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On the complexity of All ε -Best Arms Identification

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Abstract

We consider the problem introduced by [MJTN20] of identifying all the ε -optimal arms in a finite stochastic multi-armed bandit with Gaussian rewards. In the fixed confidence setting, we give a lower bound on the number of samples required by any algorithm that returns the set of ε -good arms with a failure probability less than some risk level δ . This bound writes as $T_\varepsilon^*(\boldsymbol{\mu}) \log(1/\delta)$, where $T_\varepsilon^*(\boldsymbol{\mu})$ is a characteristic time that depends on the vector of mean rewards $\boldsymbol{\mu}$ and the accuracy parameter ε . We also provide an efficient numerical method to solve the convex max-min program that defines the characteristic time. Our method is based on a complete characterization of the alternative bandit instances that the optimal sampling strategy needs to rule out, thus making our bound tighter than the one provided by [MJTN20]. Using this method, we propose a Track-and-Stop algorithm that identifies the set of ε -good arms w.h.p and enjoys asymptotic optimality (when δ goes to zero) in terms of the expected sample complexity. Finally, using numerical simulations, we demonstrate our algorithm's advantage over state-of-the-art methods, even for moderate values of the risk parameter.

1 Introduction

The problem of finding all the ε -good arms was recently introduced by [MJTN20]. For a finite family of distributions $(\nu_a)_{a \in [K]}$ with vector of mean rewards $\boldsymbol{\mu} = (\mu_a)_{a \in [K]}$, the goal is to return the set of ε -good arms, defined as $G_\varepsilon(\boldsymbol{\mu}) \triangleq \{a \in [K] : \mu_a \geq \max_i \mu_i - \varepsilon\}$ in the additive case and $G_\varepsilon(\boldsymbol{\mu}) \triangleq \{a \in [K] : \mu_a \geq (1 - \varepsilon) \max_i \mu_i\}$ in the multiplicative case. This problem is closely related to two other pure-exploration problems in the multi-armed bandit literature, namely the TOP- k arms selection and the THRESHOLD bandits. The former aims to find the k arms with the highest means, while the latter seeks to identify all arms with means larger than a given threshold s . As argued by [MJTN20], finding all the ε -good arms is a more robust objective than the TOP- k and THRESHOLD problems, which require some prior knowledge of the distributions in order to return a relevant set of solutions. Take for example drug discovery applications, where the goal is to perform an initial selection of potential drugs through *in vitro* essays before conducting more expensive clinical trials: setting the number of arms k too high or the threshold s too low may result into poorly performing solutions. Conversely, if we set k to a small number or the threshold s too high we might miss promising drugs that will prove to be more efficient under careful examination. The All- ε objective circumvents this issues by requiring to return all drugs whose efficiency lies within a certain range from the best. In this paper, we want to identify $G_\varepsilon(\boldsymbol{\mu})$ in a PAC learning framework with fixed confidence: For a risk level δ , the algorithm samples arms $a \in [K]$ in a sequential manner

to gather information about the distribution means $(\mu_a)_{a \in [K]}$ and returns an estimate \widehat{G}_ε such that $\mathbb{P}_\mu(\widehat{G}_\varepsilon \neq G_\varepsilon(\mu)) \leq \delta$. Such an algorithm is called δ -PAC and its performance is measured by the expected number of samples $\mathbb{E}[\tau_\delta]$, also called the *sample complexity*, needed to return a good answer with high probability. [MJTN20] provided two lower bounds on the sample complexity: The first bound is using a change-of-measure argument and exhibits the behavior of sample complexity in the low confidence regime ($\delta \rightarrow 0$). The second bound resorts to the Simulator technique [SJR17] and shows the dependency of the sample complexity on the number of arms K for moderate values of δ . For both bounds, they proposed algorithms with matching sample complexity, up to some numerical constants. Our contributions can be summarized as follows:

- Usual lower bounds on the sample complexity write as $f(\nu) \log(1/\delta) + g(\nu)$ for an instance ν . We propose a tight bound in terms of the first-order term which writes as $T_\varepsilon^*(\mu) \log(1/\delta)$, where the characteristic time $T_\varepsilon^*(\mu)$ is the value of a concave max-min optimization program. Our bound is tight in the sense that any lower bound of the form $f(\nu) \log(1/\delta)$ that holds for all $\delta \in (0, 1)$ is such that $f(\nu) \leq T_\varepsilon^*(\mu)$. To do so, we investigate all the possible alternative instances λ that one can obtain from the original problem μ by a change-of-measure, including the ones that were considered by [MJTN20].
- Our second contribution consists of two efficient methods to solve the minimization sub-problem (resp. the entire max-min program) that defines the characteristic time. These methods are used respectively in the stopping and sampling rule of our Track-and-Stop algorithm, whose sample complexity matches the lower bound when δ tends to 0. Additionally, our analysis proves that one does not need to compute the optimal weights exactly, as was done in the original version of Track-and-Stop. Instead, it is sufficient to compute near-optimal weights, with a decaying optimisation error, while still being asymptotically optimal in terms of the sample complexity.
- Finally, to corroborate our asymptotic results, we conduct numerical experiments for a wide range of the confidence parameters and number of arms. Empirical evaluation shows that Track-and-Stop is optimal either for a small number of arms K or when δ goes to 0, and excellent in practice for much larger values of K and δ . We believe these are significant improvements in performance to be of interest for ML practitioners seeking solutions for this kind of problem.

Similar to previous works, we restrict our attention to bandits with rewards coming from a Gaussian distribution with variance 1. Even though this assumption is not mandatory, it considerably simplifies the results presented in the paper. In Section 2 we present our lower bound on the sample complexity of identifying the set of ε -good arms and the pseudo-code of our algorithm, along with the theoretical guarantees on its sample complexity. In Sections 3 and 4, we present our method for solving the optimization program that defines the characteristic time, which is at the heart of the sampling and stopping rules of our algorithm.

2 Lower bound and Matching Algorithm

We start by proving a lower bound on the sample complexity of any δ -correct algorithm. This lower bound will later motivate the design of our algorithm.

2.1 Lower bound

Let Δ_K denote the K -dimensional simplex and $\text{kl}(\delta, 1 - \delta)$ be the KL-divergence between two Bernoulli distributions with parameters δ and $1 - \delta$. Finally, define the set of *alternative* bandit problems $\text{Alt}(\mu) = \{\lambda \in \mathbb{R}^K : G_\varepsilon(\mu) \neq G_\varepsilon(\lambda)\}$. Using the change-of-measure trick developed by [LR85] and later popularized by [GK16] for Best Arm Identification problems, we derive the following lower bound on the sample complexity in our special setting:

Proposition 1. *For any δ -correct strategy and any bandit instance μ , the expected stopping time τ_δ can be lower-bounded as*

$$\mathbb{E}[\tau_\delta] \geq T_\varepsilon^*(\mu) \text{kl}(\delta, 1 - \delta)$$

where

$$T_\varepsilon^*(\boldsymbol{\mu})^{-1} \triangleq \sup_{\boldsymbol{\omega} \in \Delta_K} T_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\omega})^{-1} \quad (1)$$

$$T_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\omega})^{-1} \triangleq \inf_{\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\mu})} \sum_{a \in [K]} \omega_a \frac{(\mu_a - \lambda_a)^2}{2} \quad (2)$$

The characteristic time $T_\varepsilon^*(\boldsymbol{\mu})$ above is an instance-specific quantity that determines the difficulty of our problem. The optimization problem in the definition of $T_\varepsilon^*(\boldsymbol{\mu})$ can be seen as a two-player game between an algorithm which samples each arm a proportionally to ω_a and an adversary who chooses an alternative instance $\boldsymbol{\lambda}$ that is difficult to distinguish from $\boldsymbol{\mu}$ under the algorithm's sampling scheme. This suggests that an optimal strategy should play the optimal allocation $\boldsymbol{\omega}^*$ that maximizes the optimization problem (1) and, as a consequence, rules out all alternative instances as fast as possible. This motivates our algorithm, presented below.

2.2 Algorithm

We propose a simple Track-and-Stop strategy similar to the one proposed by [GK16] for the problem of Best-Arm Identification. It starts by sampling once from every arm $a \in [K]$ and constructs an initial estimate $\hat{\boldsymbol{\mu}}_K$ of the vector of mean rewards $\boldsymbol{\mu}$. After this burn-in phase, the algorithm enters a loop where at every iteration it samples the arm given by the sampling rule (3), updates its estimate $\hat{\boldsymbol{\mu}}_t$. Finally, the algorithm checks if the stopping rule (4) is satisfied, in which case it stops and returns the set of empirically ε -good arms.

Sampling rule: For our sampling rule, we perform a C-tracking: First, we compute $\tilde{\boldsymbol{\omega}}(\hat{\boldsymbol{\mu}}_t)$, an allocation vector which is $\frac{1}{\sqrt{t}}$ -optimal in the lower-problem (1) for the instance $\hat{\boldsymbol{\mu}}_t$. Then we project $\tilde{\boldsymbol{\omega}}(\hat{\boldsymbol{\mu}}_t)$ on the set $\Delta_K^{\eta_t} = \Delta_K \cap [\eta_t, 1]^K$. Given the projected vector $\tilde{\boldsymbol{\omega}}^{\eta_t}(\hat{\boldsymbol{\mu}}_t)$, the next arm to sample from is defined by:

$$a_{t+1} = \arg \min_a N_a(t) - \sum_{s=1}^t \tilde{\omega}_a^{\eta_t}(\hat{\boldsymbol{\mu}}_s) \quad (3)$$

where $N_a(t)$ is the number of times arm a has been pulled up to time t . In other words, we sample the arm whose number of visits is far behind its corresponding sum of empirical optimal allocations. This insures that on the long run, as our estimate $\hat{\boldsymbol{\mu}}_t$ becomes close to the true value $\boldsymbol{\mu}$, the sampling frequency $N_a(t)/t$ of every arm a will converge to the oracle optimal allocation $\omega_a^*(\boldsymbol{\mu})$. The projection beforehand on $\Delta_K^{\eta_t}$ ensures exploration at minimal rate $\eta_t = \frac{1}{2\sqrt{(K^2+t)}}$ so that no arm is left-behind because of bad initial estimates.

Stopping rule: To be sample-efficient, the algorithm should stop as soon as the collected samples are enough to declare that $G_\varepsilon(\hat{\boldsymbol{\mu}}_t) = G_\varepsilon(\boldsymbol{\mu})$ with probability larger than $1 - \delta$. For this purpose we use the Generalized Likelihood Ratio (GLR) test [Che59]. We define the Z -statistic:

$$Z(t) = t \times T_\varepsilon\left(\hat{\boldsymbol{\mu}}_t, \frac{N(t)}{t}\right)^{-1}$$

where $N(t) = (N_a(t))_{a \in [K]}$. It can be shown [GK19] that the Z -statistic is equal to the ratio of the likelihood of observations under the most likely model where $G_\varepsilon(\hat{\boldsymbol{\mu}}_t)$ is the correct answer, i.e. $\hat{\boldsymbol{\mu}}_t$, to the likelihood of observations under the most likely model where $G_\varepsilon(\hat{\boldsymbol{\mu}})$ is not the set of ε -good arms. The algorithm rejects the hypothesis $G_\varepsilon(\hat{\boldsymbol{\mu}}_t) \neq G_\varepsilon(\boldsymbol{\mu})$ and stops as soon as this ratio of likelihoods becomes larger than a certain threshold $\beta(\delta, t)$, properly tuned to ensure that the algorithm is δ -PAC. The stopping rule is defined as:

$$\tau_\delta = \inf \{t \in \mathbb{N} : Z(t) > \beta(t, \delta)\} \quad (4)$$

One can find many suitable thresholds from the bandit literature [Gar13], [MCP14], [KK18], all of which are of the order $\beta(\delta, t) \approx \log(1/\delta) + \frac{K}{2} \log(\log(t/\delta))$ is enough to ensure that $\mathbb{P}(G_\varepsilon(\hat{\boldsymbol{\mu}}_{\tau_\delta}) \neq G_\varepsilon(\boldsymbol{\mu})) \leq \delta$, i.e. that the algorithm is δ -correct.

Algorithm 1: Track and Stop

Input: Confidence level δ , accuracy parameter ε .

- 1 Pull each arm once and observe rewards $(r_a)_{a \in [K]}$.
 - 2 Set initial estimate $\hat{\boldsymbol{\mu}}_K = (r_1, \dots, r_K)^T$.
 - 3 Set $t \leftarrow K$ and $N_a(t) \leftarrow 1$ for all arms a .
 - 4 **while** *Stopping condition (4) is not satisfied* **do**
 - 5 Compute $\tilde{\boldsymbol{\omega}}(\hat{\boldsymbol{\mu}}_t)$, a $\frac{1}{\sqrt{t}}$ -optimal vector for (1) using mirror-ascent.
 - 6 Pull next arm a_{t+1} given by (3) and observe reward r_t .
 - 7 Update $\hat{\boldsymbol{\mu}}_t$ according to r_t .
 - 8 Set $t \leftarrow t + 1$ and update $(N_a(t))_{a \in [K]}$.
 - 9 **end**
- Output:** Empirical set of ε -good arms: $G_\varepsilon(\hat{\boldsymbol{\mu}}_{\tau_\delta})$
-

Now we state our sample complexity result which we adapted from Theorem 14 in [GK16]. Notably, while their Track-and-Stop strategy relies on tracking the exact optimal weights to prove that the expected stopping time matches the lower bound when δ tends to zero, our proof shows that it is enough to track some slightly sub-optimal weights with a decreasing gap in the order of $\frac{1}{\sqrt{t}}$ to enjoy the same sample complexity guarantees.

Theorem 2. *For all $\delta \in (0, 1)$, Track-and-Stop terminates almost-surely and its stopping time τ_δ satisfies:*

$$\limsup_{\delta \rightarrow 0} \frac{\mathbb{E}[\tau_\delta]}{\log(1/\delta)} \leq T_\varepsilon^*(\boldsymbol{\mu}).$$

Remark 1. *Suppose that the arms are ordered decreasingly $\mu_1 \geq \mu_2 \geq \dots \geq \mu_K$. [MJTN20] define the gap $\alpha_\varepsilon = \min_{k \in G_\varepsilon(\boldsymbol{\mu})} \mu_k - (\mu_1 - \varepsilon)$ and provide a lower bound of the form $f(\nu) \log(1/\delta)$ where:*

$$f(\nu) \triangleq 2 \sum_{i=1}^K \max \left(\frac{1}{(\mu_1 - \varepsilon - \mu_i)^2}, \frac{1}{(\mu_1 + \alpha_\varepsilon - \mu_i)^2} \right).$$

Therefore, we deduce from Theorem 2 that $f(\nu) \leq T_\varepsilon^*(\boldsymbol{\mu})$. In a second step, they proposed (ST)², an algorithm whose sample complexity matches their bound up to some universal constant c that doesn't depend on the instance ν . From Proposition 1, we deduce that $T_\varepsilon^*(\boldsymbol{\mu}) \leq cf(\nu)$, which can be seen directly from the particular changes of measure considered in that paper. The sample complexity of our algorithm improves previous work by multiplicative constants that can possibly be large, as illustrated in Section 5.

Note that the algorithm requires to solve the best response problem, i.e. the minimization problem in (2), in order to be able to compute the Z -statistic of the stopping rule, and also to solve the entire lower bound problem in (1) to compute the optimal weights for the sampling rule. The rest of the paper is dedicated to presenting the tools necessary to solve these two problems.

3 Solving the min problem: Best response oracle

For a given vector $\boldsymbol{\omega}$, we want to compute the best response

$$\boldsymbol{\lambda}_{\varepsilon, \boldsymbol{\mu}}^*(\boldsymbol{\omega}) \triangleq \arg \min_{\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\mu})} \sum_{a \in [K]} \omega_a \frac{(\mu_a - \lambda_a)^2}{2}. \quad (5)$$

For the simplicity of the presentation, we assume that the arms are ordered decreasingly $\mu_1 \geq \mu_2 \geq \dots \geq \mu_K$ and start by presenting the additive case (i.e. $G_\varepsilon(\boldsymbol{\mu}) \triangleq \{a \in [K] : \mu_a \geq \max_i \mu_i - \varepsilon\}$).

The multiplicative case can be treated in the same fashion and is deferred to appendix A. Finally, we denote by $B_\varepsilon(\boldsymbol{\mu}) \triangleq [K] \setminus G_\varepsilon(\boldsymbol{\mu})$ the set of bad arms.

Since an alternative problem $\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\mu})$ must have a different set of ε -optimal arms than the original problem $\boldsymbol{\mu}$, we can obtain it from $\boldsymbol{\mu}$ by changing the expected reward of some arms. We have two options to create an alternative problem $\boldsymbol{\lambda}$:

- **Making one of the ε -optimal arms bad.** We can achieve it by decreasing the expectation of some ε -optimal arm k while increasing the expectation of some other arm ℓ to the point where k is no more ε -optimal. This is illustrated in Figure 1.
- **Making one of the ε -sub-optimal arms good.** We can achieve it by increasing the expectation of some sub-optimal arm k while decreasing the expectations of the arms with the largest means -as many as it takes- to the point where k becomes ε -optimal. This is illustrated in Figure 2.

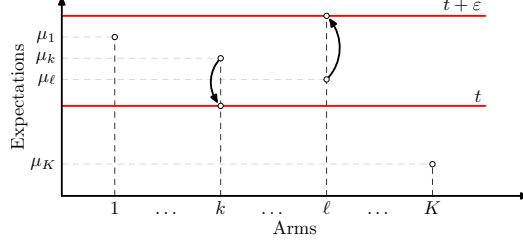


Figure 1: Making One of the ε -Optimal Arms Bad.

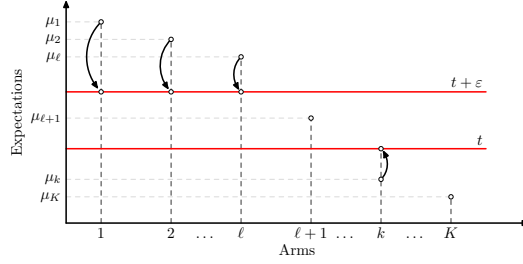


Figure 2: Making One of the ε -Sub-Optimal Arms Good.

In the following, we solve both cases separately.

Case 1: Making one of the ε -optimal arms bad. Let $k \in G_\varepsilon(\boldsymbol{\mu})$ be one of the ε -optimal arms. In order to make arm k sub-optimal, we need to set the expectation of arm k to some value $\lambda_k = t$ and the maximum expectation over all arms to $\max_a \lambda_a = t + \varepsilon$. Note that the index of the arm ℓ with maximum expectation can be chosen in $G_\varepsilon^a(\boldsymbol{\mu})$. Indeed, if we choose some arm from $B_\varepsilon(\boldsymbol{\mu})$ to become the arm with maximum expectation in λ then we would make an ε -suboptimal arm good which is covered in the other case below. The expectations of all the other arms should stay the same as in the instance $\boldsymbol{\mu}$, since changing their values would only increase the value of the objective. Now given indices k and ℓ , computing the optimal value of t is rather straightforward since the objective function simplifies to

$$\omega_k \frac{(\mu_k - t)^2}{2} + \omega_\ell \frac{(\mu_\ell - t - \varepsilon)^2}{2}$$

for which the optimal value of t is:

$$t = \bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) \triangleq \frac{\omega_k \mu_k + \omega_\ell (\mu_\ell - \varepsilon)}{\omega_k + \omega_\ell}.$$

and the corresponding alternative bandit is:

$$\boldsymbol{\lambda}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) \triangleq (\mu_1, \dots, \underbrace{\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega})}_{\text{index } k}, \dots, \underbrace{\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) + \varepsilon}_{\text{index } \ell}, \dots, \mu_K)^\top.$$

The last step is taking the pair of indices $(k, \ell) \in G_\varepsilon(\boldsymbol{\mu}) \times (G_\varepsilon(\boldsymbol{\mu}) \setminus \{k\})$ with the minimal value in the objective (2).

Case 2: Making one of the sub-optimal arms good. Let $k \in B_\varepsilon(\boldsymbol{\mu})$ be a sub-optimal arm, if such arm exists, and denote by t the value of its expectation in $\boldsymbol{\lambda}$. In order to make this arm ε -optimal, we need to decrease the expectations of all the arms that are above the threshold $t + \varepsilon$. We pay a cost of $\frac{1}{2}\omega_k(t - \mu_k)^2$ for moving arm k and of $\frac{1}{2}\omega_i(t + \varepsilon - \mu_i)^2$ for every arm i such that $\mu_i > t + \varepsilon$. Consider the functions:

$$f_k(t) = \frac{1}{2}\omega_k(t - \mu_k)^2$$

and for $i \in [K] \setminus \{k\}$

$$f_i(t) = \begin{cases} \frac{1}{2}\omega_i(t + \varepsilon - \mu_i)^2 & \text{for } t < \mu_i - \varepsilon, \\ 0 & \text{for } t \geq \mu_i - \varepsilon. \end{cases}$$

Each of these functions is convex. Therefore the function $f(t) = \sum_{i=1}^K f_i(t)$ is convex and has a unique minimizer t^* . One can easily check that $f'(\mu_k) \leq 0$ and $f'(\mu_1 - \varepsilon) \geq 0$, implying that $\mu_k - \varepsilon < \mu_k \leq t^* \leq \mu_1 - \varepsilon$. Therefore:

$$\ell = \min\{i \geq 1 : t^* > \mu_i - \varepsilon\} - 1$$

is well defined and satisfies $\ell \in \llbracket 1, k - 1 \rrbracket$. Note that by definition $\mu_{\ell+1} - \varepsilon < t^*$ and $t^* \leq \mu_a - \varepsilon$ for all $a \leq \ell$, hence:

$$0 = f'(t^*) = \omega_k(t^* - \mu_k) + \sum_{a=1}^{\ell} \omega_a(t^* + \varepsilon - \mu_a).$$

Implying that¹:

$$t^* = \bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) \triangleq \frac{\omega_k \mu_k + \sum_{a=1}^{\ell} \omega_a (\mu_a - \varepsilon)}{\omega_k + \sum_{a=1}^{\ell} \omega_a}$$

and the alternative bandit in this case writes as:

$$\boldsymbol{\lambda}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) \triangleq (\underbrace{\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) + \varepsilon}_{\text{indices 1 to } \ell}, \mu_{\ell+1}, \dots, \underbrace{\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega})}_{\text{index } k}, \dots, \mu_K)^\top.$$

Observe that since ℓ depends on t^* , we can't directly compute t^* from the expression above. Instead, we use the fact that ℓ is unique by definition. Therefore, to determine t^* one can compute $\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega})$ for all values of $\ell \in \llbracket 1, k - 1 \rrbracket$ and search for the index ℓ satisfying $\mu_{\ell+1} - \varepsilon < \bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) \leq \mu_\ell - \varepsilon$ and with minimum value in the objective (2).

As a summary, we have reduced the minimization problem over the infinite set $\text{Alt}(\boldsymbol{\mu})$ to a combinatorial search over a finite number of alternative bandit instances whose analytical expression is given in the next definition.

Definition 1. Let $\boldsymbol{\lambda}_\varepsilon^{k,\ell}(\boldsymbol{\omega})$ be a vector created from $\boldsymbol{\mu}$ by replacing elements on positions k and ℓ (resp. 1 to ℓ), defined as:

$$\boldsymbol{\lambda}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) \triangleq (\mu_1, \dots, \underbrace{\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega})}_{\text{index } k}, \dots, \underbrace{\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) + \varepsilon}_{\text{index } \ell}, \dots, \mu_K)^\top$$

for $k \in G_\varepsilon(\boldsymbol{\mu})$ and

$$\boldsymbol{\lambda}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) \triangleq (\underbrace{\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) + \varepsilon}_{\text{indices 1 to } \ell}, \mu_{\ell+1}, \dots, \underbrace{\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega})}_{\text{index } k}, \dots, \mu_K)^\top$$

for $k \in B_\varepsilon(\boldsymbol{\mu})$ where $\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega})$ is a weighted average of elements on positions k and ℓ (resp. 1 to ℓ) defined as:

$$\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) \triangleq \frac{\omega_k \mu_k + \omega_\ell (\mu_\ell - \varepsilon)}{\omega_k + \omega_\ell}$$

for $k \in G_\varepsilon(\boldsymbol{\mu})$ and

$$\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) \triangleq \frac{\omega_k \mu_k + \sum_{a=1}^{\ell} \omega_a (\mu_a - \varepsilon)}{\omega_k + \sum_{a=1}^{\ell} \omega_a}$$

for $k \in B_\varepsilon(\boldsymbol{\mu})$.

¹ $\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega})$ has a different definition depending on k being a good or a bad arm.

The next lemma then states that the best response oracle belongs to the finite set of $(\lambda_\varepsilon^{k,\ell}(\omega))_{k,\ell}$.

Lemma 3. Using the previous definition, $\lambda_{\varepsilon,\mu}^*(\omega)$ can be computed as

$$\lambda_{\varepsilon,\mu}^*(\omega) = \arg \min_{\lambda \in \Lambda_G \cup \Lambda_B} \sum_{a \in [K]} \omega_a \frac{(\mu_a - \lambda_a)^2}{2}$$

where

$$\Lambda_G = \{\lambda_\varepsilon^{k,\ell}(\omega) : k \in G_\varepsilon(\mu), \ell \in G_\varepsilon(\mu) \setminus \{k\}\}$$

and

$$\Lambda_B = \{\lambda_\varepsilon^{k,\ell}(\omega) : k \in B_\varepsilon(\mu), \ell \in [1, k-1] \text{ s.t. } \mu_\ell \geq \bar{\mu}_\varepsilon^{k,\ell}(\omega) + \varepsilon > \mu_{\ell+1}\}.$$

4 Solving the max-min problem: Optimal weights

First observe that we can rewrite $T_\varepsilon(\mu, \cdot)^{-1}$ as a minimum of linear functions:

$$T_\varepsilon(\mu, \omega)^{-1} = \inf_{\mathbf{d} \in \mathcal{D}_{\varepsilon,\mu}} \omega^\top \mathbf{d} \quad (6)$$

where

$$\mathcal{D}_{\varepsilon,\mu} \triangleq \left\{ \left(\frac{(\lambda_a - \mu_a)^2}{2} \right)_{a \in [K]}^\top \mid \lambda \in \text{Alt}(\mu) \right\}.$$

Note that by using $\mathcal{D}_{\varepsilon,\mu}$ instead of $\text{Alt}(\mu)$, the optimization function becomes simpler for the price of more complex domain (see Figure 3 for an example). As a result, $T_\varepsilon(\mu, \cdot)^{-1}$ is concave and the next lemma provides a convenient way to compute its supergradients.

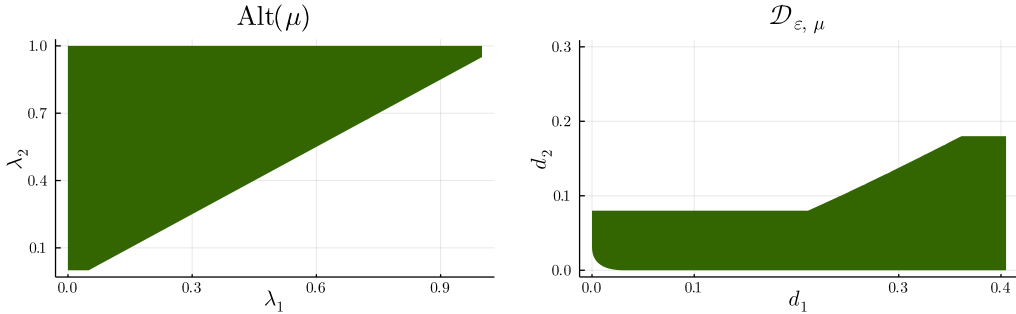


Figure 3: Comparison of $\text{Alt}(\mu)$ with Simple Linear Boundaries (First Figure) and $\mathcal{D}_{\varepsilon,\mu}$ with Non-Linear Boundaries (Second Figure) for $\mu = [0.9, 0.6]$ and $\varepsilon = 0.05$.

Lemma 4. Let $\lambda^*(\omega)$ be a best response to ω and define $\mathbf{d}^*(\omega) \triangleq \left(\frac{(\lambda^*(\omega)_a - \mu_a)^2}{2} \right)_{a \in [K]}^\top$. Then $\mathbf{d}^*(\omega)$ is a supergradient of $T_\varepsilon(\mu, \cdot)^{-1}$ at ω .

Proof. Let f denote the function $\omega \mapsto T_\varepsilon(\mu, \omega)^{-1}$. By definition, $\mathbf{d}^*(\omega)$ is a vector that realizes the infimum from the expression (6) of f . We will prove that $\mathbf{d}^*(\omega)$ is a supergradient of f at any point ω . The concavity of the function is a direct consequence of the existence of supergradients.

Let ω_1 and ω_2 be any two points from the domain of f . From the definition of $\mathbf{d}^*(\omega)$ we have

$$\omega_1^\top \mathbf{d}^*(\omega_1) \geq \omega_2^\top \mathbf{d}^*(\omega_2).$$

Adding $0 = \omega_1^\top \mathbf{d}^*(\omega_1) - \omega_1^\top \mathbf{d}^*(\omega_1)$ to the left hand side of the inequality we obtain

$$\omega_1^\top \mathbf{d}^*(\omega_1) + (\omega_2 - \omega_1)^\top \mathbf{d}^*(\omega_1) \geq \omega_2^\top \mathbf{d}^*(\omega_2).$$

Which, using the definition of f , can be further rewritten as

$$f(\omega_1) + \mathbf{d}^*(\omega_1)^\top (\omega_2 - \omega_1) \geq f(\omega_2).$$

Thus, $\mathbf{d}^*(\omega_1)$ is a supergradient of f at ω_1 and f is concave. \square

Next we prove that $T_\varepsilon(\boldsymbol{\mu}, \cdot)^{-1}$ is Lipschitz.

Lemma 5. *The function $\boldsymbol{\omega} \mapsto T_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\omega})^{-1}$ is L -Lipschitz with respect to $\|\cdot\|_1$ for any*

$$L \geq \max_{a,b \in [K]} \frac{(\mu_a - \mu_b + \varepsilon)^2}{2}.$$

Proof. As we showed in Lemma 3, the best response $\boldsymbol{\lambda}_{\varepsilon, \boldsymbol{\mu}}^*(\boldsymbol{\omega})$ to $\boldsymbol{\omega}$ is created from $\boldsymbol{\mu}$ by replacing some of the elements by $\overline{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega})$ or $\overline{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) + \varepsilon$. We also know that $\overline{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega})$ is a weighted average of an element of $\boldsymbol{\mu}$ with one or more elements of $\boldsymbol{\mu}$ decreased by ε . This means that:

$$\max_{a \in [K]} \mu_a \geq \overline{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) \geq \min_{a \in [K]} \mu_a - \varepsilon$$

and, as a consequence, we have:

$$|\mu_i - \boldsymbol{\lambda}_{\varepsilon, \boldsymbol{\mu}}^*(\boldsymbol{\omega})_i| \leq \max_{a,b \in [K]} (\mu_a - \mu_b + \varepsilon)$$

for any $i \in [K]$. Let $f(\boldsymbol{\omega}) \triangleq T_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\omega})^{-1}$. Using the last inequality and the definition of $\boldsymbol{d}^*(\boldsymbol{\omega})$, we can obtain:

$$\begin{aligned} f(\boldsymbol{\omega}) - f(\boldsymbol{\omega}') &\leq (\boldsymbol{\omega} - \boldsymbol{\omega}')^\top \boldsymbol{d}^*(\boldsymbol{\omega}') \\ &\leq \|\boldsymbol{\omega} - \boldsymbol{\omega}'\|_1 \|\boldsymbol{d}^*(\boldsymbol{\omega}')\|_\infty \\ &\leq \|\boldsymbol{\omega} - \boldsymbol{\omega}'\|_1 \max_{a,b \in [K]} \frac{(\mu_a - \mu_b + \varepsilon)^2}{2} \end{aligned}$$

for any $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \Delta_K$. □

As a summary $T_\varepsilon(\boldsymbol{\mu}, \cdot)^{-1}$ is concave, Lipschitz and we have a simple expression to compute its supergradients through the best response oracle. Therefore we have all the necessary ingredients to apply a gradient-based algorithm in order to find the optimal weights and therefore, the value of $T_\varepsilon^*(\boldsymbol{\mu})$. The algorithm of our choice is the mirror ascent algorithm which provides the following guarantees:

Proposition 6. [Bub15] *Let $\boldsymbol{\omega}_1 = (\frac{1}{K}, \dots, \frac{1}{K})^\top$ and learning rate $\alpha_n = \frac{1}{L} \sqrt{\frac{2 \log K}{n}}$. Then using mirror ascent algorithm to maximize a L -Lipschitz function f , with respect to $\|\cdot\|_1$, defined on Δ_K with generalized negative entropy $\Phi(\boldsymbol{\omega}) = \sum_{a \in [K]} \omega_a \log(\omega_a)$ as the mirror map enjoys the following guarantees:*

$$f(\boldsymbol{\omega}^*) - f\left(\frac{1}{N} \sum_{n=1}^N \boldsymbol{\omega}_n\right) \leq L \sqrt{\frac{2 \log K}{N}}.$$

5 Experiments

We conducted three experiments to compare Track-and-Stop with state-of-the-art algorithms, mainly (ST)² and FAREAST from [MJTN20]. In the first experiment, we simulate a multi-armed bandit with Gaussian rewards of means $\boldsymbol{\mu} = [1, 1, 1, 1, 0.05]$, variance one and a parameter $\varepsilon = 0.9$. We chose this particular instance $\boldsymbol{\mu}$ because its difficulty is two-fold: First, the last arm μ_5 is very close to the threshold $\max_a \mu_a - \varepsilon$. Second, the argmax is realized by more than one arm, which implies that any algorithm must estimate all the means to high precision to produce a confident guess of $G_\varepsilon(\boldsymbol{\mu})$. Indeed, a small underestimation error of $\max_a \mu_a$ would mean wrongly classifying μ_5 as a good arm. We run the three algorithms for several values of δ ranging from $\delta = 0.1$ to $\delta = 10^{-10}$, with $N = 100$ Monte-Carlo simulations for each risk level. Figure 4 shows the expected stopping time along with the 10% and 90% quantiles (shaded area) for each algorithm. Track-and-Stop consistently outperforms (ST)² and FAREAST, even for moderate values of δ . Also note that, as we pointed out in Remark 1, the sample complexity of Track-and-Stop is within some multiplicative constant of (ST)².

Next, we examine the performance of the algorithms w.r.t the number of arms. For any given K , we consider a bandit problem $\boldsymbol{\mu}$ similar to the previous instance: $\forall a \in [1, K-1], \mu_a = 1$ and

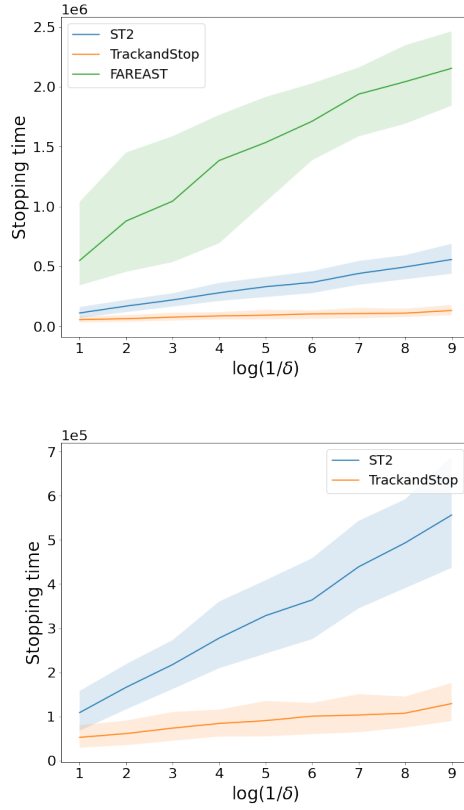


Figure 4: Expected Stopping Time on $\mu = [1, 1, 1, 1, 0.05]$. Left: All three Algorithms. Right: Track-and-Stop vs FAREAST.

$\mu_K = 0.05$. We fix $\varepsilon = 0.9$ and $\delta = 0.1$ and run $N = 30$ Monte-Carlo simulations for each K . Figure 5 shows, in log-scale, the ratio of the sample complexities of $(ST)^2$ and FAREAST w.r.t to the sample complexity of Track-and-Stop. We see that Track-and-Stop performs better than $(ST)^2$ (resp. FAREAST) for small values of K . However when the number of arms grows larger than $K = 40$ (resp. $K = 60$), $(ST)^2$ (resp. FAREAST) have a smaller sample complexity.

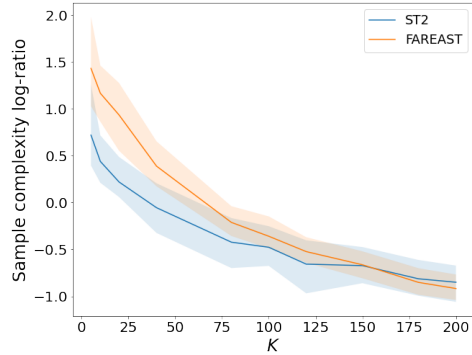


Figure 5: $\log_{10} \left(\frac{\mathbb{E}_{\text{Alg}}[\tau_\delta]}{\mathbb{E}_{\text{TaS}}[\tau_\delta]} \right)$ for $\text{Alg} \in \{(ST)^2, \text{FAREAST}\}$ and $\text{TaS} = \text{Track-and-Stop}$. $K_{\min} = 5$ arms.

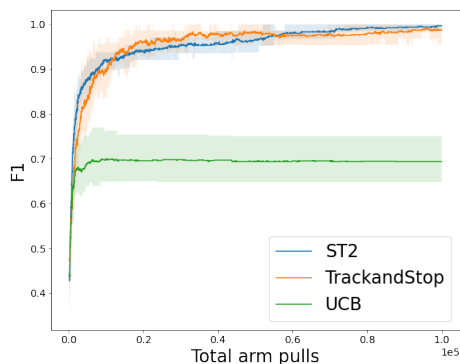


Figure 6: Cancer Drug Discovery

Finally, we rerun the Cancer Drug Discovery experiment from [MJTN20]. Note that this experiment is more adapted to a *fixed budget setting* where we fix a sampling budget and the algorithm stops once it has reached this limit, which is different from the *fixed confidence* setting that our algorithm was designed for. The goal is to find, among a list of 189 chemical compounds, potential inhibitors to **ACRVL1**, a Kinase that researchers [BSK⁺19] have linked to several forms of cancer. We use the same dataset as [MJTN20], where for each compound a percent control² is reported. We fix a budget of samples $N = 10^5$ and try to find all the ε -good compounds in the multiplicative case with $\varepsilon = 0.8$. For each algorithm, we compute the F1-score³ of its current estimate $\hat{G}_\varepsilon = \{i : \hat{\mu}_i \geq (1 - \varepsilon) \max_a \hat{\mu}_a\}$ after every iteration. The F1-score in this fixed-budget setting reflects how good is the sampling scheme of an algorithm, independently of its stopping condition. In Figure 6 we plot the average F1-score along with the 10% and 90% quantiles (shaded area). We see that (ST)² and Track-and-Stop have comparable performance and that both outperform UCB’s sampling scheme.

6 Conclusion

We investigated the Sample Complexity of finding all the ε -good arms in a multi-armed bandit with Gaussian rewards. First we derived a lower bound involving a characteristic time that reflects the true hardness of the problem. Then based on a method to solve the optimization program that defines the characteristic time, we proposed a Track-and-Stop strategy whose sample complexity matches the lower bound for small values of the risk level. Finally, we showed through numerical simulations that our algorithm outperforms state-of-the-art methods for bandits with small to moderate number of arms.

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²percent control is a metric expressing the efficiency of the compound as an inhibitor against the target Kinase.

³F1 score is the harmonic mean of precision (the proportion of arms in \hat{G} that are actually good) and recall (the proportion of arms in $G_\varepsilon(\mu)$ that were correctly returned in \hat{G}).

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A Multiplicative Case

Most of the previous analysis of the additive case can be directly applied to the multiplicative case where the set of the best arms is defined as $G_\varepsilon(\boldsymbol{\mu}) \triangleq \{a \in [K] : \mu_a \geq (1 - \varepsilon) \max_i \mu_i\}$. The only missing pieces are the best response oracle and an upper bound on the Lipschitz constant for a mirror ascent algorithm. Solving these two problems gives us all the necessary tools to produce the multiplicative version of the algorithm presented in 9.

A.1 Best Response Oracle - Multiplicative Case

Similarly, as in the additive case, split the reasoning into two parts: making a good arm bad or making a bad arm good. This provides the following variation of Lemma 3

Lemma 7. Define $\lambda_\varepsilon^{k,\ell}(\boldsymbol{\omega})$ as in Definition 1 for

$$\bar{\mu}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) \triangleq \frac{(1 - \varepsilon)^2 \omega_k \mu_k + (1 - \varepsilon) \omega_\ell \mu_\ell}{(1 - \varepsilon)^2 \omega_k + \omega_\ell}$$

in the case of $k \in G_\varepsilon(\boldsymbol{\mu})$ and

$$\bar{\mu}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) \triangleq \frac{(1 - \varepsilon)^2 \omega_k \mu_k + \sum_{a=1}^\ell (1 - \varepsilon) \omega_a \mu_a}{(1 - \varepsilon)^2 \omega_k + \sum_{a=1}^\ell \omega_a}$$

in the case of $k \in B_\varepsilon(\boldsymbol{\mu})$. Then the best response $\lambda_{\varepsilon,\boldsymbol{\mu}}^*(\boldsymbol{\omega})$ to a given $\boldsymbol{\omega}$ is defined as:

$$\lambda_{\varepsilon,\boldsymbol{\mu}}^*(\boldsymbol{\omega}) = \arg \min_{\lambda \in \Lambda_G \cup \Lambda_B} \sum_{a \in [K]} \omega_a \frac{(\mu_a - \lambda_a)^2}{2}.$$

where

$$\Lambda_G = \{\lambda_\varepsilon^{k,\ell}(\boldsymbol{\omega}) : k \in G_\varepsilon(\boldsymbol{\mu}), \ell \in G_\varepsilon(\boldsymbol{\mu}) \setminus \{k\}\}$$

and

$$\Lambda_B = \{\lambda_\varepsilon^{k,\ell}(\boldsymbol{\omega}) : k \in B_\varepsilon(\boldsymbol{\mu}), \ell \in [1, k - 1]\} \\ \text{s.t. } \mu_\ell \geq \bar{\mu}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) / (1 - \varepsilon) > \mu_{\ell+1}\}.$$

Proof.

Case 1: Making one of the ε -optimal arms bad. Selecting a good arm $k \in G_\varepsilon(\boldsymbol{\mu})$ and any other arm $\ell \in G_\varepsilon(\boldsymbol{\mu})$ we can set the expectation of arm k to t and the expectation of arm ℓ to $t/(1 - \varepsilon)$. Using calculus, we can find the optimal value

$$t = \frac{(1 - \varepsilon)^2 \omega_k \mu_k + (1 - \varepsilon) \omega_\ell \mu_\ell}{(1 - \varepsilon)^2 \omega_k + \omega_\ell}.$$

Case 2: Making one of the sub-optimal arms good. Selecting a bad arm $k \in B_\varepsilon(\boldsymbol{\mu})$ and the first ℓ arms we can set the expectation of arm k to t and the expectation of the first ℓ arms to $t/(1 - \varepsilon)$. Note that ℓ is a unique integer such that $\mu_\ell \geq t/(1 - \varepsilon) > \mu_{\ell+1}$ thanks to the same argument as in the additive case. Using calculus, we can find the optimal value

$$t = \frac{(1 - \varepsilon)^2 \omega_k \mu_k + \sum_{a=1}^\ell (1 - \varepsilon) \omega_a \mu_a}{(1 - \varepsilon)^2 \omega_k + \sum_{a=1}^\ell \omega_a}.$$

□

A.2 Upper bound on the Lipschitz constant

The final ingredient is a constant L for a mirror ascent algorithm, characterized in the following lemma.

Lemma 8. *The function $\omega \mapsto T_\varepsilon(\boldsymbol{\mu}, \omega)^{-1}$ is L -Lipschitz with respect to $\|\cdot\|_1$ for any*

$$L \geq \max_{a,b \in [K]} \frac{(\mu_a - \mu_b(1 - \varepsilon))^2}{2(1 - \varepsilon)^2}.$$

Proof. Lemma 7 shows that the expectation of the selected arm k should be in equal to

$$t = \frac{(1 - \varepsilon)^2 \omega_k \mu_k + \sum (1 - \varepsilon) \omega_a \mu_a}{(1 - \varepsilon)^2 \omega_k + \sum \omega_a}$$

where the sum is taken either over a single arm ℓ or over the first ℓ arms. In both cases we can show

$$(1 - \varepsilon) \mu_{\min} \leq \frac{(1 - \varepsilon)^2 \omega_k \mu_k + \sum (1 - \varepsilon) \omega_a \mu_a}{(1 - \varepsilon)^2 \omega_k + \sum \omega_a}$$

using $(1 - \varepsilon) < 1$, $\mu_{\min} \leq \mu_a$ for any $a \in [K]$ and

$$\frac{(1 - \varepsilon)^2 \omega_k \mu_k + \sum (1 - \varepsilon) \omega_a \mu_a}{(1 - \varepsilon)^2 \omega_k + \sum \omega_a} \leq \mu_{\max}$$

using $(1 - \varepsilon) < 1$ and $\mu_{\max} \geq \mu_a$ for any $a \in [K]$. Using this inequality we can show that $(\mu_a - \lambda_a)^2$ can be upper-bounded by

$$\left(\frac{\mu_{\max}}{1 - \varepsilon} - \mu_{\min} \right)^2$$

for any best response $\boldsymbol{\lambda}$. Applying this bound to the optimization problem concludes the proof of the lemma. \square

B Proof of Proposition 1

Let $\delta \in (0, 1)$ be a confidence parameter and $\boldsymbol{\mu}$ be a bandit problem. Considering a δ -PAC strategy, we can apply transportation Lemma 1 of [KCG15] to obtain

$$\sum_{a \in [K]} \frac{(\mu_a - \lambda_a)^2}{2} \mathbb{E}[N_a] \geq \text{kl}(\delta, 1 - \delta)$$

for $N_a \triangleq N_a(\tau_\delta)$. This holds for any alternative problem $\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\mu})$. Therefore, we have

$$\begin{aligned} \text{kl}(\delta, 1 - \delta) &\leq \inf_{\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\mu})} \sum_{a \in [K]} \mathbb{E}[\tau_\delta] \frac{(\mu_a - \lambda_a)^2}{2} \frac{\mathbb{E}[N_a]}{\mathbb{E}[\tau_\delta]} \\ &\leq \mathbb{E}[\tau_\delta] \sup_{\boldsymbol{\omega} \in \Delta_K} \inf_{\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\mu})} \sum_{a \in [K]} \omega_a \frac{(\mu_a - \lambda_a)^2}{2}. \end{aligned}$$

Fraction $\mathbb{E}[N_a]/\mathbb{E}[\tau_\delta]$ represents empirical proportion of arm draws which enables us to upper-bound it by a supremum over all possible arm allocations.

C Proof of Theorem 2

We start with a few technical lemmas. The first two are adapted from [GK16]:

Lemma 9. (LEMMA 7, [GK16]) *For all $t \geq 1$, the C-Tracking with weights $(\tilde{\omega}(\hat{\boldsymbol{\mu}}_s))_{s \in \mathbb{N}^*}$ ensures that $N_a(t) \geq \sqrt{t + K^2} - 2K$ and that*

$$\max_{1 \leq i \leq K} \left| N_a(t) - \sum_{s=1}^t \tilde{\omega}_a(\hat{\boldsymbol{\mu}}_s) \right| \leq K(1 + \sqrt{t})$$

Lemma 10. (LEMMA 19, [GK16]) For $\xi > 0$, define $I_\xi \triangleq [\mu_1 - \xi, \mu_1 + \xi] \times \dots \times [\mu_K - \xi, \mu_K + \xi]$.

And for $T \geq 1$, consider the event: $\mathcal{E}_T = \bigcap_{t=\lfloor T^{1/4} \rfloor}^T (\hat{\boldsymbol{\mu}}_t \in I_\xi)$. Then there exists two constants B, C that only depend on $\boldsymbol{\mu}$ and ξ such that

$$\mathbb{P}(\mathcal{E}_T^c) \leq BT \exp(-CT^{1/8})$$

where \mathcal{E}_T^c is the complementary event of \mathcal{E}_T .

The last lemma states that $\boldsymbol{\mu} \mapsto T_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\omega})^{-1}$ is Lipschitz. Its proof is deferred to the end.

Lemma 11. For all vectors $\boldsymbol{\omega}$ in the simplex, for all instances $\boldsymbol{\mu}, \boldsymbol{\mu}'$ in $[\mu_{\min}, \mu_{\max}]^K$ we have:

$$|T_\varepsilon(\boldsymbol{\mu}', \boldsymbol{\omega})^{-1} - T_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\omega})^{-1}| \leq 4(\mu_{\max} - \mu_{\min} + \varepsilon) \|\boldsymbol{\mu}' - \boldsymbol{\mu}\|_\infty.$$

Now we are ready to prove the Theorem. We denote by $L_1([\mu_{\min}, \mu_{\max}]^K) \triangleq 4(\mu_{\max} - \mu_{\min} + \varepsilon)$ the Lipschitz constant of the mapping $\boldsymbol{\mu} \mapsto T_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\omega})^{-1}$ in the domain $[\mu_{\min}, \mu_{\max}]^K$ and by $L_2(\boldsymbol{\mu}) \triangleq \max_{a,b \in [K]} \frac{(\mu_a - \mu_b + \varepsilon)^2}{2}$ the Lipschitz constant of the mapping $\boldsymbol{\omega} \mapsto T_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\omega})^{-1}$.

We will prove a lower bound on $T_\varepsilon\left(\hat{\boldsymbol{\mu}}_t, \frac{N(t)}{t}\right)^{-1}$ under \mathcal{E}_T which will result into an upper bound on the stopping time $\tau_\delta = \inf \{t \in \mathbb{N} : tT_\varepsilon\left(\hat{\boldsymbol{\mu}}_t, \frac{N(t)}{t}\right)^{-1} \geq \beta(\delta, t)\}$. First observe that under \mathcal{E}_T , the L_1 constant is upper bounded: $L_1(I_\xi) \leq L_{1,\max} \triangleq 4(\max_a \mu_a - \min_b \mu_b + \varepsilon + 2\xi)$. Similarly, we have for all $\lfloor T^{1/4} \rfloor \leq t \leq T$, $L_2(\hat{\boldsymbol{\mu}}_t) \leq L_{2,\max} \triangleq \frac{(\max_a \mu_a - \min_b \mu_b + 2\xi + \varepsilon)^2}{2}$. Now applying Lemma 7 and the Lipschitz property w.r.t the weights, we have for all $\lfloor T^{1/4} \rfloor \leq t \leq T$:

$$\begin{aligned} T_\varepsilon\left(\hat{\boldsymbol{\mu}}_t, \frac{N(t)}{t}\right)^{-1} &\geq T_\varepsilon\left(\hat{\boldsymbol{\mu}}_t, \frac{\sum_{s=1}^t \tilde{\boldsymbol{\omega}}(\hat{\boldsymbol{\mu}}_s)}{t}\right)^{-1} - L_{2,\max} \frac{K(1 + \sqrt{t})}{t} \\ &\geq \frac{\sum_{s=1}^t T_\varepsilon(\hat{\boldsymbol{\mu}}_t, \tilde{\boldsymbol{\omega}}(\hat{\boldsymbol{\mu}}_s))^{-1}}{t} - L_{2,\max} \frac{K(1 + \sqrt{t})}{t} \\ &\geq \frac{\sum_{s=\lfloor T^{1/4} \rfloor}^t T_\varepsilon(\hat{\boldsymbol{\mu}}_t, \tilde{\boldsymbol{\omega}}(\hat{\boldsymbol{\mu}}_s))^{-1}}{t} - L_{2,\max} \frac{K(1 + \sqrt{t})}{t} \end{aligned}$$

Where we used the fact that the mapping $\boldsymbol{\omega} \mapsto T_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\omega})^{-1}$ is concave (resp. non-negative) in the second (resp. last) inequality. Now observe that for all $s, t \geq \lfloor T^{1/4} \rfloor$, $\|\hat{\boldsymbol{\mu}}_t - \hat{\boldsymbol{\mu}}_s\|_\infty \leq 2\xi$. Therefore the Lipschitz property w.r.t $\boldsymbol{\mu}$ implies that:

$$\begin{aligned} T_\varepsilon\left(\hat{\boldsymbol{\mu}}_t, \frac{N(t)}{t}\right)^{-1} &\geq \frac{\sum_{s=\lfloor T^{1/4} \rfloor}^t T_\varepsilon(\hat{\boldsymbol{\mu}}_s, \tilde{\boldsymbol{\omega}}(\hat{\boldsymbol{\mu}}_s))^{-1}}{t} - \frac{2\xi L_{1,\max}(t - \lfloor T^{1/4} \rfloor)}{t} - L_{2,\max} \frac{K(1 + \sqrt{t})}{t} \\ &\geq \frac{\sum_{s=\lfloor T^{1/4} \rfloor}^t T_\varepsilon^*(\hat{\boldsymbol{\mu}}_s)^{-1}}{t} - \frac{\sum_{s=\lfloor T^{1/4} \rfloor}^t \frac{1}{\sqrt{s}}}{t} - 2\xi L_{1,\max} - L_{2,\max} \frac{K(1 + \sqrt{t})}{t} \quad (7) \end{aligned}$$

where in the second inequality we used the fact that by definition $\tilde{\boldsymbol{\omega}}(\hat{\boldsymbol{\mu}}_s)$ is at most $\frac{1}{\sqrt{s}}$ sub-optimal.

Now observe that:

$$T_\varepsilon^*(\hat{\boldsymbol{\mu}}_s)^{-1} \xrightarrow{s \rightarrow \infty} T_\varepsilon^*(\boldsymbol{\mu})^{-1} \quad \text{almost surely (since } N_a(t) \geq \sqrt{t + K^2} - 2K\text{)}.$$

$$\frac{\sum_{s=\lfloor T^{1/4} \rfloor}^t \frac{1}{\sqrt{s}}}{t} \xrightarrow{t \rightarrow \infty} \frac{\int_1^t \frac{dx}{\sqrt{x}}}{t} \rightarrow 0.$$

$$\frac{K(1 + \sqrt{t})}{t} \rightarrow 0.$$

Therefore for $\eta > 0$, there exists t_η such that for all $t \geq t_\eta$:

$$\frac{\sum_{s=\lfloor T^{1/4} \rfloor}^t T_\varepsilon^*(\hat{\boldsymbol{\mu}}_s)^{-1}}{t} - \frac{\sum_{s=\lfloor T^{1/4} \rfloor}^t \frac{1}{\sqrt{s}}}{t} - L_{2,max} \frac{K(1+\sqrt{t})}{t} \geq T_\varepsilon^*(\boldsymbol{\mu})^{-1} - \eta. \quad (8)$$

Summing up (7) and (8), we get for all $t \geq t_\eta$:

$$T_\varepsilon\left(\hat{\boldsymbol{\mu}}_t, \frac{N(t)}{t}\right)^{-1} \geq T_\varepsilon^*(\boldsymbol{\mu})^{-1} - 2\xi L_{1,max} - \eta.$$

Therefore for every $T \geq \max\left(t_\eta, \frac{\beta(\delta, t)}{T_\varepsilon^*(\boldsymbol{\mu})^{-1} - 2\xi L_{1,max} - \eta}\right)$, we have $\mathcal{E}_T \subset (\tau_\delta \leq T)$ thus $\mathbb{P}(\tau_\delta > T) \leq \mathbb{P}(\mathcal{E}_T^c) \leq BT \exp(-CT^{1/8})$. Hence:

$$\begin{aligned} \mathbb{E}[\tau_\delta] &= \sum_{T=1}^{\infty} \mathbb{P}(\tau_\delta > T) \\ &\leq \max\left(t_\eta, \frac{\beta(\delta, t)}{T_\varepsilon^*(\boldsymbol{\mu})^{-1} - 2\xi L_{1,max} - \eta}\right) + \sum_{T=1}^{\infty} BT \exp(-CT^{1/8}) \end{aligned}$$

Since $\lim_{\delta \rightarrow 0} \frac{\beta(\delta, t)}{\log(1/\delta)} = 1$ the last inequality implies that $\limsup_{\delta \rightarrow 0} \frac{\mathbb{E}[\tau_\delta]}{\log(1/\delta)} \leq \frac{1}{T_\varepsilon^*(\boldsymbol{\mu})^{-1} - 2\xi L_{1,max} - \eta}$. We conclude by letting η and ξ go to zero.

C.1 Proof of Lemma 11

First case: arms in $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ have the same order

Without loss of generality, suppose that $\mu_1 \geq \mu_2 \geq \dots \geq \mu_K$ and $\mu'_1 \geq \mu'_2 \geq \dots \geq \mu'_K$. Then we see that for all $k \neq l \in [K]$, $\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega})$ and $\bar{\boldsymbol{\mu}}_\varepsilon'^{k,\ell}(\boldsymbol{\omega})$ have the same formula and: $|\bar{\boldsymbol{\mu}}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) - \bar{\boldsymbol{\mu}}_\varepsilon'^{k,\ell}(\boldsymbol{\omega})| \leq \|\boldsymbol{\mu}' - \boldsymbol{\mu}\|_\infty$, which implies that $|\boldsymbol{\lambda}_\varepsilon^{k,\ell}(\boldsymbol{\omega}) - \boldsymbol{\lambda}_\varepsilon'^{k,\ell}(\boldsymbol{\omega})| \leq \|\boldsymbol{\mu}' - \boldsymbol{\mu}\|_\infty$. Therefore, letting f denote the function $f(\boldsymbol{\mu}, \boldsymbol{\lambda}) \triangleq \sum_{a \in [K]} \omega_a \frac{(\mu_a - \lambda_a)^2}{2}$, we have:

$$\begin{aligned} |f(\boldsymbol{\mu}', \boldsymbol{\lambda}_\varepsilon'^{k,\ell}(\boldsymbol{\omega})) - f(\boldsymbol{\mu}, \boldsymbol{\lambda}_\varepsilon^{k,\ell}(\boldsymbol{\omega}))| &\leq \sum_{a \in [K]} \frac{\omega_a (\mu'_a - \mu_a + \boldsymbol{\lambda}_\varepsilon^{k,\ell}(\boldsymbol{\omega})_a - \boldsymbol{\lambda}_\varepsilon'^{k,\ell}(\boldsymbol{\omega})_a) (\mu'_a + \mu_a - \boldsymbol{\lambda}_\varepsilon^{k,\ell}(\boldsymbol{\omega})_a - \boldsymbol{\lambda}_\varepsilon'^{k,\ell}(\boldsymbol{\omega})_a)}{2} \\ &\leq \frac{\omega_a \times 2 \|\boldsymbol{\mu}' - \boldsymbol{\mu}\|_\infty \times 2(\mu_{max} - \mu_{min} + \varepsilon)}{2} \\ &= 2(\mu_{max} - \mu_{min} + \varepsilon) \|\boldsymbol{\mu}' - \boldsymbol{\mu}\|_\infty. \end{aligned}$$

where in the second inequality we used the fact that $\boldsymbol{\lambda}_\varepsilon^{k,\ell}(\boldsymbol{\omega})$ (resp. $\boldsymbol{\lambda}_\varepsilon'^{k,\ell}(\boldsymbol{\omega})$) is a weighted average of some arm in $\boldsymbol{\mu}$ (resp. $\boldsymbol{\mu}'$) with one or more arms of $\boldsymbol{\mu}$ (resp. $\boldsymbol{\mu}'$) decreased by ε and therefore lies in $[\mu_{min} - \varepsilon, \mu_{max}]^K$. Let (k_0, l_0) be such that $\boldsymbol{\lambda}_\varepsilon^*, \boldsymbol{\mu}(\boldsymbol{\omega}) = \boldsymbol{\lambda}_\varepsilon^{k_0, l_0}(\boldsymbol{\omega})$ then:

$$T_\varepsilon(\boldsymbol{\mu}', \boldsymbol{\omega})^{-1} - T_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\omega})^{-1} = T_\varepsilon(\boldsymbol{\mu}', \boldsymbol{\omega})^{-1} - f(\boldsymbol{\omega}, \boldsymbol{\lambda}_\varepsilon^{k_0, l_0}(\boldsymbol{\omega})) \quad (9)$$

$$\leq f(\boldsymbol{\omega}, \boldsymbol{\lambda}_\varepsilon'^{k_0, l_0}(\boldsymbol{\omega})) - f(\boldsymbol{\omega}, \boldsymbol{\lambda}_\varepsilon^{k_0, l_0}(\boldsymbol{\omega})) \quad (10)$$

$$\leq 2(\mu_{max} - \mu_{min} + \varepsilon) \|\boldsymbol{\mu}' - \boldsymbol{\mu}\|_\infty. \quad (11)$$

By symmetry we get for all instances $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ with the same arm ordering:

$$|T_\varepsilon(\boldsymbol{\mu}', \boldsymbol{\omega})^{-1} - T_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\omega})^{-1}| \leq 2(\mu_{max} - \mu_{min} + \varepsilon) \|\boldsymbol{\mu}' - \boldsymbol{\mu}\|_\infty.$$

Second case: arms in $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ have a different order

Then for n large enough we can find a sequence $(\mu^i)_{0 \leq i \leq 2^n}$ of instances in the segment $[\boldsymbol{\mu}, \boldsymbol{\mu}']$ such that $\mu^0 = \boldsymbol{\mu}$, $\mu^{2^n} = \boldsymbol{\mu}'$ and:

$$\forall i \in [0, 2^n - 1], \mu^i \text{ and } \mu^{i+1} \text{ have the same arm ordering and } \|\mu^{i+1} - \mu^i\|_\infty \leq \frac{\|\boldsymbol{\mu}' - \boldsymbol{\mu}\|_\infty}{2^{n-1}}.$$

We can construct such a sequence in the following way: Split $[\mu_{min}, \mu_{max}]^K$ into $K!$ regions such that any two instances in the same region share the same arm ordering. The boundaries between these regions correspond to instances where two or more arms are equal. Starting from $\mu^0 \triangleq \boldsymbol{\mu}$, span the segment $[\boldsymbol{\mu}, \boldsymbol{\mu}']$ and define μ^{i+1} to be the first instance where: either the L^∞ distance from μ^i is equal to $\frac{\|\boldsymbol{\mu}' - \boldsymbol{\mu}\|_\infty}{2^{n-1}}$, or we cross a boundary between two regions. Since there can be at most $K! - 1$ changes in the arm ordering, for n large enough such sequence always exists. Now we have:

$$\begin{aligned} |T_\varepsilon(\boldsymbol{\mu}', \boldsymbol{\omega})^{-1} - T_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\omega})^{-1}| &\leq \sum_{i=0}^{2^n-1} |T_\varepsilon(\boldsymbol{\omega}, \boldsymbol{\mu}^{i+1})^{-1} - T_\varepsilon(\boldsymbol{\omega}, \boldsymbol{\mu}^i)^{-1}| \\ &\leq \sum_{i=0}^{2^n-1} 2(\mu_{max} - \mu_{min} + \varepsilon) \frac{\|\boldsymbol{\mu}' - \boldsymbol{\mu}\|_\infty}{2^{n-1}} \\ &\leq 4(\mu_{max} - \mu_{min} + \varepsilon) \|\boldsymbol{\mu}' - \boldsymbol{\mu}\|_\infty. \end{aligned}$$

where in the second inequality we use the first case and the fact that μ^i and μ^{i+1} have the same arm ordering. As a summary, we always have:

$$|T_\varepsilon(\boldsymbol{\mu}', \boldsymbol{\omega})^{-1} - T_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\omega})^{-1}| \leq 4(\mu_{max} - \mu_{min} + \varepsilon) \|\boldsymbol{\mu}' - \boldsymbol{\mu}\|_\infty.$$