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A common fixed point theorem for compatible mappings satisfying a contractive condition involving altering distance functions

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Abstract

Our goal in this paper is to obtain a generalization of Jungck's fixed point theorem for a pair of compatible mappings satisfying a contractive inequality which involves altering distance functions. Also, we use functions as contractive parameters. Our theorem extends some well-known results in the literature. At the end, we provide an example in support of our result.

Keywords: Complete metric space; common fixed point; altering distance function; compatible function; Picard iteration.

Resumen

El propósito de este artículo es dar una generalización del teorema del punto fijo de Jungck para un par de funciones compatibles satisfaciendo una desigualdad contractiva que involucra funciones que alteran distancia. Como parámetros contractivos consideramos cierto tipo de funciones. Para finalizar, se da un ejemplo en soporte de nuestro resultado.

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Palabras claves: Espacio métrico completo; punto fijo en común; función que altera distancia; funciones compatibles; iteración de Picard.

1 Introduction

The fixed point theory is one of the most important and powerful tool in nonlinear analysis since it provides theorems to solve problems in several mathematics branches such as, differential equations, games theory, fractals theory, economy, optimization theory, approximation theory, integral equations, among others.

In 1922, the polish mathematician S. Banach [1], proved a fixed point theorem, now known as the Banach contraction principle, which under certain conditions guarantees the existence and uniqueness of a fixed point for a function on a complete metric space. This result has been studied and generalized in many different directions, giving rise to a wide range of very interesting results (see, e.g., [2–9]).

The study of common fixed points begins in 1936 with the results proved independently by Markov and Kakutani (see [10]), where they establish that a family of commutative linear and continuous mappings defined on a nonempty convex compact subset of a linear topological space on itself, has a common fixed point.

By 1975 began the interest for the study of common fixed point theorems for a pair (or families) of mappings satisfying certain contractive conditions in metric spaces. In 1976, G. Jungck [4] proved a common fixed point theorem for a pair of commutative mappings, which generalizes the Banach contraction principle. In 1986, G. Jungck [5] introduced the concept of compatible mappings, this notion allowed to obtain common fixed point results by weakening the commutativity of the involved mappings. In 2012, Morales and Rojas [7] generalized Jungck's fixed point theorem for commuting mappings by means of the concepts of altering distance functions and compatible pair of mappings.

From now on, \mathbb{N} will represent the set of natural numbers (starting from 1), \mathbb{R} the set of real numbers and \mathbb{R}^+ the set of nonnegative real numbers.

Definition 1. *Let $f, g : X \rightarrow X$ be two functions. A point $z \in X$ is called a common fixed point of f and g if $f(z) = g(z) = z$.*

The following theorem was proved by G. Jungck. This result is known as Jungck's Fixed Point Theorem. (See [4].)

Theorem 1. *Let $f, g : X \rightarrow X$ be two mappings defined on a complete metric space (X, d) . If,*

1. $g(X) \subseteq f(X)$,
2. f is a continuous function,
3. there is $c \in [0, 1)$ such that for all x, y in X , we have

$$d(g(x), g(y)) \leq c \cdot d(f(x), f(y)), \quad (1)$$

4. f and g are commutative functions, i.e. $f(g(x)) = g(f(x))$, $\forall x \in X$.

Then there exists a unique $z \in X$ such that $f(z) = g(z) = z$.

Our attention is focused in generalizing the previous theorem. For doing this, we use both the notion of compatible pair of mappings given in [5] and the concept of altering distance functions. Also, we will replace the contraction constant by a function.

Definition 2 ([5], Definition 2.1). *Self mappings f and g of a metric space (X, d) are compatible iff $\lim_{n \rightarrow \infty} d(f(g(x_n)), g(f(x_n))) = 0$ whenever $(x_n)_n$ is a sequence in X such that $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} g(x_n) = t$, for some $t \in X$.*

Example 1. *Let $X = [1, +\infty)$ be endowed with the Euclidean metric $d(\cdot, \cdot) = |\cdot|$, and let $f, g : X \rightarrow X$ be two mappings defined by $f(x) = x^2$ and $g(x) = 2x - 1$, $\forall x \in X$. Then, if $(x_n)_n$ is a sequence in X such that*

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} g(x_n) = t, \text{ for some } t \in X$$

we get that

$$\lim_{n \rightarrow \infty} (2x_n - 1) = \lim_{n \rightarrow \infty} x_n^2 = t.$$

Therefore

$$\lim_{n \rightarrow \infty} x_n = \frac{t+1}{2} \text{ and } \lim_{n \rightarrow \infty} x_n = \sqrt{t}.$$

From uniqueness of limit, we conclude that $t = 1$. Hence, $\lim_{n \rightarrow \infty} x_n = 1$.

On the other hand, we have that

$$\lim_{n \rightarrow \infty} f(g(x_n)) = \lim_{n \rightarrow \infty} (2x_n - 1)^2 = 1,$$

and

$$\lim_{n \rightarrow \infty} g(f(x_n)) = \lim_{n \rightarrow \infty} 2x_n^2 - 1 = 1.$$

Hence,

$$\lim_{n \rightarrow \infty} |f(g(x_n)) - g(f(x_n))| = 0.$$

Therefore, f and g are compatible mappings.

It is easy to verify that all pair of commutative functions are compatible, nevertheless, the reciprocal is not true, (see [5]). In this order of ideas we present the following result.

Proposition 1 ([5], Proposition 2.2). *Let $f, g : X \rightarrow X$ be compatible. If $f(t) = g(t)$ then $f(g(t)) = g(f(t))$.*

In 1984, M. S. Khan, S. Swaleh y S. Sessa [6], introduced a version of the notion of control function which they called an *altering distance function* and proved a common fixed point theorem which generalizes the result presented by E. Rakotch (see [11]).

Definition 3. *A function $\psi : [0, +\infty) \rightarrow [0, +\infty)$, is called an altering distance function if the following properties are satisfied:*

1. ψ is a continuous and increasing function,
2. $\psi(t) = 0 \Leftrightarrow t = 0$.

We set $\Psi = \{\psi : \psi \text{ alter distance}\}$.

The following result is an immediate consequence of Definition 3.

Lemma 1. *If $\psi_1, \psi_2 \in \Psi$, then $\psi_1 \circ \psi_2 \in \Psi$.*

When one has functions defined on complete metric spaces satisfying some type of contraction, the Picard iterations play an important role in the construction of a Cauchy sequence converging to the fixed point (in the case when it exists). The following result is essential in this constructive process. The proof can be found in [12].

Lemma 2. *Suppose (X, d) is a metric space. Let $(x_n)_n$ be a sequence in X such that $d(x_n, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$. If $(x_n)_n$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $(m_k)_k$ and $(n_k)_k$ with $m_k > n_k > k$, such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ and*

1. $\lim_{k \rightarrow \infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$;
2. $\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k}) = \epsilon$;
3. $\lim_{k \rightarrow \infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$;
4. $\lim_{k \rightarrow \infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon$.

Theorem 2. *Let $\alpha : \mathbb{R}^+ \rightarrow [0, 1)$ be a function. If $(x_n)_n$ is a sequence in $\mathbb{R}^+ \setminus \{a\}$ such that $\lim_{n \rightarrow \infty} x_n = a$, then*

$$\limsup_{n \rightarrow \infty} \alpha(x_n) \leq \limsup_{t \rightarrow a} \alpha(t), \quad \forall a > 0.$$

2 Main Result

In this section we are going to prove the existence of a unique common fixed point for a pair of compatible mappings satisfying a contractive condition which involves altering distance functions. Further, we use the ideas of S. Reich [9], that is to say, we replace the contraction constant by a function $\alpha : \mathbb{R}^+ \rightarrow [0, 1)$ such that $\limsup_{t \rightarrow a} \alpha(t) < 1$, for all $a > 0$. As consequence of this result, we extend Theorem 1, Theorem 3 of [6] and Theorem 2.1 of [2]. Finally, we combine the ideas of Jungck and Branciari to obtain a common fixed point result for a pair of compatible mappings, satisfying a contractive condition of integral type.

Definition 4. *Let (X, d) be a metric space and let $f, g : X \rightarrow X$ be two mappings. The pair (f, g) is called a (ψ, α) -Jungck contraction, denoted by $(\psi\text{-}\alpha\text{-}JC)$, if there exists both a function $\alpha : \mathbb{R}^+ \rightarrow [0, 1)$ with $\limsup_{t \rightarrow a} \alpha(t) < 1$ for all $a > 0$ and $\psi \in \Psi$, such that*

$$\psi(d(g(x), g(y))) \leq \alpha(d(f(x), f(y))) \cdot \psi(d(f(x), f(y))), \quad \forall x, y \in X.$$

Theorem 3. *Let (X, d) be a complete metric space, let $f, g : X \rightarrow X$ be two mappings and suppose that $\psi \in \Psi$. If,*

1. $g(X) \subseteq f(X)$,
2. f is a continuous function,
3. (f, g) is $(\psi$ - α - $JC)$,
4. f and g are compatible functions,

then, f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ fixed. Since $g(X) \subseteq f(X)$, there exists $x_1 \in X$ such that $f(x_1) = g(x_0)$. In this way we define inductively a sequence $(y_n)_n \subseteq X$, as

$$y_n = f(x_n) = g(x_{n-1}), \quad \forall n \in \mathbb{N}.$$

Let us prove that $\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0$. Indeed, for each $n \in \mathbb{N}$ we have,

$$\begin{aligned} \psi(d(y_{n+1}, y_{n+2})) &= \psi(d(g(x_n), g(x_{n+1}))) \\ &\leq \alpha(d(f(x_n), f(x_{n+1}))) \cdot \psi(d(f(x_n), f(x_{n+1}))) \\ &< \psi(d(y_n, y_{n+1})). \end{aligned}$$

Therefore,

$$\psi(d(y_{n+1}, y_{n+2})) < \psi(d(y_n, y_{n+1})).$$

Since ψ is an increasing function, we have

$$d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1}), \quad \forall n \in \mathbb{N}.$$

Hence, $(d(y_n, y_{n+1}))_n$ is a monotone sequence (decreasing) and bounded from below of non negative real numbers. This implies that there exists $r \geq 0$ such that, $\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = r$. Now, we are going to prove that $r = 0$. Suppose that $r > 0$. Then, for each $n \in \mathbb{N}$, we have that,

$$\begin{aligned} 0 < \psi(r) &\leq \psi(d(y_{n+1}, y_{n+2})) \\ &= \psi(d(g(x_n), g(x_{n+1}))) \\ &\leq \alpha(d(f(x_n), f(x_{n+1}))) \cdot \psi(d(f(x_n), f(x_{n+1}))). \end{aligned}$$

Hence,

$$\psi(r) \leq \alpha(d(f(x_n), f(x_{n+1}))) \cdot \psi(d(f(x_n), f(x_{n+1}))). \quad (2)$$

Now, taking limit superior in both sides of inequality (2), we have

$$\limsup_{n \rightarrow \infty} \psi(r) \leq \limsup_{n \rightarrow \infty} [\alpha(d(f(x_n), f(x_{n+1}))) \cdot \psi(d(f(x_n), f(x_{n+1})))]. \quad (3)$$

By using both the continuity of the function ψ and Theorem 2 in (3) it follows that,

$$\psi(r) \leq \limsup_{t \rightarrow r} \alpha(t) \cdot \psi(r).$$

Thus,

$$\psi(r) < \psi(r)$$

which is a contradiction. Therefore, $r = 0$.

Now, Let us prove that $(y_n)_n$ is a Cauchy sequence. In fact, suppose by the contrary that $(y_n)_n$ is not a Cauchy sequence, then by Lemma 2 there exists an $\epsilon > 0$ and sequences of positive numbers $(m_k)_k$ and $(n_k)_k$ with $m_k > n_k > k$, for all $k \in \mathbb{N}$, such that,

1. $d(y_{m_k}, y_{n_k}) \geq \epsilon$,
2. $\lim_{n \rightarrow \infty} d(y_{m_{k-1}}, y_{n_{k-1}}) = \epsilon$.

Hence, for each $k \in \mathbb{N}$ we have

$$\begin{aligned} \psi(\epsilon) &\leq \psi(d(y_{m_k}, y_{n_k})) \\ &= \psi(d(g(x_{m_k-1}), g(x_{n_k-1}))) \\ &\leq \alpha(d(f(x_{m_k-1}), f(x_{n_k-1}))) \cdot \psi(f(x_{m_k-1}), f(x_{n_k-1})), \end{aligned}$$

that is,

$$\psi(\epsilon) \leq \alpha(d(f(x_{m_k-1}), f(x_{n_k-1}))) \cdot \psi(d(f(x_{m_k-1}), f(x_{n_k-1}))). \quad (4)$$

Now, taking limit superior in both sides of (4) we get,

$$\limsup_{k \rightarrow \infty} \psi(\epsilon) \leq \limsup_{k \rightarrow \infty} [\alpha(d(f(x_{m_k-1}), f(x_{n_k-1}))) \cdot \psi(d(f(x_{m_k-1}), f(x_{n_k-1})))]. \quad (5)$$

As before, by using the continuity of the function ψ in (5) it follows that

$$\psi(\epsilon) \leq \limsup_{t \rightarrow \epsilon} \alpha(t) \cdot \psi(\epsilon)$$

from which we conclude that,

$$\psi(\epsilon) < \psi(\epsilon)$$

getting a contradiction. Thus, $(y_n)_n$ is a Cauchy sequence in the complete metric space (X, d) , hence there exists $z \in X$ such that $\lim_{n \rightarrow \infty} y_n = z$. Thus,

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} g(x_n) = z. \quad (6)$$

Now, since f and g are compatible, it follows that

$$\lim_{n \rightarrow \infty} d(f(g(x_n)), g(f(x_n))) = 0. \quad (7)$$

On the other hand, given that f is a continuous function, from (6) we get

$$\lim_{n \rightarrow \infty} f(g(x_n)) = f(z), \quad (8)$$

$$\lim_{n \rightarrow \infty} f(f(x_n)) = f(z). \quad (9)$$

Hence from (7) and (8) we obtain

$$\lim_{n \rightarrow \infty} g(f(x_n)) = f(z).$$

Now, we notice that for each $n \in \mathbb{N}$

$$\psi(d(g(f(x_n)), g(x_n))) \leq \alpha(d(f(f(x_n)), f(x_n))) \cdot \psi(d(f(f(x_n)), f(x_n))). \quad (10)$$

Taking limit superior in both sides of (10), using the continuity of ψ and equality (9) it follows that

$$\psi(p) \leq \limsup_{t \rightarrow p} \alpha(t) \cdot \psi(p), \quad (11)$$

being $p = d(f(z), z)$. Therefore, from (11) we obtain

$$(1 - \limsup_{t \rightarrow p} \alpha(t)) \cdot \psi(d(f(z), z)) \leq 0. \quad (12)$$

Since $\limsup_{t \rightarrow p} \alpha(t) < 1$, then from (12) it follows that $\psi(d(f(z), z)) = 0$ and thus, $d(f(z), z) = 0$, therefore,

$$f(z) = z. \quad (13)$$

Now, as before, for each $n \in \mathbb{N}$ we have that

$$\psi(d(g(x_n), g(z))) \leq \alpha(d(f(x_n), f(z))) \cdot \psi(d(f(x_n), f(z))). \quad (14)$$

Taking limit superior in both sides of (14) and using the continuity of ψ , we get that

$$\psi(d(z, g(z))) \leq \limsup_{t \rightarrow 0} \alpha(t) \cdot \psi(d(z, f(z))). \quad (15)$$

Then, from (13) it follows that $\psi(d(z, f(z))) = 0$ and from (15) we have

$$\psi(d(z, g(z))) \leq 0. \quad (16)$$

Since $\psi \in \Psi$ then, from (16) we conclude that

$$g(z) = z. \quad (17)$$

Finally, from (13) and (17), it follows that z is a common fixed point for f and g .

To prove the uniqueness, let us suppose that there exists $w \in X$ such that $f(w) = g(w) = w$. Then

$$\begin{aligned} \psi(d(w, z)) &= \psi(d(g(w), g(z))) \\ &\leq \alpha(d(f(w), f(z))) \cdot \psi(d(f(w), f(z))) \\ &= \alpha(d(w, z)) \cdot \psi(d(w, z)). \end{aligned}$$

Therefore, $\psi(d(w, z)) \leq \alpha(d(w, z)) \cdot \psi(d(w, z))$. Hence

$$(1 - \alpha(d(w, z))) \cdot \psi(d(w, z)) \leq 0.$$

Then, since $1 - \alpha(d(w, z)) > 0$ it follow that $\psi(d(w, z)) = 0$, which imply that $d(w, z) = 0$. That is $w = z$. Thus, z is the unique common fixed point of f and g . \square

Now, we show some consequences of the previous result. The following corollary corresponds to the so-called Jungck Fixed Point Theorem.

Corollary 1. *Let (X, d) be a complete metric space and let $f, g : X \rightarrow X$ be two mappings. Suppose that*

1. $g(X) \subset f(X)$,
2. there is $k \in [0, 1)$ such that $d(g(x), g(y)) \leq kd(f(x), f(y))$, $\forall x, y \in X$,
3. f is a continuous function,
4. f and g are commutative functions, that is, $f \circ g = g \circ f$,

then, there exists a unique point $z \in X$ such that $f(z) = g(z) = z$.

Proof. This is an immediate consequence of Theorem 3, by considering

$$\begin{aligned}\psi(t) &= t, \quad \forall t \in [0, +\infty), \\ \alpha(t) &= k, \quad \forall t \in \mathbb{R}^+.\end{aligned}$$

□

The following example shows that Theorem 3 is a proper generalization of Theorem 1.

Example 2. *Let $X = [1, +\infty)$ be endowed with the Euclidean metric $d(\cdot, \cdot) = |\cdot|$. We define the selfmaps f, g on X by*

$$f(x) = x^2 \text{ and } g(x) = 2x - 1.$$

Let $\alpha : \mathbb{R}^+ \rightarrow [0, 1)$ defined as follow,

$$\alpha(t) = \begin{cases} 0, & t = 0 \\ \frac{2}{\sqrt{t} + 2}, & t > 0, \end{cases}$$

and consider $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ given by the formula $\psi(t) = \frac{t}{2}$. Then, it is simple to see that $g(X) \subseteq f(X)$ and f is a continuous mapping. Also, from Example 1 we know that f and g are compatible mappings. Now, notice that

$$\psi(|gx - gy|) = \frac{1}{2}|gx - gy| = |x - y|$$

and

$$\alpha(|fx - fy|)\psi(|fx - fy|) = \frac{2}{\sqrt{|x^2 - y^2|} + 2} \cdot \frac{|x^2 - y^2|}{2}.$$

Since $X = [1, +\infty)$, is not difficult to show that

$$1 \leq \frac{|x + y|}{\sqrt{|x^2 - y^2|} + 2}, \quad \forall x, y \in X. \quad (18)$$

Therefore, from (18) we have

$$|x - y| \leq \frac{2}{\sqrt{|x^2 - y^2|} + 2} \cdot \frac{|x^2 - y^2|}{2}, \quad \forall x, y \in X,$$

which mean that (f, g) is (ψ, α) -JC. That is, the conditions of Theorem 3 are fulfilled. It follows from Theorem 3 that f and g have a unique common fixed point $z = 1$.

Finally, we cannot apply Theorem 1 to show that f and g have a common fixed point in X because (1) do not hold. Also, f and g are not commutative mappings.

The following corollary corresponds to Theorem 3 of [6].

Corollary 2. *Let (X, d) be a complete metric space and let $g : X \rightarrow X$ be a function. Furthermore let $\psi \in \Psi$ and let $\alpha : \mathbb{R}^+ \rightarrow [0, 1)$ be a decreasing monotone function. If*

$$\psi(d(g(x), g(y))) \leq \alpha(d(x, y))\psi(d(x, y)), \quad \forall x, y \in X,$$

then, there exists a unique point $z \in X$ such that $g(z) = z$.

Proof. In this case we apply Theorem 3 with

$$f(t) = t, \quad \forall t \in X.$$

Moreover, using the fact that if $\alpha : \mathbb{R}^+ \rightarrow [0, 1)$ is a monotone decreasing function, then for all $a > 0$, we have that $\limsup_{t \rightarrow a} \alpha(t) < 1$. \square

The following definition was introduced by A. Branciari in [2].

Definition 5. We define by Φ as the set of all functions $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying

1. φ is Lebesgue integrable,
2. for each $K \subseteq [0, +\infty)$ one has $\int_K \varphi(t) dt < +\infty$,
3. for all $\epsilon > 0$, $\int_0^\epsilon \varphi(t) dt > 0$.

The following result is an immediate consequence of Definition 5, the proof can be found in [8].

Lemma 3. Let $\varphi \in \Phi$ and let us define $\varphi_0(t) = \int_0^t \varphi(x) dx$, $\forall t \in \mathbb{R}^+$. Then, $\varphi_0 \in \Psi$.

Corollary 3. If we consider in Theorem 3,

$$\begin{aligned} f(x) &= x, \forall x \in X, \\ \psi(x) &= \int_0^x \varphi(t) dt, \forall x \in [0, +\infty), \\ \alpha(t) &= c, \forall t \in \mathbb{R}^+, \end{aligned}$$

being $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $(0, +\infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t) dt > 0$. Then, we obtain Theorem 2.1 of [2].

The following result is a combination between the main Theorem of [4] and Theorem 2.1 of [2]. We replace the contraction constant by a function α .

Theorem 4. Let (X, d) be a complete metric space, $f, g : X \rightarrow X$ two mappings, $\psi \in \Psi$ and $\varphi \in \Phi$. Furthermore, let $\alpha : \mathbb{R}^+ \rightarrow [0, 1)$ be as in Definition 4. If

1. $g(X) \subseteq f(X)$,
2. f is a continuous function,

$$3. \int_0^{\psi[d(g(x),g(y))]} \varphi(t) dt \leq \alpha[d(f(x), f(y))] \int_0^{\psi[d(f(x),f(y))]} \varphi(t) dt, \text{ for each } x, y \text{ in } X,$$

with $\limsup_{t \rightarrow a} \alpha(t) < 1, \forall a \in (0, +\infty),$

4. f and g are compatible functions,

then, there exists a unique point $z \in X$ such that $f(z) = g(z) = z$.

Proof. For any $w \in \mathbb{R}^+$ we consider $\psi_0(w) = \int_0^w \varphi(t) dt$. From Lemma 3 we have that $\psi_0 \in \Psi$. Now, for each $x, y \in X$ we get,

$$\begin{aligned} (\psi_0 \circ \psi)(d(g(x), g(y))) &= \int_0^{\psi(d(g(x),g(y)))} \varphi(t) dt \\ &\leq \alpha(d(f(x), f(y))) \int_0^{\psi(d(f(x),f(y)))} \varphi(t) dt \\ &= \alpha(d(f(x), f(y))) \cdot (\psi_0 \circ \psi)(d(f(x), f(y))), \end{aligned}$$

that is,

$$(\psi_0 \circ \psi)(d(g(x), g(y))) \leq \alpha(d(f(x), f(y))) \cdot (\psi_0 \circ \psi)(d(f(x), f(y))).$$

From Lemma 1 it follows that $\psi_0 \circ \psi \in \Psi$, therefore applying Theorem 3, we conclude that there exists a unique $z \in X$ such that $f(z) = g(z) = z$. \square

3 Conclusions

Since the appearance of Jungck's Fixed Point Theorem, a lot of works devoted to extend and generalized it have been appeared in the literature. In this paper we maintain the same approach by considering a more general contraction type, in which the set of functions that alter distance between points is involved. This fact along with the compatibility condition and the replacement of the contraction constant by a function had allowed us to obtain a common fixed point results for two mappings, extending in this way the state of the art of this line of research.

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