$$\|T\|_\omega=\min\Big\{\max_{i\in\{1,\dots,n\}}\|e_i\|_{V,\omega}\cdot\|f_i\|_{W,\omega}\ :\ \begin{array}{l}n\in\mathbb{N},\ (e_i)_{i=1}^n\in V_\omega^n,\ (f_i)_{i=1}^n\in W_\omega^n\\ T=e_1\otimes f_1+\cdots+e_n\otimes f_n\end{array}\Big\}.$$

In the case where $|\cdot|_\omega$ is Archimedean, let $\|\cdot\|_\omega$ be $\pi$-tensor product of $\|\cdot\|_{V,\omega}$ of $\|\cdot\|_{W,\omega}$. Recall that for any $T\in V_\omega\otimes_{K_\omega}W_\omega$, one has

$$\|T\|_\omega=\min\Big\{\sum_{i=1}^n\|e_i\|_{V,\omega}\cdot\|f_i\|_{W,\omega}\ :\ \begin{array}{l}n\in\mathbb{N},\ (e_i)_{i=1}^n\in V_\omega^n,\ (f_i)_{i=1}^n\in W_\omega^n\\ T=e_1\otimes f_1+\cdots+e_n\otimes f_n\end{array}\Big\}.$$

The pair

$$\overline{V}\otimes_{\varepsilon,\pi}\overline{W}=(V\otimes_K W,(\|\cdot\|_\omega)_{\omega\in\Omega})$$

is called the *$\varepsilon,\pi$-tensor product* of $\overline{V}$ and $\overline{W}$.

Assume that $\overline{V}$ and $\overline{W}$ are Hermitian. If $|\cdot|_\omega$ is non-Archimedean, let $\|\cdot\|_\omega^H$ be the $\varepsilon$-tensor product of $\|\cdot\|_{V,\omega}$ and $\|\cdot\|_{W,\omega}$; otherwise let $\|\cdot\|_\omega^H$ be the orthogonal tensor of the Euclidean or Hermitian norms $\|\cdot\|_{V,\omega}$ and $\|\cdot\|_{W,\omega}$. Then the pair

$$\overline{V}\otimes\overline{W}=(V\otimes_K W,(\|\cdot\|_\omega^H)_{\omega\in\Omega})$$

is called the *Hermitian tensor product* of $\overline{V}$ and $\overline{W}$.

**2.7.** — Let $(k,|\cdot|)$ be a field equipped with a complete absolute value, $X$ be a projective scheme over $\operatorname{Spec}k$. We denote by $X^{\mathrm{an}}$ the analytic space associated with $X$. Recall that a point $x$ of $X^{\mathrm{an}}$ is of the form $(j(x),|\cdot|_x)$, where $j(x)$ is a scheme point of $X$, $|\cdot|_x$ is an absolute value on the residue field of $j(x)$, which extends the absolute value $|\cdot|$ on the base field $k$. We denote by $\widehat{\kappa}(x)$ the completion of the residue field of $j(x)$ with respect to the absolute value $|\cdot|_x$, on which $|\cdot|_x$ extends by continuity.

## 3. Metric families on vector bundles

The purpose of this section is to generalize dominancy and measurability conditions in [**13**, Chapter 6] to metrized locally free modules, and to develop related topics. Let $S=(K,(\Omega,\mathcal{A},\nu),\phi)$ be an adelic curve as introduced in §2.1.

**3.1. Metric family.** — Let $p:X\to\operatorname{Spec}K$ be a quasi-projective scheme over $\operatorname{Spec}K$. Let $E$ be a vector bundle on $X$, that is, a locally free $\mathcal{O}_X$-module of finite rank. For any $\omega\in\Omega$, let $\psi_\omega$ be a metric on $E_\omega$ (see §2.7). By definition $\psi_\omega$ is a family $(|\cdot|_{\psi_\omega}(x))_{x\in X_\omega^{\mathrm{an}}}$ parametrized by $X_\omega^{\mathrm{an}}$, where each $|\cdot|_{\psi_\omega}$ is a norm on $E_\omega(x):=E_\omega\otimes_{\mathcal{O}_{X_\omega}}\widehat{\kappa}(x)$. We assume that the norm $|\cdot|_{\psi_\omega}(x)$ is ultrametric if the absolute value

$|\cdot|_\omega$ is non-Archimedean. Moreover, we assume that the metric $\psi_\omega$ is continuous, namely, for any section $s$ of $E$ over a Zariski open subset $U$ of $X_\omega$, the function

$$(x \in U^{\mathrm{an}}) \longmapsto |s|_{\psi_\omega}(x)$$

is continuous. The data $\psi = (\psi_\omega)_{\omega\in\Omega}$ is called a *metric family* on the vector bundle $E$.

Assume that $X$ is projective and geometrically reduced. For any $\omega \in \Omega$, we denote by $\|\cdot\|_{\psi_\omega}$ the supremum norm on $H^0(X_\omega, E_\omega)$, which is defined as

$$\forall\, s \in H^0(X_\omega, E_\omega), \quad \|s\|_{\psi_\omega} = \sup_{x\in X_\omega^{\mathrm{an}}} |s(x)|_{\psi_\omega}(x).$$

We denote by $p_*(E,\psi)$ the couple $(H^0(X,E), (\|\cdot\|_{\psi_\omega})_{\omega\in\Omega})$.

If $\varphi$ and $\psi$ are two metric families of $E$. For any $\omega \in \Omega$ we denote by $d_\omega(\varphi,\psi)$ the element

$$\sup_{x\in X_\omega^{\mathrm{an}}} \sup_{s\in E_\omega(x)\setminus\{0\}} \left| \ln|s|_{\varphi_\omega}(x) - \ln|s|_{\psi_\omega}(x)\right| \in [0,+\infty],$$

which is called the *local distance* at $\omega$ between $\varphi$ and $\psi$.

We denote by $\mathcal{O}_E(1)$ the universal invertible sheaf on the projective bundle $\pi : \mathbb{P}(E) \to \operatorname{Spec} K$. For any $\omega \in \Omega$, the metric $\psi_\omega$ induces by passing to quotient a continuous metric on $\mathcal{O}_E(1)_\omega \cong \mathcal{O}_{E_\omega}(1)$, which we denote by $\psi_\omega^{\mathrm{FS}}$. Recall that, if $y$ is an element of $\mathbb{P}(E_\omega)^{\mathrm{an}}$ and $x = \pi_\omega^{\mathrm{an}}(y)$, then the norm $|\cdot|_{\psi_\omega^{\mathrm{FS}}}$ on $\mathcal{O}_E(1)_y$ is the quotient metric induced by the universal surjective homomorphism

$$E_\omega(x) \otimes_{\widehat{\kappa}(x)} \widehat{\kappa}(y) \longrightarrow \mathcal{O}_E(1)_y,$$

where we consider the $\varepsilon$-extension of $|\cdot|_{\psi_\omega}(x)$ to $E_\omega(x) \otimes_{\widehat{\kappa}(x)} \widehat{\kappa}(y)$ if $|\cdot|_\omega$ is non-Archimedean, and $\pi$-extension of $|\cdot|_{\psi_\omega}(x)$ if $|\cdot|_\omega$ is Archimedean (see [**13**, §1.3 and §2.2.3]). Note that, if $\varphi$ and $\psi$ are two metric families of $E$, then one has (see [**13**, Proposition 2.2.20])

$$\forall\, \omega \in \Omega, \quad d_\omega(\varphi^{\mathrm{FS}}, \psi^{\mathrm{FS}}) \leqslant d_\omega(\varphi,\psi). \tag{3.1}$$

**3.2. Dominancy and measurability.** — In this subsection, we fix a projective scheme $X$ over $\operatorname{Spec} K$.

***Definition 3.1.*** — Let $E$ be a vector bundle on $X$.

(1) We say the metric family $\psi = (\psi_\omega)_{\omega\in\Omega}$ on the locally free $\mathcal{O}_X$-module $E$ is *dominated* (resp. *measurable*) if the metric family $\psi^{\mathrm{FS}} = (\psi_\omega^{\mathrm{FS}})_{\omega\in\Omega}$ on $\mathcal{O}_E(1)$ is dominated (resp. measurable). We refer the readers to [**13**, Definitions 6.1.9 and 6.1.27] for the dominancy and measurability conditions of metrized line bundles.

(2) We say $(E,\psi)$ is an *adelic locally free $\mathcal{O}_X$-module* or an *adelic vector bundle* if $\psi$ is dominated and measurable, or equivalently, $(\mathcal{O}_E(1), \psi^{\mathrm{FS}})$ is an adelic line bundle on $\mathbb{P}(E)$.

***Proposition 3.2.*** — (1) *If $\psi$ is dominated, then the norm family $\xi_\psi = (\|\cdot\|_{\psi_\omega})_{\omega\in\Omega}$ on $H^0(X,E)$ is strongly dominated.*
(2) *If the metric family $\psi$ on $E$ is measurable, then the norm family $\xi_\psi$ on $H^0(X,E)$ is measurable.*

*Proof.* — If we identify $H^0(X,E)$ with $H^0(\mathbb{P}(E),\mathcal{O}_E(1))$, then for any $\omega\in\Omega$ one has $\|\cdot\|_{\psi_\omega} = \|\cdot\|_{\psi_\omega^{\mathrm{FS}}}$ by [**13**, Remark 2.2.14]. Therefore the assertions follow from [**13**, Theorems 6.1.13 and 6.1.32]. □

***Proposition 3.3.*** — *Let $E$ be a vector bundle on $X$, and $\varphi$ and $\psi$ be two metric families of $E$. Suppose that $\varphi$ is dominated and that the local distance function*

$$(\omega\in\Omega)\longmapsto d_\omega(\varphi,\psi)$$

*is bounded from above by an integrable function. Then the metric family $\psi$ is also dominated.*

*Proof.* — This is a consequence of [**13**, Proposition 6.1.12] and (3.1). □

***Definition 3.4.*** — Let $f: Y\to X$ be a projective $K$-morphism from a geometrically reduced projective $K$-scheme $Y$ to $X$. Let $E$ be a vector bundle on $X$ and $\psi = (\psi_\omega)_{\omega\in\Omega}$ be a metric family on $E$. We denote by $f^*(\psi)$ the metric family on $E$ such that, for any $y\in Y_\omega^{\mathrm{an}}$, the norm $|\cdot|_{f^*(\psi)_\omega}(y)$ on

$$f^*(E)_\omega(y) = E_\omega(x)\otimes_{\widehat{\kappa}(x)}\widehat{\kappa}(y)$$

is induced by $|\cdot|_{\psi_\omega}(f^{\mathrm{an}}(y))$ by $\varepsilon$-extension of scalars if $|\cdot|_\omega$ is non-Archimedean, and by $\pi$-extension of scalars if $|\cdot|_\omega$ is Archimedean.

***Proposition 3.5.*** — *We keep the notation and the assumptions of Definition 3.4. Suppose that the metric family $\psi$ is dominated (resp. measurable), then its pull-back $f^*(\psi)$ is also dominated (resp. measurable).*

*Proof.* — The universal property of projective bundle induces a projective morphism $F:\mathbb{P}(f^*(E))\to\mathbb{P}(E)$ such that the following diagramme is cartesian.

$$\begin{array}{ccc}\mathbb{P}(f^*(E)) & \xrightarrow{F} & \mathbb{P}(E)\\ {\scriptstyle\pi_{f^*(E)}}\downarrow & & \downarrow{\scriptstyle\pi_E}\\ Y & \xrightarrow[f]{} & X\end{array}$$

Moreover, one has $\mathcal{O}_{f^*(E)}(1)\cong F^*(\mathcal{O}_E(1))$ and $F^*(\psi^{\mathrm{FS}}) = f^*(\psi)^{\mathrm{FS}}$. Hence the assertion follows from [**13**, Propositions 6.1.12 and 6.1.28]. □

***Definition 3.6.*** — Let $E$ be a vector bundle on $X$ and $\psi = (\psi_\omega)_{\omega\in\Omega}$ be a metric family of $E$. If $F$ is a vector subbundle of $E$, for any $\omega\in\Omega$ and any $x\in X_\omega^{\mathrm{an}}$, we denote by $|\cdot|_{\psi_{F,\omega}}(x)$ the restriction of $|\cdot|_{\psi_\omega}(x)$ to $F_\omega(x)$. Note that $\psi_F = (\psi_{F,\omega})_{\omega\in\Omega}$ forms a metric family of $F$, called the *restriction of $\psi$ to $F$*. Similarly, if $G$ is a

quotient vector bundle of $E$, we denote by $|\cdot|_{\psi_G,\omega}(x)$ the quotient norm of $|\cdot|_{\psi_\omega}(x)$ on $G_\omega(x)$. Then $\psi_G = (\psi_{G,\omega})_{\omega\in\Omega}$ is a metric family of $G$, called the *quotient metric family* of $\psi$ on $G$.

***Proposition 3.7.*** — *Let $E$ be a vector bundle on $X$ and $G$ be a quotient vector bundle of $E$. Let $\psi$ be a metric family on $E$. If $\psi$ is dominated (resp. measurable), then the quotient metric family $\psi_G$ is also dominated (resp. measurable).*

*Proof.* — Let $i : \mathbb{P}(G) \to \mathbb{P}(E)$ be the closed embedding induced by the quotient homomorphism $E \to G$. Then one has $i^*(\psi^{\mathrm{FS}}) = \psi_G^{\mathrm{FS}}$. Hence the assertion of the proposition follows from [**13**, Propositions 6.1.12 and 6.1.28]. □

***Definition 3.8.*** — Let $E$ and $F$ be vector bundles on $X$, equipped with metric families $\psi_E$ and $\psi_F$, respectively. For any $\omega \in \Omega$ and any $x \in X_\omega^{\mathrm{an}}$, if $|\cdot|_\omega$ is non-Archimedean, we denote by $|\cdot|_{(\psi_E\otimes\psi_F)_\omega}(x)$ the $\varepsilon$-tensor product of the norms $|\cdot|_{\psi_{E,\omega}}(x)$ and $|\cdot|_{\psi_{F,\omega}}(x)$, if $|\cdot|_\omega$ is Archimedean, we denote by $|\cdot|_{(\psi_E\otimes\psi_F)_\omega}(x)$ the $\pi$-tensor product of the norms $|\cdot|_{\psi_{E,\omega}}(x)$ and $|\cdot|_{\psi_{F,\omega}}(x)$. Thus we obtain a metric family $\psi_E \otimes \psi_F$ on the vector bundle $E \otimes F$, called the tensor product of metric families $\psi_E$ and $\psi_F$. In the case where one of the vector bundles $E$ and $F$ is of rank 1, we also write the tensor product metric family of $\psi_E$ and $\psi_F$ in an additive way as $\psi_E + \psi_F$.

***Proposition 3.9.*** — *Let $E$ and $F$ be vector bundles on $X$, equipped with metric families $\psi_E$ and $\psi_F$ respectively. We assume that $E$ is a line bundle. If both metric families $\psi_E$ and $\psi_F$ are dominated (resp. measurable), then $\psi_E+\psi_F$ is also dominated (resp. measurable).*

*Proof.* — Since $E$ is of rank 1, we can identify $\mathbb{P}(E \otimes F)$ with $\mathbb{P}(F)$. Moreover, if we denote by $\pi : \mathbb{P}(F) \to X$ the structural morphism, one has $\mathcal{O}_{E\otimes F}(1) = \pi^*(E)\otimes\mathcal{O}_F(1)$, and the metric family $(\psi_E + \psi_F)^{\mathrm{FS}}$ identifies with the tensor product of $\pi^*(\psi_E)$ and $\psi_F^{\mathrm{FS}}$. Hence the assertions follow from [**13**, Propositions 6.1.12 and 6.1.28]. □

***Proposition 3.10.*** — *Let $E$ be a vector bundle on $X$. Then there exists a dominated and measurable metric family of $E$.*

*Proof.* — Let $L$ be an ample line bundle on $X$ and $\varphi$ be a dominated and measurable metric family of $L^\vee$. Then, one can find a positive integer $m$ such that $L^m \otimes E$ is ample and generated by global sections. If $L^m \otimes E$ has a dominated and measurable metric family $\psi'$, then the tensor product of $m\varphi$ with $\psi'$ is a dominated and measurable metric family of $E$ by Proposition 3.9, so we may assume that $E$ is ample and generated by global sections.

Let $H^0(X,E)\otimes\mathcal{O}_X\to E$ be the natural surjective homomorphism. Fix a basis $e_1,\dots,e_N$ of $H^0(X,E)$ and, for each $\omega\in\Omega$ and $(a_1,\dots,a_N)\in K_\omega^N$, we set

$$\|a_1e_1+\cdots+a_Ne_N\|_\omega=\begin{cases}\sqrt{|a_1|_\omega^2+\cdots+|a_N|_\omega^2} & \text{if } \omega\in\Omega_\infty,\\ \max\{|a_1|_\omega,\dots,|a_N|_\omega\} & \text{if } \omega\in\Omega\setminus\Omega_\infty,\end{cases}$$

and $\xi$ be the norm family $(\|\cdot\|_\omega)_{\omega\in\Omega}$. Let $\psi$ be a metric family of $E$ induced by $H^0(X,E)\otimes\mathcal{O}_X\to E$ and $\xi$. Let $\pi:\mathbb{P}(E)\to X$ be the projective bundle of $E$ and $\mathcal{O}_E(1)$ be the tautological line bundle of $\mathbb{P}(E)$. Note that the metric family $\psi^{\mathrm{FS}}$ of $\mathcal{O}_E(1)$ is induced by $H^0(X,E)\otimes\mathcal{O}_{\mathbb{P}(E)}\to\mathcal{O}_E(1)$ and $\xi$, so it is dominated and measurable. Thus the assertion follows. □

**3.3. Dual metric family.** — In this subsection, let $X$ be a projective scheme over $\operatorname{Spec}K$.

***Definition 3.11.*** — Let $E$ be a vector bundle on $X$, equipped with a metric family $\psi=(\psi_\omega)_{\omega\in\Omega}$. For any $\omega\in\Omega$ and any $x\in X_\omega^{\mathrm{an}}$, the norm $|\cdot|_{\psi_\omega}(x)$ on $E_\omega(x)$ induces a dual norm on $E_\omega(x)^\vee$, which we denote by $|\cdot|_{\psi_\omega^\vee}(x)$. It turns out that $\psi_\omega^\vee=(|\cdot|_{\psi_\omega^\vee}(x))_{x\in X_\omega^{\mathrm{an}}}$ forms a continuous metric on $E_\omega^\vee$. Hence $\psi^\vee=(\psi_\omega^\vee)_{\omega\in\Omega}$ is a metric family on $E^\vee$, called the *dual metric family of* $\psi$.

***Proposition 3.12.*** — *Let $E$ be a vector bundle on $X$ and $\psi$ be a metric family of $E$. If $\psi$ is dominated, then the dual metric family $\psi^\vee$ is also dominated.*

*Proof.* — Let $\pi_E:\mathbb{P}(E)\to X$ and $\pi_{E^\vee}:\mathbb{P}(E^\vee)\to X$ be the projective bundles associated with $E$ and $E^\vee$ respectively. We consider the fiber product $\mathbb{P}(E)\times_X\mathbb{P}(E^\vee)$ of projective bundles and denote by

$$p_1:\mathbb{P}(E)\times_X\mathbb{P}(E^\vee)\longrightarrow\mathbb{P}(E)\quad\text{and}\quad p_2:\mathbb{P}(E)\otimes_X\mathbb{P}(E^\vee)\longrightarrow\mathbb{P}(E^\vee)$$

the morphisms of projection. Let

$$\mathcal{O}_E(1)\boxtimes\mathcal{O}_{E^\vee}(1):=p_1^*(\mathcal{O}_E(1))\otimes p_2^*(\mathcal{O}_{E^\vee}(1))$$

and let

$$s\in H^0(\mathbb{P}(E)\times_X\mathbb{P}(E^\vee),\mathcal{O}_E(1)\boxtimes\mathcal{O}_{E^\vee}(1))$$

be the trace section of $\mathcal{O}_E(1)\boxtimes\mathcal{O}_{E^\vee}(1)$, which corresponds to the composition of the following universal homomorphisms

$$p_2^*(\mathcal{O}_{E^\vee}(-1))\longrightarrow p_2^*(\pi_{E^\vee}^*(E))\cong p_1^*(\pi_E^*(E))\longrightarrow p_1^*(\mathcal{O}_E(1)).$$

***Claim 3.13.*** — *Let $\psi_1=(\psi_{1,\omega})_{\omega\in\Omega}$ and $\psi_2=(\psi_{2,\omega})_{\omega\in\Omega}$ be metric families on $E$ and $E^\vee$ respectively. We equip $\mathcal{O}_E(1)\boxtimes\mathcal{O}_{E^\vee}(1)$ with the metric family $\varphi=(\varphi_\omega)_{\omega\in\Omega}$*

*which is the tensor product of the metric families* $p_1^*(\psi_1^{\mathrm{FS}})$ *and* $p_2^*(\psi_2^{\mathrm{FS}})$*. Then, for any* $\omega \in \Omega$ *and* $x \in X_\omega^{\mathrm{an}}$*, one has*

$$\sup_{f \in E_\omega^\vee(x)\setminus\{0\}} \frac{|f|_{\psi_{1,\omega}^\vee}(x)}{|f|_{\psi_{2,\omega}}(x)} \leqslant \|s\|_{\varphi_\omega}.$$

*Proof.* — Let $f$ be a non-zero element of $E_\omega^\vee(x)$. The one-dimensional $\widehat{\kappa}(x)$-vector space of $E_\omega^\vee(x)$ spanned by $f$ determines a point $P_f$ of $P(E_\omega)^{\mathrm{an}}$ valued in $(\widehat{\kappa}(x), |\cdot|_x)$ which lies over $x \in X_\omega^{\mathrm{an}}$. Suppose $Q$ is a point of $\mathbb{P}(E_\omega^\vee)^{\mathrm{an}}$ valued in $(\widehat{\kappa}(x), |\cdot|_x)$ which lies over $x$. Then $s(P_f, Q)$ corresponds to the following composition of universal homomorphisms

$$\mathcal{O}_{E^\vee}(-1)(Q) \longrightarrow E_\omega(x) \longrightarrow \mathcal{O}_E(1)(P_f), \tag{3.2}$$

and $|s|_{\varphi_\omega}(P_f, Q)$ is the operator norm of this homomorphism. We pick an arbitrary non-zero element $\ell$ of $\mathcal{O}_{E^\vee}(-1)(Q)$. The dual element in $\mathcal{O}_E(-1)(P_f)$ of the image of $\ell$ by (3.2) is $f(\ell)^{-1}f$. Therefore one has

$$|s|_{\varphi_\omega}(P_f, Q) = \frac{|f(\ell)|_x}{|\ell|_{\psi_{1,\omega}}(x) \cdot |f|_{\psi_{2,\omega}}(x)} \leqslant \|s\|_{\varphi_\omega}.$$

Taking the supremum with respect to $\ell$, we obtain the required inequality. □

In the above claim, if both metric families $\psi_1$ and $\psi_2$ are dominated, then the metric family $\varphi$ on $\mathcal{O}_E(1) \boxtimes \mathcal{O}_{E^\vee}(1)$ is also dominated. In particular, the function

$$(\omega \in \Omega) \longmapsto \ln \|s\|_{\varphi_\omega}$$

is bounded from above by an integrable function. Then the claim shows that the function

$$(\omega \in \Omega) \longmapsto \sup_{x \in X_\omega^{\mathrm{an}}} \sup_{f \in E_\omega^\vee(x)\setminus\{0\}} \Big( \ln |f|_{\psi_{1,\omega}^\vee}(x) - \ln |f|_{\psi_{2,\omega}}(x) \Big)$$

is bounded from above by an integrable function. Therefore, the function

$$(\omega \in \Omega) \longmapsto \sup_{Q \in \mathbb{P}(E^\vee)^{\mathrm{an}}} \sup_{\substack{f \in \mathcal{O}_{E^\vee}(1)(Q) \\ f \neq 0}} \Big( \ln |f|_{\psi_{1,\omega}^{\vee,\mathrm{FS}}}(Q) - \ln |f|_{\psi_{2,\omega}^{\mathrm{FS}}}(Q) \Big)$$

is bounded from above by an integrable function. For the same reason, by exchanging the roles of $E$ and $E^\vee$ we obtain that the function

$$(\omega \in \Omega) \longmapsto \sup_{P \in \mathbb{P}(E)^{\mathrm{an}}} \sup_{\substack{t \in \mathcal{O}_E(1)(P) \\ t \neq 0}} \Big( \ln |t|_{\psi_{2,\omega}^{\vee,\mathrm{FS}}}(P) - \ln |t|_{\psi_{1,\omega}^{\mathrm{FS}}}(P) \Big)$$

is also bounded from above by an integrable function. In particular, if we denote by $\widetilde{\varphi}$ the tensor product of the metric families $p_1^*(\psi_2^{\vee,\mathrm{PS}})$ and $p_2^*(\psi_1^{\vee,\mathrm{PS}})$, then the function

$$(\omega \in \Omega) \longmapsto \ln \|s\|_{\widetilde{\varphi}_\omega}$$

is still bounded from above by an integrable function. Hence the above claim implies that the function

$$(\omega \in \Omega) \longmapsto \sup_{x \in X_\omega^{\mathrm{an}}} \sup_{f \in E_\omega^\vee(x) \setminus \{0\}} \Big( \ln |f|_{\psi_{2,\omega}}(x) - \ln |f|_{\psi_{1,\omega}^\vee}(x) \Big)$$

is bounded from above by an integrable function. Therefore we obtain that the local distance function

$$(\omega \in \Omega) \longmapsto d_\omega(\psi_1^\vee, \psi_2)$$

is bounded from above by an integrable function. By Proposition 3.3, the metric family $\psi_1^\vee$ is dominated. By Proposition 3.10, there exists at least a dominated metric family on $E^\vee$, the assertion is thus proved. □

***Definition 3.14.*** — Let $E$ be a vector bundle on $X$, $\psi = (\psi_\omega)_{\omega \in \Omega}$ be a metric family of $E$. Let $K'/K$ be a finite extension and let $P : \operatorname{Spec} K' \to X$ be a $K$-morphism. Let

$$(K', (\Omega', \mathcal{A}', \nu'), \phi') = S \otimes_K K'.$$

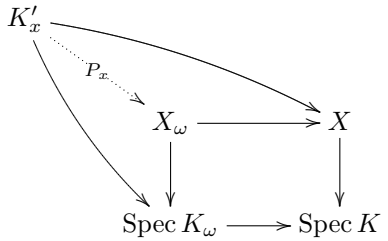
Recall that $\Omega'$ is a disjoint union

$$\Omega' = \coprod_{\omega \in \Omega} \Omega'_\omega,$$

where $\Omega'_\omega$ denotes the set of all absolute values on $\Omega'$ extending $|\cdot|_\omega$. For any $\omega \in \Omega$ and any $x \in \Omega'_\omega$, we let $P_x : \operatorname{Spec} K'_x \to X_\omega$ be the morphism induced by

$$\operatorname{Spec} K'_x \longrightarrow \operatorname{Spec} K' \xrightarrow{P} X$$

and the canonical morphism $\operatorname{Spec} K'_x \to \operatorname{Spec} K_\omega$.




We denote by $\|\cdot\|_x$ the norm on

$$(E \otimes_K K') \otimes_{K'} K'_x \cong E_\omega \otimes_{K_\omega} K'_x$$

which is induced by $|\cdot|_{\psi_\omega}(P_x)$ by $\varepsilon$-extension of scalars if $|\cdot|_\omega$ is non-Archimedean, and by $\pi$-extension of scalars if $|\cdot|_\omega$ in Archimedean. Then, $(\|\cdot\|_x)_{x \in \Omega'}$ forms a norm family of $P^*(E)$, which we denote by $P^*(\psi)$.

***Definition 3.15.*** — Let $\Omega_0$ be the set of $\omega \in \Omega$ such that the absolute value $|\cdot|_\omega$ is trivial. Let $x = (K_x, |\cdot|_x, P_x)$ be a triplet, where $(K_x, |\cdot|_x)$ is a valued extension of the trivially valued field $(K, |\cdot|_0)$, and $P_x : \operatorname{Spec} K_x \to X$ is a $K$-morphism. Assume

that $E$ is a vector bundle on $X$ and $\psi = (\psi_\omega)_{\omega\in\Omega}$ be a metric family of $E$. Denote by $E_x$ the $K_x$-vector space $P_x^*(E)$. We consider the following adelic curve

$$(K_x, (\Omega_0, \mathcal{A}_0, \nu_0), (|\cdot|_x)_{\omega\in\Omega_0}),$$

where $\mathcal{A}_0$ is the restriction of the $\sigma$-algebra $\mathcal{A}$ to $\Omega_0$, and $\nu_0$ is the restriction of $\nu$ to $(\Omega_0, \mathcal{A}_0)$. We denote by $x^*(\psi)$ the norm family $(|\cdot|_{\psi_\omega}(P_x^\omega))_{\omega\in\Omega_0}$ on $E_x$, where $P_x^\omega$ denotes the point of $X_\omega^{\mathrm{an}}$ determined by $(P_x, |\cdot|_x)$.

Assume that the transcendence degree of $K_x/K$ is $\leqslant 1$. Then $|\cdot|_x$ is a discrete absolute value on $K_x$. Let $\mathrm{ord}_x(\cdot) : K_x \to \mathbb{Z}\cup\{+\infty\}$ be the corresponding discrete valuation, which is defined as

$$\mathrm{ord}_x(a) = \sup\{n\in\mathbb{Z} \,:\, a\in\mathfrak{m}_x^n\},$$

where

$$\mathfrak{m}_x = \{b\in K_x \,:\, |b|_x < 1\}.$$

Then there is a non-negative real number $q$ such that

$$|\cdot|_x = \exp(-q\,\mathrm{ord}_x(\cdot)).$$

This non-negative real number is called the *exponent* of $x$.

***Proposition 3.16.*** — *Let $E$ be a vector bundle on $X$ and $\psi = (\psi_\omega)_{\omega\in\Omega}$ be a metric family of $E$. Then the metric family $\psi$ is measurable if and only if both of the following conditions are satisfied:*

(1) *for any finite extension $K'/K$ and any $K$-morphism $P : \mathrm{Spec}\, K' \to X$, the norm family $P^*(\psi)$ is measurable,*
(2) *for any triplet $x = (K_x, |\cdot|_x, P_x)$, where $(K_x, |\cdot|_x)$ is a valued extension of transcendence degree $\leqslant 1$ and of rational exponent of the trivially valued field $(K, |\cdot|_0)$ and $P_x : \mathrm{Spec}\, K_x \to X$ is a $K$-morphism, the norm family $x^*(\psi)$ is measurable.*

*Proof.* — It suffices to treat the case where the field $K$ is countable. Recall that the measurability of the metric family $\psi$ signifies that the following two conditions are satisfied:

(1') for any finite extension $K'/K$ and any $K$-morphism $Q : \mathrm{Spec}\, K' \to \mathbb{P}(E)$, the norm family $Q^*(\psi^{\mathrm{FS}})$ is measurable,
(2') for any triplet $y = (K_y, |\cdot|_y, Q_y)$, where $(K_y, |\cdot|_y)$ is a valued extension of transcendence degree $\leqslant 1$ and of rational exponent of the trivially valued field $(K, |\cdot|_0)$, and $Q_y : \mathrm{Spec}\, K_y \to \mathbb{P}(E)$ is a $K$-morphism, the norm family $Q_y^*(\psi^{\mathrm{FS}})$ is measurable.

Let $K'/K$ be a finite extension. Any $K$-morphism $Q : \mathrm{Spec}\, K' \to \mathbb{P}(E)$ corres ponds to a $K$-morphisme $P : \mathrm{Spec}\, K' \to X$ together with a one-dimensional quotient vector space $L$ of $P^*(E)$, which identifies with $Q^*(\mathcal{O}_E(1))$. Moreover, the norm family $Q^*(\psi^{\mathrm{FS}})$ identifies with the quotient norm family of $P^*(\psi)$. If the norm family $P^*(\psi)$ is measurable, by [**13**, Proposition 4.1.24], we obtain that $Q^*(\psi^{\mathrm{FS}})$ is also measurable.

Conversely, if for any one-dimensional quotient vector space of $P^*(E)$, the quotient norm family of $P^*(\psi)$ on it is measurable, by passing to dual we obtain from [**13**, Proposition 4.1.24] that $P^*(\psi)^\vee$ is measurable and therefore $P^*(\psi)$ is also measurable.

Let $x = (K_x, |\cdot|_x, P_x)$ be a triplet, where $(K_x, |\cdot|_x)$ is a valued extension of transcendence degree $\leqslant 1$ and rational exponent of the trivially valued field $(K, |\cdot|_0)$, and $P_x : \operatorname{Spec} K_x \to X$ be a $K$-morphism. Note that the field $K_x$ is countable. Similarly to the above argument, the norm family $P_x^*(\psi)$ is measurable if and only if all its quotient norm families on one-dimensional quotient subspaces are measurable. The proposition is thus proved. □

***Proposition 3.17.*** — *Let $E$ be a vector bundle on $X$ and $\psi = (\psi_\omega)_{\omega\in\Omega}$ be a metric family on $E$. If $\psi$ is measurable, then the dual metric family $\psi^\vee$ of $E^\vee$ is also measurable.*

*Proof.* — Let $K'/K$ be a finite extension and $P : \operatorname{Spec} K' \to X$ be a $K$-morphism. If $P^*(\psi)$ is measurable, by [**13**, Proposition 4.1.24] we obtain that $P^*(\psi^\vee) = P^*(\psi)^\vee$ is measurable. Similarly, for any triplet $x = (K_x, |\cdot|_x, P_x)$, where $(K_x, |\cdot|_x)$ is a valued extension of transcendence degree $\leqslant 1$ and of rational exponent of the trivially valued field $(K, |\cdot|_0)$ and $P_x : \operatorname{Spec} K_x \to X$ is a $K$-morphism, if the norm family $P_x^*(\psi)$ is measurable, then $P_x^*(\psi^\vee) = P_x^*(\psi)^\vee$ is also measurable. The proposition is thus proved. □

***Corollary 3.18.*** — *Let $E$ be a vector bundle on $X$, $F$ be a vector subbundle of $E$, $\psi_E$ be a metric family of $E$, and $\psi_F$ be the restriction of $\psi_E$ to $F$. If the metric family $\psi_E$ is dominated (resp. measurable), then the restricted metric family $\psi_F$ is also dominated (resp. measurable).*

*Proof.* — The homomorphism of inclusion $F \to E$ induces by passing to dual a surjective homomorphism $E^\vee \to F^\vee$. Thus $F^\vee$ can be considered as a quotient vector bundle of $E^\vee$. Note that $\psi_F^\vee$ identifies with the quotient metric family of $\psi_E^\vee$. Hence the assertion follows from Propositions 3.12, 3.17 and 3.7. □

**3.4. Metric families on torsion-free sheaves.** — In this subsection, we assume that the $K$-scheme $X$ is geometrically integral.

***Definition 3.19.*** — Let $E$ be a torsion free $\mathcal{O}_X$-module and $U$ be a non-empty Zariski open set of $X$ such that $E$ is locally free over $U$. For any $\omega \in \Omega$, let $\psi_\omega$ be a continuous metric of $E_\omega$ over $U_\omega^{\mathrm{an}}$ such that, for any $s \in H^0(X_\omega, E_\omega)$,

$$\|s\|_{\psi_\omega} := \sup\{|s|_{\psi_\omega}(x) \,:\, x \in U_\omega^{\mathrm{an}}\} < +\infty.$$

We set $\psi = (\psi_\omega)_{\omega\in\Omega}$ and $\xi_\psi = (\|\cdot\|_{\psi_\omega})_{\omega\in\Omega}$. We say $(E, U, \psi)$ is a *sectionally adelic torsion free $\mathcal{O}_X$-module* if $(H^0(X, E), \xi_\psi)$ is an adelic vector bundle on $S$. By Proposition 3.9, an adelic locally free $\mathcal{O}_X$-module is sectionally adelic.

***Definition 3.20.*** — Let $E$ be a torsion free sheaf on $X$ and $U$ be a non-empty Zariski open set of $X$ such that $E|_U$ is locally free. Let $\psi = (\psi_\omega)_{\omega\in\Omega}$ be a metric family of $E|_U$. We say $(E, U, \psi)$ is a *birationally adelic torsion free $\mathcal{O}_X$-module* if it satisfies the following properties:

(1) There exist a birational morphism $\mu : X' \to X$ of geometrically integral projective schemes over $K$ such that $\mu^{-1}(U) \to U$ is an isomorphism, an adelic vector bundle $(E', \psi')$ on $X'$, and an injective morphism of $\mathcal{O}_{X'}$-modules $E \to \mu_*(E')$ whose restriction to $U$ gives an isomorphism $E|_U \to \mu_*(E')|_U \cong E'|_{\mu^{-1}(U)}$.

(2) The isomorphism $E|_U \to E'|_{\mu^{-1}(U)}$ yields an isometry

$$(E, \psi)|_U \longrightarrow (E', \psi')|_{\mu^{-1}(U)}\,.$$

By definition, for $s \in H^0(X, E)$ and each $\omega \in \Omega$,

$$\|s\|_{\psi_\omega} := \sup\{|s|_{\psi_\omega}(\xi) \,:\, \xi \in U_\omega^{\mathrm{an}}\}$$

exists. Note that $\|\cdot\|_{\psi_\omega}$ is the restriction of $\|\cdot\|_{\psi'_\omega}$ to $H^0(X, E)$ by using the injective homomorphism $H^0(X, E) \to H^0(X', E')$, so that $(H^0(X, E), (\|\cdot\|_{\psi_\omega})_{\omega\in\Omega})$ is an adelic vector bundle on $S$, that is, a birationally adelic torsion free $\mathcal{O}_X$-module is sectionally adelic in the sense of Definition 3.19.

***Lemma 3.21.*** — *Let $\pi : X \to Y$ be a continuous map of locally compact Hausdorff spaces such that $\pi$ is open and proper. Let $f : X \to \mathbb{R}$ be a continuous function on $X$ and $\tilde{f} : Y \to \mathbb{R}$ be a function on $Y$ given by*

$$\tilde{f}(y) = \max\{f(x) \,:\, \pi(x) = y\}.$$

*Then $\tilde{f}$ is continuous on $Y$.*

*Proof.* — Fix $y_0 \in Y$. Since $\pi^{-1}(y_0)$ is compact, for $\varepsilon > 0$, there exist $x_1, \ldots, x_n \in \pi^{-1}(y_0)$ and open subsets $U_1, \ldots, U_n$ of $X$ such that

$$\pi^{-1}(y_0) \subseteq U_1 \cup \cdots \cup U_n,$$

$x_i \in U_i$ for all $i \in \{1, \ldots, n\}$ and $|f(x) - f(x_i)| \leqslant \varepsilon$ for all $i \in \{1, \ldots, n\}$ and $x \in U_i$. If we set $Z = X \setminus U_1 \cup \cdots \cup U_n$, then $\pi(Z)$ is closed and $y_0 \notin \pi(Z)$. We choose an open set $W$ of $Y$ such that $y_0 \in W$ and

$$W \subseteq \pi(U_1) \cap \cdots \cap \pi(U_n) \cap (Y \setminus \pi(Z)).$$

Note that $\pi^{-1}(W) \subseteq U_1 \cup \cdots \cup U_n$. Let $y \in W$ and

$$\lambda_i = \sup\{f(x) \,:\, x \in U_i \text{ and } y = \pi(x)\}.$$

Then $\tilde{f}(y) = \max\{\lambda_1, \ldots, \lambda_n\}$ and $\lambda_i - \varepsilon \leqslant f(x_i) \leqslant \lambda_i + \varepsilon$ for all $i \in \{1, \ldots, n\}$, so that

$$\tilde{f}(y) - \varepsilon \leqslant \tilde{f}(y_0) \leqslant \tilde{f}(y) + \varepsilon,$$

as required. □

Let $\pi : X \to Y$ be a generically finite morphism of geometrically integral projective schemes over $\operatorname{Spec} K$ and $(M, U, \psi)$ be a sectionally adelic torsion free $\mathcal{O}_X$-module. Note that $\pi_*(M)$ is a torsion free $\mathcal{O}_Y$-module. The pushforward $\pi_*(\psi)$ is defined as follows: We choose a non-empty Zariski open set $V$ of $Y$ such that

$$\pi|_{\pi^{-1}(V)} : \pi^{-1}(V) \longrightarrow V$$

is étale and $\pi^{-1}(V) \subseteq U$. Note that $\pi_*(M)$ is locally free over $V$. For $y \in V_\omega^{\mathrm{an}}$ and $s \in \pi_*(M) \otimes \hat{\kappa}(y)$, $|s|_{\pi_*(\psi)_\omega}(y)$ is defined to be

$$|s|_{\pi_*(\psi_\omega)}(y) := \max\{|s|_{\psi_\omega}(x) \,:\, x \in (\pi_\omega^{\mathrm{an}})^{-1}(y)\}.$$

Since $\pi^{-1}(V)_\omega^{\mathrm{an}} \to V_\omega^{\mathrm{an}}$ is proper and open (c.f. [**2**, Lemma 3.2.4]), by Lemma 3.21, $\pi_*(\psi)_\omega$ yields a continuous metric of $\pi_*(M)_\omega$ over $V_\omega^{\mathrm{an}}$. We denote $(\pi_*(\psi)_\omega)_{\omega\in\Omega}$ by $\pi_*(\psi)$. For $s \in H^0(Y, \pi_*(M)) = H^0(X, M)$, as

$$\sup\{|s|_{\pi_*(\psi)_\omega}(y) \,:\, y \in V_\omega^{\mathrm{an}}\} = \sup\{|s|_{\psi_\omega}(x) \,:\, x \in \pi^{-1}(V)_\omega^{\mathrm{an}}\},$$

one has $\|s\|_{\pi_*(\psi)_\omega} = \|s\|_{\psi_\omega} < \infty$, so that $(\pi_*(M), V, \pi_*(\psi))$ forms a sectionally adelic torsion free $\mathcal{O}_Y$-module and $(H^0(Y, \pi_*(M)), (\|\cdot\|_{\pi_*(\psi)_\omega})_{\omega\in\Omega})$ is isometric to $(H^0(X, M), (\|\cdot\|_{\psi_\omega})_{\omega\in\Omega})$. We call $V$ an *open subscheme of definition* of $\pi_*(\psi)$.

## 4. Volumes of normed graded linear series

In this section, we let $k$ be a commutative field and we denote by $|\cdot|_0$ the trivial absolute value on $k$. Recall that $|a|_0 = 1$ for any $a \in k^\times$. Moreover, $S_0 = (k, \{0\}, |\cdot|_0)$ forms an adelic curve.

**4.1. Adelic vector bundle on $S_0$.** — Adelic vector bundles on $S_0$ are just finite-dimensional ultrametrically normed vector space over $k$. If $\overline{V} = (V, \|\cdot\|)$ is an adelic vector bundle on $S_0$, then the function $\|\cdot\|$ only takes finitely many values. Moreover, if the vector space $V$ is non-zero, then one has (see [**13**, Remark 4.3.63])

$$\widehat{\mu}_{\max}(\overline{V}) = -\min_{s\in V\setminus\{0\}} \ln\|s\|, \quad \widehat{\mu}_{\min}(\overline{V}) = -\max_{s\in V} \ln\|s\|.$$

The Harder-Narasimhan $\mathbb{R}$-filtration of $\overline{V}$ is give by

$$\forall\, t \in \mathbb{R}, \quad \mathcal{F}^t(\overline{V}) = \{s \in V \,:\, \|s\| \leqslant \mathrm{e}^{-t}\}.$$

Note that

$$\widehat{\deg}_+(\overline{V}) := \sup_{W\subset V} \widehat{\deg}(\overline{W}) = \int_0^{+\infty} \dim_k(\mathcal{F}^t(\overline{V}))\,\mathrm{d}t,$$
$$\widehat{\deg}(\overline{V}) = \int_{\mathbb{R}} t\,\mathrm{d}\dim_k(\mathcal{F}^t(\overline{V}))\,\mathrm{d}t.$$

**4.2. Normed graded algebra.** — Let $V_\bullet = \bigoplus_{n\in\mathbb{N}} V_n$ be a graded $k$-algebra. We assume that each $V_n$ is a finite-dimensional vector space over $k$. For any $n \in \mathbb{N}_{\geqslant 1}$, let $\|\cdot\|_n$ be an ultrametric norm on $V_n$. Then the pair $\overline{V}_\bullet = (V_\bullet, (\|\cdot\|_n)_{n\in\mathbb{N}_{\geqslant 1}})$ is called a *normed graded algebra* over $(k, |\cdot|_0)$. Let $f : \mathbb{N}_{\geqslant 1} \to \mathbb{R}_{\geqslant 0}$ be a function. If, for all $\ell \in \mathbb{N}_{\geqslant 2}$, $(n_1, \ldots, n_\ell) \in \mathbb{N}_{\geqslant 1}^\ell$ and $(s_1, \ldots, s_\ell) \in V_{n_1} \times \cdots \times V_{n_\ell}$, one has

$$\|s_1 \cdots s_\ell\|_{n_1+\cdots+n_\ell} \leqslant \mathrm{e}^{f(n_1)+\cdots+f(n_\ell)} \|s_1\|_{n_1} \cdots \|s_\ell\|_{n_\ell}, \tag{4.1}$$

we say that $\overline{V}_\bullet$ is *$f$-sub-multiplicative*. In the particular case where $f$ is the constant function taking value 0, we just say the $\overline{V}_\bullet$ is *sub-mutiplicative*. If there exists two constant $C_1$ and $C_2$ such that, for any $n \in \mathbb{N}$ and any $s \in V_n \setminus \{0\}$, one has

$$\mathrm{e}^{C_1 n} \leqslant \|s\|_n \leqslant \mathrm{e}^{C_2 n}, \tag{4.2}$$

we say that $\overline{V}_\bullet$ is *bounded*.

***Proposition 4.1.*** — *Let $\overline{V}_\bullet$ be a normed graded algebra over $(k, |\cdot|_0)$ and $f : \mathbb{N}_{\geqslant 1} \to \mathbb{R}_{\geqslant 0}$ be a function such that*

$$\lim_{n\to+\infty} \frac{f(n)}{n} = 0.$$

*Assume that $V_\bullet$ is an integral domain and that $\overline{V}_\bullet$ is $f$-sub-multiplicative and bounded.*

(1) *For any $n \in \mathbb{N}_{\geqslant 1}$ and any $s \in V_n$, the sequence*

$$\|s^N\|_{nN}^{1/N}, \quad N \in \mathbb{N},\ N \geqslant 1$$

*converges.*

(2) *For any $n \in \mathbb{N}_{\geqslant t1}$, the map*

$$\|\cdot\|_{\mathrm{sp},n} : V_n \longrightarrow \mathbb{R}_{\geqslant 0}, \quad s \longmapsto \lim_{N\to+\infty} \|s^N\|_{nN}^{1/N}$$

*is an ultrametric norm on $V_n$.*

(3) *The family of norms $(\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}}$ satisfies the following sub-multiplicativity condition: for any $(n, m) \in \mathbb{N}^2$ and any $(s_n, s_m) \in V_n \times V_m$,*

$$\|s_n s_m\|_{\mathrm{sp},n+m} \leqslant \|s_n\|_{\mathrm{sp},n} \cdot \|s_m\|_{\mathrm{sp},m}.$$

(4) *For any $n \in \mathbb{N}_{\geqslant 1}$ and any $s \in V_n \setminus \{0\}$, one has*

$$\|s\|_{\mathrm{sp},n} \leqslant \mathrm{e}^{f(n)} \|s\|_n. \tag{4.3}$$

*Proof.* — (1) It suffices to treat the case where $s \neq 0$. By (4.1), for $\ell \in \mathbb{N}_{\geqslant 2}$, and $(N_1, \ldots, N_\ell) \in \mathbb{N}_{\geqslant 1}^\ell$, one has

$$\ln \|s^{N_1+\cdots+N_\ell}\|_{n(N_1+\cdots+N_\ell)} \leqslant \sum_{i=1}^{\ell} \ln \|s^{N_i}\|_{nN_1} + f(nN_i).$$

Moreover, by (4.2), the sequence

$$\frac{1}{N} \ln \|s^N\|_{nN}, \quad N \in \mathbb{N},\ N \geqslant 1$$

is bounded. Therefore this sequence converges in $\mathbb{R}$ (see [**9**, Proposition 1.3.1]), which shows that the sequence

$$\|s^N\|_{nN}^{1/N}, \quad N \in \mathbb{N},\ N \geqslant 1$$

converges to a positive real number.

(2) It suffices to show that $\|\cdot\|_{\mathrm{sp},n}$ satisfies the strong triangle inequality. Let $s$ and $t$ be two elements of $V_n$. For any $N \in \mathbb{N}_{\geqslant 1}$, one has

$$(s+t)^N = \sum_{i=0}^{N} \binom{N}{i} s^i t^{N-i}$$

and hence

$$\|(s+t)^N\|_{nN} \leqslant \max_{i\in\{0,\dots,N\}} \|s^i t^{N-i}\|_{nN}.$$

Let

$$M = \max_{j\in\mathbb{N},\, j\geqslant 1} \frac{1}{j} \max\{\ln\|s^j\|_{nj}, \ln\|t^j\|_{nj}, 0\}.$$

Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a sequence of real numbers in $[0, \frac{1}{2}]$ such that

$$\lim_{j\to+\infty} \varepsilon_j = 0, \quad \lim_{j\to+\infty} j\varepsilon_j = +\infty, \quad \lim_{j\to+\infty} (j - j\varepsilon_j) = +\infty.$$

If $i/N \leqslant \varepsilon_N$, one has

$$\frac{1}{N}\ln\|s^i t^{N-i}\|_{nN} \leqslant \varepsilon_N M + \frac{N-i}{N}\cdot\frac{1}{N-i}\ln\|t^{N-i}\|_{n(N-i)} + \frac{f(ni)}{N} + \frac{f(n(N-i))}{N}.$$

Similarly, if $(N-i)/N \leqslant \varepsilon_N$, one has

$$\frac{1}{N}\ln\|s^i t^{N-i}\|_{nN} \leqslant \frac{i}{N}\cdot\frac{1}{i}\ln\|s^i\|_{ni} + \varepsilon_N M + \frac{f(ni)}{N} + \frac{f(n(N-i))}{N}.$$

If $N\varepsilon_N < i < N - N\varepsilon_N$, one has

$$\frac{1}{N}\ln\|s^i t^{N-i}\|_{nN} \leqslant \frac{i}{N}\cdot\frac{1}{i}\ln\|s^i\|_{ni} + \frac{N-i}{N}\cdot\frac{1}{N-i}\ln\|t^{N-i}\|_{n(N-i)} + \frac{f(ni)}{N} + \frac{f(n(N-i))}{N}.$$

Taking the superior limit when $N \to +\infty$, we obtain that

$$\limsup_{N\to+\infty} \max_{i\in\{0,\dots,N\}} \frac{1}{N}\ln\|s^i t^{N-i}\|_{nN} \leqslant \max\{\|s\|_{\mathrm{sp},n}, \|t\|_{\mathrm{sp},n}\}.$$

(3) Let $(n,m) \in \mathbb{N}^2$ and $(s_n, s_m) \in V_n \times V_m$. For any $N \in \mathbb{N}$ such that $N \geqslant 1$, one has

$$\|(s_n s_m)^N\|_{(n+m)N} \leqslant \mathrm{e}^{f(nN)+f(mN)} \|s_n^N\|_{nN} \cdot \|s_m^N\|_{mN}.$$

Taking the $N$-th root and letting $N \to +\infty$ we obtain

$$\|s_n s_m\|_{\mathrm{sp},n+m} \leqslant \|s_n\|_{\mathrm{sp},n} \cdot \|s_m\|_{\mathrm{sp},m}.$$

(4) For any $N \in \mathbb{N}_{\geqslant 1}$, the following inequality holds:

$$\|s^N\|_{nN} \leqslant \mathrm{e}^{Nf(n)} \|s\|_n^N.$$

Taking the $N$-th root and then letting $N \to +\infty$, we obtain

$$\|s\|_{\mathrm{sp},n} \leqslant \mathrm{e}^{f(n)}\|s\|_n.$$

□

**4.3. Reminder on graded linear series.** — In this subsection, we let $k'/k$ be a finitely generated extension of fields. As *graded linear series* of $k'/k$, we refer to a graded sub-$k$-algebra $V_\bullet$ of

$$k'[T] = \bigoplus_{n\in\mathbb{N}} k'T^n$$

such that $V_0 = k$. We denote by $\mathbb{N}(V_\bullet)$ the set of $n \in \mathbb{N}$ such that $V_n \neq \mathbf{0}$. If $V_\bullet$ is a graded linear series and $\mathbb{N}(V_\bullet) \neq \{0\}$, we denote by $k(V_\bullet)$ the sub-extension of $k'/k$ generated by

$$\bigcup_{n\in\mathbb{N}(V_\bullet)\setminus\{0\}} \{f/g \,|\, (f,g) \in V_n \times (V_n \setminus \{0\})\}$$

over $k$. If $\mathbb{N}(V_\bullet) \neq \{0\}$, then we denote by $\dim(V_\bullet)$ the transcendence degree of the extension $k(V_\bullet)/k$, and call it the *Kodaira-Iitaka dimension* of $V_\bullet$. In the case where $V_n = \{0\}$ for any $n \in \mathbb{N}_{\geqslant 1}$, by convention $\dim(V_\bullet)$ is defined to be $-\infty$. If $\mathbb{N}(V_\bullet) \neq \{0\}$ and if the field $k(V_\bullet)$ coincides with $k'$, we say that the graded linear series $V_\bullet$ is *birational*.

We say that $V_\bullet$ is *of sub-finite type* if there exists a graded linear series $W_\bullet$ of $k'/k$ which is a $k$-algebra of finite type and contains $V_\bullet$ as a sub-$k$-algebra. By [**12**, Theorem 3.7], there exists a graded sub-$k$-algebra of finite type $W_\bullet$ of the polynomial ring

$$k(V_\bullet)[T] = \bigoplus_{n\in\mathbb{N}} k(V_\bullet)T^n$$

such that $k(W_\bullet) = k(V_\bullet)$, which contains $V_\bullet$ as a sub-$k$-algebra. In other words, $V_\bullet$ viewed as a graded linear series of $k(V_\bullet)/k$ is sub-finite.

Let $V_\bullet$ be a graded linear series of sub-finite type, and $d$ be its Kodaria-Iitaka dimension. If $\mathbb{N}(V_\bullet) \neq \{0\}$, we define the *volume* of $V_\bullet$ as the limit (see [**12**, Theorem 6.2] for the convergence)

$$\mathrm{vol}(V_\bullet) := \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{\dim_k(V_n)}{n^d/d!}.$$

Note that $V_\bullet$ satisfies the Fujita approximation property, namely, one has

$$\mathrm{vol}(V_\bullet) = \sup_{\substack{W_\bullet\subset V_\bullet \\ \dim(W_\bullet)=\dim(V_\bullet)}} \mathrm{vol}(W_\bullet),$$

where $W_\bullet$ runs over the set of all graded sub-$k$-algebras of $V_\bullet$ such that $\dim(W_\bullet) = \dim(V_\bullet)$.

**4.4. Normed graded series.** — In this subsection, we fix a finitely generated extension $k'/k$, a graded linear series $V_\bullet$ of $k'/k$ which is of sub-finite type, and a $f : \mathbb{N}_{\geqslant 1} \to \mathbb{R}_{\geqslant 0}$ such that

$$\lim_{n\to+\infty} \frac{f(n)}{n} = 0.$$

Let $d$ be the Kodaira-Iitaka dimension of $V_\bullet$. We assume that $d \geqslant 0$ (namely $\mathbb{N}(V_\bullet) = \{n \in \mathbb{N} : V_n \neq \mathbf{0}\} \neq \{0\}$) and we equip the graded algebra $V_\bullet$ with a family of norms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ such that $\overline{V}_\bullet = (V_\bullet, (\|\cdot\|_n)_{n\in\mathbb{N}_{\geqslant 1}})$ forms a normed graded algebra which is $f$-sub-multiplicative and bounded (see §4.2). For any $n \in \mathbb{N}_{\geqslant 1}$, let $\|\cdot\|_{\mathrm{sp},n} : V_n \to \mathbb{R}_{\geqslant 0}$ be the map defined as

$$\|s\|_{\mathrm{sp},n} := \lim_{N\to+\infty} \|s^N\|_{nN}^{1/N}.$$

Then $(V_\bullet, (\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}_{\geqslant 1}})$ forms a normed graded algebra which is sub-multiplicative and bounded. Moreover, we denote by $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{V}_\bullet)$ the *asymptotic maximal slope* of $\overline{V}_\bullet$, which is defined as

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{V}_\bullet) = -\lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \min_{s\in V_n\setminus\{0\}} \frac{1}{n} \ln \|s\|_n = \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \widehat{\mu}_{\max}(V_n, \|\cdot\|_n).$$

Note that the existence of the limite is ensured by the inequality (4.1), which implies that

$$\widehat{\mu}_{\max}(V_{n_1+\cdots+n_\ell}, \|\cdot\|_{n_1+\cdots+n_\ell}) \geqslant \sum_{i=1}^{\ell} \Big(\widehat{\mu}_{\max}(V_{n_i}, \|\cdot\|_{n_i}) - f(n_i)\Big).$$

We refer the readers to [**9**, Corollary 1.3.2] for a proof of the convergence.

***Proposition 4.2.*** — *The following equality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{V}_\bullet) = \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{1}{n}\widehat{\mu}_{\max}(V_n, \|\cdot\|_{\mathrm{sp},n}).$$

*Proof.* — By Proposition 4.1, one has

$$\|\cdot\|_{\mathrm{sp},n} \leqslant \mathrm{e}^{f(n)}\|\cdot\|_n$$

and hence for $n \in \mathbb{N}(V_\bullet)$ the following inequality holds

$$\widehat{\mu}_{\max}(V_n, \|\cdot\|_{\mathrm{sp},n}) \geqslant \widehat{\mu}_{\max}(V_n, \|\cdot\|_n) - f(n).$$

This implies

$$\lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{1}{n}\widehat{\mu}_{\max}(V_n, \|\cdot\|_{\mathrm{sp},n}) \geqslant \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{1}{n}\widehat{\mu}_{\max}(V_n, \|\cdot\|_n).$$

Conversely, for any fixed $n \in \mathbb{N}(V_\bullet)$ and $s \in V_n \setminus \{0\}$ such that

$$\ln \|s\|_{\mathrm{sp},n} = -\widehat{\mu}_{\max}(V_n, \|\cdot\|_{\mathrm{sp},n}),$$

one has

$$\begin{aligned}\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{V}_\bullet) &= \lim_{N\to+\infty}\frac{1}{nN}\widehat{\mu}_{\max}(V_{nN},\|\cdot\|_{nN})\\ &\geqslant \lim_{N\to+\infty}\frac{-1}{nN}\ln\|s^N\|_{nN} = -\frac{1}{n}\ln\|s\|_{\mathrm{sp},n} = \frac{1}{n}\widehat{\mu}_{\max}(V_n,\|\cdot\|_{\mathrm{sp},n}).\end{aligned}$$

Taking the limit when $n\to+\infty$, we obtain

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{V}_\bullet)\geqslant\widehat{\mu}^{\mathrm{asy}}_{\max}(V_\bullet,(\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}_{\geqslant 1}}).$$

□

***Definition 4.3.*** — We define the *arithmetic volume* of $\overline{V}_\bullet$ as (see §2.4 for the definition of $\widehat{\deg}_+$)

$$\widehat{\mathrm{vol}}(\overline{V}_\bullet) := \limsup_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty}\frac{\widehat{\deg}_+(V_n,\|\cdot\|_n)}{n^{d+1}/(d+1)!}. \tag{4.4}$$

***Theorem 4.4.*** — *The superior limit in the formula* (4.4) *defining the arithmetic volume function is actually a limite. Moreover, the following equalities hold:*

$$\widehat{\mathrm{vol}}(\overline{V}_\bullet) = \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty}\frac{\widehat{\deg}_+(V_n,\|\cdot\|_{\mathrm{sp},n})}{n^{d+1}/(d+1)!} = (d+1)\int_0^{+\infty}\mathrm{vol}(V^t_\bullet)\,\mathrm{d}t,$$

*where for* $t\in\mathbb{R}$,

$$V^t_\bullet := k\oplus\bigoplus_{n\in\mathbb{N},\,n\geqslant 1}\mathrm{Vect}_k(\{s\in V_n\,:\,\|s\|_{\mathrm{sp},n}\leqslant \mathrm{e}^{-nt}\}).$$

*Proof.* — By replacing $k'$ by $k(V_\bullet)$, we may assume that the graded linear series $V_\bullet$ is birational. For simplifying the notation, we let $M$ be the asymptotic maximal slope of $\overline{V}_\bullet$. Note that $M$ is also the asymptotic maximal slope of $(V_\bullet,(\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}})$ (see Proposition 4.2). Moreover, since $\overline{V}_\bullet$ is bounded, there exists a constant $A\geqslant 0$ such that $\|s\|_n\leqslant \mathrm{e}^{nA}$ for any $n\in\mathbb{N}_{\geqslant 1}$ and any $s\in V_n$.

By the same argument as the proof of [**12**, Proposition 6.6], we obtain that, for any $t<M$, one has $k(V^t_\bullet)=k(V_\bullet)$. Moreover, for any $t>M$ and any $n\in\mathbb{N}_{\geqslant 1}$, one has $V^t_n=\mathbf{0}$. Therefore, combining the construction of Newton-Okounkov bodies in [**11**, Theorem 1.1] and that of the concave transform developed in [**5**, §1.3], we obtain, in a similar way as [**5**, Corollary 1.13] that

$$\widehat{\mathrm{vol}}(V_\bullet,(\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}_{\geqslant 1}}) = \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty}\frac{\widehat{\deg}_+(V_n,\|\cdot\|_{\mathrm{sp},n})}{n^{d+1}/(d+1)!} = (d+1)\int_0^{+\infty}\mathrm{vol}(V^t_\bullet)\,\mathrm{d}t.$$

Moreover, by (4.3) we obtain that

$$\widehat{\deg}_+(V_n,\|\cdot\|_{\mathrm{sp},n})\geqslant\widehat{\deg}_+(V_n,\|\cdot\|_n)-\dim_k(V_n)f(n),$$

which leads to

$$\limsup_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty}\frac{\widehat{\deg}_+(V_n,\|\cdot\|_n)}{n^{d+1}/(d+1)!}\leqslant\widehat{\mathrm{vol}}(V_\bullet,(\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}_{\geqslant 1}})$$

since $\dim_k(V_n) = O(n^d)$ when $n \in \mathbb{N}(V_\bullet)$, $n \to +\infty$.

Let $\varepsilon$ be an element of $]0, M[$, $t$ be an element of $[\varepsilon, M[$. Let $W_\bullet^t$ be a graded sub-$k$-algebra of finite type of $V_\bullet^t$, which is generated by a family of homogeneous elements $s_1, \dots, s_\ell$ of homogeneous degrees $n_1, \dots, n_\ell$ respectively. For any $i \in \{1, \dots, \ell\}$, there exists $a_i \in \mathbb{N}_{\geqslant 1}$ such that the inequalities

$$\|s_i^N\|_{n_i N} \leqslant \mathrm{e}^{n_i N\varepsilon/2} \|s_i\|_{\mathrm{sp},n_i}^N \leqslant \mathrm{e}^{n_i N(\varepsilon/2 - t)} \tag{4.5}$$

hold for any integer $N \geqslant a_i$. Therefore, by the inequality (4.1) we obtain that, for any $(N_1, \dots, N_\ell) \in \mathbb{N}_{\geqslant 1}^\ell$, one has

$$\ln \|s_1^{N_1} \cdots s_\ell^{N_\ell}\|_{n_1 N_1 + \cdots + n_\ell N_\ell} \leqslant \sum_{i=1}^{\ell} \big( \ln \|s_i^{N_i}\|_{n_i N_i} + f(n_i N_i) \big).$$

By (4.5), we obtain that

$$\begin{aligned}\ln \|s_1^{N_1} \cdots s_\ell^{N_\ell}\|_{n_1 N_1 + \cdots + n_\ell N_\ell} &\leqslant \sum_{\substack{i \in \{1,\dots,\ell\} \\ N_i \geqslant a_i}} n_i N_i \Big( \frac{\varepsilon}{2} - t \Big) + \sum_{\substack{i \in \{1,\dots,\ell\} \\ N_i < a_i}} n_i N_i A \\ &\leqslant \Big( \frac{\varepsilon}{2} - t \Big) \sum_{i=1}^{\ell} n_i (N_i - a_i) + \sum_{i=1}^{\ell} n_i a_i A \leqslant \Big( \frac{\varepsilon}{2} - t \Big) \sum_{i=1}^{\ell} n_i N_i + \sum_{i=1}^{\ell} n_i a_i (A + M).\end{aligned}$$

Therefore, for $(N_1, \dots, N_\ell) \in \mathbb{N}_{\geqslant 1}^\ell$ such that $n_1 N_1 + \cdots + n_\ell N_\ell$ is sufficiently large, one has

$$\|s_1^{N_1} \cdots s_\ell^{N_\ell}\|_{n_1 N_1 + \cdots + n_\ell N_\ell} \leqslant \mathrm{e}^{(\varepsilon - t)(n_1 N_1 + \cdots + n_\ell N_\ell)}.$$

In particular, for $n \in \mathbb{N}(V_\bullet)$ sufficiently large, one has $W_n^t \subset \mathcal{F}^{(t-\varepsilon)n}(V_n, \|\cdot\|_n)$, which leads to

$$\liminf_{n \in \mathbb{N}(V_\bullet),\, n \to +\infty} \frac{\dim_k(\mathcal{F}^{(t-\varepsilon)n}(V_n, \|\cdot\|_n))}{n^d/d!} \geqslant \mathrm{vol}(W_\bullet^t).$$

Taking the supremum when $W_\bullet^t$ varies, by the Fujita approximation property of $V_\bullet^t$ we obtain that

$$\liminf_{n \in \mathbb{N}(V_\bullet),\, n \to +\infty} \frac{\dim_k(\mathcal{F}^{(t-\varepsilon)n}(V_n, \|\cdot\|_n))}{n^d/d!} \geqslant \mathrm{vol}(V_\bullet^t). \tag{4.6}$$

Note that

$$\begin{aligned}\widehat{\deg}_+(V_n, \|\cdot\|_n) &= \int_0^{+\infty} \dim_k(\mathcal{F}^t(V_n, \|\cdot\|_n))\, \mathrm{d}t = n \int_0^{+\infty} \dim_k(\mathcal{F}^{nt}(V_n, \|\cdot\|_n))\, \mathrm{d}t \\ &\geqslant n \int_\varepsilon^M \dim_k(\mathcal{F}^{n(t-\varepsilon)}(V_n, \|\cdot\|_n))\, \mathrm{d}t.\end{aligned}$$

Taking the integral with respect to $t$, by Fatou's lemma we deduce from (4.6) that

$$\begin{aligned}\liminf_{n \in \mathbb{N}(V_\bullet), n \to +\infty} \frac{\widehat{\deg}_+(V_n, \|\cdot\|_n)}{n^{d+1}/(d+1)!} &\geqslant \liminf_{n \in \mathbb{N}(V_\bullet),\, n \to +\infty} \frac{(d+1)!}{n^d} \int_\varepsilon^M \dim_k(\mathcal{F}^{n(t-\varepsilon)}(V_n, \|\cdot\|_n)) \\ &\geqslant (d+1) \int_\varepsilon^M \mathrm{vol}(V_\bullet^t)\, \mathrm{d}t = (d+1) \int_\varepsilon^{+\infty} \mathrm{vol}(V_\bullet^t).\end{aligned}$$

Finally, taking the supremum with respect to $\varepsilon$, we obtain the inequality

$$\liminf_{n\in\mathbb{N}(V_\bullet),n\to+\infty}\frac{\widehat{\deg}_+(V_n,\|\cdot\|_n)}{n^{d+1}/(d+1)!}\geqslant\widehat{\mathrm{vol}}(V_\bullet,(\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}_{\geqslant 1}}).$$

The theorem is thus proved. □

***Corollary 4.5.*** — *The sequences*

$$\frac{\widehat{\deg}(V_n,\|\cdot\|_n)}{n^{d+1}/(d+1)!},\quad n\in\mathbb{N}(V_\bullet)$$

*and*

$$\frac{\widehat{\deg}(V_n,\|\cdot\|_{\mathrm{sp},n})}{n^{d+1}/(d+1)!},\quad n\in\mathbb{N}(V_\bullet)$$

*converge to the same real number, which is equal to*

$$-\int_{\mathbb{R}}t\,\mathrm{d}\,\mathrm{vol}(V_\bullet^t).$$

*Proof.* — Let $A$ be a positive constant such that $\|s\|_n\leqslant \mathrm{e}^{nA}$ for any $n\in\mathbb{N}_{\geqslant 1}$ and any $s\in V_n$. For any $n\in\mathbb{N}_{\geqslant 1}$, let $\|\cdot\|'_n=\mathrm{e}^{-nA}\|\cdot\|_n$. Then, $(V_\bullet,(\mathrm{e}^{-nA}\|\cdot\|_n)_{n\in\mathbb{N}_{\geqslant 1}})$ forms a normed graded algebra over $(k,|\cdot|_0)$, which is $f$-sub-multiplicative and bounded. Moreover, for any $n\in\mathbb{N}_{\geqslant 1}$, one has

$$\widehat{\deg}(V_n,\|\cdot\|'_n)=\widehat{\deg}_+(V_n,\|\cdot\|'_n)=nA\dim_k(V_n)+\widehat{\deg}(V_n,\|\cdot\|_n),$$

where the first equality comes from the fact that the image of $\|\cdot\|'_n$ is contained in $[0,1]$. For any $n\in\mathbb{N}$ one has

$$\|\cdot\|'_{\mathrm{sp},n}=\mathrm{e}^{-nA}\|\cdot\|_{\mathrm{sp},n}.$$

By (4.3), for any $n\in\mathbb{N}_{\geqslant 1}$ and any $s\in V_n$, one has

$$\forall\, N\in\mathbb{N}_{\geqslant 1},\quad \|s\|_{\mathrm{sp},n}=\|s^N\|^{1/N}_{\mathrm{sp},nN}\leqslant \mathrm{e}^{f(nN)/N}\|s^N\|^{1/N}_{nN}\leqslant \mathrm{e}^{f(nN)/N+nA}.$$

Taking the limite when $N\to+\infty$, we obtain $\|s\|_{\mathrm{sp},n}\leqslant\mathrm{e}^{nA}$ and hence $\|\cdot\|'_{\mathrm{sp},n}$ also takes value in $[0,1]$. Therefore, for any $n\in\mathbb{N}_{\geqslant 1}$, one has

$$\widehat{\deg}(V_n,\|\cdot\|'_{\mathrm{sp},n})=\widehat{\deg}_+(V_n,\|\cdot\|'_{\mathrm{sp},n})=nA\dim_k(V_n)+\widehat{\deg}(V_n,\|\cdot\|_{\mathrm{sp},n}),$$

Hence Theorem 4.4 leads to the convergence of the sequences

$$\frac{\widehat{\deg}(V_n,\|\cdot\|_n)+nA\dim_k(V_n)}{n^{d+1}/(d+1)!},\quad n\in\mathbb{N}(V_\bullet)$$

and

$$\frac{\widehat{\deg}(V_n,\|\cdot\|_{\mathrm{sp},n})+nA\dim_k(V_n)}{n^{d+1}/(d+1)!},\quad n\in\mathbb{N}(V_\bullet)$$

to the same limite, which is equal to

$$(d+1)\int_0^{+\infty}\mathrm{vol}(V_\bullet^{t-A})\,\mathrm{d}t=(d+1)\int_{-A}^{+\infty}\mathrm{vol}(V_\bullet^t)\,\mathrm{d}t$$
$$=A(d+1)\,\mathrm{vol}(V_\bullet)-\int_{\mathbb{R}}t\,\mathrm{d}\,\mathrm{vol}(V_\bullet^t),$$

where the last equality comes from the fact that $V_\bullet^t=V_\bullet$ when $t\leqslant -A$. By the formula

$$\lim_{n\in\mathbb{N}(V_\bullet),\,n\to+\infty}\frac{\dim_k(V_n)}{n^d/d!}=\mathrm{vol}(V_\bullet),$$

we obtain the assertion. □

***Definition 4.6.*** — We define the *$\chi$-volume* of the normed graded linear series $\overline{V}_\bullet$ as

$$\widehat{\mathrm{vol}}_\chi(\overline{V}_\bullet)=\lim_{n\in\mathbb{N}(V_\bullet),\,n\to+\infty}\frac{\widehat{\deg}(V,\|\cdot\|_n)}{n^{d+1}/(d+1)!}.$$

By Corollary 4.5, we obtain that $\widehat{\mathrm{vol}}_\chi(\overline{V}_\bullet)=\widehat{\mathrm{vol}}_\chi(V_\bullet,(\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}_{\geqslant 1}})$.

## 5. Arithmetic volumes over a general adelic curve

In this section, we use the results of the previous section to study the volume functions of a normed graded algebra over a general adelic curve. Let $S=(K,(\Omega,\mathcal{A},\nu),\phi)$ be the adelic curve defined in §2.1. We let $|\cdot|_0$ be the trivial absolute value on $K$, and denote by $S_0=(K,\{0\},|\cdot|_0)$ the adelic curve consisting of a single copy of the trivial absolute value $|\cdot|_0$ on $K$.

**5.1. Graded algebra of adelic vector bundles.** — In this subsection, we consider basic facts on graded algebras of adelic vector bundles.

***Definition 5.1.*** — Let $E_\bullet=\bigoplus_{n\in\mathbb{N}}E_n$ be a graded $K$-algebra. We assume that each vector space $E_n$ is finite-dimensional over $K$. For any $n\in\mathbb{N}$, let $\xi_n=(\|\cdot\|_{n,\omega})_{\omega\in\Omega}$ be a norm family on $E_n$ such that $\overline{E}_n=(E_n,\xi_n)$ forms an adelic vector bundle on $S$. We call $\overline{E}_\bullet=(\overline{E}_n)_{n\in\mathbb{N}}$ a *graded algebra of adelic vector bundles on $S$*. For any $n\in\mathbb{N}$ such that $n\geqslant 1$, let $(\mathcal{F}^t(\overline{E}_n))_{t\in\mathbb{R}}$ be the Harder-Narasimhan $\mathbb{R}$-filtration of $\overline{V}_n$ (see §2.5). We denote by $\|\cdot\|_n^{\mathrm{HN}}$ the norm on $E_n$ (viewed as a vector space over $(K,|\cdot|_0)$) defined as

$$\forall\, s\in E_n,\quad \|s\|_n^{\mathrm{HN}}=\exp\big(-\sup\{t\in\mathbb{R}\,:\,s\in\mathcal{F}^t(\overline{E}_n)\}\big).$$

Then, the couple $(E_\bullet,(\|\cdot\|_n^{\mathrm{HN}})_{n\in\mathbb{N}_{\geqslant 1}})$ forms a normed graded algebra over $(K,|\cdot|_0)$ (see §4.2). Moreover if we view $(E_n,\|\cdot\|_n^{\mathrm{HN}})$ as an adelic vector bundle on $S_0$, then its

Harder-Narasimhan filtration coincides with that of $(E_n, \|\cdot\|_n)$. In particular, by the results recalled in §2.5, the following estimates holds:

$$0 \leqslant \widehat{\deg}(E_n, \|\cdot\|_n) - \widehat{\deg}(E_n, \|\cdot\|_n^{\mathrm{HN}}) \leqslant \frac{1}{2}\nu(\Omega_\infty)\dim_K(E_n)\ln(\dim_K(E_n)), \tag{5.1}$$

$$0 \leqslant \widehat{\deg}_+(E_n, \|\cdot\|_n) - \widehat{\deg}_+(E_n, \|\cdot\|_n^{\mathrm{HN}}) \leqslant \frac{1}{2}\nu(\Omega_\infty)\dim_K(E_n)\ln(\dim_K(E_n)). \tag{5.2}$$

Let $b = (b_n)_{n\in\mathbb{N}_{\geqslant 1}}$ be a sequence of non-negative integrable functions on $(\Omega, \mathcal{A}, \nu)$. We say that a graded algebra of adelic vector bundles $\overline{E}_\bullet$ is *b-sub-multiplicative* if for all $\omega \in \Omega$, $\ell \in \mathbb{N}_{\geqslant 2}$, $(n_1, \ldots, n_\ell) \in \mathbb{N}_{\geqslant 1}^\ell$ and $(s_1, \ldots, s_\ell) \in E_{n_1,\omega} \times \cdots \times E_{n_\ell,\omega}$, the following inequality holds

$$\|s_1 \cdots s_\ell\|_{n_1+\cdots+n_\ell,\omega} \leqslant \mathrm{e}^{b_{n_1}(\omega)+\cdots+b_{n_\ell}(\omega)}\|s_1\|_{n_1,\omega}\cdots\|s_\ell\|_{n_\ell,\omega}. \tag{5.3}$$

If for any $n$, $b_n$ is the constant function taking 0 as its value, we simply say that $\overline{E}_\bullet$ is *sub-multiplicative*.

***Proposition 5.2.*** — *Assume that the field $K$ is perfect. Let $b = (b_n)_{n\in\mathbb{N}_{\geqslant 1}}$ be a sequence of non-negative integrable functions on $(\Omega, \mathcal{A}, \nu)$, and $\overline{E}_\bullet$ be a graded algebra of adelic vector bundles on $S$, which is b-sub-multiplicative. Let $f : \mathbb{N}_{\geqslant 1} \to \mathbb{R}_{\geqslant 0}$ be the function defined as*

$$f(n) = \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E_n)) + \int_\Omega b_n(\omega)\,\nu(\mathrm{d}\omega).$$

*Then the normed graded algebra $(E_\bullet, (\|\cdot\|_n^{\mathrm{HN}})_{n\in\mathbb{N}_{\geqslant 1}})$ is $f$-sub-multiplicative.*

*Proof.* — Let $\ell \in \mathbb{N}_{\geqslant 1}$ and $(n_1, \ldots, n_\ell) \in \mathbb{N}_{\geqslant 1}^\ell$. For any $i \in \{1, \ldots, \ell\}$, let $F_{n_i}$ be a $K$-vector subspace of $E_{n_i}$. For any $\omega \in \Omega$, we consider the $K_\omega$-linear map

$$F_{n_1,\omega} \otimes \cdots \otimes F_{n_\ell,\omega} \longrightarrow E_{n_1+\cdots+n_\ell,\omega}$$

induced by the $K$-algebra structure of $E_\bullet$. If we equip with $F_{n_1,\omega} \otimes \cdots \otimes F_{n_\ell,\omega}$ with the $\varepsilon$-tensor product of the norms $\|\cdot\|_{n_1,\omega}, \ldots, \|\cdot\|_{n_\ell,\omega}$ when $|\cdot|_\omega$ is non-Archimedean, and with the $\pi$-tensor product when $|\cdot|_\omega$ is Archimedean, then the operator norm of the above map is bounded from above by $\exp(b_{n_1}(\omega) + \cdots + b_{n_\ell}(\omega))$. Moreover, by **[13]**, Corollary 5.6.2] (although this result has been stated under the assumption that $\mathrm{char}(K) = 0$, this assumption is only used in the application of Theorem 5.4.3 of **[13]**, which actually applies to any perfect field), one has

$$\widehat{\mu}_{\min}(\overline{F}_{n_1} \otimes_{\varepsilon,\pi} \cdots \otimes_{\varepsilon,\pi} \overline{F}_{n_\ell}) \geqslant \sum_{i=1}^{\ell}\Big(\widehat{\mu}_{\min}(\overline{F}_{n_i}) - \frac{3}{2}\nu(\Omega_\infty)\ln(E_{n_i})\Big).$$

Let $F_{n_1+\cdots+n_\ell}$ be the image of the map By **[13]**, Proposition 4.3.31], we obtain that

$$\widehat{\mu}_{\min}(\overline{F}_{n_1+\cdots+n_\ell}) \geqslant \sum_{i=1}^{\ell}\Big(\widehat{\mu}_{\min}(\overline{F}_{n_i}) - \frac{3}{2}\nu(\Omega_\infty)\ln(E_{n_i}) - \int_\Omega b_{n_i}(\omega)\,\nu(\mathrm{d}\omega)\Big). \tag{5.4}$$

Therefore, we obtain that, for any $(t_1, \dots, t_\ell) \in \mathbb{R}^\ell$, one has

$$\mathcal{F}^{t_1}(\overline{E}_{n_1}) \cdots \mathcal{F}^{t_\ell}(E_{n_\ell}) \subset \mathcal{F}^{t_1+\cdots+t_\ell-f(n_1)-\cdots-f(n_\ell)}(E_{n_1+\cdots+n_\ell}),$$

which shows that the normed graded algebra $(E_\bullet, (\|\cdot\|_n^{\mathrm{HN}})_{n\in\mathbb{N}_{\geqslant 1}})$ is $f$-sub-multiplicative. □

***Corollary-Definition 5.3.*** — *Assume that the field $K$ is perfect. Let $b = (b_n)_{n\in\mathbb{N}_{\geqslant 1}}$ be a sequence of non-negative integrable functions on $(\Omega, \mathcal{A}, \nu)$ such that*

$$\lim_{n\to+\infty} \frac{1}{n} \int_\Omega b_n(\omega)\, \nu(\mathrm{d}\omega) = 0.$$

*Let $\overline{E}_\bullet$ be a graded algebra of adelic vector bundles on $S$, which is $b$-sub-multiplicative. Denote by $\mathbb{N}(E_\bullet)$ the set of $n \in \mathbb{N}$ such that $E_n \neq \mathbf{0}$. Assume that*

1. *$E_\bullet$ is isomorphic to a graded linear series of sub-finite type of a finitely generated extension of $K$, which is of Kodaira-Iitaka dimension $d \geqslant 0$,*
2. *there exists $C > 0$ such that, for any $n \in \mathbb{N}(E_\bullet)$,*

$$-Cn \leqslant \widehat{\mu}_{\min}(\overline{E}_n) \leqslant \widehat{\mu}_{\max}(\overline{E}_n) \leqslant Cn.$$

*Then the sequences*

$$\frac{\widehat{\deg}(\overline{E}_n)}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N}(E_\bullet)$$

*and*

$$\frac{\widehat{\deg}_+(\overline{E}_n)}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N}(E_\bullet)$$

*converge to two real numbers $\widehat{\mathrm{vol}}_\chi(\overline{E}_\bullet)$ and $\widehat{\mathrm{vol}}(\overline{E}_\bullet)$, which we call* $\chi$-volume *and* volume *of $\overline{E}_\bullet$, respectively.*

*Proof.* — These results follow from Proposition 5.2, Theorem 4.4, Corollary 4.5 and the comparisons (5.1), (5.2) and the convergence of the sequence

$$\frac{\dim_K(E_n)}{n^d/d!}, \quad n \in \mathbb{N}(E_\bullet).$$

□

***Remark 5.4.*** — Assume that the field $K$ is perfect. Let $b = (b_n)_{n\in\mathbb{N}_{\geqslant 1}}$ be a sequence of non-negative integrable functions on $(\Omega, \mathcal{A}, \nu)$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \int_\Omega b_n(\omega)\, \nu(\mathrm{d}\omega) = 0.$$

Let $\overline{E}_\bullet$ be a graded algebra of adelic vector bundles on $S$, which is $b$-sub-multiplicative. We assume that $n_1, \dots, n_\ell$ are elements of $\mathbb{N}(E_\bullet) \setminus \{0\}$ such that

$$K \oplus \bigoplus_{n\in\mathbb{N},\, n\geqslant 1} E_n$$

is generated as $K$-algebra by $E_{n_1} \cup \cdots \cup E_{n_\ell}$. By (5.4) we obtain that, for any $(a_1, \ldots, a_\ell) \in \mathbb{N}^\ell \setminus \{(0, \ldots, 0)\}$, the canonical image of

$$E_{n_1}^{\otimes a_1} \otimes \cdots \otimes E_{n_\ell}^{\otimes a_\ell}$$

in $E_{a_1 n_1 + \cdots + a_\ell n_\ell}$ has a minimal slope

$$\geqslant \sum_{i=1}^{\ell} a_i \biggl( \widehat{\mu}_{\min}(\overline{E}_i) - \frac{3}{2} \nu(\Omega_\infty) \ln(E_{n_i}) - \int_\Omega b_{n_i}(\omega)\, \nu(\mathrm{d}\omega) \biggr).$$

Therefore we deduce that, for any $n \in \mathbb{N}(E_\bullet) \setminus \{0\}$, the minimal slope of $\overline{E}_n$ is bounded from below by

$$\min_{\substack{(a_1, \ldots, a_\ell) \in \mathbb{N}^\ell \\ n = a_1 n_1 + \cdots + a_\ell n_\ell}} \sum_{i=1}^{\ell} a_i \biggl( \widehat{\mu}_{\min}(\overline{E}_i) - \frac{3}{2} \nu(\Omega_\infty) \ln(E_{n_i}) - \int_\Omega b_{n_i}(\omega)\, \nu(\mathrm{d}\omega) \biggr).$$

Hence there exists $C > 0$ such that $\widehat{\mu}_{\min}(\overline{E}_n) \geqslant -Cn$ holds for any $n \in \mathbb{N}(E_\bullet)$.

**5.2. Arithmetic $\chi$-volumes of adelic line bundles.** — In this subsection, we introduce the arithmetic $\chi$-volume of an adelic line bundle.

***Theorem-Definition 5.5.*** — *Let $p : X \to \operatorname{Spec} K$ be a geometrically integral projective scheme over $\operatorname{Spec} K$, $d$ be the dimension of $X$, and $\overline{L} = (L, \varphi)$ be an adelic line bundle on $X$. Assume that $L$ is big and the graded $K$-algebra*

$$\bigoplus_{n \in \mathbb{N}} H^0(X, L^{\otimes n})$$

*is of finite type. We denote the adelic vector bundle $\bigl(H^0(X, L^{\otimes n}), (\|\cdot\|_{n\varphi_\omega})_{\omega \in \Omega}\bigr)$ over $S$ by $p_*(\overline{L}^{\otimes n})$. Then the sequence*

$$\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}))}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N},\ n \geqslant 1 \tag{5.5}$$

*converges to a real number, which we denote by $\widehat{\mathrm{vol}}_\chi(\overline{L})$ and which we call the $\chi$-volume of $\overline{L}$.*

*Proof.* — Let $K^{\mathrm{pc}}$ be the perfect closure of $K$. Recall that, if $K^{\mathrm{ac}}$ denotes the algebraic closure of $K$, then $K^{\mathrm{pc}}$ is the intersection of all subfields of $K^{\mathrm{ac}}$ containing $K$ which are perfect fields. Note that $K^{\mathrm{pc}}/K$ is a purely inseparable algebraic extension of $K$. Therefore, for any $\omega \in \Omega$, the absolute value $|\cdot|_\omega$ extends in a unique way to $K^{\mathrm{pc}}/K$. In other words, the measure space in the adelic curve structure of $S \otimes_K K^{\mathrm{pc}}$ coincides with $(\Omega, \mathcal{A}, \nu)$.

For any $n \in \mathbb{N}$, let

$$E_n = H^0(X, L^{\otimes n}) \otimes_K K^{\mathrm{pc}} = H^0(X_{K^{\mathrm{pc}}}, L_{K^{\mathrm{pc}}}^{\otimes n}).$$

The norm family of $p_*(\overline{L}^{\otimes n})$ induces by extension of scalars a norm family on $E_n$, which we denote by $\xi_n$. By [**13**, Proposition 4.3.14], the equality

$$\widehat{\deg}(E_n, \xi_n) = \widehat{\deg}(p_*(\overline{L}^{\otimes n}))$$

holds. Moreover,

$$E_\bullet = \bigoplus_{n\in\mathbb{N}} E_n$$

is a graded $K^{\mathrm{pc}}$-algebra of finite type, which is isomorphic to a graded linear series of the function field of $X_{K^{\mathrm{pc}}}$ over $K^{\mathrm{pc}}$. As a graded $K^{\mathrm{pc}}$-algebra of adelic vector bundles on $S \otimes_K K^{\mathrm{pc}}$, $\overline{E}_\bullet = (\overline{E}_n)_{n\in\mathbb{N}}$ is sub-multiplicative. By [**13**, Proposition 6.2.7], we obtain, following the proof of [**13**, Proposition 6.4.4], that the sequence

$$\frac{\widehat{\mu}_{\max}(\overline{E}_n)}{n}, \quad n \in \mathbb{N},\ n \geqslant 1$$

is bounded from above. Therefore the assertion follows from Corollary-Definition 5.3 (see also Remark 5.4). □

***Remark 5.6.*** — Under the notation and the assumption of the above theorem-definition, the following relation holds

$$\lim_{n\to+\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}))}{n \dim_K(H^0(X, L^{\otimes n}))} = \frac{\widehat{\mathrm{vol}}_\chi(\overline{L})}{(d+1)\,\mathrm{vol}(L)}.$$

**5.3. Normed graded module.** — Let $\overline{R}_\bullet = (\overline{R}_n)_{n\in\mathbb{N}}$ be a graded algebra of adelic vector bundles on $S$, where $\overline{R}_n = (R_n, (\|\cdot\|_{n,\omega})_{\omega\in\Omega})$. Let $M_\bullet = \bigoplus_{n\in\mathbb{N}} M_n$ be a graded module over $R_\bullet = \bigoplus_{n\in\mathbb{N}} R_n$. If each $M_n$ is a finite-dimensional vector space over $K$ and is equipped with a norm family $(\|\cdot\|^M_{n,\omega})_{\omega\in\Omega}$ such that $\overline{M}_n = (M_n, (\|\cdot\|^M_{n,\omega})_{\omega\in\Omega})$ is an adelic vector bundle on $S$, we say that $\overline{M}_\bullet = (\overline{M}_n)_{n\in\mathbb{N}}$ is *a graded $\overline{R}_\bullet$-module of adelic vector bundles on $S$*.

Assume that $\overline{R}_\bullet$ is sub-multiplicative (see Definition 5.1). If, for all $(n, m) \in \mathbb{N}^2$, $\omega \in \Omega$ and $(a, s) \in R_{n,\omega} \times M_{m,\omega}$, one has

$$\|as\|^M_{n+m,\omega} \leqslant \|a\|_{n,\omega} \cdot \|s\|^M_{m,\omega},$$

we say that $\overline{M}_\bullet$ is *sub-multiplicative*.

***Lemma 5.7.*** — *Let $\overline{M}_\bullet = ((M_n, \xi_{M_n}))_{n\in\mathbb{N}}$ be a graded $\overline{R}_\bullet$-module of adelic vector bundle on $S$. Let $Q = \bigoplus_{n=0}^\infty Q_n$ be a graded quotient $R$-module of $M$, that is, $Q_n$ is a quotient vector space of $M_n$ over $K$ for all $n$ and $a_\ell \cdot : M_n \to M_{n+\ell}$ induces by passing to quotient $a_\ell \cdot : Q_n \to Q_{n+\ell}$ for $a_\ell \in R_\ell$. Let $\xi_{Q_n}$ be the quotient norm family of $Q_n$ induced by $M_n \to Q_n$ and $\xi_{M_n}$. Then $\overline{Q}_\bullet = ((Q_n, \xi_{Q_n}))_{n\in\mathbb{N}}$ is a graded $\overline{R}_\bullet$-algebra.*

*Proof.* — Assume that $\xi_{M_n}$ and $\xi_{Q_n}$ are of the form $(\|\cdot\|^M_{n,\omega})_{\omega\in\Omega}$ and $(\|\cdot\|^Q_{n,\omega})_{\omega\in\Omega}$, respectively. Let $(n, n') \in \mathbb{N}^2$, $\omega \in \Omega$, $a \in R_{n,\omega}$ and $q \in Q_{n',\omega}$. For any $s \in M_{n',\omega}$

which represents the class $q \in Q_{n',\omega}$, one has

$$\|aq\|^Q_{n+n',\omega} \leqslant \|as\|^M_{n+n',\omega} \leqslant \|a\|_{n,\omega} \cdot \|s\|^M_{n',\omega}.$$

Taking the infimum with respect to $s$, we obtain

$$\|aq\|^Q_{n+n',\omega} \leqslant \|a\|_{n,\omega} \cdot \|s\|^Q_{n',\omega},$$

as required. □

***Proposition 5.8.*** — *Suppose that $R_\bullet$ is a $K$-algebra of finite type. Let $\overline{M}_\bullet = ((M_n, \xi_{M_n}))_{n\in\mathbb{N}}$ be a graded $\overline{R}_\bullet$-module of adelic vector bundle on $S$, such that $M_\bullet$ is an $R_\bullet$-module of finite type. Suppose that*

$$\liminf_{n\to\infty} \frac{\dim_K(M_n)}{n^d} = 0$$

*for some non-negative integer $d$, then*

$$\liminf_{n\to\infty} \frac{\widehat{\deg}(M_n, \xi_{M_n})}{n^{d+1}} \geqslant 0.$$

*Proof.* — Let $x_1, \ldots, x_r$ be homogeneous elements of $R$ which generate $R$ as $K$-algebra. We choose non-zero homogeneous elements $m_1, \ldots, m_\ell$ of $M$ such that $M$ is generated by $m_1, \ldots, m_\ell$ over $R$. We set $e_i = \deg(x_i)$ and $f_i = \deg(m_i)$ for $i \in \{1, \ldots, r\}$. For $\alpha = (a_1, \ldots, a_r) \in \mathbb{N}^r$, we denote $x_1^{a_1} \cdots x_r^{a_r}$ by $x^\alpha$. If we set $d_n = \dim_K(M_n)$, then, for $n \geqslant \max\{f_1, \ldots, f_r\}$, we can find $\alpha_1, \ldots, \alpha_{d_n} \in \mathbb{N}^r$ and $m_{i_1}, \ldots, m_{i_{d_n}} \in \{m_1, \ldots, m_\ell\}$ such that $x^{\alpha_1} m_{i_1}, \ldots, x^{\alpha_{d_n}} m_{i_{d_n}}$ form a basis of $M_n$. Note that

$$\begin{aligned}
\|(x^{\alpha_1} m_{i_1}) \wedge \cdots \wedge (x^{\alpha_{d_n}} m_{i_{d_n}})\|^M_{n,\omega,\det} &\leqslant \|x^{\alpha_1} m_{i_1}\|_{M_n,\omega} \cdots \|x^{\alpha_{d_n}} m_{i_{d_n}}\|^M_{n,\omega} \\
&\leqslant \|x^{\alpha_1}\|_{n-f_{i_1},\omega} \cdots \|x^{\alpha_{d_n}}\|_{n-f_{i_{d_n}},\omega} \cdot \|m_{i_1}\|^M_{f_{i_1},\omega} \cdots \|m_{i_{d_n}}\|^M_{f_{i_{d_n}},\omega} \\
&\leqslant \max\{1, \|x_1\|_{e_1,\omega}, \ldots, \|x_r\|_{e_r,\omega}\}^{nd_n} \max\{1, \|m_1\|^M_{f_1,\omega}, \ldots, \|m_\ell\|^M_{f_\ell,\omega}\}^{d_n},
\end{aligned}$$

so that

$$\begin{aligned}
\widehat{\deg}(M_n, \xi_{M_n}) \geqslant nd_n \int_\Omega \min\{0, -\ln\|x_1\|_{e_1,\omega}, \ldots, -\ln\|x_r\|_{e_r,\omega}\}\, \nu(\mathrm{d}\omega) \\
+ d_n \int_\Omega \min\{0, -\ln\|m_1\|^M_{f_1,\omega}, \ldots, -\ln\|m_\ell\|^M_{f_\ell,\omega}\}\, \nu(\mathrm{d}\omega).
\end{aligned}$$

Thus the assertion follows. □

## 6. Bounds of $\chi$-volume with auxiliary torsion free module

Let us begin with the following lemma.

***Lemma 6.1.*** — *Let $X$ be an integral projective scheme over a field $k$, $L$ be an invertible $\mathcal{O}_X$-module and $F$ be a coherent $\mathcal{O}_X$-module. We assume that there exist a surjective morphism $f : X \to Y$ of integral projective schemes over $k$ and an*

*ample invertible $\mathcal{O}_Y$-module $A$ such that $f^*(A) = L$. Then $R = \bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})$ is a finitely generated algebra over $k$ and $M = \bigoplus_{n=0}^{\infty} H^0(X, F \otimes L^{\otimes n})$ is a finitely generated $R$-module.*

*Proof.* — By [**20**, §1.8], there exist positive integers $d$ and $n_0$ such that

$$H^0(Y, A^{\otimes d}) \otimes H^0(Y, A^{\otimes n} \otimes f_*(F)) \longrightarrow H^0(Y, A^{\otimes (d+n)} \otimes f_*(F))$$

is surjective for all $n \geqslant n_0$, and hence

$$H^0(X, L^{\otimes d}) \otimes H^0(X, L^{\otimes n} \otimes F) \to H^0(X, L^{\otimes (d+n)} \otimes F)$$

is surjective for all $n \geqslant n_0$ because $f_*(L^{\otimes n}) = A^{\otimes n} \otimes f_*(\mathcal{O}_X)$, $f_*(L^{\otimes n} \otimes F) = A^{\otimes n} \otimes f_*(F)$, $\mathcal{O}_Y \subseteq f_*(\mathcal{O}_X)$. Thus, by the arguments in [**20**, §1.8], one can see the assertion. □

In the rest of the section, let $p : X \to \operatorname{Spec} K$ be a $d$-dimensional geometrically integral projective variety over $K$. Let $\overline{L} = (L, \varphi)$ be an adelic invertible $\mathcal{O}_X$-module. Let $E$ be a torsion free $\mathcal{O}_X$-module and $U$ be a non-empty Zariski open set of $X$ such that $E$ is locally free over $U$. Let $\psi = (\psi_\omega)_{\omega \in \Omega}$ be a metric family of $E|_U$. We assume that $(L^{\otimes n} \otimes E, U, n\varphi + \psi)$ is a sectionally adelic torsion free $\mathcal{O}_X$-module (see Definition 3.19) for all $n \in \mathbb{N}$. Note that, if the sectional algebra $\bigoplus_{n \in \mathbb{N}} H^0(X, L^{\otimes n})$ is of finite type over $K$ (this condition is true notably when $L$ satisfies the hypothesis of Lemma 6.1), by Theorem-Definition 5.5, the sequence

$$\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}))}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N},\ n \geqslant 1$$

converges to a real number denoted by $\widehat{\mathrm{vol}}_\chi(\overline{L})$.

***Theorem 6.2.*** — *If there are a birational morphism $f : X \to Z$ of geometrically integral projective schemes over $\operatorname{Spec} K$ and an ample invertible $\mathcal{O}_Z$-module $A$ such that $L = f^*(A)$, then the following inequality holds:*

$$\mathrm{rk}(E)\, \widehat{\mathrm{vol}}_\chi(\overline{L}) \leqslant \liminf_{n \to \infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{E}))}{n^{d+1}/(d+1)!}.$$

*Proof.* — Let $r$ be the rank of $E$. Note that $p_*(\overline{L}^{\otimes n} \otimes \overline{E})$ forms an adelic vector bundle over $S$ for any $n \in \mathbb{N}$. For a sufficiently large positive integer $n_0$, shrinking $U$ if necessarily, we can find $e_1, \ldots, e_r \in H^0(X, L^{\otimes n_0} \otimes E)$ such that $e_1, \ldots, e_r$ yield a basis of $L^{\otimes n_0} \otimes E$ over $U$. Indeed, there is a positive integer $n_0$ such that

$$H^0(Z, A^{\otimes n_0} \otimes f_*(E)) \otimes \mathcal{O}_Z \longrightarrow A^{\otimes n_0} \otimes f_*(E)$$

is surjective, and hence

$$H^0(X, L^{\otimes n_0} \otimes E) \otimes \mathcal{O}_X \longrightarrow L^{\otimes n_0} \otimes E$$

is surjective on some non-empty Zariski open subset of $X$. Thus the assertion follows. Let $\mathcal{O}_X^{\oplus r} \to L^{\otimes n_0} \otimes E$ be the homomorphism given by

$$(a_1, \ldots, a_r) \longmapsto a_1 e_1 + \cdots + a_r e_r.$$

Let $Q$ be the cokernel of $\mathcal{O}_X^{\oplus r} \to L^{\otimes n_0} \otimes E$. The sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow L^{\otimes n_0} \otimes E \longrightarrow Q \longrightarrow 0$$

is exact, and so is

$$0 \longrightarrow (L^{\otimes n})^{\oplus r} \longrightarrow L^{\otimes n+n_0} \otimes E \longrightarrow L^{\otimes n} \otimes Q \longrightarrow 0.$$

Thus

$$0 \longrightarrow H^0(X, L^{\otimes n})^{\oplus r} \longrightarrow H^0(X, L^{\otimes n+n_0} \otimes E) \longrightarrow H^0(X, L^{\otimes n} \otimes Q)$$

is also exact for all $n \geqslant 0$. Let $Q_n$ be the image of

$$H^0(X, L^{\otimes n+n_0} \otimes E) \longrightarrow H^0(X, L^{\otimes n} \otimes Q).$$

We equip $H^0(X, L^{\otimes n+n_0} \otimes E)$ with the norm family

$$\xi_{(n+n_0)\varphi+\psi} = (\|\cdot\|_{(n+n_0)\varphi_\omega+\psi_\omega})_{\omega\in\Omega}.$$

Let $\xi_n^L = (\|\cdot\|_{n,\omega}^L)_{\omega\in\Omega}$ be its restricted norm family on $H^0(X, L^{\otimes n})^{\oplus r}$ induced by the injection

$$H^0(X, L^{\otimes n})^{\oplus r} \longrightarrow H^0(X, L^{\otimes n+n_0} \otimes E).$$

Let $\xi_n^Q = (\|\cdot\|_{n,\omega}^Q)_{\omega\in\Omega}$ be its quotient family on $Q_n$ induced by the surjection

$$H^0(X, L^{\otimes n+n_0} \otimes E) \longrightarrow Q_n.$$

Then, by [**13**, Proposition 4.3.13, (4.26)],

$$\widehat{\deg}(H^0(X, L^{\otimes n})^{\oplus r}, \xi_n^L) + \widehat{\deg}(Q_n, \xi_n^Q) \leqslant \widehat{\deg}(H^0(X, L^{\otimes n+n_0} \otimes E), \xi_{(n+n_0)\varphi+\psi}).$$

Since $\dim \mathrm{Supp}(Q) < \dim X$, by Proposition 5.8,

$$\liminf_{n\to\infty} \frac{\widehat{\deg}(Q_n, (\|\cdot\|_{n,\omega}^Q)_{\omega\in\Omega})}{n^{d+1}} \geqslant 0.$$

Therefore, by the super-additivity of inferior limit, we obtain

$$\liminf_{n\to\infty} \frac{\widehat{\deg}(H^0(X, L^{\otimes n})^{\oplus r}, \xi_n^L)}{n^{d+1}/(d+1)!} \leqslant \liminf_{n\to\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes E))}{n^{d+1}/(d+1)!}. \tag{6.1}$$

Let us consider the homomorphism of identity

$$(H^0(X, L^{\otimes n})^{\oplus r}, (\|\cdot\|_{n\varphi_\omega}^{\oplus r})_{\omega\in\Omega}) \longrightarrow (H^0(X, L^{\otimes n})^{\oplus r}, (\|\cdot\|_{n,\omega}^L)_{\omega\in\Omega}),$$

where

$$\|(s_1, \ldots, s_r)\|_{n\varphi_\omega}^{\oplus r} = \begin{cases} \max\limits_{i\in\{1,\ldots,r\}} \|s_i\|_{n\varphi_\omega} & \text{if } \omega \in \Omega \setminus \Omega_\infty, \\ (\|s_1\|_{n\varphi_\omega}^2 + \cdots + \|s_r\|_{n\varphi_\omega}^2)^{1/2} & \text{if } \omega \in \Omega_\infty. \end{cases}$$

If $\omega \in \Omega \setminus \Omega_\infty$, then

$$\begin{aligned}\|(s_1,\dots,s_r)\|_{n,\omega}^L &\leqslant \|s_1e_1+\cdots+s_re_r\|_{(n+n_0)\varphi_\omega+\psi_\omega}\\ &\leqslant \max_{i\in\{1,\dots,r\}}\|s_i\|_{n\varphi_\omega}\|e_i\|_{n_0\varphi_\omega+\psi_\omega}\\ &\leqslant \|(s_1,\dots,s_r)\|_{n\varphi_\omega}^{\oplus r}\Big(\max_{i\in\{1,\dots,r\}}\|e_i\|_{n_0\varphi_\omega+\psi_\omega}\Big).\end{aligned}$$

Moreover, if $\omega \in \Omega_\infty$, then by Cauchy-Schwarz inequality

$$\begin{aligned}\|(s_1,\dots,s_r)\|_{n,\omega}^L &\leqslant \|s_1e_1+\cdots+s_re_r\|_{(n+n_0)\varphi_\omega+\psi_\omega}\\ &\leqslant \sum\nolimits_{i=1}^r\|s_i\|_{n\varphi_\omega}\|e_i\|_{n_0\varphi_\omega+\psi_\omega}\\ &\leqslant \Big(\sum\nolimits_{i=1}^r\|s_i\|_{n\varphi_\omega}\Big)\Big(\max_{i\in\{1,\dots,r\}}\|e_i\|_{n_0\varphi_\omega+\psi_\omega}\Big)\\ &\leqslant \sqrt{r}\|(s_1,\dots,s_r)\|_{n\varphi_\omega}^{\oplus r}\max_{i\in\{1,\dots,r\}}\|e_i\|_{n_0\varphi_\omega+\psi_\omega}.\end{aligned}$$

Therefore,

$$h(f_n) \leqslant \int_\Omega \max_{i\in\{1,\dots,r\}}\log\|e_i\|_{n_0\varphi_\omega+\psi_\omega}\nu(d\omega)+\frac{1}{2}\log(r)\operatorname{vol}(\Omega_\infty),$$

and hence, by [**13**, Proposition 4.3.18],

$$\begin{aligned}&r\,\widehat{\deg}(H^0(X,L^{\otimes n}),\xi_{n\varphi}) = \widehat{\deg}(H^0(X,L^{\otimes n})^{\oplus r},\xi_{n\varphi}^{\oplus r})\\ &\quad\leqslant \widehat{\deg}(H^0(X,L^{\otimes n})^{\oplus r},\xi_n^L)+\\ &\quad r\dim_k H^0(X,L^{\otimes n})\left(\int_\Omega \max_{i\in\{1,\dots,r\}}\log\|e_i\|_{n_0\varphi_\omega+\psi_\omega}\,\nu(d\omega)+\frac{1}{2}\log(r)\operatorname{vol}(\Omega_\infty)\right),\end{aligned}$$

where

$$\xi_{n\varphi} = (\|\cdot\|_{n\varphi_\omega})_{\omega\in\Omega},\quad \xi_{n\varphi}^{\oplus r} = (\|\cdot\|_{n\varphi_\omega}^{\oplus r})_{\omega\in\Omega}.$$

Thus,

$$r\,\widehat{\operatorname{vol}}_\chi(\overline{L}) \leqslant \liminf_{n\to\infty}\frac{\widehat{\deg}(H^0(X,L^{\otimes n})^{\oplus r},\xi_n^L)}{n^{d+1}/(d+1)!}.$$

Combining this inequality with (6.1), we obtain the assertion. □

***Corollary 6.3.*** — *Let $\pi : Y \to X$ be a generically finite morphism of geometrically integral projective varieties over $K$, $\overline{L} = (L,\varphi)$ be an adelic invertible $\mathcal{O}_X$-module and $\overline{M} = (M,\psi)$ be an adelic invertible $\mathcal{O}_Y$-module. If there are a birational morphism $f : X \to Z$ of geometrically integral projective varieties over $K$ and an ample invertible $\mathcal{O}_Z$-module $A$ such that $L = f^*(A)$, then*

$$\deg(\pi)\,\widehat{\operatorname{vol}}_\chi(\overline{L}) \leqslant \liminf_{n\to\infty}\frac{\widehat{\deg}\big((p\circ\pi)_*(\pi^*(\overline{L})^{\otimes n}\otimes\overline{M})\big)}{n^{d+1}/(d+1)!}.$$

*In particular,* $\deg(\pi)\,\widehat{\operatorname{vol}}_\chi(\overline{L}) \leqslant \widehat{\operatorname{vol}}_\chi(\pi^*(\overline{L}))$.

*Proof.* — Since $\pi^*(\overline{L}^{\otimes n}) \otimes \overline{M}$ is an adelic invertible $\mathcal{O}_Y$-module, one can see that

$$(L^{\otimes n} \otimes \pi_*(M), \pi_*(n\pi^*(\varphi) + \psi))$$

is sectionally adelic for all $n \geqslant 0$ (see the last paragraph of Section 3). Note that

$$\pi_*(n\pi^*(\varphi) + \psi) = n\varphi + \pi_*(\psi) \quad \text{and} \quad \mathrm{rk}(\pi_* M) = \deg(\pi).$$

Thus, by Theorem 6.2,

$$\deg(\pi)\, \widehat{\mathrm{vol}}_\chi(\overline{L}) \leqslant \liminf_{n\to+\infty} \frac{\widehat{\deg}(H^0(X, L^{\otimes n} \otimes \pi_*(M)), (\|\cdot\|_{n\varphi_\omega + \pi_*(\psi)_\omega})_{\omega\in\Omega})}{n^{d+1}/(d+1)!}.$$

Moreover,

$$(H^0(X, L^{\otimes n} \otimes \pi_*(M)), (\|\cdot\|_{n\varphi_\omega + \pi_*(\psi)_\omega})_{\omega\in\Omega})$$

is isometric to

$$(H^0(Y, \pi^*(L^{\otimes n}) \otimes M), (\|\cdot\|_{n\pi_\omega^*(\varphi_\omega) + \psi_\omega})_{\omega\in\Omega}).$$

Thus we obtain the required inequality. □

***Theorem 6.4.*** — *Let $\overline{L} = (L, \varphi)$ be an adelic invertible $\mathcal{O}_X$-module and $\overline{E} = (E, U, \psi)$ be a birationally adelic torsion free $\mathcal{O}_X$-module. We assume that there are a birational morphism $f : X \to Z$ of geometrically integral projective varieties over $K$ and an ample invertible $\mathcal{O}_Z$-module $A$ with $L = f^*(A)$. If either $(E, \psi)$ is an adelic invertible $\mathcal{O}_X$-module or $X$ is normal, then the sequence*

$$\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{E}))}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N},\ n \geqslant 1$$

*is convergent to* $\mathrm{rk}(E)\, \widehat{\mathrm{vol}}_\chi(\overline{L})$.

*Proof.* — In view of Theorem 6.2, it suffices to establish the following inequality

$$\limsup_{n\to\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{E}))}{n^{d+1}/(d+1)!} \leqslant \mathrm{rk}(E)\, \widehat{\mathrm{vol}}_\chi(\overline{L}).$$

First we assume that $(E, \psi)$ is an adelic invertible $\mathcal{O}_X$-module. Let us begin with the following claim:

***Claim 6.5.*** — *One has the following inequality:*

$$\limsup_{n\to\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{E}))}{n^{d+1}/(d+1)!} \leqslant \limsup_{n\to\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes (n+n_0)}))}{n^{d+1}/(d+1)!}$$

*for some positive integer $n_0$.*

*Proof.* — Since $L$ is nef and big, we can choose a positive integer $n_0$ and $s_0 \in H^0(X, L^{\otimes n_0} \otimes E^\vee) \setminus \{0\}$. Note that $s_0$ gives rise to an injective homomorphism

$$H^0(X, L^{\otimes n} \otimes E) \longrightarrow H^0(X, L^{\otimes (n+n_0)}).$$

Let $\xi_{\mathrm{sub},n} = (\|\cdot\|_{\mathrm{sub},n,\omega})_{\omega\in\Omega}$ be the restricted norm family of $H^0(X, L^{\otimes n}\otimes E)$ induced by the above injective homomorphism and $\xi_{(n+n_0)\varphi} = (\|\cdot\|_{(n+n_0)\varphi_\omega})_{\omega\in\Omega}$. In order to show Claim 6.5, it is sufficient to see the following two inequalities:

$$\limsup_{n\to\infty} \frac{\widehat{\deg}(H^0(X, L^{\otimes n}\otimes E), \xi_{\mathrm{sub},n})}{n^{d+1}/(d+1)!} \leqslant \limsup_{n\to\infty} \frac{\widehat{\deg}(H^0(X, L^{\otimes (n+n_0)}), \xi_{(n+n_0)\varphi})}{n^{d+1}/(d+1)!}$$

and

$$\limsup_{n\to\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{E}))}{n^{d+1}/(d+1)!} \leqslant \limsup_{n\to\infty} \frac{\widehat{\deg}(H^0(X, L^{\otimes n}\otimes E), \xi_{\mathrm{sub},n})}{n^{d+1}/(d+1)!}.$$

The first inequality is a consequence of Lemma 5.7, Proposition 5.8, [**15**, Lemma 1.2.16] and [**13**, Proposition 4.3.13, (4.26)]. Let us consider the homomorphism of identity

$$f : \Big(H^0(X, L^{\otimes n}\otimes E), (\|\cdot\|_{n\varphi_\omega+\psi_\omega})_{\omega\in\Omega}\Big) \longrightarrow \Big(H^0(X, L^{\otimes n}\otimes E), \xi_{\mathrm{sub},n}\Big).$$

For $s \in H^0(X, E\otimes L^{\otimes n})\setminus\{0\}$,

$$\begin{aligned}\frac{\|s\|_{\mathrm{sub},n,\omega}}{\|s\|_{n\varphi_\omega+\psi_\omega}} &= \frac{\|ss_0\|_{(n+n_0)\varphi_\omega}}{\|s\|_{n\varphi_\omega+\psi_\omega}}\\ &\leqslant \frac{\|s\|_{n\varphi_\omega+\psi_\omega}\|s_0\|_{n_0\varphi_\omega-\psi_\omega}}{\|s\|_{n\varphi_\omega+\psi_\omega}} = \|s_0\|_{n_0\varphi_\omega-\psi_\omega},\end{aligned}$$

so that $\|f\|_\omega \leqslant \|s_0\|_{n_0\varphi_\omega-\psi_\omega}$. Therefore, by [**13**, Proposition 4.3.18],

$$\begin{aligned}\widehat{\deg}(H^0(X, L^{\otimes n}\otimes E), (\|\cdot\|_{n\varphi_\omega+\psi_\omega})_{\omega\in\Omega}) &\leqslant \widehat{\deg}(H^0(X, L^{\otimes n}\otimes E), \xi_{\mathrm{sub},n})\\ &\quad + \dim H^0(X, L^{\otimes n}\otimes E)\int_\Omega \log\|s_0\|_{n_0\varphi_\omega-\psi_\omega}\nu(d\omega).\end{aligned}$$

Thus the second inequality follows. □

By Lemma 6.1, Theorem-Definition 5.5 and the relation

$$\lim_{n\to+\infty}\frac{(n+n_0)^{d+1}}{n^{d+1}} = 1,$$

we obtain that

$$\lim_{n\to+\infty}\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes(n+n_0)}))}{n^{d+1}/(d+1)!} = \widehat{\mathrm{vol}}_\chi(\overline{L}).$$

Hence Claim 6.5 leads to

$$\widehat{\mathrm{vol}}_\chi(\overline{L};\overline{E}) \leqslant \widehat{\mathrm{vol}}_\chi(\overline{L}),$$

as required.

Next we assume that $X$ is normal. We prove the assertion by induction on $r := \mathrm{rk}(E)$. Let $\mu : X' \to X$, $(E', \psi')$ and $U$ be a birational morphism, an adelic invertible $\mathcal{O}_{X'}$-module and a non-empty Zariski open set of $X$, respectively, as in Definition 3.20. First we suppose that $r = 1$.

***Claim 6.6.*** — *One has the following inequality:*

$$\limsup_{n\to\infty}\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{E}))}{n^{d+1}/(d+1)!}\leqslant\limsup_{n\to\infty}\frac{\widehat{\deg}((p\circ\mu)_*(\mu^*(\overline{L})^{\otimes n}\otimes\overline{E'}))}{n^{d+1}/(d+1)!}$$

*Proof.* — This is a consequence of Lemma 5.7, Proposition 5.8, Lemma 6.1 and [**13**, (4.26) in Proposition 4.3.13]. □

By Claim 6.6 together with the case where $(E,\psi)$ is an adelic invertible $\mathcal{O}_X$-module, one has

$$\limsup_{n\to\infty}\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{E}))}{n^{d+1}/(d+1)!}\leqslant\widehat{\mathrm{vol}}_\chi(\mu^*(\overline{L})).$$

On the other hand, since $X$ is normal, one can see that $\widehat{\mathrm{vol}}_\chi(\mu^*(\overline{L}))=\widehat{\mathrm{vol}}_\chi(\overline{L})$, as desired.

In the case where $r\geqslant 2$, considering a birational morphism $X''\to X'$ if necessarily, we may assume that there exists an exact sequence $0\to F'\to E'\to Q'\to 0$ on $X'$ such that $F'$ and $Q'$ are locally free, $\mathrm{rk}(F')=1$ and $\mathrm{rk}(Q')=r-1$. Let $\psi_{F'}$ be the submetric of $F'$ over $X'$ and $\psi_{Q'}$ be the quotient metric of $Q'$ over $X'$. Let $Q$ be the image of $E\to\mu_*(E')\to\mu_*(Q')$ and $F$ be the kernel of $E\to Q$. Shrinking $U$ if necessarily, $\psi_{Q'}$ and $\psi_{F'}$ descent to metric families $\psi_Q$ and $\psi_F$ of $Q|_U$ and $F|_U$. Note that $\overline{Q}=(Q,\psi_Q)$ and $\overline{F}=(F,\psi_F)$ are birationally adelic torsion free $\mathcal{O}_X$-modules by Proposition 3.7 and Corollary 3.18. Therefore, by hypothesis of induction,

$$\begin{cases}\limsup_{n\to\infty}\dfrac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{F}))}{n^{d+1}/(d+1)!}\leqslant\widehat{\mathrm{vol}}_\chi(L,\varphi),\\ \limsup_{n\to\infty}\dfrac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{Q}))}{n^{d+1}/(d+1)!}\leqslant(r-1)\,\widehat{\mathrm{vol}}_\chi(L,\varphi).\end{cases}$$

For any $n\in\mathbb{N}$, one has an exact sequence

$$0\to H^0(X,L^n\otimes)\to H^0(X,L^n\otimes E)\to H^0(X,L^n\otimes Q)\to H^1(X,L^{\otimes n}\otimes F).\quad(6.2)$$

Let $Q_n$ be the image of

$$H^0(X,L^{\otimes n}\otimes E)\longrightarrow H^0(X,L^{\otimes n}\otimes Q).$$

Let $\xi_{n,\mathrm{sub}}=(\|\cdot\|_{n,\mathrm{sub},\omega})_{\omega\in\Omega}$ be the restricted norm family of $\xi_{n\varphi+\psi}=(\|\cdot\|_{n\varphi_\omega+\psi_\omega})_{\omega\in\Omega}$ on $H^0(X,L^n\otimes F)$ and $\xi_{n,\mathrm{quot}}=(\|\cdot\|_{n,\mathrm{quot},\omega})_{\omega\in\Omega}$ be the quotient norm family of $\xi_{n\varphi+\psi}$ on $H^0(X,L^n\otimes Q)$. By [**13**, (4.28)],

$$\begin{aligned}&\widehat{\deg}(H^0(X,L^n\otimes E),\xi_{n\varphi+\psi})-\delta(H^0(X,L^n\otimes E),\xi_{n\varphi+\psi})\\&\quad\leqslant\Big(\widehat{\deg}(H^0(X,L^n\otimes F),\xi_{n,\mathrm{sub}})-\delta(H^0(X,L^n\otimes F),\xi_{n,\mathrm{sub}})\Big)\\&\qquad+\Big(\widehat{\deg}(H^0(X,L^n\otimes Q),\xi_{n,\mathrm{quot}})-\delta(H^0(X,L^n\otimes Q),\xi_{n,\mathrm{quot}})\Big),\end{aligned}$$

where for any adelic vector bundle $\overline{V}$ on $S$, $\delta(\overline{V})$ denotes the sum $\widehat{\deg}(\overline{V}) + \widehat{\deg}(\overline{V}^\vee)$. Let $\xi_{n\varphi+\psi_Q,\mathrm{sub}} = (\|\cdot\|_{n\varphi_\omega+\psi_{Q,\omega},\mathrm{sub}})_{\omega\in\Omega}$ be the restriction of $\xi_{n\varphi+\psi_Q} = (\|\cdot\|_{n\varphi_\omega+\psi_{Q,\omega}})_{\omega\in\Omega}$ to $Q_n$. It is easy to see that, for any $\omega \in \Omega$,

$$\|\cdot\|_{n,\mathrm{sub},\omega} = \|\cdot\|_{n\varphi_\omega+\psi_{F,\omega}}, \quad \|\cdot\|_{n,\mathrm{quot},\omega} \geqslant \|\cdot\|_{n\varphi_\omega+\psi_{Q,\omega},\mathrm{sub}}.$$

Thus, by [**13**, Proposition 4.3.18],

$$\widehat{\deg}(Q_n, \xi_{n,\mathrm{quot}}) \leqslant \widehat{\deg}(Q_n, \xi_{n\varphi+\psi_Q,\mathrm{sub}}),$$

so that

$$\begin{aligned}\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{E})) - \delta(p_*(\overline{L}^{\otimes n} \otimes \overline{E})) \\ \leqslant \Big(\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{F})) - \delta(H^0(X, L^n \otimes F), \xi_{n,\mathrm{sub}})\Big) \\ + \Big(\widehat{\deg}(Q_n, \xi_{n\varphi+\psi_Q,\mathrm{sub}}) - \delta(Q_n, \xi_{n,\mathrm{quot}})\Big).\end{aligned}$$

Moreover, by [**13**, Proposition 4.3.10],

$$\begin{cases} \lim\limits_{n\to\infty} \dfrac{\delta(p_*(\overline{L}^{\otimes n} \otimes \overline{E}))}{n^{d+1}} = 0, \\ \lim\limits_{n\to\infty} \dfrac{\delta(H^0(X, L^n \otimes F), \xi_{n,\mathrm{sub}})}{n^{d+1}} = 0, \\ \lim\limits_{n\to\infty} \dfrac{\delta(Q_n, \xi_{n,\mathrm{quot}})}{n^{d+1}} = 0, \end{cases}$$

so that one obtains

$$\limsup_{n\to+\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{E}))}{n^{d+1}/(d+1)!} \leqslant \widehat{\mathrm{vol}}_\chi(L, \varphi) + \limsup_{n\to+\infty} \frac{\widehat{\deg}(Q_n, \xi_{n\varphi+\psi_Q,\mathrm{sub}})}{n^{d+1}/(d+1)!},$$

and hence it is sufficient to show that

$$\limsup_{n\to+\infty} \frac{\widehat{\deg}(Q_n, \xi_{n\varphi+\psi_Q,\mathrm{sub}})}{n^{d+1}/(d+1)!} \leqslant \limsup_{n\to+\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{Q}))}{n^{d+1}/(d+1)!}. \tag{6.3}$$

***Claim 6.7.*** — *If we set $T_n = H^0(X, L^{\otimes n} \otimes Q)/Q_n$, then*

$$\lim_{n\to+\infty} \dim_K(T_n)/n^d = 0.$$

*Proof.* — By the Leray spectral sequence

$$E_2^{p,q} = H^p(Z, A^{\otimes n} \otimes R^q f_*(F)) \Longrightarrow H^{p+q}(X, L^{\otimes n} \otimes F),$$

if $n$ is sufficiently large, then one has an injective homomorphism

$$H^1(X, L^{\otimes n} \otimes F) \longrightarrow H^0(Z, A^{\otimes n} \otimes R^1 f_*(F))$$

so that

$$\lim_{n\to+\infty} \frac{\dim_K(H^1(X, L^{\otimes n} \otimes F))}{n^d} = 0$$

because $\mathrm{Supp}(R^1 f_*(F))$ has Krull dimension $< d$. Thus the assertion follows by (6.2). □

By Lemma 6.1, $\bigoplus_{n=0}^{\infty} H^0(X, Q \otimes L^n)$ is finitely generated over $\bigoplus_{n=0}^{\infty} H^0(X, L^n)$, so that $\bigoplus_{n=0}^{\infty} T_n$ is also finitely generated over it. Let $\xi_{T_n}$ be the quotient norm family of $\xi_{n\varphi+\psi_Q}$ on $T_n$. Then by Claim 6.7 together with Proposition 5.8, we obtain that

$$\liminf_{n\to+\infty} \frac{\widehat{\deg}(T_n, \xi_{T_n})}{n^{d+1}} \geqslant 0,$$

that is, for any $\varepsilon > 0$,

$$\frac{\widehat{\deg}(T_n, \xi_{T_n})}{n^{d+1}} \geqslant -\varepsilon$$

for sufficiently large $n$. Moreover, by [**13**, Proposition 4.3.13, (4.26)],

$$\frac{\widehat{\deg}(Q_n, \xi_{n\varphi+\psi_Q,\mathrm{sub}})}{n^{d+1}} + \frac{\widehat{\deg}(T_n, \xi_{T_n})}{n^{d+1}} \leqslant \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{Q}))}{n^{d+1}},$$

so that

$$\frac{\widehat{\deg}(Q_n, \xi_{n\varphi+\psi_Q,\mathrm{sub}})}{n^{d+1}} - \varepsilon \leqslant \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes Q))}{n^{d+1}}$$

for sufficiently large $n$. Thus,

$$\limsup_{n\to+\infty} \frac{\widehat{\deg}(Q_n, \xi_{n\varphi+\psi_Q,\mathrm{sub}})}{n^{d+1}} - \varepsilon \leqslant \limsup_{n\to+\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes Q))}{n^{d+1}}.$$

Since $\varepsilon$ is arbitrary, we obtain the inequality (6.3). □

***Corollary 6.8.*** — *Let $(E, U, \psi)$ be a birational adelic torsion free $\mathcal{O}_X$-module. If $X$ is normal and $L$ is ample, then*

$$\lim_{n\to+\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{E}))}{n^{d+1}/(d+1)!} = \mathrm{rk}(E)\, \widehat{\mathrm{vol}}_\chi(L, \varphi).$$

*Proof.* — This is a consequence of Theorem 6.2 and Theorem 6.4. □

## 7. Hilbert-Samuel property

Let $f : X \to \operatorname{Spec} K$ be a geometrically integral projective scheme over $\operatorname{Spec} K$, $d$ be the dimension of $X$ and $L$ be an ample invertible $\mathcal{O}_X$-module. We denote by $\mathscr{M}(L)$ the set of metrics families $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ such that all metrics $\varphi_\omega$ are semi-positive and that $(L, \varphi)$ forms an adelic line bundle on $X$.

***Definition 7.1.*** — We say that $\varphi \in \mathscr{M}(L)$ satisfies the *Hilbert-Samuel property* if the equality

$$\widehat{\mathrm{vol}}_\chi(L, \varphi) = ((L, \varphi)^{d+1})$$

holds, namely the $\chi$-volume and the self-intersection number of $(L, \varphi)$ coincides.

***Remark 7.2.*** — Note that Theorem-Definition 5.5 shows that, for any positive integer $n$, one has

$$\widehat{\mathrm{vol}}_\chi(L^{\otimes n}, n\varphi) = n^{d+1}\widehat{\mathrm{vol}}_\chi(L,\varphi).$$

Therefore, if $\varphi$ satisfies the Hilbert-Samuel property, then for any positive integer $n$, the metric family $n\varphi$ also satisfies the Hilbert-Samuel property. Conversely, if there exists a positive integer $n$ such that $n\varphi$ satisfies the Hilbert-Samuel property, then so does the metric family $\varphi$.

**7.1. Reduction to a special metric family.** — The purpose of this subsection is to show that, in order to show the Hilbert-Samuel property for all metrics families in $\mathscr{M}(L)$, it suffices to check the property for one arbitrary metric family in $\mathscr{M}(L)$.

***Lemma 7.3.*** — *Let $E$ be a finite-dimensional vector space over $K$. If $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ and $\xi' = (\|\cdot\|'_\omega)_{\omega\in\Omega}$ are two norm families on $E$, then one has*

$$d_\omega(\det(\xi), \det(\xi')) \leqslant r d_\omega(\xi,\xi'). \tag{7.1}$$

*In particular, if $\xi$ is strongly dominated, so is $\det(\xi)$.*

*Proof.* — Let $r$ be the dimension of $E$ over $K$. If $\eta$ is a non-zero element of $\det(E_\omega)$, then one has

$$\begin{aligned}\ln\|\eta\|_{\omega,\det} - \ln\|\eta\|'_{\omega,\det} &= \sup_{\substack{(s_1,\dots,s_r)\in E_\omega^r\\ \eta = s_1\wedge\cdots\wedge s_r}} \ln\|s_1\wedge\cdots\wedge s_r\|_{\omega,\det} - \sum_{i=1}^r \ln\|s_i\|'_\omega\\ &\leqslant \sup_{\substack{(s_1,\dots,s_r)\in E_\omega^r\\ \eta = s_1\wedge\cdots\wedge s_r}} \sum_{i=1}^r \ln\|s_i\|_\omega - \ln\|s_i\|'_\omega \leqslant r d_\omega(\xi,\xi').\end{aligned}$$

Interchanging $\xi$ and $\xi'$, the above inequality leads to

$$\ln\|\eta\|'_{\omega,\det} - \ln\|\eta\|_{\omega,\det} \leqslant r d_\omega(\xi,\xi').$$

Therefore, the inequality (7.1) holds. □

***Proposition 7.4.*** — *Assume that there exists a metric family $\psi \in \mathscr{M}(L)$ which satisfies the Hilbert-Samuel property. Then any metric family $\varphi \in \mathscr{M}(L)$ satisfies the Hilbert-Samuel property.*

*Proof.* — For any $n\in\mathbb{N}$, let $E_n$ be the $K$-vector space $H^0(X, L^{\otimes n})$ and $r_n$ be the dimension of $E_n$ of $K$. For any $\omega\in\Omega$, let $E_{n,\omega} = E_n \otimes_K K_\omega$,

$$d_{n,\omega} = \sup_{s\in E_{n,\omega}\setminus\{0\}} \left|\ln\|s\|_{n\varphi_\omega} - \ln\|s\|_{n\psi_\omega}\right|$$

be the distance of $\|\cdot\|_{n\varphi_\omega}$ and $\|\cdot\|_{n\psi_\omega}$, and

$$\delta_{n,\omega} = \sup_{\eta\in\det(E_{n,\omega})\setminus\{0\}} \ln\|\eta\|_{n\varphi_\omega,\det} - \ln\|\eta\|_{n\psi_\omega,\det}.$$

Note that the function $(\omega \in \Omega) \mapsto \delta_{n,\omega}$ is $\nu$-integrable, and one has

$$\int_\Omega \delta_{n,\omega}\, \nu(\mathrm{d}\omega) = \widehat{\deg}(p_*(L^{\otimes n}, n\psi)) - \widehat{\deg}(p_*(L^{\otimes n}, n\varphi)).$$

By Lemma 7.3, one has

$$|\delta_{n,\omega}| \leqslant r_n d_{n,\omega} \leqslant n r_n d_\omega(\varphi, \psi).$$

Note that the function

$$(\omega \in \Omega) \longrightarrow d_\omega(\varphi, \psi)$$

is dominated (see [**13**, Proposition 6.1.12]). Moreover, by [**7**, Theorem 1.7], one has

$$\lim_{n\to+\infty} \frac{\delta_{n,\omega}}{n^{d+1}/(d+1)!} = \sum_{j=0}^{d} \int_{X_\omega^{\mathrm{an}}} f_\omega(x)\, \mu_{(L_\omega,\varphi_\omega)^j (L_\omega,\psi_\omega)^{d-j}}(\mathrm{d}x),$$

where $f_\omega$ is the continuous function on $X_\omega^{\mathrm{an}}$ such that

$$\mathrm{e}^{f_\omega(\omega)} |\cdot|_{\psi_\omega}(x) = |\cdot|_{\varphi_\omega}$$

for any $x \in X_\omega^{\mathrm{an}}$. Hence Theorem-Definition 5.5 and Lebesgue's dominated convergence theorem leads to (see Remark 5.6)

$$\begin{aligned}
\widehat{\mathrm{vol}}_\chi(L,\psi) - \widehat{\mathrm{vol}}_\chi(L,\varphi) &= \lim_{n\to+\infty} \frac{1}{n^{d+1}/(d+1)!} \int_\Omega \delta_{n,\omega}\, \nu(d\omega) \\
&= \sum_{j=0}^{d} \int_\Omega \int_{X_\omega^{\mathrm{an}}} f_\omega(x)\, \mu_{(L_\omega,\varphi_\omega)^j (L_\omega,\psi_\omega)^{d-j}}(\mathrm{d}x)\, \nu(\mathrm{d}\omega) \\
&= ((L,\psi)^{d+1}) - ((L,\varphi)^{d+1}).
\end{aligned}$$

The proposition is thus proved. □

***Definition 7.5.*** — Let $X$ be a geometrically integral projective scheme over $\operatorname{Spec} K$ and $L$ be an ample invertible $\mathcal{O}_X$-module. If there exists a metric family $\varphi \in \mathscr{M}(L)$ which satisfies the Hilbert-Samuel property, or equivalently, any metric family $\varphi \in \mathscr{M}(L)$ satisfies the Hilbert-Samuel property (see Proposition 7.4), we say that the ample invertible $\mathcal{O}_X$-module $L$ *satisfies the Hilbert-Samuel property.*

***Remark 7.6.*** — The proof of Proposition 7.4 actually shows a more precise result: the function

$$(\varphi \in \mathscr{M}(L)) \longrightarrow \widehat{\mathrm{vol}}_\chi(L,\varphi) - ((L,\varphi)^{d+1})$$

is constant.

**7.2. Case of a projective space.** — In this subsection, we assume that $X = \mathbb{P}^d_K$ is the projective case and $L = \mathcal{O}_{\mathbb{P}^d_K}(1)$ is the universal line bundle. We show that any metric family in $\mathscr{M}(L)$ satisfies the Hilbert-Samuel property. Without loss of generality (by Proposition 7.4), we consider a particular case as follows. Let $E$ be a $(d+1)$-dimensional vector space over $K$ and $(e_i)_{i=0}^d$ be a basis of $E$. Let $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ be the Hermitian norm family on $E$ such that $(e_i)_{i=0}^d$ forms an orthonormal basis of $E$ with respect to $\|\cdot\|_\omega$. We then identify $\mathbb{P}^d_K$ with $\mathbb{P}(E)$ and let $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ be the quotient metric family on $L$ induced by $\xi$. Note that, for any integer $n \in \mathbb{N}$, the vector space $H^0(X, L^{\otimes n})$ is isomorphic to the symmetric power $S^n(E)$. We denote by $r_n$ the dimension of $S^n(E)$. One has

$$r_n = \binom{n+d}{d}.$$

***Definition 7.7.*** — Let $\omega \in \Omega$ such that $|\cdot|_\omega$ is non-Archimedean. Let $x$ be the point in $\mathbb{P}(E_\omega)^{\mathrm{an}}$ which consists of the generic scheme point of $\mathbb{P}(E_\omega)$ equipped with the absolute value

$$|\cdot|_x : k\big(\tfrac{e_0}{e_r}, \ldots, \tfrac{e_{r-1}}{e_r}\big) \longrightarrow \mathbb{R}_{\geqslant 0}$$

such that, for any

$$P = \sum_{\boldsymbol{a}=(a_0,\ldots,a_{r-1})\in\mathbb{N}^d} \lambda_{\boldsymbol{a}} \Big(\frac{e_0}{e_r}\Big)^{a_0} \cdots \Big(\frac{e_{r-1}}{e_r}\Big)^{a_{r-1}} \in k\big[\tfrac{e_0}{e_r}, \ldots, \tfrac{e_{r-1}}{e_r}\big],$$

one has

$$|P|_x = \max_{\boldsymbol{a}\in\mathbb{N}^d} |\lambda_{\boldsymbol{a}}|_\omega.$$

Note that the point $x$ does not depend on the choice of the orthonormal basis $(e_j)_{j=0}^r$. In fact, the norm $\|\cdot\|$ induces a symmetric algebra norm on $K_\omega[E_\omega]$ (which is often called a *Gauss norm*) and hence defines an absolute value on the fraction field of $K_\omega[E_\omega]$. The restriction of this absolute value to the field of rational functions on $\mathbb{P}(E_\omega)$ identifies with $|\cdot|_x$. Hence $x$ is called the *Gauss point* of $\mathbb{P}(E_\omega)^{\mathrm{an}}$.

***Lemma 7.8.*** — *Let $\omega$ be an element of $\Omega$ such that $|\cdot|_\omega$ is non-Archimedean, and $n \in \mathbb{N}$. Let $\|\cdot\|_{n,\omega}$ be the $\varepsilon$-tensor power of $\|\cdot\|_\omega$ on the tensor power space $E_\omega^{\otimes n}$ and let $\|\cdot\|'_{n,\omega}$ be the quotient norm of $\|\cdot\|_{n,\omega}$ by the quotient homomorphism $E_\omega^{\otimes n} \to S^n(E_\omega)$. Then the norm $\|\cdot\|'_{n,\omega}$ coincides with the supremum norm $\|\cdot\|_{n\varphi_\omega}$ of the metric $n\varphi_\omega$ on $L_\omega^{\otimes n}$.*

*Proof.* — For any $\omega \in \Omega$, we denote by $E_\omega$ the $K_\omega$-vector space $E \otimes_K K_\omega$. By **[13**, Propositions 1.3.16 and 1.2.36], if we consider the Segre embedding $\mathbb{P}(E_\omega) \to \mathbb{P}(E_\omega^{\otimes n})$, then the metric $n\varphi_\omega$ identifies with the quotient metric induced by the norm $\|\cdot\|_{n,\omega}$. Moreover, if we denote by $\mathcal{O}_{E_\omega^{\otimes n}}(1)$ the universal invertible sheaf of $\mathbb{P}(E_\omega^{\otimes n})$ and by $\psi_\omega$ the quotient metric on this invertible sheaf induced by the norm $\|\cdot\|_{n,\omega}$. By **[13**, Proposition 2.2.22], the supremum norm $\|\cdot\|_{\psi_\omega}$ on

$$H^0(\mathbb{P}(E_\omega^{\otimes n}), \mathcal{O}_{E_\omega^{\otimes n}}(1)) = E_\omega^{\otimes n}$$

of the metric $\psi_\omega$ coincides with $\|\cdot\|_{n,\omega}$. Since $L^{\otimes n}$ is the restriction of $\mathcal{O}_{E_\omega^{\otimes n}}(1)$ to $X$ and the restriction map

$$H^0(\mathbb{P}(E_\omega^{\otimes n}), \mathcal{O}_{E_\omega^{\otimes n}}(1)) \longrightarrow H^0(\mathbb{P}(E_\omega), L_\omega^{\otimes n})$$

identifies with the quotient homomorphism $E_\omega^{\otimes n} \to S^n(E_\omega)$. In particular, the supremum norm $\|\cdot\|_{\varphi_\omega^{\otimes n}}$ is bounded from above by the quotient norm $\|\cdot\|'_{n,\omega}$.

Let $x$ be the Gauss point of the Berkovich analytic space $\mathbb{P}(E_\omega)^{\mathrm{an}}$ (see Definition 7.7). If

$$F = \sum_{\substack{I=(a_0,\dots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}} \lambda_I e_0^{a_0}\cdots e_d^{a_d}$$

is an element of $S^n(E)$, then the relation

$$F(x) = \Bigg(\sum_{\substack{I=(a_0,\dots,a_d)\in\mathbb{N}^{d}\\ a_0+\cdots+a_d=n}} \lambda_I \Big(\frac{e_1}{e_0}\Big)^{a_1}\cdots\Big(\frac{e_d}{e_0}\Big)^{a_d}\Bigg) e_0(x)^{\otimes n}$$

holds. In particular, one has

$$\|F\|_{n\varphi_\omega} \geqslant |F|_{n\varphi_\omega}(x) = \max_{\substack{I=(a_0,\dots,a_d)\in\mathbb{N}^{d}\\ a_0+\cdots+a_n=d}} |\lambda_I|_\omega.$$

Since $F$ is the image of the element

$$\widetilde{F} = \sum_{\substack{I=(a_0,\dots,a_d)\in\mathbb{N}^{d}\\ a_0+\cdots+a_d=n}} \lambda_I e_0^{\otimes a_0}\otimes\cdots\otimes e_d^{\otimes a_d}$$

by the quotient map $E_\omega^{\otimes n} \to S^n(E_\omega)$, we obtain that

$$\|F\|_{n\varphi_\omega} \geqslant \|\widetilde{F}\|_{n,\omega} \geqslant \|F\|'_{n,\omega}.$$

Therefore the equality $\|\cdot\|_{n\varphi_\omega} = \|F\|'_{n,\omega}$ holds. □

***Remark 7.9.*** — As a byproduct, the proof of the above lemma shows that, for any

$$F = \sum_{\substack{I=(a_0,\dots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}} \lambda_I e_0^{a_0}\cdots e_d^{a_d} \in S^n(E_\omega),$$

one has

$$\|F\|_{n\varphi_\omega} = \max_{\substack{I=(a_0,\dots,a_d)\in\mathbb{N}^{d}\\ a_0+\cdots+a_n=d}} |\lambda_I|_\omega.$$

In other words, the family

$$(e_0^{a_0}\cdots e_d^{a_d})_{\substack{(a_0,\dots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}}$$

forms an orthonormal basis of $(S^d(E_\omega), \|\cdot\|_{n\varphi_\omega})$.

***Lemma 7.10.*** — *For any integer $d \in \mathbb{N}$ and any any $x > 0$, let*

$$P_{d+1,x} = \{(t_0, \dots, t_d) \in \mathbb{R}^{d+1}_{\geqslant 0} \,|\, t_0 + \cdots + t_d \leqslant x\},$$
$$\Delta_{d,x} = \{(t_0, \dots, t_d) \in \mathbb{R}^{d+1}_{\geqslant 0} \,|\, t_0 + \cdots + t_d = x\}.$$

*We denote by $\mathrm{vol}_{d+1}$ the Lebesgue measure on $\mathbb{R}^d$. For any affine hyperplane of $\mathbb{R}^d$, we denote by $\nu_d$ the translate of the Haar measure on the underlying hyperplane which is normalized with respect to the canonical Euclidean norm on $\mathbb{R}^{d+1}$ (namely the parallelotope spanned by an orthonormal basis has volume 1).*

(1) *The volume of $P_{d+1,x}$ with respect to $\mathrm{vol}_{d+1}$ is $x^{d+1}/(d+1)!$.*
(2) *The volume of $\Delta_{d,x}$ with respect to $\nu_d$ is $x^d\sqrt{d+1}/d!$.*
(3) *Let $\mu_d$ be the uniform probability distribution on $\Delta_{d,x}$. One has*

$$\int_{\Delta_{d,1}} t_0 \ln(t_0) + \cdots + t_d \ln(t_d)\, \mu_d(\mathrm{d}t) = -\frac{1}{d+1} \sum_{m=1}^{d} \sum_{\ell=1}^{m} \frac{1}{\ell}.$$

*Proof.* — (1) We reason by induction on $d$. The case where $d = 0$ is trivial. In the following we assume the induction hypothesis that the lemma holds for $\mathbb{R}^d$. By Fubini's theorem, we have

$$\mathrm{vol}_{d+1}(P_{d+1,x}) = \int_0^x \mathrm{vol}_d(P_{d,x-t})\, \mathrm{d}t = \int_0^x \frac{(x-t)^d}{d!} = \frac{x^{d+1}}{(d+1)!}.$$

(2) The distance from the origin to the affine hyperplane containing $\Delta_{d,x}$ is $x/\sqrt{d+1}$. Therefore, by the equality

$$\mathrm{vol}_{d+1}(P_{d+1,x}) = \frac{1}{d+1} \cdot \frac{x}{\sqrt{d+1}} \nu_d(\Delta_{d,x}),$$

we obtain

$$\nu_d(\Delta_{d,x}) = \sqrt{d+1}\frac{x^d}{d!}.$$

(3) By Fubini's theorem, one has

$$\begin{aligned}
\int_{P_{d+1,x}} t_0 \ln(t_0)\ \mathrm{vol}_{d+1}(\mathrm{d}t_0, \dots, \mathrm{d}t_d) &= \int_0^x t \ln(t)\, \mathrm{vol}_d(P_{d,x-t})\, \mathrm{d}t \\
&= \frac{1}{d!} \int_0^x t(x-t)^d \ln(t)\, \mathrm{d}t = \frac{1}{d!} \sum_{i=0}^{d} (-1)^i \binom{d}{i} x^{d-i} \int_0^x t^{i+1} \ln(t)\, \mathrm{d}t \\
&= \frac{1}{d!} \sum_{i=0}^{d} (-1)^i \binom{d}{i} x^{d-i} \frac{1}{i+2} \left( x^{i+2} \ln(x) - \frac{1}{i+2} x^{i+2} \right) \\
&= \frac{x^{d+2} \ln(x)}{d!} \sum_{i=0}^{d} (-1)^i \binom{d}{i} \frac{1}{i+2} - \frac{x^{d+2}}{d!} \sum_{i=0}^{d} (-1)^i \binom{d}{i} \frac{1}{(i+2)^2}.
\end{aligned}$$

By a change of variables, we obtain

$$\int_{P_{d+1,x}} t_0 \ln(t_0)\ \mathrm{vol}_{d+1}(\mathrm{d}t_0, \dots, \mathrm{d}_{t_d}) = \frac{1}{\sqrt{d+1}} \int_0^x \int_{\Delta_{d,u}} t_0 \ln(t_0)\, \nu_d(\mathrm{d}t)\, \mathrm{d}u.$$

Taking the derivative with respect to $x$, we obtain

$$\frac{(d+2)x^{d+1}\ln(x)+x^{d+1}}{d!}\sum_{i=0}^{d}(-1)^i\binom{d}{i}\frac{1}{i+2}-\frac{(d+2)x^{d+1}}{d!}\sum_{i=0}^{d}(-1)^i\binom{d}{i}\frac{1}{(i+2)^2}$$
$$=\frac{1}{\sqrt{d+1}}\int_{\Delta_{d,x}}t_0\ln(t_0)\,\nu_d(\mathrm{d}t)=\frac{\nu_d(\Delta_{d,x})}{\sqrt{d+1}}\int_{\Delta_{d,x}}t_0\ln(t_0)\,\mu_d(\mathrm{d}t).$$

In particular, one has

$$\int_{\Delta_{d,1}}t_0\ln(t_0)\,\mu_d(\mathrm{d}t)=\sum_{i=0}^{d}(-1)^i\binom{d}{i}\frac{1}{i+2}\Big(1-\frac{d+2}{i+2}\Big)$$
$$=\sum_{i=0}^{d}(-1)^i\frac{d!}{i!(d-i)!}\cdot\frac{i-d}{(i+2)^2}=-\frac{1}{d+1}\sum_{i=0}^{d-1}(-1)^i\frac{(d+1)!}{(i+2)!(d-i-1)!}\cdot\frac{i+1}{i+2}.$$

Therefore

$$(d+1)\int_{\Delta_{d,1}}t_0\ln(t_0)\,\mu_d(\mathrm{d}t)-d\int_{\Delta_{d-1,1}}t_0\ln(t_0)\,\mu_{d-1}(\mathrm{d}t)$$
$$=-\sum_{i=0}^{d-1}(-1)^i\frac{(d+1)!}{(i+2)!(d-i-1)!}\cdot\frac{i+1}{i+2}+\sum_{i=0}^{d-2}(-1)^i\frac{d!}{(i+2)!(d-i-2)!}\cdot\frac{i+1}{i+2}$$
$$=-\sum_{i=0}^{d-1}(-1)^i\frac{d!}{(i+2)!(d-i-1)!}\cdot\frac{i+1}{i+2}(d+1-(d-i-1))$$
$$=-\sum_{i=0}^{d-1}(-1)^i\frac{d!}{(i+2)!(d-i-1)!}((i+2)-1)$$
$$=-\sum_{i=0}^{d-1}(-1)^i\left(\binom{d}{i+1}-\frac{1}{d+1}\binom{d+1}{i+2}\right)$$
$$=\sum_{i=1}^{d}(-1)^i\binom{d}{i}+\frac{1}{d+1}\sum_{i=2}^{d+1}(-1)^i\binom{d+1}{i}$$
$$=-1+\frac{1}{d+1}(-1+(d+1))=-\frac{1}{d+1}.$$

Combining with

$$2\int_{\Delta_{1,1}}t_0\ln(t_0)\,\mu_1(\mathrm{d}t)=2\int_0^1 t\ln(t)\,\mathrm{d}t=-\int_0^1 t\,\mathrm{d}t=-\frac{1}{2},$$

by induction we obtain

$$(d+1)\int_{\Delta_{d,1}}t_0\ln(t_0)\,\mu_d(\mathrm{d}t)=-\sum_{i=1}^{d}\frac{1}{i+1}=-\sum_{\ell=2}^{d+1}\frac{1}{\ell}.$$

By symmetry of $(t_0, \dots, t_d)$, we get

$$(d+1)\sum_{i=0}^{d}\int_{\Delta_{d,1}} t_i \ln(t_i)\,\mu_d(\mathrm{d}t) = -(d+1)\sum_{\ell=2}^{d+1}\frac{1}{\ell}.$$

Since

$$\begin{aligned}\sum_{m=1}^{d}\sum_{\ell=1}^{m}\frac{1}{\ell} &= \sum_{\ell=1}^{d}\sum_{m=\ell}^{d}\frac{1}{\ell} = \sum_{\ell=1}^{d}\frac{d+1-\ell}{\ell} = (d+1)\sum_{\ell=1}^{d}\frac{1}{\ell} - d\\ &= (d+1)\sum_{\ell=2}^{d+1}\frac{1}{\ell} + (d+1) - \frac{d+1}{d+1} - d = (d+1)\sum_{\ell=2}^{d+1}\frac{1}{\ell},\end{aligned}$$

we obtain the desired result. □

***Proposition 7.11.*** — *The universal invertible sheaf $\mathcal{O}_{\mathbb{P}^d_K}(1)$ satisfies the Hilbert-Samuel property.*

*Proof.* — By Proposition 7.4, it suffices to prove that the particular quotient metric family $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ defined in the beginning of the subsection satisfies the Hilbert-Samuel property. For any $n \in \mathbb{N}$, let

$$\eta_n = \bigwedge_{\substack{(a_0,\dots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}} e_0^{a_0}\cdots e_d^{a_d} \in \det(S^n(E)).$$

By Lemma 7.8 and [**13**, Proposition 1.2.23], for any $\omega \in \Omega$ such that $|\cdot|_\omega$ is non-Archimedean, one has

$$\|\eta_n\|_{n\varphi_\omega,\det} = 1.$$

Let $\omega$ be an element of $\Omega$ such that $|\cdot|_\omega$ is Archimedean. Similarly to Lemma 7.8, for each $n \in \mathbb{N}$, we let $\|\cdot\|_{n,\omega}$ be the orthogonal tensor power norm on $E_\omega^{\otimes n}$ and $\|\cdot\|'_{n,\omega}$ be its quotient norm on $S^n(E_\omega)$. Note that

$$(e_0^{a_0}\cdots e_d^{a_d})_{\substack{(a_0,\dots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}}$$

forms an orthogonal basis of $(S^d(E_\omega), \|\cdot\|'_{n,\omega})$ and

$$\|e_0^{a_0}\cdots e_d^{a_d}\|'_{n,\omega} = \Big(\frac{a_0!\cdots a_d!}{n!}\Big)^{\frac{1}{2}}.$$

By [**13**, Proposition 1.2.25], one has

$$\|\eta_n\|'_{n,\omega,\det} = \prod_{\substack{(a_0,\dots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}} \Big(\frac{a_0!\cdots a_d!}{n!}\Big)^{\frac{1}{2}}.$$

In particular, using Stirling's formula one obtains

$$\lim_{n\to+\infty}\frac{\ln\|\eta_n\|'_{n,\omega,\det}}{nr_n}=-\frac{1}{2}\int_\Delta(t_0\ln(t_0)+\cdots+t_d\ln(t_d))\,\mathrm{d}\mu$$
$$=\frac{1}{2(d+1)}\sum_{m=1}^{d}\sum_{\ell=1}^{m}\frac{1}{\ell},$$

where $\mu$ denotes the uniform probability measure on the simplex

$$\Delta=\{(t_0,\ldots,t_d)\in\mathbb{R}^{d+1}_{\geqslant 0}\mid t_0+\cdots+t_d=1\},$$

and the second equality comes from Lemma 7.10.

By [**4**, Lemma 4.3.6] and [**18**, Lemma 30] (see also [**23**, VIII.2.5 lemma 2]), one has

$$\sup_{s\in S^n(E_\omega)\setminus\{0\}}\left|\ln(r_n^{-\frac{1}{2}}\|s\|'_{n,\omega})-\ln\|s\|_{\varphi_\omega^{\otimes n}}\right|=O(\ln(n)).$$

Moreover,

$$\ln(r_n^{-\frac{1}{2}})=-\frac{1}{2}\ln r_n=O(\ln(n)).$$

Hence by Lemma 7.3 we obtain

$$\lim_{n\to+\infty}\frac{\ln\|\eta_n\|_{\varphi_\omega^{\otimes n},\det}}{nr_n}=\lim_{n\to+\infty}\frac{\ln\|\eta_n\|'_{n,\omega,\det}}{nr_n}.$$

The proposition is thus proved. □

## 8. Trivial valuation case

In this section, we show the Hilbert-Samuel property in the trivial valuation case. Let $v=(k,|\cdot|)$ be a trivially valued field. Let us begin with the following Lemma:

***Lemma 8.1.*** — *Let $X$ be a projective integral scheme of dimension $d$ over* $\operatorname{Spec}k$ *and $L$ be a very ample invertible $\mathcal{O}_X$-module. Let $\|\cdot\|$ be the trivial norm on $H^0(X,L)$, that is, $\|e\|=1$ for $e\in H^0(X,L)\setminus\{0\}$. Let $\varphi$ be the Fubuni-Study metric of $L$ induced by the surjective homomorphism $H^0(X,L)\otimes\mathcal{O}_X\to L$ and $\|\cdot\|$. Then we have*

$$\widehat{\mathrm{vol}}_\chi(L,\varphi)=((L,\varphi)^{d+1})_v=0,$$

*where in the construction of $\widehat{\mathrm{vol}}_\chi(L,\varphi)$ we consider the adelic curve consisting of one copy of the trivial absolute value on $k$ and the counting measure.*

*Proof.* — Let $X\hookrightarrow\mathbb{P}^\ell_k$ be the embbedding given by $L$, where $\ell=\dim_k H^0(X,L)-1$. We can find a positive integer $n_0$ such that $H^0(\mathbb{P}^\ell_k,\mathcal{O}_{\mathbb{P}^\ell_k}(n))\to H^0(X,L^{\otimes n})$ is surjective for all $n\geqslant n_0$. In order to see $\widehat{\mathrm{vol}}_\chi(L,\varphi)=0$, it is sufficient to show that the norm $\|\cdot\|_{n\varphi}$ is trivial for all $n\geqslant n_0$. As $H^0(\mathbb{P}^\ell_k,\mathcal{O}_{\mathbb{P}^\ell_k}(n))=\mathrm{Sym}^n(H^0(X,L))$, one

has that $\mathrm{Sym}^n(H^0(X,L)) \to H^0(X,L^{\otimes n})$ is surjective for all $n \geqslant n_0$. Let $(T_0,\dots,T_\ell)$ be a homogeneous coordinate of $\mathbb{P}^\ell_k$. For $n \geqslant n_0$ and $s \in H^0(X,L^{\otimes n})$, if

$$s \equiv \sum_{\substack{(i_0,\dots,i_\ell)\in\mathbb{N}^{\ell+1}\\ i_0+\cdots+i_\ell=n}} a_{i_0,\dots,i_\ell} T_0^{i_0}\cdots T_\ell^{i_\ell} \mod \mathrm{Ker}(\mathrm{Sym}^n(H^0(X,L)) \to H^0(X,L^{\otimes n})),$$

then

$$\|s\|_{n\varphi} = \sup_{x\in(X\cap U_0)^{\mathrm{an}}} \frac{\Big|\sum_{\substack{(i_0,\dots,i_\ell)\in\mathbb{N}^{\ell+1}\\ i_0+\cdots+i_\ell=n}} a_{i_0,\dots,i_\ell} z_1^{i_1}\cdots z_\ell^{i_\ell}\Big|_x}{\Big(\max\{1,|z_1|_x,\dots,|z_\ell|_x\}\Big)^n},$$

where $z_i = T_i/T_0$ and $U_0 = \{(T_0,\dots,T_\ell)\in\mathbb{P}^\ell_k : T_0 \neq 0\}$. Note that

$$\begin{aligned}\Big|\sum_{\substack{(i_0,\dots,i_\ell)\in\mathbb{N}^{\ell+1}\\ i_0+\cdots+i_\ell=n}} a_{i_0,\dots,i_\ell} z_1^{i_1}\cdots z_\ell^{i_\ell}\Big|_x &\leqslant \max\{|z_1|_x^{i_1}\cdots|z_\ell|_x^{i_\ell} : (i_0,\dots,i_\ell)\in\mathbb{N}^{\ell+1}, i_0+\cdots+i_\ell=n\}\\ &\leqslant \Big(\max\{1,|z_1|_x,\dots,|z_\ell|_x\}\Big)^n,\end{aligned}$$

and hence $\|s\|_{n\varphi} \leqslant 1$. Let $k^{\mathrm{ac}}$ be an algebraic closure of $k$. We assume $s \neq 0$. We choose $\xi = (1,\xi_1,\dots,\xi_n) \in X(k^{\mathrm{ac}})$ such that $s(\xi) \neq 0$. Then, as $\sum_{\substack{(i_0,\dots,i_\ell)\in\mathbb{N}^{\ell+1}\\ i_0+\cdots+i_\ell=n}} a_{i_0,\dots,i_\ell}\xi_1^{i_1}\cdots\xi_\ell^{i_\ell} \in k^{\mathrm{an}}\setminus\{0\}$ and $\xi_1,\dots,\xi_\ell \in k^{\mathrm{ac}}$, one has

$$\Big|\sum_{\substack{(i_0,\dots,i_\ell)\in\mathbb{N}^{\ell+1}\\ i_0+\cdots+i_\ell=n}} a_{i_0,\dots,i_\ell}\xi_1^{i_1}\cdots\xi_\ell^{i_\ell}\Big|_{v'} = 1 \quad\text{and}\quad \max\{1,|\xi_1|_{v'},\dots,|\xi_\ell|_{v'}\} = 1$$

where $v'$ is the pair of $k^{\mathrm{ac}}$ and its trivial absolute value. Therefore, $\|s\|_{n\varphi} = 1$.

Next let us see that $((L,\varphi)^{d+1})_v = 0$. Note that

$$H^0(\mathbb{P}^\ell_k,\mathcal{O}_{\mathbb{P}^\ell_k}(1)) = H^0(X,L) \quad\text{and}\quad \mathrm{Sym}^n(H^0(\mathbb{P}^\ell_k,\mathcal{O}_{\mathbb{P}^\ell_k}(1))) = H^0(\mathbb{P}^\ell_k,\mathcal{O}_{\mathbb{P}^\ell_k}(n))$$

for $n \geqslant 1$. Let $\psi$ be the Fubuni-Study metric of $\mathcal{O}_{\mathbb{P}^\ell_k}(1)$ induced by the surjective homomorphism $H^0(\mathbb{P}^\ell_k,\mathcal{O}_{\mathbb{P}^\ell_k}(1))\otimes\mathcal{O}_{\mathbb{P}^\ell_k} \to \mathcal{O}_{\mathbb{P}^\ell_k}(1)$ and $\|\cdot\|$. Then $\psi|_{X^{\mathrm{an}}} = \varphi$. In the same way as before, $\|\cdot\|_{n\psi}$ on $H^0(\mathbb{P}^\ell_k,\mathcal{O}_{\mathbb{P}^\ell_k}(n))$ is trivial for $n \geqslant 1$. Therefore, the induced norm on $H^0(\check{\mathbb{P}}^\ell_k\times\cdots\times\check{\mathbb{P}}^\ell_k,\mathcal{O}_{\check{\mathbb{P}}^\ell_k}(\delta)\boxtimes\cdots\boxtimes\mathcal{O}_{\check{\mathbb{P}}^\ell_k}(\delta))$ is also trivial, where $\delta = (L^d)$. Thus the assertion follows. □

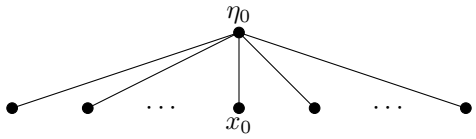
***Theorem 8.2.*** — *Assume that, for any $\omega\in\Omega$, $|\cdot|_\omega$ is the trivial absolute value on $K$. Then any ample line bundle $L$ on $X$ satisfies the Hilbert-Samuel property.*

*Proof.* — By Remark 7.2, we may assume that $L$ is very ample. Let $E$ be the vector space $H^0(X,L)$. For any $\omega\in\Omega$, we denote by $\|\cdot\|_\omega$ the trivial norm on $E = E_\omega$. Let $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ and $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ be the quotient metric family on $L$ induced by $\xi$ and the canonical closed embedding $X\to\mathbb{P}(E)$. Then, Lemma 8.1 implies

$$\mathrm{vol}_\chi(L,\varphi) = ((L,\varphi)^{d+1}) = 0.$$

Therefore, by Proposition 7.4 we obtain that the invertible sheaf $L$ satisfies the Hilbert-Samuel property. □

***Remark 8.3.*** — In **[14]**, an intersection product of metrized divisors has been introduced in the setting of curves over a trivially valued field $(k, |\cdot|)$. Let $X$ be a regular projective curve over $\operatorname{Spec} k$. Recall that the Berkovich space $X^{\mathrm{an}}$ is an infinite tree




where the root point $\eta_0$ corresponds to the generic point of $X$ together with the trivial absolute value on $\kappa(\eta)$, and each leaf $x_0$ corresponds to the closed point $x$ together with the trivial absolute value on $\kappa(x)$. Moreover, each branch $]\eta_0, x_0[$ is parametrized by $]0, +\infty[$, where $t \in ]0, +\infty[$ corresponds to the generic point $\eta$ together with the absolute value

$$|\cdot|_{x,t} = \exp(-t \operatorname{ord}_x(\cdot)).$$

We denote by $t(\cdot) : X^{\mathrm{an}} \to [0, +\infty]$ the parametrization map, where $t(\eta_0) = 0$ and $t(x_0) = +\infty$. Let $D$ be a Cartier divisor on $X$. Recall that a Green function $g$ of $D$ is of the form

$$g = g_D + \varphi_g,$$

where $g_D$ is the canonical Green function of $D$, which is defined as

$$g_D(\xi) = \operatorname{ord}_x(D) t(\xi),$$

and $\varphi_g$ is a continuous real-valued function on $X^{\mathrm{an}}$ (which is hence bounded since $X^{\mathrm{an}}$ is compact). Then, the intersection number of two integrable metrized Cartier divisor $\overline{D}_0 = (D_0, g_0)$ and $\overline{D}_1 = (D_1, g_1)$ has been defined as

$$g_1(\eta_0) \deg(D_0) + g_0(\eta_0) \deg(D_1) - \sum_{x \in X^{(1)}} [\kappa(x) : k] \int_0^{+\infty} \varphi'_{g_0 \circ \xi_x}(t) \varphi'_{g_1 \circ \xi_x}(t) \, \mathrm{d}t, \quad (8.1)$$

where $X^{(1)}$ is the set of closed points of $X$, $\xi_x : [0, +\infty] \to [\eta_0, x_0]$ is the map sending $t \in [0, +\infty]$ to the point in $[\eta_0, x_0]$ of parameter $t$, and the function $\varphi'_{g_1 \circ \xi_x}(\cdot)$ should be considered as right-continuous version of the Radon-Nikodym density of the function $\varphi_{g_1 \circ \xi_x}(\cdot)$ with respect to the Lebesgue measure.

Let $(L, \varphi_0)$ and $(L_1, \varphi_1)$ be integrable metrized invertible $\mathcal{O}_X$-modules. By **[14**, Remark 7.3**]**, the above intersection number with respect to $(L, \varphi_0)$ and $(L_1, \varphi_1)$ is well-defined. To destinguish this intersection number with the intersection number defined in **[15**, Definition 3.10.1**]** it is denoted by $((L_0, \varphi_0) \cdot (L_1, \varphi_2))'$. Then one can see

$$((L_0, \varphi_0) \cdot (L_1, \varphi_1)) = ((L_0, \varphi_0) \cdot (L_1, \varphi_1))'. \quad (8.2)$$

Indeed, by using the linearity of $(\,\cdot\,)$ and $(\,\cdot\,)'$, we may assume that $L_0$ and $L_1$ are ample, and $\varphi_0$ and $\varphi_1$ are semipositive. Moreover, as

$$\begin{cases} ((L_0,\varphi_0)\cdot(L_1,\varphi_1)) = \dfrac{(((L_0,\varphi_0)+(L_1,\varphi_1))^2)-((L_0,\varphi_0)^2)-((L_1,\varphi_1)^2)}{2}, \\ ((L_0,\varphi_0)\cdot(L_1,\varphi_1))' = \dfrac{(((L_0,\varphi_0)+(L_1,\varphi_1))^2)'-((L_0,\varphi_0)^2)'-((L_1,\varphi_1)^2)'}{2}, \end{cases}$$

we may further assume that $(L_0,\varphi_0)=(L_1,\varphi_1)$, say $(L,\varphi)$. Then, by [**14**, Theorem 7.4],

$$\lim_{n\to\infty}\frac{-\ln\|s_1\wedge\cdots\wedge s_{r_n}\|_{n\varphi,\det}}{n^2/2} = ((L,\varphi)\cdot(L,\varphi))',$$

where $\{s_1,\dots,s_{r_n}\}$ is a basis of $H^0(X,L^{\otimes n})$. On the other hand,

$$\lim_{n\to\infty}\frac{-\ln\|s_1\wedge\cdots\wedge s_{r_n}\|_{n\varphi,\det}}{n^2/2} = ((L,\varphi)\cdot(L,\varphi))$$

by Theorem 8.2 (the Hilbert-Samuel formula over a trivially valued field), as required.

## 9. Casting to the trivial valuation case

In this section, we assume that $K$ is perfect. Let $X$ be a projective $K$-scheme, $d$ be the dimension of $X$, $E$ be a finite-dimensional vector space over $K$, $f: X\to\mathbb{P}(E)$ be a closed embedding, and $L$ be the restriction of the universal invertible sheaf $\mathcal{O}_E(1)$ to $X$. We assume that, for any positive integer $n$, the restriction map

$$S^n(E)=H^0(\mathbb{P}(E),\mathcal{O}_E(n))\longrightarrow H^0(X,L^{\otimes n})$$

is surjective. We equip $E$ with a Hermitian norm family $\xi=(\|\cdot\|_\omega)_{\omega\in\Omega}$ such that the couple $\overline{E}=(E,\xi)$ forms a strongly adelic vector bundle on the adelic curve $S$. Denote by $\varphi=(\varphi_\omega)_{\omega\in\Omega}$ the quotient metric family on $L$ induced by $\xi$ and the closed embedding $f$.

Let $\mathcal{F}=(\mathcal{F}^t(\overline{E}))_{t\in\mathbb{R}}$ be the Harder-Narasimhan $\mathbb{R}$-filtration of $\overline{E}$. Recall that

$$\mathcal{F}^t(\overline{E})=\sum_{\substack{0\neq F\subset E\\ \widehat{\mu}_{\min}(\overline{F})\geqslant t}} F.$$

Note that this $\mathbb{R}$-filtration actually defines an ultrametric norm $\|\cdot\|_0$ on $E$, where we consider the trivial absolute value $|\cdot|_0$ on the field $K$. More precisely, for any $s\in E$, one has

$$\|s\|_0=\exp(-\{t\in\mathbb{R}\,:\,s\in\mathcal{F}^t(\overline{E})\}).$$

Denote by $\varphi_0$ the quotient metric on $L$ induced by $\|\cdot\|_0$. If we consider the adelic curve $S_0$ consisting of a single copy of the trivial absolute value on $K$, then $(L,\varphi_0)$ becomes an adelic line bundle on $X$.

***Lemma 9.1.*** — *Let $(k,|\cdot|)$ be a complete non-Archimedean valued field and $(V,\|\cdot\|)$ be a finite-dimensional ultrametrically normed vector space over $(k,|\cdot|)$. Let $\alpha\in]0,1]$, and $(e_i)_{i=1}^n$ be an $\alpha$-orthogonal basis of $(V,\|\cdot\|)$, namely a basis of $V$ such that*

$$\forall\,(\lambda_1,\ldots,\lambda_n)\in k^n,\quad \|\lambda_1e_1+\cdots+\lambda_ne_n\|\geqslant\alpha\max_{i\in\{1,\ldots,n\}}|\lambda_i|\cdot\|e_i\|.$$

*For any $I=(i_1,\ldots,i_n)\in\mathbb{N}^n$, we let*

$$|I|=i_1+\cdots+i_n,\quad e^I=e_1^{i_1}\cdots e_n^{i_n}\in S^{|I|}(V).$$

*Then, for any $\delta\in\mathbb{N}$, $(e^I)_{I\in\mathbb{N}^n,\,|I|=\delta}$ is an $\alpha^\delta$-orthogonal basis of $S^\delta(V)$ with respect to the symmetric product norm $\|\cdot\|_{S^\delta}$ (namely the quotient norm of the $\varepsilon$-tensor product norm induced by the quotient map $V^{\otimes\delta}\to S^\delta(V)$), and for any $I=(i_1,\ldots,i_n)\in\mathbb{N}^n$ such that $|I|=\delta$, one has*

$$\|e_1\|^{i_1}\cdots\|e_n\|^{r_n}\geqslant\|e^I\|_{S^\delta}\geqslant\alpha^\delta\|e_1\|^{i_1}\cdots\|e_n\|^{r_n}.$$

*Proof.* — Denote by $f:\{1,\ldots,n\}^\delta\to\mathbb{N}^n$ the map which sends $(a_1,\ldots,a_\delta)$ to the vector

$$\Big(\operatorname{card}\big(\{j\in\{1,\ldots,\delta\}\,|\,a_j=i\}\big)\Big)_{i=1}^n.$$

Let $\pi:V^{\otimes\delta}\to S^\delta(V)$ be the projection map. For any $a=(a_1,\ldots,a_\delta)\in\{1,\ldots,n\}^\delta$, denote by $e_a$ the split tensor $e_{a_1}\otimes\cdots\otimes e_{a_\delta}\in V^{\otimes\delta}$.

For $I=(i_1,\ldots,i_n)\in\mathbb{N}^n$ such that $|I|=\delta$, one has

$$\|e^I\|_{S^\delta}=\inf\left\{\left\|\sum_{a\in f^{-1}(\{I\})}\lambda_ae_a\right\|\ :\ \sum_{a\in f^{-1}(\{I\})}\lambda_a=1,\ \right\}.$$

Hence (see [**13**, Remark 1.1.56])

$$\|e^I\|_{S^\delta}\leqslant\|e_1\|^{i_1}\cdots\|e_n\|^{i_n}.$$

Since $(e_i)_{i=1}^n$ is an $\alpha$-orthogonal basis, $(e_a)_{a\in\{1,\ldots,n\}^\delta}$ is an $\alpha^\delta$-orthogonal basis of $V^{\otimes\delta}$ (see [**13**, Proposition 1.2.19]). For any $(\lambda_a)_{a\in f^{-1}(\{I\})}\in k^{f^{-1}(\{I\})}$ such that

$$\sum_{a\in f^{-1}(\{I\})}\lambda_a=1,$$

one has

$$\|e_1\|^{i_1}\cdots\|e_n\|^{i_n}\leqslant\|e_1\|^{i_1}\cdots\|e_n\|^{i_n}\max_{a\in f^{-1}(\{I\})}|\lambda_a|\leqslant\alpha^{-\delta}\left\|\sum_{a\in f^{-1}(\{I\})}\lambda_ae_a\right\|,$$

which leads to $\|e^I\|\geqslant\alpha^{-\delta}\|e_1\|^{i_1}\cdots\|e_n\|^{i_n}$.

For any

$$s=\sum_{a\in\{1,\ldots,n\}^\delta}\mu_ae_a\in E^{\otimes\delta},$$

one has

$$\pi(s)=\sum_{I\in\mathbb{N}^n,\,|I|=\delta}\Big(\sum_{a\in f^{-1}(\{I\})}\mu_a\Big)e^I.$$

Moreover,

$$\begin{aligned}\|s\| &\geqslant \alpha^{\delta} \max_{\substack{I=(i_1,\dots,i_n)\in\mathbb{N}^n\\ |I|=\delta}} \|e_1\|^{i_1}\cdots\|e_n\|^{i_n} \max_{a\in f^{-1}(\{I\})}|\mu_a| \\ &\geqslant \alpha^{\delta} \max_{\substack{I=(i_1,\dots,i_n)\in\mathbb{N}^n\\ |I|=\delta}} \|e_1\|^{i_1}\cdots\|e_n\|^{i_n} \Big|\sum_{a\in f^{-1}(\{I\})}\mu_a\Big| \\ &\geqslant \alpha^{\delta} \max_{\substack{I=(i_1,\dots,i_n)\in\mathbb{N}^n\\ |I|=\delta}} \|e^I\|_{S^\delta}\cdot\Big|\sum_{a\in f^{-1}(\{I\})}\mu_a\Big|.\end{aligned}$$

Therefore, we obtain that $(e^I)_{I\in\mathbb{N}^n,\,|I|=\delta}$ forms an $\alpha^\delta$-orthogonal basis of $(S^\delta(V),\|\cdot\|_{S^\delta})$. □

***Lemma 9.2.*** — *Let $(k,|\cdot|)$ be the field $\mathbb{R}$ or $\mathbb{C}$ equipped with the usual absolute value. Let $(V,\|\cdot\|)$ be a finite-dimensional normed vector space over $(k,|\cdot|)$. We assume that $\|\cdot\|$ is induced by an inner product and let $(e_i)_{i=1}^n$ be an orthonormal basis of $(V,\|\cdot\|)$. For any $\delta\in\mathbb{N}$, let $\|\cdot\|_{S^\delta}^H$ be the orthogonal symmetric power norm of $\|\cdot\|$ on $S^\delta(V)$ (namely the quotient norm of the orthogonal tensor product norm induced by the quotient map $V^{\otimes\delta}\to S^\delta(V)$). Then $(e^I)_{I\in\mathbb{N}^n,\,|I|=\delta}$ is an orthogonal basis of $(S^\delta(V),\|\cdot\|_{S^\delta}^H)$. Moreover, for any $I=(i_1,\dots,i_n)\in\mathbb{N}^n$ such that $|I|=\delta$, one has*

$$\|e^I\|_{S^\delta}^H = \Big(\frac{\delta!}{i_1!\cdots i_n!}\Big)^{-\frac{1}{2}}.$$

*Proof.* — Let $f:\{1,\dots,n\}^\delta\to\mathbb{N}^n$ be the map sending $(a_1,\dots,a_d)$ to

$$\Big(\operatorname{card}\big(\{j\in\{1,\dots,\delta\}\,|\,a_j=i\}\big)\Big)_{i=1}^n$$

For $I=(i_1,\dots,i_n)\in\mathbb{N}^n$ such that $|I|=\delta$, one has

$$\|e^I\|_{S^\delta}^H = \inf\Big\{\Big(\sum_{a\in f^{-1}(I)}|\lambda_a|^2\Big)^{1/2} : \sum_{a\in f^{-1}(\{I\})}\lambda_a=1\Big\}.$$

Note that the cardinal of $f^{-1}(\{I\})$ is

$$\frac{\delta!}{i_1!\cdots i_n!},$$

by Cauchy-Schwarz inequality we obtain

$$\|e^I\|_{S^\delta}^H = \Big(\frac{\delta!}{i_1!\cdots i_n!}\Big)^{-1/2}.$$

For any

$$s=\sum_{a\in\{1,\dots,n\}^\delta}\mu_a e_a\in E^{\otimes\delta},$$

one has

$$\pi(s)=\sum_{I\in\mathbb{N}^n,\,|I|=\delta}\Big(\sum_{a\in f^{-1}(\{I\})}\mu_a\Big)e^I.$$

Moreover, by Cauchy-Schwarz inequality,

$$\|s\|^2 = \sum_{a\in\{1,\dots,n\}^\delta} |\mu_a|^2 \geqslant \sum_{I\in\mathbb{N}^n,\,|I|=\delta} \Big|\sum_{a\in f^{-1}(\{I\})} \mu_a\Big|^2 \cdot \Big(\frac{\delta!}{i_1!\cdots i_n!}\Big)^{-1}.$$

Therefore, we obtain that $(e^I)_{I\in\mathbb{N}^n,\,|I|=\delta}$ forms an orthogonal basis of $(S^\delta(V), \|\cdot\|^H_{S^\delta})$. □

***Proposition 9.3.*** — *The following inequality holds:*

$$((L,\varphi)^{d+1}) \geqslant ((L,\varphi_0)^{d+1}) - \frac{1}{2}\nu(\Omega_\infty)(d+1)\delta\ln(r), \tag{9.1}$$

*where $r$ denotes the dimension of $E$ over $K$ and $\delta$ is the degree of $X$ with respect to the line bundle $L$, that is, $\delta = (L^d)$.*

*Proof.* — For any $\omega\in\Omega$, let $\|\cdot\|_{\omega,*}$ be the dual norm on $E^\vee_\omega$ and let $\|\cdot\|_{\omega,*,\delta}$ be the $\delta$-th symmetric power of the norm $\|\cdot\|_{\omega,*}$, that is the quotient norm of the $\varepsilon$-tensor power (resp. orthogonal tensor power) of $\|\cdot\|_{\omega,*}$ by the canonical quotient map if $|\cdot|_\omega$ is non-Archimedean (resp. Archimedean). Let $\|\cdot\|'_{\omega,*}$ be the $\varepsilon$-tensor product (resp. orthogonal tensor product) of $d+1$ copies of the norm $\|\cdot\|_{\omega,*,\delta}$ if $|\cdot|_\omega$ is non-Archimedean (resp. Archimedean). By [**13**, Proposition 1.2.36], this norm also identifies with the quotient of the tensor power of $\|\cdot\|_{\omega,*}$ by the quotient map

$$p_\omega : E_\omega^{\vee\otimes\delta(d+1)} \cong (E_\omega^{\vee\otimes\delta})^{\otimes(d+1)} \longrightarrow S^\delta(E_\omega^\vee)^{\otimes(d+1)}. \tag{9.2}$$

We denote by $\xi'$ the norm family $(\|\cdot\|'_{\omega,*})_{\omega\in\Omega}$. It turns out that $(S^\delta(E^\vee)^{\otimes(d+1)}, \xi')$ forms a strongly adelic vector bundle on $S$. Moreover, if we let $R\in S^\delta(E^\vee)^{\otimes(d+1)}$ be a resultant of $X$ with respect to $d+1$ copies of the closed embedding $f: X\to\mathbb{P}(E)$, then the following inequality holds:

$$((L,\varphi)^{d+1}) \geqslant -\widehat{\deg}_{\xi'}(R) - \frac{1}{2}\nu(\Omega_\infty)(d+1)\ln\binom{r+\delta-1}{\delta}, \tag{9.3}$$

where $r$ is the dimension of $E$ over $K$. This is a consequence of [**15**, Theorem 3.9.7] and [**4**, Corollary 1.4.3, formula (1.4.10) and Lemma 4.3.6]. Note that in the case where $\Omega_\infty = \varnothing$, the equality

$$((L,\varphi)^{d+1}) = -\widehat{\deg}_{\xi'}(R) \tag{9.4}$$

holds.

We now consider the trivial absolute value $|\cdot|_0$ on $K$ and we let $\xi'_0$ be the ultrametric norm on $S^\delta(E^\vee_\omega)^{\otimes(d+1)}$ defined as the quotient norm of the $\varepsilon$-tensor power of $\|\cdot\|_{0,*}$ by the quotient map

$$p : E^{\vee\otimes\delta(d+1)} \cong (E^{\vee\otimes\delta})^{\otimes(d+1)} \longrightarrow S^\delta(E^\vee)^{\otimes(d+1)}.$$

Similarly to (9.4), the following equality holds:

$$((L,\varphi_0)^{d+1}) = -\widehat{\deg}_{\xi'_0}(R).$$

Note that the dual norm $\|\cdot\|_{0,*}$ corresponds to the Harder-Narasimhan $\mathbb{R}$-filtration of the dual adelic vector bundle $\overline{E}^\vee = (E^\vee, \xi^\vee)$, where $\xi^\vee = (\|\cdot\|_{\omega,*})_{\omega\in\Omega}$ (see the proof of [**13**, Proposition 4.3.41]). Therefore, if we denote by $\Psi$ the quotient vector space of $S^\delta(E^\vee)^{\otimes(d+1)}$ by the one-dimensional vector subspace spanned by the resultants of $X$ with respect to $d+1$ copies of $f : X \to \mathbb{P}(E)$, then [**13**, Theorem 5.6.1] (this theorem still holds when $K$ is perfect) leads to

$$\widehat{\deg}(\Psi, \xi'') \geqslant \widehat{\deg}(\Psi, \xi''_0) \tag{9.5}$$

by considering $\Psi$ as a quotient vector space of $E^{\vee\otimes\delta(d+1)}$ where $\xi''$ denotes the quotient norm family of $\xi'$, and $\xi''_0$ denotes the quotient norm of $\xi'_0$. By Lemmas 9.1 and 9.2, one has

$$\begin{aligned}&\widehat{\mu}(S^\delta(E^\vee)^{\otimes(d+1)}, \xi') = (d+1)\,\widehat{\mu}(S^\delta(E^\vee), (\|\cdot\|_{\omega,*,\delta})_{\omega\in\Omega})\\ &= \delta(d+1)\,\widehat{\mu}(E^\vee, \xi^\vee) + \frac{1}{2}\nu(\Omega_\infty)\frac{d+1}{\binom{r+\delta-1}{\delta}}\sum_{\substack{(i_1,\dots,i_r)\in\mathbb{N}^r\\ i_1+\cdots+i_r=\delta}} \ln\Big(\frac{\delta!}{i_1!\cdots i_r!}\Big).\end{aligned}$$

Similarly,

$$\widehat{\mu}(S^\delta(E^\vee)^{\otimes(d+1)}, \xi'_0) = \delta(d+1)\widehat{\mu}(E^\vee, \|\cdot\|_{0,*}) = \delta(d+1)\widehat{\mu}(E^\vee, \xi^\vee).$$

Therefore, we obtain

$$\begin{aligned}\widehat{\deg}(S^\delta(E^\vee)^{\otimes(d+1)}, \xi') + \frac{1}{2}\nu(\Omega_\infty)(d+1)\sum_{\substack{(i_1,\dots,i_r)\in\mathbb{N}^r\\ i_1+\cdots+i_r=\delta}} \ln\Big(\frac{\delta!}{i_1!\cdots i_r!}\Big)&\\ = \widehat{\deg}(S^\delta(E^\vee)^{\otimes(d+1)}, \xi'_0).&\end{aligned}$$

By [**13**, Proposition 4.3.13], the inequality (9.5) leads to

$$\begin{aligned}((L,\varphi)^{d+1}) &\geqslant ((L,\varphi_0)^{d+1})\\ &\quad + \frac{1}{2}\nu(\Omega_\infty)(d+1)\Bigg(\sum_{\substack{(i_1,\dots,i_r)\in\mathbb{N}^r\\ i_1+\cdots+i_r=\delta}} \ln\Big(\frac{\delta!}{i_1!\cdots i_r!}\Big) - \ln\binom{r+\delta-1}{\delta}\Bigg)\\ &\geqslant ((L,\varphi_0)^{d+1}) - \frac{1}{2}\nu(\Omega_\infty)(d+1)\delta\ln(r),\end{aligned}$$

by using the inequality

$$\binom{r+\delta-1}{\delta} \leqslant r^\delta.$$

□

## 10. Arithmetic Hilbert-Samuel theorem

The purpose of this section is to prove the following theorem.

***Theorem 10.1.*** — *Let $X$ be a geometrically integral projective $K$-scheme, $d$ be the dimension of $X$ and $L$ be an ample invertible $\mathcal{O}_X$-module. Then for any metric family $\varphi \in \mathscr{M}(L)$, the following equality holds*

$$\widehat{\mathrm{vol}}_\chi(L,\varphi) = ((L,\varphi)^{d+1}). \tag{10.1}$$

*Proof.* — **Step 1:** *We first prove the inequality* $\widehat{\mathrm{vol}}_\chi(L,\varphi) \leqslant ((L,\varphi)^{d+1})$.

By taking a tensor power of $L$ we may assume that $L$ is very ample and the canonical $K$-linear map

$$S^n(H^0(X,L)) \longrightarrow H^0(X,L^{\otimes n}) \tag{10.2}$$

is surjective for any integer $n \geqslant 1$. Moreover, by Remark 7.6, the difference

$$\widehat{\mathrm{vol}}_\chi(L,\varphi) - ((L,\varphi)^{d+1})$$

does not depend on the choice of the metric family $\varphi$. Therefore, we may assume that $\varphi$ identifies with the quotient metric family induced by the norm family $\xi_1 = (\|\cdot\|_{\varphi_\omega})_{\omega\in\Omega}$. By [**13**, Proposition 2.2.22 (2)], for any positive integer $n$, the metric $n\varphi$ identifies with the quotient metric family induced by the norm family $\xi_n = (\|\cdot\|_{n\varphi_\omega})_{\omega\in\Omega}$. Moreover, by changing metrics we may also assume that the minimal slope of $(H^0(X,L),\xi_1)$ is non-negative. Since the $K$-linear map (10.2) is surjective, by [**13**, Proposition 6.3.25], we obtain that the minimal slope of $(H^0(X,L^{\otimes n}),\xi_n)$ is non-negative for any positive integer $n$. By [**13**, Theorem 4.1.26], there exists a Hermitian norm family $\xi'_n = (\|\cdot\|'_{n,\omega})$ of $H^0(X,L^{\otimes n})$ such that $\|\cdot\|_{n,\omega} = \|\cdot\|_{n\varphi_\omega}$ when $|\cdot|_\omega$ is non-Archimedean and

$$\|\cdot\|'_{n,\omega} \leqslant \|\cdot\|_{n\varphi_\omega} \leqslant (2r_n)^{1/2}\|\cdot\|'_{n,\omega} \tag{10.3}$$

when $|\cdot|_\omega$ is Archimedean, where $r_n$ denotes the dimension of $H^0(X,L^{\otimes n})$.

For any positive integer $n$, let $\|\cdot\|_n$ be the ultrametric norm on $H^0(X,L^{\otimes n})$ corresponding to the Harder-Narasimhan $\mathbb{R}$-filtration of $(H^0(X,L^{\otimes n}),\xi'_n)$, where we consider the trivial absolute value $|\cdot|_0$ on $K$. Let $\widetilde{\varphi}_n$ be the continuous metric on $L$ (where we still consider the trivial absolute value on $K$) such that $n\widetilde{\varphi}_n$ identifies with the quotient metric on $L^{\otimes n}$ induced by $\|\cdot\|_n$. By [**13**, Proposition 2.2.22 (2)], one has $\|\cdot\|_{n\widetilde{\varphi}_n} = \|\cdot\|_n$ on $H^0(X,L^{\otimes n})$ and hence

$$\widehat{\deg}(H^0(X,L^{\otimes n}),\|\cdot\|_{n\widetilde{\varphi}_n}) = \widehat{\deg}(H^0(X,L^{\otimes n}),\|\cdot\|_n) = \widehat{\deg}(H^0(X,L^{\otimes n}),\xi'_n). \tag{10.4}$$

By Proposition 9.3 and the second inequality of (10.3) we obtain that

$$\begin{aligned}&((nL,n\varphi)^{d+1}) + \frac{1}{2}\nu(\Omega_\infty)(d+1)n^d(L^d)\ln(2r_n)\\ &\quad\geqslant ((nL,n\widetilde{\varphi}_n)^{d+1}) - \frac{1}{2}\nu(\Omega_\infty)(d+1)n^d(L^d)\ln(r_n),\end{aligned} \tag{10.5}$$

where we consider $X$ as an arithmetic variety over the adelic curve $S$ (resp. as an arithmetic variety over the adelic curve consisting of a single copy of the trivial absolute value on $K$) in the computation of the arithmetic intersection number on the

left-hand side (resp. right-hand). Moreover, by Theorem 8.2, the following equality holds:

$$\widehat{\mathrm{vol}}_\chi(L,\widetilde{\varphi}_n)=((L,\widetilde{\varphi}_n)^{d+1}). \tag{10.6}$$

By [**10**, Corollary 5.2] (see also the proof of Theorem 7.3 of *loc. cit.*), there exists a positive constant $C$ such that, for any positive integer $n$, one has

$$\widehat{\deg}(H^0(X,L^{\otimes n}),\|\cdot\|_{n\widetilde{\varphi}_n})\leqslant\frac{\widehat{\mathrm{vol}}_\chi(nL,n\widetilde{\varphi}_n)}{(d+1)!}+Cn^d.$$

The constant $C$ can be taken in the form an invariant of the graded linear series $\bigoplus_{m\in\mathbb{N}}H^0(X,L^{\otimes m})$ multiplied by

$$\sup_{m\in\mathbb{N},\,m\geqslant 1}\frac{\widehat{\mu}_{\max}(H^0(X,L^{\otimes m}),\xi'_m)}{m}.$$

By (10.4), (10.5) and (10.6), we deduce that

$$\widehat{\deg}(H^0(X,L^{\otimes n}),\xi_n)\leqslant\frac{n^{d+1}}{(d+1)!}((L,\varphi)^{d+1})+Cn^d+\frac{1}{2}\nu(\Omega_\infty)(d+1)n^d(L^d)\ln(2r_n^2).$$

Dividing the two sides of the inequality by $n^{d+1}/(d+1)!$ and then taking the limit when $n\to+\infty$, we obtain

$$\widehat{\mathrm{vol}}_\chi(L,\varphi)\leqslant((L,\varphi)^{d+1}).$$

**Step 2:** the inequality $\widehat{\mathrm{vol}}_\chi(L,\varphi)\geqslant((L,\varphi)^{d+1})$.

By replacing $L$ by a tensor power, we may assume that $L$ is very ample. Moreover, by the normalization of Noether (cf. [**15**, Proposition 1.7.4]), we may also assume that there is a finite $K$-morphism $\pi:X\to\mathbb{P}^d_K$ such that $L\cong\pi^*(\mathcal{O}_{\mathbb{P}^d_K}(1))$. By Remark 7.6, we may further assume that there exists an element $\psi=(\psi_\omega)_{\omega\in\Omega}$ of $\mathscr{M}(\mathcal{O}_{\mathbb{P}^d_K}(1))$ such that $\varphi$ equals the pull-back of $\psi$ by $\pi$. Then, by Corollary 6.3, Proposition 7.11 and [**15**, Theorem 4.4.9], one has

$$\widehat{\mathrm{vol}}_\chi(L,\varphi)\geqslant\deg(\pi)\widehat{\mathrm{vol}}_\chi(\mathcal{O}_{\mathbb{P}^d_K}(1),\psi)=\deg(\pi)((\mathcal{O}_{\mathbb{P}^d_K}(1),\psi)^{d+1})=((L,\varphi)^{d+1}),$$

as required. □

***Definition 10.2.*** — Let $(L,\varphi)$ be an adelic invertible $\mathcal{O}_X$-module. We say $(L,\varphi)$ is *relatively ample* if $L$ is ample and $\varphi$ is semipositve. Moreover, we say $(L,\varphi)$ is *relatively nef* if there exist a relatively ample adelic invertible $\mathcal{O}_X$-module $(A,\psi)$ and a sequence $(a_n)_{n=1}^\infty$ of positive integers such that $\lim_{n\to\infty}a_n=\infty$ and $a_n(L,\varphi)+(A,\psi)$ is relatively ample for $n\geqslant 1$.

***Corollary 10.3*** **(Generalized Hodge index theorem)**
*Let $(L,\varphi)$ be a relatively nef adelic invertible $\mathcal{O}_X$-module. Then one has*

$$\widehat{\mathrm{vol}}(L,\varphi)\geqslant((L,\varphi)^{d+1}). \tag{10.7}$$

*Proof.* — Let $(A,\psi)$ be a relatively ample adelic invertible $\mathcal{O}_X$-module and $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers such that $\lim_{n\to\infty} a_n = \infty$ and $a_n(L,\varphi)+(A,\psi)$ is relatively ample for $n \geqslant 1$. Then, by Theorem 10.1,

$$\widehat{\mathrm{vol}}(a_n(L,\varphi)+(A,\psi)) \geqslant \widehat{\mathrm{vol}}_{\chi}(a_n(L,\varphi)+(A,\psi)) \geqslant ((a_n(L,\varphi)+(A,\psi))^{d+1})$$

for $n \geqslant 1$, and hence by [**13**, Theorem 6.4.14],

$$\widehat{\mathrm{vol}}((L,\varphi)+(1/a_n)(A,\psi)) \geqslant (((L,\varphi)+(1/a_n)(A,\psi))^{d+1}).$$

Thus, by using the continuity of $\widehat{\mathrm{vol}}$ (cf. [**13**, Theorem 6.4.24]),

$$\begin{aligned}\widehat{\mathrm{vol}}(L,\varphi) &= \lim_{n\to\infty} \widehat{\mathrm{vol}}((L,\varphi)+(1/a_n)(A,\psi)) \\ &\geqslant \lim_{n\to\infty} (((L,\varphi)+(1/a_n)(A,\psi))^{d+1}) = ((L,\varphi)^{d+1}),\end{aligned}$$

as desired. □

***Corollary 10.4.*** — *Let $(L,\varphi)$ be a relatively nef adelic invertible $\mathcal{O}_X$-module. If $((L,\varphi)^{d+1}) > 0$, then $L$ is big.*

*Proof.* — By Corollary 10.3, $\widehat{\mathrm{vol}}(L,\varphi) > 0$. Let $(A,\psi)$ be a relatively ample adelic invertible $\mathcal{O}_X$-module. By the continuity of $\widehat{\mathrm{vol}}$ (see [**13**, Theorem 6.4.24]), there is a positive integer $n$ such that $\widehat{\mathrm{vol}}((L,\varphi)-(1/n)(A,\psi)) > 0$, that is, $\widehat{\mathrm{vol}}(n(L,\varphi)-(A,\psi)) > 0$, so that, for some positive integer $m$, $H^0(X,(L^{\otimes n}\otimes A^{-1})^{\otimes m}) \neq \{0\}$. Therefore $L$ is big. □

***Corollary 10.5.*** — *Let $X$ be a geometrically integral projective scheme over* $\operatorname{Spec} K$*, $d$ be the dimension of $X$, $\overline{L}=(L,\varphi)$ be an adelic line bundle on $X$ and $\overline{E}=(E,U,\psi)$ be a birational adelic torsion free $\mathcal{O}_X$-module. Assume that $L$ is ample and the metrics in $\varphi$ are semi-positive. Moreover we suppose that either $(E,\psi)$ is an adelic invertible $\mathcal{O}_X$-module or $X$ is normal. Then one has*

$$\lim_{n\to+\infty} \frac{\widehat{\deg}(H^0(X,L^{\otimes n}\otimes E),(\|\cdot\|_{n\varphi_\omega+\psi_\omega})_{\omega\in\Omega})}{n^{d+1}/(d+1)!} = \mathrm{rk}(E)(\overline{L}^{d+1}).$$

*Proof.* — This is a consequence of Theorem 10.1 together with Theorem 6.4. □

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