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# MAXIMAL MEASURE AND ENTROPIC CONTINUITY OF LYAPUNOV EXONENTS FOR $C^r$ SURFACE DIFFEOMORPHISMS WITH LARGE ENTROPY

DAVID BURGNET

ABSTRACT. We prove a finite smooth version of the entropic continuity of Lyapunov exponents proved recently by Buzzi, Crovisier and Sarig for  $C^\infty$  surface diffeomorphisms [10]. As a consequence we show that any  $C^r$ ,  $r > 1$ , smooth surface diffeomorphism  $f$  with  $h_{top}(f) > \frac{1}{r} \limsup_n \frac{1}{n} \log^+ \|d f^n\|_\infty$  admits a measure of maximal entropy. We also prove the  $C^r$  continuity of the topological entropy at  $f$ .

## INTRODUCTION

The entropy of a dynamical system quantifies the dynamical complexity by counting distinct orbits. There are topological and measure theoretical versions which are related by a variational principle : the topological entropy of a continuous map on a compact space is equal to the supremum of the entropy of the invariant (probability) measures. An invariant measure is said to be of maximal entropy (or a maximal measure) when its entropy is equal to the topological entropy, i.e. this measure realizes the supremum in the variational principle. In general a topological system may not admit a measure of maximal entropy. But such a measure exists for dynamical systems satisfying some expansiveness properties. In particular Newhouse [15] has proved their existence for  $C^\infty$  systems by using Yomdin's theory. In the present paper we show the existence of a measure of maximal entropy for  $C^r$ ,  $1 < r < +\infty$ , smooth surface diffeomorphisms with large entropy.

Other important dynamical quantities for smooth systems are given by the Lyapunov exponents which estimate the exponential growth of the derivative. For  $C^\infty$  surface diffeomorphisms, J. Buzzi, S. Crovisier and O. Sarig proved recently a property of continuity in the entropy of the Lyapunov exponents with many statistical applications [10]. More precisely, they showed that for a  $C^\infty$  surface diffeomorphism  $f$ , if  $\nu_k$  is a converging sequence of ergodic measures with  $\lim_k h(\nu_k) = h_{top}(f)$ , then the Lyapunov exponents of  $\nu_k$  are going to the (average) Lyapunov exponents of the limit (which is a measure of maximal entropy). We prove a  $C^r$  version of this fact for  $1 < r < +\infty$ .

## 1. STATEMENTS

We define now some notations to state our main results. For a  $C^r$ ,  $r \geq 1$ , diffeomorphism  $f$  on a compact Riemannian surface  $(\mathbf{M}, \|\cdot\|)$  we let  $F : \mathbb{P}T\mathbf{M} \circlearrowleft$  be the induced map on the projective tangent bundle  $\mathbb{P}T\mathbf{M} = T^1\mathbf{M}/\pm 1$  and we denote by  $\phi, \psi : \mathbb{P}T\mathbf{M} \rightarrow \mathbb{R}$  the continuous observables on  $\mathbb{P}T\mathbf{M}$  given respectively by  $\phi : (x, v) \mapsto \log \|d_x f(v)\|$  and  $\psi : (x, v) \mapsto \log \|d_x f(v)\| - \frac{1}{r} \log^+ \|d_x f\|$  with  $\|d_x f\| = \sup_{v \in T_x \mathbf{M} \setminus \{0\}} \frac{\|d_x f(v)\|}{\|v\|}$ . For  $k \in$

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$\mathbb{N}^*$  we define more generally  $\phi_k : (x, v) \mapsto \log \|d_x f^k(v)\|$  and  $\psi_k : (x, v) \mapsto \phi_k(x, v) - \frac{1}{r} \sum_{l=0}^{k-1} \log^+ \|d_{f^l x} f\|$ . Then we let  $\lambda^+(x)$  and  $\lambda^-(x)$  be the pointwise Lyapunov exponents given by  $\lambda^+(x) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \|d_x f^n\|$  and  $\lambda^-(x) = \liminf_{n \rightarrow -\infty} \frac{1}{n} \log \|d_x f^n\|$  for any  $x \in \mathbf{M}$  and  $\lambda^+(\mu) = \int \lambda^+(x) d\mu(x)$ ,  $\lambda^-(\mu) = \int \lambda^-(x) d\mu(x)$ , for any  $f$ -invariant measure  $\mu$ .

Also we put  $\lambda^+(f) := \lim_n \frac{1}{n} \log^+ \|df^n\|_\infty$  with  $\|df^n\|_\infty = \sup_{x \in \mathbf{M}} \|d_x f^n\|$ . The function  $f \mapsto \lambda^+(f)$  is upper semi-continuous in the  $\mathcal{C}^1$  topology on the set of  $\mathcal{C}^1$  diffeomorphisms on  $\mathbf{M}$ . For an  $f$ -invariant measure  $\mu$  with  $\lambda^+(x) > 0 \geq \lambda^-(x)$  for  $\mu$  a.e.  $x$ , there are by Oseledets\* theorem one-dimensional invariant vector spaces  $\mathcal{E}_+(x)$  and  $\mathcal{E}_-(x)$ , resp. called the unstable and stable Oseledets bundle, such that

$$\forall \mu \text{ a.e. } x \forall v \in \mathcal{E}_\pm(x) \setminus \{0\}, \quad \lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|d_x f^n(v)\| = \lambda^\pm(x).$$

Then we let  $\hat{\mu}^+$  be the  $F$ -invariant measure given by the lift of  $\mu$  on  $\mathbb{P}\mathbf{T}\mathbf{M}$  with  $\hat{\mu}^+(\mathcal{E}_+) = 1$ . When writing  $\hat{\mu}^+$  we assume implicitly that the push-forward measure  $\mu$  on  $\mathbf{M}$  satisfies  $\lambda^+(x) > 0 \geq \lambda^-(x)$  for  $\mu$  a.e.  $x$ .

A sequence of  $\mathcal{C}^r$ , with  $r > 1$ , surface diffeomorphisms  $(f_k)_k$  on  $\mathbf{M}$  is said to converge  $\mathcal{C}^r$  weakly to a diffeomorphism  $f$ , when  $f_k$  goes to  $f$  in the  $\mathcal{C}^1$  topology and the sequence  $(f_k)_k$  is  $\mathcal{C}^r$  bounded. In particular  $f$  is  $\mathcal{C}^{r-1}$ .

**Theorem** (Buzzi-Crovisier-Sarig, Theorem C [10]). *Let  $(f_k)_{k \in \mathbb{N}}$  be a sequence of  $\mathcal{C}^r$ , with  $r > 1$ , surface diffeomorphisms converging  $\mathcal{C}^r$  weakly to a diffeomorphism  $f$ . Let  $(F_k)_{k \in \mathbb{N}}$  and  $F$  be the lifts of  $(f_k)_{k \in \mathbb{N}}$  and  $f$  to  $\mathbb{P}\mathbf{T}\mathbf{M}$ . Assume there is a sequence  $(\hat{\nu}_k^+)_{k \in \mathbb{N}}$  of ergodic  $F_k$ -invariant measures converging to  $\hat{\mu}$ .*

*Then there are  $\beta \in [0, 1]$  and  $F$ -invariant measures  $\hat{\mu}_0$  and  $\hat{\mu}_1^+$  with  $\hat{\mu} = (1 - \beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$ , such that:*

$$\limsup_{k \rightarrow +\infty} h(\nu_k) \leq \beta h(\mu_1) + \frac{\lambda^+(f) + \lambda^+(f^{-1})}{r - 1}.$$

In particular when  $f$  ( $= f_k$  for all  $k$ ) is  $\mathcal{C}^\infty$  and  $h(\nu_k)$  goes to the topological entropy of  $f$ , then  $\beta$  is equal to 1 and therefore  $\lambda^+(\nu_k)$  goes to  $\lambda^+(\mu)$ :

**Corollary** (Entropic continuity of Lyapunov exponents [10]). *Let  $f$  be a  $\mathcal{C}^\infty$  surface diffeomorphism with  $h_{top}(f) > 0$ .*

*Then if  $(\nu_k)_k$  is a sequence of ergodic measures converging to  $\mu$  with  $\lim_k h(\nu_k) = h_{top}(f)$ , then*

- $h(\mu) = h_{top}(f)^\dagger$ ,
- $\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)$ .

We state an improved version of Buzzi-Crovisier-Sarig Theorem, which allows to prove the same entropy continuity of Lyapunov exponents for  $\mathcal{C}^r$ ,  $1 < r < +\infty$ , surface diffeomorphisms with large enough entropy (see Corollary 1).

**Main Theorem.** *Let  $(f_k)_{k \in \mathbb{N}}$  be a sequence of  $\mathcal{C}^r$ , with  $r > 1$ , surface diffeomorphisms converging  $\mathcal{C}^r$  weakly to a diffeomorphism  $f$ . Let  $(F_k)_{k \in \mathbb{N}}$  and  $F$  be the lifts of  $(f_k)_{k \in \mathbb{N}}$  and  $f$*

\*We refer to [16] for background on Lyapunov exponents and Pesin theory.

†This follows from the upper semi-continuity of the entropy function  $h$  on the set of  $f$ -invariant probability measures for a  $\mathcal{C}^\infty$  diffeomorphism  $f$  (in any dimension), which was first proved by Newhouse in [15].

to  $\mathbb{P}TM$ . Assume there is a sequence  $(\hat{\nu}_k^+)_k$  of ergodic  $F_k$ -invariant measures converging to  $\hat{\mu}$ .

Then for any  $\alpha > \frac{\lambda^+(f)}{r}$ , there are  $\beta = \beta_\alpha \in [0, 1]$  and  $F$ -invariant measures  $\hat{\mu}_0 = \hat{\mu}_{0,\alpha}$  and  $\hat{\mu}_1^+ = \hat{\mu}_{1,\alpha}^+$  with  $\hat{\mu} = (1 - \beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$ , such that:

$$\limsup_{k \rightarrow +\infty} h(\nu_k) \leq \beta h(\mu_1) + (1 - \beta)\alpha.$$

The Main Theorem implies Buzzi-Crovisier-Sarig statement. Indeed, either  $\lim_k \lambda^+(\nu_k) = \int \phi d\hat{\mu} \leq \frac{\lambda^+(f)}{r}$  and we get by Ruelle inequality,  $\limsup_k h(\nu_k) \leq \frac{\lambda^+(f)}{r}$  or there exists  $\alpha \in \left] \frac{\lambda^+(f)}{r}, \min \left( \int \phi d\hat{\mu}, \frac{\lambda^+(f)}{r-1} \right) \right[$ . By applying our Main Theorem with respect to  $\alpha$ , there is a decomposition  $\hat{\mu} = (1 - \beta_\alpha)\hat{\mu}_{0,\alpha} + \beta_\alpha\hat{\mu}_{1,\alpha}^+$  satisfying  $\limsup_{k \rightarrow +\infty} h(\nu_k) \leq \beta_\alpha h(\mu_{1,\alpha}) + (1 - \beta_\alpha)\alpha$ . But it follows from the proofs that  $\beta_\alpha\mu_{1,\alpha}$  is a component of  $\beta\mu_1$  with  $\beta$  and  $\mu_1$  being as in Buzzi-Crovisier-Sarig's statement (see Remark 6). In particular  $\beta_\alpha h(\mu_{1,\alpha}) \leq \beta h(\mu_1)$ , therefore  $\limsup_{k \rightarrow +\infty} h(\nu_k) \leq \beta h(\mu_1) + \frac{\lambda^+(f) + \lambda^+(f-1)}{r-1}$ . In Theorem C [10], the authors also proved  $\int \phi d\hat{\mu}_0 = 0$  whenever  $\beta \neq 1$ . Therefore we get here  $(1 - \beta_\alpha) \int \phi d\hat{\mu}_{0,\alpha} \geq (1 - \beta) \int \phi d\hat{\mu}_0 = 0$ , then  $\int \phi d\hat{\mu}_{0,\alpha} \geq 0$ . But maybe we could have  $\int \phi d\hat{\mu}_{0,\alpha} > 0$ .

**Corollary 1** (Existence of maximal measures and entropic continuity of Lyapunov exponents). *Let  $f$  be a  $C^r$ , with  $r > 1$ , surface diffeomorphism satisfying  $h_{top}(f) > \frac{\lambda^+(f)}{r}$ .*

*Then  $f$  admits a measure of maximal entropy. More precisely, if  $(\nu_k)_k$  is a sequence of ergodic measures converging to  $\mu$  with  $\lim_k h(\nu_k) = h_{top}(f)$ , then*

- $h(\mu) = h_{top}(f)$ ,
- $\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)$ .

It was proved in [9] that any  $C^r$  surface diffeomorphism satisfying  $h_{top}(f) > \frac{\lambda^+(f)}{r}$  admits at most finitely many ergodic measures of maximal entropy. On the other hand, J. Buzzi has built examples of  $C^r$  surface diffeomorphisms for any  $+\infty > r > 1$  with  $\frac{h_{top}(f)}{\lambda^+(f)}$  arbitrarily close to  $1/r$  without a measure of maximal entropy [7]. Such results were already known for interval maps [3, 6, 8].

*Proof.* We consider the constant sequence of diffeomorphisms equal to  $f$ . By taking a subsequence, we can assume that  $(\hat{\nu}_k^+)_k$  is converging to a lift  $\hat{\mu}$  of  $\mu$ . By using the notations of the Main Theorem with  $h_{top}(f) > \alpha > \frac{\lambda^+(f)}{r}$ , we have

$$\begin{aligned} h_{top}(f) &= \lim_{k \rightarrow +\infty} h(\nu_k), \\ &\leq \beta h(\mu_1) + (1 - \beta)\alpha, \\ &\leq \beta h_{top}(f) + (1 - \beta)\alpha, \\ (1 - \beta)h_{top}(f) &\leq (1 - \beta)\alpha. \end{aligned}$$

But  $h_{top}(f) > \alpha$ , therefore  $\beta = 1$ , i.e.  $\hat{\mu}_1^+ = \hat{\mu}$  and  $\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)$ . Moreover  $h_{top}(f) = \lim_{k \rightarrow +\infty} h(\nu_k) \leq \beta h(\mu_1) + (1 - \beta)\alpha = h(\mu)$ . Consequently  $\mu$  is a measure of maximal entropy of  $f$ . □

**Corollary 2** (Continuity of topological entropy and maximal measures). *Let  $(f_k)_k$  be a sequence of  $C^r$ , with  $r > 1$ , surface diffeomorphisms converging  $C^r$  weakly to a diffeomorphism*

$f$  with  $h_{top}(f) \geq \frac{\lambda^+(f)}{r}$ .

Then

$$h_{top}(f) = \lim_k h_{top}(f_k).$$

Moreover if  $h_{top}(f) > \frac{\lambda^+(f)}{r}$  and  $\nu_k$  is a maximal measure of  $f_k$  for large  $k$ , then any limit measure of  $(\nu_k)_k$  for the weak-\* topology is a maximal measure of  $f$ .

*Proof.* By Katok's horseshoes theorem [14], the topological entropy is lower semi-continuous for the  $\mathcal{C}^1$  topology on the set of  $\mathcal{C}^r$  surface diffeomorphisms. Therefore it is enough to show the upper semi-continuity.

By the variational principle there is a sequence of probability measures  $(\nu_k)_{k \in K}$ ,  $K \subset \mathbb{N}$  with  $\#K = \infty$ , such that :

- $\nu_k$  is an ergodic  $f_k$ -invariant measure for each  $k$ ,
- $\lim_{k \in K} h(\nu_k) = \limsup_{k \in \mathbb{N}} h_{top}(f_k)$ .

By extracting a subsequence we can assume  $(\nu_k^+)_k$  is converging to a  $F$ -invariant measure  $\hat{\mu}$  in the weak-\* topology. We can then apply the Main Theorem for any  $\alpha > \frac{\lambda^+(f)}{r}$  to get for some  $f$ -invariant measures  $\mu_1, \mu_0$  and  $\beta \in [0, 1]$  (depending on  $\alpha$ ) with  $\mu = (1 - \beta)\mu_0 + \beta\mu_1$ :

$$(1.1) \quad \begin{aligned} \limsup_k h_{top}(f_k) &= \lim_k h(\nu_k), \\ &\leq \beta h(\mu_1) + (1 - \beta)\alpha, \\ &\leq \beta h_{top}(f) + (1 - \beta)\alpha, \\ &\leq \max(h_{top}(f), \alpha). \end{aligned}$$

By letting  $\alpha$  go to  $\frac{\lambda^+(f)}{r}$  we get

$$\limsup_k h_{top}(f_k) \leq h_{top}(f).$$

If  $h_{top}(f) > \frac{\lambda^+(f)}{r}$ , we can fix  $\alpha \in \left] \frac{\lambda^+(f)}{r}, h_{top}(f) \right[$  and the inequalities (1.1) may be then rewritten as follows :

$$\begin{aligned} \limsup_k h_{top}(f_k) &\leq \beta h(\mu_1) + (1 - \beta)\alpha, \\ &\leq h_{top}(f). \end{aligned}$$

By the lower semi-continuity of the topological entropy, we have  $h_{top}(f) \leq \limsup_k h_{top}(f_k)$  and therefore these inequalities are equalities, which implies  $\beta = 1$ , then  $\mu_1 = \mu$ , and  $h(\mu) = h_{top}(f)$ .  $\square$

The corresponding result was proved for interval maps in [5] by using a different method. We also refer to [5] for counterexamples of the upper semi-continuity property for interval maps  $f$  with  $h_{top}(f) < \frac{\lambda^+(f)}{r}$ . Finally, in [7], the author built, for any  $r > 1$ , a  $\mathcal{C}^r$  surface diffeomorphism  $f$  with  $\limsup_{g \xrightarrow{\mathcal{C}^r} f} h_{top}(g) = \frac{\lambda^+(f)}{r} > h_{top}(f) = 0$ . We recall also that upper semi-continuity of the topological entropy in the  $\mathcal{C}^\infty$  topology was established in any dimension by Y. Yomdin in [18].

Newhouse proved that for a  $\mathcal{C}^\infty$  system  $(\mathbf{M}, f)$ , the entropy function  $h : \mathcal{M}(\mathbf{M}, f) \rightarrow \mathbb{R}^+$  is an upper semi-continuous function on the set  $\mathcal{M}(\mathbf{M}, f)$  of  $f$ -invariant probability measure.

It follows from our Main Theorem, that the entropy function is upper semi-continuous at ergodic measures with entropy larger than  $\frac{\lambda^+(f)}{r}$  for a  $C^r$ ,  $r > 1$ , surface diffeomorphism  $f$ .

**Corollary 3** (Upper semi-continuity of the entropy function at ergodic measures with large entropy). *Let  $f : \mathbf{M} \circlearrowleft$  be a  $C^r$ ,  $r > 1$ , surface diffeomorphism.*

*Then for any ergodic measure  $\mu$  with  $h(\mu) \geq \frac{\lambda^+(f)}{r}$ , we have*

$$\limsup_{\nu \rightarrow \mu} h(\nu) \leq h(\mu).$$

*Proof.* By continuity of the ergodic decomposition at ergodic measures and by harmonicity of the entropy function, we have for any ergodic measure  $\mu$  (see e.g. Lemma 8.2.13 in [12]):

$$\limsup_{\nu \text{ ergodic}, \nu \rightarrow \mu} h(\nu) = \limsup_{\nu \rightarrow \mu} h(\mu).$$

Let  $(\nu_k)_{k \in \mathbb{N}}$  be a sequence of ergodic  $f$ -invariant measures with  $\lim_k h(\nu_k) = \limsup_{\nu \rightarrow \mu} h(\nu)$ . By extracting a subsequence we can assume that the sequence  $(\hat{\nu}_k^+)_{k \in \mathbb{N}}$  is converging to some lift  $\hat{\mu}$  of  $\mu$ . Take  $\alpha$  with  $\alpha > \frac{\lambda^+(f)}{r}$ . Then, in the decomposition  $\hat{\mu} = (1 - \beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$  given by the Main Theorem, we have  $\mu_1 = \mu_0$  by ergodicity of  $\mu$ . Therefore

$$\lim_k h(\nu_k) \leq \beta h(\mu) + (1 - \beta)\alpha.$$

By letting  $\alpha$  go to  $\frac{\lambda^+(f)}{r}$  we get

$$\begin{aligned} \lim_k h(\nu_k) &\leq \beta h(\mu) + (1 - \beta) \frac{\lambda^+(f)}{r}, \\ &\leq h(\mu). \end{aligned}$$

□

## 2. MAIN STEPS OF THE PROOF

We follow the strategy of the proof of [10]. We point out below the main differences:

- *Geometric and neutral empirical component.* For  $\lambda^+(\nu_k) > \frac{\lambda^+(f)}{r}$  we split the orbit of a  $\nu_k$ -typical point  $x$  into two parts. We consider the empirical measures from  $x$  at times lying between to  $M$ -close consecutive times where the unstable manifold has a "bounded geometry". We take their limit in  $k$ , then in  $M$ . In this way we get an invariant component of  $\hat{\mu}$ . In [10] the authors consider rather such empirical measures for  $\alpha$ -hyperbolic times and then take the limit when  $\alpha$  go to zero.
- *Entropy computations.* To compute the asymptotic entropy of the  $\nu_k$ 's, we use the static entropy w.r.t. partitions and its conditional version. Instead the authors in [10] used Katok's like formulas.
- *$C^r$  Reparametrizations.* Finally we use here reparametrization methods from [4] and [2] respectively rather than Yomdin's reparametrizations of the projective action  $F$  as done in [10]. This is the principal difference with [10].

**2.1. Empirical measures.** Let  $(X, T)$  be a topological system. For a fixed Borel measurable subset  $G$  of  $X$  we let  $E(x) = E_G(x)$  be the set of times of visits in  $G$  from  $x$ :

$$E(x) = \{n \in \mathbb{Z}, T^n x \in G\}.$$

When  $a < b$  are two consecutive times in  $E(x)$ , then  $[a, b[$  is called a *neutral block* (by following the terminology of [9]). For all  $M$  we let then

$$E^M(x) = \bigcup_{a < b \in E(x), |a-b| \leq M} [a, b[.$$

The complement of  $E^M(x)$  is made of disjoint neutral blocks of length larger than  $M$ . We consider the associated empirical measures :

$$\forall n, \mu_{x,n}^M = \frac{1}{n} \sum_{k \in E^M(x) \cap [0, n[} \delta_{T^k x}.$$

Let  $\nu$  be an ergodic measure. We denote by  $\chi^M$  the indicator function of  $\{x, 0 \in E^M(x)\}$ . By the Birkhoff ergodic theorem, there is a set  $\mathbf{G}$  of full  $\nu$ -measure such that the empirical measures  $(\mu_{x,n}^M)_n$  are converging for any  $x \in \mathbf{G}$  and any  $M \in \mathbb{N}^*$  to  $\xi^M := \chi^M \nu$  in the weak-\* topology. We also let  $\eta^M = \nu - \xi^M$ . Moreover we put  $\beta_M = \int \chi^M d\nu$ , then  $\xi^M = \beta_M \cdot \underline{\xi}^M$  when  $\beta_M \neq 0$  and  $\eta^M = (1 - \beta_M) \cdot \underline{\eta}^M$  when  $\beta_M \neq 1$  with  $\underline{\xi}^M, \underline{\eta}^M$  being thus probability measures. Following partially [10], the measures  $\xi^M$  and  $\eta^M$  are respectively called here the *geometric and neutral components* of  $\nu$ . In general these measures are not  $T$ -invariant. From the definition one easily checks that  $\xi^M \geq \xi^N$  for  $M \geq N$ .

**2.2. Pesin unstable manifolds.** We consider a smooth compact riemannian manifold  $(\mathbf{M}, \|\cdot\|)$ . Let  $\exp_x$  be the exponential map at  $x$  and let  $R_{inj}$  be the radius of injectivity of  $(\mathbf{M}, \|\cdot\|)$ . We consider the distance  $d$  on  $\mathbf{M}$  induced by the Riemannian structure. Let  $f : \mathbf{M} \circlearrowleft$  be a  $\mathcal{C}^r$ ,  $r > 1$ , surface diffeomorphism. We denote by  $\mathcal{R}$  the set of Lyapunov regular points with  $\lambda^+(x) > 0 > \lambda^-(x)$ . For  $x \in \mathbf{M}$  we let  $W^u(x)$  denote the unstable manifold at  $x$  :

$$W^u(x) := \left\{ y \in \mathbf{M}, \lim_n \frac{1}{n} \log d(f^n x, f^n y) < 0 \right\}.$$

By Pesin unstable manifold theorem, the set  $W^u(x)$  for  $x \in \mathcal{R}$  is a  $\mathcal{C}^r$  submanifold tangent to  $\mathcal{E}_+(x)$  at  $x$ .

For  $x \in \mathcal{R}$ , we let  $\hat{x}$  be the vector in  $\mathbb{P}T\mathbf{M}$  associated to the unstable Oseledets bundle  $\mathcal{E}_+(x)$ . For  $\delta > 0$  the point  $x$  is said  *$\delta$ -hyperbolic* with respect to  $\phi$  (resp.  $\psi$ ) when we have  $\phi_l(F^{-l}\hat{x}) \geq \delta l$  (resp.  $\psi_l(F^{-l}\hat{x}) \geq \delta l$ ) for all  $l > 0$ . Note that if  $x$  is  $\delta$ -hyperbolic with respect to  $\psi$  then it is  $\delta$ -hyperbolic with respect to  $\phi$ .

Let  $\nu$  be an ergodic measure with  $\lambda^+(\nu) - \frac{\log^+ \|df\|_\infty}{r} > \delta > 0 > \lambda^-(\nu)$ . By applying the Ergodic Maximal Inequality (see e.g. Theorem 1.1 in [1]) to the measure preserving system  $(F^{-1}, \hat{\nu}^+)$  with the observable  $\psi^\delta = \delta - \psi \circ F^{-1}$ , we get with  $A_\delta = \{\hat{x} \in \mathbb{P}T\mathbf{M}, \exists k \geq 0 \text{ s.t. } \sum_{l=0}^k \psi^\delta(F^{-l}\hat{x}) > 0\}$ :

$$\int_{A_\delta} \psi^\delta d\hat{\nu}^+ \geq 0.$$

But the set  $H_\delta := \{\hat{x} \in \mathbb{P}T\mathbf{M}, \forall l > 0 \psi_l(F^{-l}\hat{x}) \geq \delta l\}$  of  $\delta$ -hyperbolic points w.r.t.  $\psi$  is just the complement set  $\mathbb{P}T\mathbf{M} \setminus A_\delta$  of  $A_\delta$ . Therefore  $\int_{H_\delta} (\delta - \psi \circ F^{-1}) d\hat{\nu}^+ \leq \int (\delta - \psi \circ F^{-1}) d\hat{\nu}^+ = \delta - \lambda^+(\nu) + \frac{1}{r} \int \frac{\log^+ \|df\|}{r} d\nu < 0$ . In particular we have  $\hat{\nu}^+(H_\delta) > 0$ .

A point  $x \in \mathcal{R}$  is said to have  $\kappa$ -bounded geometry for  $\kappa > 0$  when  $\exp_x^{-1} W^u(x)$  contains the graph of an  $\kappa$ -admissible map at  $x$ , which is defined as a 1-Lipschitz map  $f : I \rightarrow \mathcal{E}_+(x)^\perp \subset T_x \mathbf{M}$ , with  $I$  being an interval of  $\mathcal{E}_+(x)$  containing 0 with length  $\kappa$ . We let  $G_\kappa$  be the subset of points in  $\mathcal{R}$  with  $\kappa$ -bounded geometry.

**Lemma 1.** *The set  $G_\kappa$  is Borel measurable.*

*Proof.* For  $x \in \mathcal{R}$  we have  $W^u(x) = \bigcup_{n \in \mathbb{N}} f^n W_{loc}^u(f^{-n}x)$  with  $W_{loc}^u$  being the Pesin unstable local manifold at  $x$ . The sequence  $(f^{-n} W_{loc}^u(f^n x))_n$  is increasing in  $n$  for the inclusion. Therefore, if we let  $G_\kappa^n$  be the subset of points  $x$  in  $G_\kappa$ , such that  $\exp_x^{-1} f^n W_{loc}^u(f^{-n}x)$  contains the graph of a  $\kappa$ -admissible map, then we have

$$G_\kappa = \bigcup_n G_\kappa^n.$$

There are closed subsets,  $(\mathcal{R}_l)_{l \in \mathbb{N}}$ , called the Pesin blocks, such that  $\mathcal{R} = \bigcup_l \mathcal{R}_l$  and  $x \mapsto W_{loc}^u(x)$  is continuous on  $\mathcal{R}_l$  for each  $l$  (see e.g. [16]). Let  $(x_p)_p$  be sequence in  $G_\kappa^n \cap \mathcal{R}_l$  which converges to  $x \in \mathcal{R}_l$ . By extracting a subsequence we can assume that the associated sequence of  $\kappa$ -admissible maps  $f_p$  at  $x_p$  is converging pointwisely to a  $\kappa$ -admissible map at  $x$ , when  $p$  goes to infinity. In particular  $G_\kappa^n \cap \mathcal{R}_l$  is a closed set and therefore  $G_\kappa = \bigcup_{l,n} (G_\kappa^n \cap \mathcal{R}_l)$  is Borel measurable.  $\square$

**2.3. Entropy of conditional measures.** We consider an ergodic hyperbolic measure  $\nu$ , i.e an ergodic measure with  $\nu(\mathcal{R}) = 1$ . A measurable partition  $\zeta$  is *subordinated* to the Pesin unstable local lamination  $W_{loc}^u$  of  $\nu$  if the atom  $\zeta(x)$  of  $\zeta$  containing  $x$  is a neighborhood of  $x$  inside the curve  $W_{loc}^u(x)$  and  $f^{-1}\zeta \succ \zeta$ . By Rokhlin's disintegration theorem, there are a measurable set  $Z$  of full  $\nu$ -measure and probability measures  $\nu_x$  on  $\zeta(x)$  for  $x \in Z$ , called the *conditional measures* on unstable manifolds, satisfying  $\nu = \int \nu_x d\nu(x)$ . Moreover  $\nu_y = \nu_x$  for  $x, y \in Z$  in the same atom of  $\zeta$ . Ledrappier and Young [13] proved the existence of such subordinated measurable partitions and showed that for  $\nu$ -a.e.  $x$ , we have with  $B_n(x, \rho)$  being the Bowen ball  $B_n(x, \rho) := \bigcap_{0 \leq k < n} f^{-k} B(f^k x, \rho)$  (where  $B(f^k x, \rho)$  denotes the ball for  $d$  at  $f^k x$  with radius  $\rho$ ):

$$(2.1) \quad \lim_{\rho \rightarrow 0} \liminf_n -\frac{1}{n} \log \nu_x(B_n(x, \rho)) = h(\nu).$$

Fix an error term  $\iota > 0$  depending<sup>‡</sup> on  $\nu$ . There is  $\rho > 0$  and a measurable set  $\mathbf{F} \subset Z \cap \mathcal{R}$  with  $\nu(\mathbf{F}) > 0$  such that

$$\forall x \in \mathbf{F}, \liminf_n -\frac{1}{n} \log \nu_x(B_n(x, \rho)) \geq h(\nu) - \iota.$$

We fix  $x_* \in \mathbf{F}$  with  $\nu_{x_*}(\mathbf{F}) > 0$  and we let  $\zeta = \frac{\nu_{x_*}(\cdot)}{\nu_{x_*}(\mathbf{F})}$  be the probability measure induced by  $\nu_{x_*}$  on  $\mathbf{F}$ . Observe that  $\nu_x = \nu_{x_*}$  for  $\zeta$  a.e.  $x$ . We let  $D$  be the  $C^r$  curve given by the Pesin local unstable manifold  $W_{loc}^u(x_*)$  at  $x_*$ . For a finite measurable partition  $P$  and a Borel probability measure  $\mu$  we let  $H_\mu(P)$  be the static entropy,  $H_\mu(P) = -\sum_{A \in P} \mu(A) \log \mu(A)$ . Moreover we let  $P^n = \bigvee_{k=0}^{n-1} f^{-k} P$  be the  $n$ -iterated partition,  $n \in \mathbb{N}$ . We also denote by  $P_x^n$  the atom of  $P^n$  containing the point  $x \in \mathbf{M}$ .

<sup>‡</sup>In the proof of the Main Theorem we will take  $\iota = \iota(\nu_k) \xrightarrow{k} 0$  for the converging sequence of ergodic measures  $(\nu_k)_k$ .

**Lemma 2.** *For any (finite measurable) partition  $P$  with diameter less than  $\rho$ , we have*

$$\liminf_n \frac{1}{n} H_\zeta(P^n) \geq h(\nu) - \iota.$$

*Proof.*

$$\begin{aligned} \liminf_n \frac{1}{n} H_\zeta(P^n) &= \liminf_n \int -\frac{1}{n} \log \zeta(P_x^n) d\zeta(x), \text{ by the definition of } H_\zeta, \\ &\geq \int \liminf_n -\frac{1}{n} \log \zeta(P_x^n) d\zeta(x), \text{ by Fatou's Lemma,} \\ &\geq \int \liminf_n -\frac{1}{n} \log \nu_{x^*}(P_x^n) d\zeta(x), \text{ by the definition of } \zeta, \\ &\geq \int \liminf_n -\frac{1}{n} \log \nu_x(P_x^n) d\zeta(x), \text{ as } \nu_x = \nu_{x^*} \text{ for } \zeta \text{ a.e. } x, \\ &\geq \int \liminf_n -\frac{1}{n} \log \nu_x(B_n(x, \rho)) d\zeta(x), \text{ as } \text{diam}(P) < \rho, \\ &\geq h(\nu) - \iota, \text{ by the choice of } \mathbf{F}. \end{aligned}$$

□

**2.4. Entropy splitting of the neutral and the geometric component.** The natural projection from  $\mathbb{P}\mathbf{T}\mathbf{M}$  to  $\mathbf{M}$  is denoted by  $\pi$ . We consider a distance  $\hat{d}$  on the projective tangent bundle  $\mathbb{P}\mathbf{T}\mathbf{M}$ , such that  $\hat{d}(\hat{x}, \hat{y}) \geq d(\pi\hat{x}, \pi\hat{y})$  for all  $\hat{x}, \hat{y} \in \mathbb{P}\mathbf{T}\mathbf{M}$ . In this section we split the entropy contribution of the neutral and geometric components  $\hat{\eta}^M$  and  $\hat{\xi}^M$  of the ergodic  $F$ -invariant measure  $\hat{\nu}^+$  associated to  $G = H_\delta \cap \pi^{-1}G_\kappa \subset \mathbb{P}\mathbf{T}\mathbf{M}$ , where the parameters  $\delta$  and  $\kappa$  will be fixed later on. We also consider their projections  $\eta^M$  and  $\xi^M$  on  $\mathbf{M}$ . Let  $\mathbf{F}$  and  $P$  as in the previous subsection. Without loss of generality we can assume

- $\{\hat{x}, x \in \mathbf{F}\} \subset \mathbf{G}$  with  $\mathbf{G}$  being the set of full  $\hat{\nu}^+$ -measure of points  $\hat{x}$  such that the empirical measures  $\mu_{\hat{x}, n}^M$  are converging to  $\hat{\xi}^M$  for any  $M$  (see Subsection 2.1),
- the boundary of  $P$  has zero  $\nu$ -measure,
- for any  $M \in \mathbb{N}$  and for any continuous function  $\varphi : \mathbb{P}\mathbf{T}\mathbf{M} \rightarrow \mathbb{R}$ ,

$$(2.2) \quad \frac{1}{n} \sum_{k \in E^M(x) \cap [1, n[} \varphi(F^k \hat{x}) \xrightarrow{n} \int \varphi d\hat{\xi}^M \text{ uniformly in } x \in \mathbf{F}.$$

- for any continuous function  $\vartheta : \mathbf{M} \rightarrow \mathbb{R}$ ,

$$(2.3) \quad \frac{1}{n} \sum_{k \in [1, n[} \vartheta(f^k x) \xrightarrow{n} \int \vartheta d\nu \text{ uniformly in } x \in \mathbf{F}.$$

Let us detail the proof of the third item. If  $\mathcal{F} = (\varphi_k)_{k \in \mathbb{N}}$  is a dense countable family in the set  $\mathcal{C}^0(\mathbb{P}\mathbf{T}\mathbf{M}, \mathbb{R})$  of real continuous functions on  $\mathbb{P}\mathbf{T}\mathbf{M}$  endowed with the supremum norm  $\|\cdot\|_\infty$ , then for all  $k, M$ , by Egorov's theorem applied to the pointwise converging sequence  $(f_n : \mathbf{F} \rightarrow \mathbb{R})_n = \left(x \mapsto \int \varphi_k d\mu_{\hat{x}, n}^M\right)_n$ , there is a subset  $\mathbf{F}_k^M$  of  $\mathbf{F}$  with  $\nu(\mathbf{F}_k^M) > \nu(\mathbf{F}) \left(1 - \frac{1}{2^{k+M+3}}\right)$  such that  $\int \varphi_k d\mu_{\hat{x}, n}^M$  converges to  $\int \varphi_k d\hat{\xi}^M$  uniformly in  $x \in \mathbf{F}_k^M$ . Let  $\mathbf{F}' = \bigcap_{k, M} \mathbf{F}_k^M$ . We have  $\nu(\mathbf{F}') \geq \frac{\nu(\mathbf{F})}{2}$ . Then, if  $\varphi \in \mathcal{C}^0(\mathbb{P}\mathbf{T}\mathbf{M}, \mathbb{R})$ , we may find for any  $\epsilon > 0$  a function  $\varphi_k \in \mathcal{F}$

with  $\|\varphi - \varphi_k\|_\infty < \epsilon$ . Let  $M \in \mathbb{N}$ . Take  $N = N_\epsilon^{k,M}$  such that  $|\int \varphi_k d\mu_{\hat{x},n}^M - \int \varphi_k d\xi^M| < \epsilon$  for  $n > N$  and for all  $x \in \mathbf{F}_k^M$ . In particular for all  $x \in \mathbf{F}'$  we have for  $n > N$

$$\begin{aligned} \left| \int \varphi d\mu_{\hat{x},n}^M - \int \varphi d\xi^M \right| &\leq \left| \int \varphi_k d\mu_{\hat{x},n}^M - \int \varphi d\mu_{\hat{x},n}^M \right| + \left| \int \varphi_k d\mu_{\hat{x},n}^M - \int \varphi_k d\xi^M \right| \\ &\quad + \left| \int \varphi_k d\xi^M - \int \varphi d\xi^M \right|, \\ &\leq 2\|\varphi - \varphi_k\|_\infty + \left| \int \varphi_k d\mu_{\hat{x},n}^M - \int \varphi_k d\xi^M \right|, \\ &< 3\epsilon. \end{aligned}$$

This proves (2.2) by taking  $\mathbf{F}'$  in the place of  $\mathbf{F}$ . One proves similarly (2.3).

Fix now  $M$ . For each  $n \in \mathbb{N}$  and  $x \in \mathbf{F}$  we let  $E_n(x) = E(\hat{x}) \cap [0, n[$  and  $E_n^M(x) = E^M(\hat{x}) \cap [0, n[$ . We also let  $\mathbf{E}_n^M$  be the partition of  $\mathbf{F}$  with atoms  $A_E := \{x \in D, E_n^M(x) = E\}$  for  $E \subset [0, n[$ . Given a partition  $Q$  of  $\mathbb{P}\mathbf{T}\mathbf{M}$ , we also let  $Q^{\mathbf{E}_n^M}$  be the partition of  $\hat{\mathbf{F}} := \{\hat{x}, x \in \mathbf{F} \cap D\}$  finer than  $\pi^{-1}\mathbf{E}_n^M$  with atoms  $\{\hat{x} \in \hat{\mathbf{F}}, E_n^M(x) = E$  and  $\forall k \in E, F^k \hat{x} \in Q_k\}$  for  $E \subset [0, n[$  and  $(Q_k)_{k \in E} \in Q^E$ . We let  $\partial Q$  be the boundary of the partition  $Q$ , which is the union of the boundaries of its atoms. For a measure  $\eta$  and a subset  $A$  of  $\mathbf{M}$  with  $\eta(A) > 0$  we denote by  $\eta_A = \frac{\eta(A \cap \cdot)}{\eta(A)}$  the induced probability measure on  $A$ . Moreover, for two sets  $A, B$  we let  $A \Delta B$  denote the symmetric difference of  $A$  and  $B$ , i.e.  $A \Delta B = (A \setminus B) \cup (B \setminus A)$ . Finally, let  $H : ]0, 1[ \rightarrow \mathbb{R}^+$  be the map  $t \mapsto -t \log t - (1-t) \log(1-t)$ . Recall that  $\hat{\zeta}^+$  is the lift of  $\zeta$  on  $\mathbb{P}\mathbf{T}\mathbf{M}$  to the unstable Oseledets bundle (with  $\zeta$  as in Subsection 2.3).

**Lemma 3.** *For any finite partition  $Q$  and any  $m \in \mathbb{N}^*$  with  $\hat{\xi}^M(\partial Q^m) = 0$  we have*

$$(2.4) \quad h(\nu) \leq \beta_M \frac{1}{m} H_{\hat{\xi}^M}(Q^m) + \limsup_n \frac{1}{n} H_{\hat{\zeta}^+}(\pi^{-1} P^n | Q^{\mathbf{E}_n^M}) + H(2/M) + \frac{12 \log \#Q}{M} + \iota.$$

Before the proof of Lemma 3, we first recall a technical lemma from [2].

**Lemma 4** (Lemma 6 in [2]). *Let  $(X, T)$  be a topological system. Let  $\mu$  be a Borel probability measure on  $X$  and let  $E$  be a finite subset of  $\mathbb{N}$ . For any finite partition  $Q$  of  $X$ , we have with  $\mu^E := \frac{1}{\#E} \sum_{k \in E} T_*^k \mu$  and  $Q^E := \bigvee_{k \in E} T^{-k} Q$ :*

$$\frac{1}{\#E} H_\mu(Q^E) \leq \frac{1}{m} H_{\mu_E}(Q^m) + 6m \frac{\#(E+1)\Delta E}{\#E} \log \#Q.$$

*Proof of Lemma 3.* As the complement of  $E_n^M(x)$  is the disjoint union of neutral blocks with length larger than  $M$ , there are at most  $A_n^M = \sum_{k=0}^{\lfloor 2n/M \rfloor + 1} \binom{n}{k}$  possible values for  $E_n^M(x)$  so that

$$\begin{aligned} \frac{1}{n} H_\zeta(P^n) &= \frac{1}{n} H_\zeta(P^n | \mathbf{E}_n^M) + H_\zeta(\mathbf{E}_n^M), \\ &\leq \frac{1}{n} H_\zeta(P^n | \mathbf{E}_n^M) + \log A_n^M, \\ \liminf_n \frac{1}{n} H_\zeta(P^n) &\leq \limsup_n \frac{1}{n} H_\zeta(P^n | \mathbf{E}_n^M) + H(2/M) \text{ by using Stirling's formula.} \end{aligned}$$

Moreover

$$\begin{aligned} \frac{1}{n}H_\zeta(P^n|\mathbf{E}_n^M) &= \frac{1}{n}H_{\hat{\zeta}_+}(\pi^{-1}P^n|\pi^{-1}\mathbf{E}_n^M), \\ &\leq \frac{1}{n}H_{\hat{\zeta}_+}(Q^{\mathbf{E}_n^M}|\pi^{-1}\mathbf{E}_n^M) + \frac{1}{n}H_{\hat{\zeta}_+}(\pi^{-1}P^n|Q^{\mathbf{E}_n^M}). \end{aligned}$$

For  $E \subset [0, n[$  we let  $\hat{\zeta}_{E,n}^+ = \frac{n}{\#E} \int \mu_{\hat{x},n}^M d\zeta_{A_E}(x)$ , which may be also written as  $\left(\hat{\zeta}_{\pi^{-1}A_E}^+\right)^E$  by using the notations of Lemma 4. By Lemma 4 applied to the system  $(\mathbb{P}\mathbf{T}\mathbf{M}, F)$  and the measures  $\mu := \hat{\zeta}_{\pi^{-1}A_E}^+$  for  $A_E \in \mathbf{E}_n^M$  we have for all  $n > m \in \mathbb{N}^*$ :

$$\begin{aligned} H_{\hat{\zeta}_+}(Q^{\mathbf{E}_n^M}|\pi^{-1}\mathbf{E}_n^M) &= \sum_E \zeta(A_E)H_{\hat{\zeta}_{\pi^{-1}A_E}^+}(Q^E), \\ &\leq \sum_E \zeta(A_E)\#E \left( \frac{1}{m}H_{\hat{\zeta}_{E,n}^+}(Q^m) + 6m \frac{\#(E+1)\Delta E}{\#E} \log \#Q \right). \end{aligned}$$

Recall again that if  $E = E_n^M(x)$  for some  $x$  then the complement set of  $E$  in  $[1, n[$  is made of neutral blocks of length larger than  $M$ , therefore  $\#(E+1)\Delta E \leq \frac{2M}{n}$ . Moreover it follows from  $\xi^M(\partial Q^m) = 0$  and (2.2), that  $\mu_{\hat{x},n}^M(A^m)$  for  $A^m \in Q^m$  and  $\#E_n^M(x)/n$  are converging to  $\hat{\xi}^M(A^m)$  and  $\beta_M$  respectively uniformly in  $x \in \mathbf{F}$  when  $n$  goes to infinity. Then we get by taking the limit in  $n$ :

$$\begin{aligned} \limsup_n \frac{1}{n}H_{\hat{\zeta}_+}(Q^{\mathbf{E}_n^M}|\pi^{-1}\mathbf{E}_n^M) &\leq \beta_M \frac{1}{m}H_{\hat{\xi}^M}(Q^m) + \frac{12m \log \#Q}{M}, \\ h(\nu) - \iota &\leq \liminf_n \frac{1}{n}H_\zeta(P^n) \leq \beta_M \frac{1}{m}H_{\hat{\xi}^M}(Q^m) + \limsup_n \frac{1}{n}H_{\hat{\zeta}_+}(\pi^{-1}P^n|Q^{\mathbf{E}_n^M}) \\ &\quad + H(2/M) + \frac{12m \log \#Q}{M}. \end{aligned}$$

□

**2.5. Bounding the entropy of the neutral component.** For a  $\mathcal{C}^1$  diffeomorphism  $f$  on  $\mathbf{M}$  we put  $C(f) := 2A_f H(A_f^{-1}) + \frac{\log^+ \|df\|_\infty}{r} + B_r$  with  $A_f = \log^+ \|df\|_\infty + \log^+ \|df^{-1}\|_\infty + 1$  and a universal constant  $B_r$  depending only  $r$  precised later on. Clearly  $f \mapsto C(f)$  is continuous in the  $\mathcal{C}^1$  topology and  $\frac{\lambda^+(f)}{r} = \lim_{\mathbb{N} \ni p \rightarrow +\infty} \frac{C(f^p)}{p}$  whenever  $\lambda^+(f) > 0$  (indeed  $A_{f^p} \xrightarrow{p} +\infty$ , therefore  $H(A_{f^p}^{-1}) \xrightarrow{p} 0$ ). In particular, if  $\frac{\lambda^+(f)}{r} < \alpha$  and  $f_k \xrightarrow{k} f$  in the  $\mathcal{C}^1$  topology, then there is  $p$  with  $\lim_k \frac{C(f_k^p)}{p} < \alpha$ .

In this section we consider the empirical measures associated to an ergodic hyperbolic measure  $\nu$  with  $\lambda^+(\nu) > \frac{\log \|df\|_\infty}{r} + \delta$ ,  $\delta > 0$ . Without loss of generality we can assume  $\delta < \frac{r-1}{r} \log 2$ . Then as observed in Subsection 2.2 we have  $\hat{\nu}^+(H_\delta) > 0$ . For  $x \in \mathcal{R}$  we let  $m_n(x) = \max\{k < n, F^k \hat{x} \in H_\delta\}$ . By a standard application of the ergodic theorem we have

$$\frac{m_n(x)}{n} \xrightarrow{n} 1 \text{ for } \nu \text{ a.e. } x.$$

By taking a smaller subset  $\mathbf{F}$ , we can assume the above convergence of  $m_n$  is uniform on  $\mathbf{F}$  and that  $\sup_{x \in \mathbf{F}} \min\{k \leq n, F^k \hat{x} \in H_\delta\} \leq N$  for some positive integer  $N$ .

We bound the term  $\limsup_n \frac{1}{n} H_{\hat{\zeta}_+}(\pi^{-1} P^n | Q_n^{E_n^M})$  in the right member of (2.4) Lemma 3, which corresponds to the local entropy contribution plus the entropy in the neutral part.

**Lemma 5.** *There is  $\kappa > 0$  such that the empirical measures associated to  $G := \pi^{-1} G_\kappa \cap H_\delta$  satisfy the following properties. For all  $q, M \in \mathbb{N}^*$ , there are  $\epsilon_q > 0$  (depending only on  $\|d^k(f^q)\|_\infty$ ,  $2 \leq q \leq r$  §) and  $\gamma_{q,M}(f) > 0$  with*

$$(2.5) \quad \forall K > 0 \limsup_q \limsup_M \left( \sup_f \{ \gamma_{q,M}(f) \mid \|df\|_\infty \vee \|df^{-1}\|_\infty < K \} \right) = 0$$

such that for any partition  $Q$  of  $\mathbb{P}TM$  with diameter less than  $\epsilon_q$ , we have:

$$\begin{aligned} \limsup_n \frac{1}{n} H_{\hat{\zeta}_+}(\pi^{-1} P^n | Q_n^{E_n^M}) &\leq (1 - \beta_M) C(f) \\ &\quad + \left( \log 2 + \frac{1}{r-1} \right) \left( \int \frac{\log^+ \|df^q\|}{q} d\zeta^M - \int \phi d\hat{\zeta}^M \right) \\ &\quad + \gamma_{q,M}(f). \end{aligned}$$

The proof of Lemma 5 appears after the statement of Proposition 4, which is a *semi-local Reparametrization Lemma*.

**Proposition 4.** *There is  $\kappa > 0$  such that the empirical measures associated to  $G := \pi^{-1} G_\kappa \cap H_\delta$  satisfy the following properties. For all  $q \in \mathbb{N}^*$  there are  $\epsilon_q > 0$  (depending only on  $\|d^k(f^q)\|_\infty$ ,  $2 \leq q \leq r$ ) and  $\gamma_{q,M}(f) > 0$  with*

$$\forall K > 0 \limsup_q \limsup_M \left( \sup_f \{ \gamma_{q,M}(f) \mid \|df\|_\infty \vee \|df^{-1}\|_\infty < K \} \right) = 0$$

such that for any partition  $Q$  with diameter less than  $\epsilon < \epsilon_q$ , the following property holds for  $n$  large enough.

Any atom  $F_n$  of the partition  $Q_n^{E_n^M}$  may be covered by a family  $\Psi_{F_n}$  of  $C^r$  curves  $\psi : [-1, 1] \rightarrow \mathbf{M}$  satisfying  $\|d(f^k \circ \psi)\|_\infty \leq 1$  for any  $k = 0, \dots, n-1$ , such that

$$\begin{aligned} \frac{1}{n} \log \#\Psi_{F_n} &\leq \left( 1 - \frac{\#E_n^M}{n} \right) C(f) \\ &\quad + \left( \log 2 + \frac{1}{r-1} \right) \left( \int \frac{\log^+ \|d_x f^q\|_\epsilon}{q} d\zeta_{F_n}^M(x) - \int \phi d\hat{\zeta}_{F_n}^M \right) \\ &\quad + \gamma_{q,M}(f) + \tau_n, \end{aligned}$$

where  $\lim_n \tau_n = 0$ ,  $E_n^M = E_n^M(x)$  for  $x \in F_n$ ,  $\hat{\zeta}_{F_n}^M = \int \mu_{\hat{x},n}^M d\zeta_{F_n}^M(x)$  and  $\zeta_{F_n}^M = \pi_* \hat{\zeta}_{F_n}^M$  its push-forward on  $\mathbf{M}$ .

The proof of Proposition 4 is given in the last section. Proposition 4 is very similar to the Reparametrization Lemma in [4]. Here we reparametrize an atom  $F_n$  of  $Q_n^{E_n^M}$  instead of  $Q^n$  in [4].

---

§Here

$$\|d^k(f^q)\|_\infty = \sup_{\alpha \in \mathbb{N}^2, |\alpha|=k} \sup_{x,y} \left\| \partial_y^\alpha \left( \exp_{f(x)}^{-1} \circ f \circ \exp_x \right) (\cdot) \right\|_\infty$$

*Proof of Lemma 5 assuming Proposition 4.* We take  $\kappa > 0$  and  $\epsilon_q > 0$  as in Proposition 4. Observe that

$$H_{\hat{\zeta}^+}(\pi^{-1}P^n|Q^{\mathbb{E}_n^M}) \leq \sum_{F_n \in Q^{\mathbb{E}_n^M}} \hat{\zeta}^+(F_n) \log \#\{A^n \in P^n, \pi^{-1}(A^n) \cap \hat{\mathbf{F}} \cap F_n \neq \emptyset\}.$$

As  $\nu(\partial P) = 0$ , for all  $\gamma > 0$ , there is  $\chi > 0$  and a continuous function  $\vartheta : \mathbf{M} \rightarrow \mathbb{R}^+$  equal to 1 on the  $\chi$ -neighborhood  $\partial P^\chi$  of  $\partial P$  satisfying  $\int \vartheta d\nu < \gamma$ . Then we have uniformly in  $x \in \mathbf{F}$  by (2.3):

$$(2.6) \quad \limsup_n \frac{1}{n} \#\{0 \leq k < n, f^k x \in \partial P^\chi\} \leq \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \vartheta(f^k x) = \int \vartheta d\nu < \gamma.$$

Assume that for arbitrarily large  $n$  there is  $F_n \in Q^{\mathbb{E}_n^M}$  and  $\psi \in \Psi_{F_n}$  with  $\#\{A^n \in P^n, A^n \cap \psi([-1, 1]) \cap \mathbf{F} \neq \emptyset\} > ([\chi^{-1}] + 1) \#P^{\gamma n}$ . We reparametrize  $\psi$  on  $\mathbf{F}$  by  $[\chi^{-1}] + 1$  affine contractions  $\theta$  so that the length of  $f^k \circ \psi \circ \theta$  is less than  $\chi$  for all  $0 \leq k < n$  and  $(\psi \circ \theta)([-1, 1]) \cap \mathbf{F} \neq \emptyset$ . Then we have  $\#\{0 \leq k < n, \partial P \cap f^k \circ \psi \circ \theta([-1, 1]) \neq \emptyset\} > \gamma n$  for some  $\theta$ . In particular we get  $\#\{0 \leq k < n, f^k x \in \partial P^\chi\} > \gamma n$  for any  $x \in \psi \circ \theta([-1, 1])$ , which contradicts (2.6). Therefore we have

$$\limsup_n \sup_{F_n, \psi \in \Psi_{F_n}} \frac{1}{n} \log \{A^n \in P^n, A^n \cap \psi([-1, 1]) \cap \mathbf{F} \neq \emptyset\} = 0.$$

Together with Proposition 4 we get

$$\begin{aligned} \limsup_n \frac{1}{n} H_{\hat{\zeta}^+}(\pi^{-1}P^n|Q^{\mathbb{E}_n^M}) &\leq \limsup_n \sum_{F_n \in Q^{\mathbb{E}_n^M}} \hat{\zeta}^+(F_n) \frac{1}{n} \log \#\Psi_{F_n}, \\ &\leq \limsup_n \sum_{F_n \in Q^{\mathbb{E}_n^M}} \hat{\zeta}^+(F_n) \left(1 - \frac{\#E_n^M}{n}\right) C(f) + \\ &\quad + \limsup_n \sum_{F_n \in Q^{\mathbb{E}_n^M}} \hat{\zeta}^+(F_n) \left(\log 2 + \frac{1}{r-1}\right) \left(\int \frac{\log^+ \|df^q\|}{q} d\zeta_{F_n}^M - \int \phi d\hat{\zeta}_{F_n}^M\right) \\ &\quad + \gamma_{q,M}(f), \\ &\leq (1 - \beta_M)C(f) + \left(\log 2 + \frac{1}{r-1}\right) \left(\int \frac{\log^+ \|df^q\|}{q} d\xi^M - \int \phi d\hat{\xi}^M\right) + \gamma_{q,M}(f). \end{aligned}$$

This concludes the proof of Lemma 5.  $\square$

**2.6. Proof of the Main Theorem.** We first reduce the Main Theorem to the following statement.

**Proposition 5.** *Let  $(f_k)_{k \in \mathbb{N}}$  be a sequence of  $\mathcal{C}^r$ , with  $r > 1$ , surface diffeomorphisms converging  $\mathcal{C}^r$  weakly to a diffeomorphism  $f$ . Assume there is a sequence  $(\hat{\nu}_k^+)_{k \in \mathbb{N}}$  of ergodic  $F_k$ -invariant measures converging to  $\hat{\mu}$  with  $\lim_k \lambda^+(\nu_k) > \frac{\log^+ \|df\|_\infty}{r}$ .*

*Then, there are  $F$ -invariant measures  $\hat{\mu}_0$  and  $\hat{\mu}_1^+$  with  $\hat{\mu} = (1 - \beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$ ,  $\beta \in [0, 1]$ , such that:*

$$\limsup_{k \rightarrow +\infty} h(\nu_k) \leq \beta h(\mu_1) + (1 - \beta)C(f).$$

*Proof of the Main Theorem assuming Proposition 5.* Let  $(\hat{\nu}_k^+)_k$  be a sequence of ergodic  $F_k$ -invariant measures converging to  $\hat{\mu}$ .

As previously mentioned, for any  $\alpha > \lambda^+(f)/r$  there is  $p \in \mathbb{N}^*$  with  $\alpha > \frac{C(f^p)}{p}$ . We can also assume  $\frac{\log \|df^p\|_\infty}{pr} < \alpha$ . Let  $\hat{\nu}_k^{+,p}$  be an ergodic component of  $\hat{\nu}_k^+$  for  $F_k^p$  and let us denote by  $\nu_k^p$  its push forward on  $\mathbf{M}$ . We have  $h_{f_k^p}(\nu_k^p) = ph_{f_k}(\nu_k)$  for all  $k$ . By taking a subsequence we can assume that  $(\hat{\nu}_k^{+,p})_k$  is converging. Its limit  $\hat{\mu}^p$  satisfies  $\frac{1}{p} \sum_{0 \leq l < p} F_*^l \hat{\mu}^p = \hat{\mu}$ . If  $\lim_k \lambda^+(\nu_k^p) < \frac{\log^+ \|df^p\|_\infty}{r} < p\alpha$ , then by Ruelle's inequality we get

$$\begin{aligned} \limsup_{k \rightarrow +\infty} h_{f_k}(\nu_k) &= \limsup_{k \rightarrow +\infty} \frac{1}{p} h_{f_k^p}(\nu_k^p), \\ &\leq \lim_{k \rightarrow +\infty} \frac{1}{p} \lambda^+(\nu_k^p), \\ &\leq \alpha. \end{aligned}$$

This proves the Main Theorem with  $\beta = 1$ .

We consider then the case  $\lim_k \lambda^+(\nu_k^p) > \frac{\log^+ \|df^p\|_\infty}{r}$ . By applying Proposition 4 to the  $p$ -power systems, we get  $F^p$ -invariant measure  $\hat{\mu}_0^p$  and  $\hat{\mu}_1^{+,p}$  with  $\hat{\mu}^p = (1 - \beta)\hat{\mu}_0^p + \beta\hat{\mu}_1^{+,p}$ ,  $\beta \in [0, 1]$ , such that we have with  $\mu_1^p = \pi_* \hat{\mu}_1^{+,p}$  :

$$\limsup_{k \rightarrow +\infty} h_{f_k^p}(\nu_k^p) \leq \beta h_{f^p}(\mu_1^p) + (1 - \beta)C(f^p).$$

But  $h_{f^p}(\mu_1^p) = ph_f(\mu_1)$  with  $\mu_1 = \frac{1}{p} \sum_{0 \leq l < p} f^l \mu_1^p$ . One easily checks that  $\hat{\mu}_1^+ = \frac{1}{p} \sum_{0 \leq l < p} F_*^l \hat{\mu}_1^{+,p}$ . Moreover we have :

$$\begin{aligned} \limsup_{k \rightarrow +\infty} h_{f_k}(\nu_k) &= \limsup_{k \rightarrow +\infty} \frac{1}{p} h_{f_k^p}(\nu_k^p), \\ &\leq \beta \frac{1}{p} h_{f^p}(\mu_1^p) + (1 - \beta) \frac{C(f^p)}{p}, \\ &\leq \beta h_f(\mu_1) + (1 - \beta)\alpha. \end{aligned}$$

□

We show now Proposition 5 by using Lemma 5. Without loss of generality we can assume  $\liminf_k h(\nu_k) > 0$ . For  $\mu$  a.e.  $x$ , we have  $\lambda^-(x) \leq 0$ . If not, some ergodic component  $\tilde{\mu}$  of  $\mu$  would have two positive Lyapunov exponents and therefore should be the periodic measure at a source  $S$  (see e.g. Proposition 4.4 in [17]). But then for large  $k$  the probability  $\nu_k$  would give positive measure to the basin of attraction of the sink  $S$  for  $f^{-1}$  and therefore  $\nu_k$  would be equal to  $\tilde{\mu}$  contradicting  $\liminf_k h(\nu_k) > 0$ .

Let  $\delta > 0$  with  $\lim_k \lambda^+(\nu_k) > \frac{\log \|df\|_\infty}{r} + \delta$ . Then take  $\kappa$  as in Lemma 5. We consider the empirical measures associated to  $G = \pi^{-1}G_\kappa \cap H_\delta$ . By a diagonal argument, there is a subsequence in  $k$  such that the geometric component  $\hat{\xi}_k^M$  of  $\hat{\nu}_k^+$  is converging to some  $\hat{\xi}_\infty^M$  for all  $M \in \mathbb{N}$ . Let us also denote by  $\beta_M^\infty$  the limit in  $k$  of  $\beta_M^k$ . Then consider a subsequence in  $M$  such that  $\hat{\xi}_\infty^M$  is converging to  $\beta \hat{\mu}_1$  with  $\beta = \lim_M \beta_M^\infty$ . We also let  $(1 - \beta)\hat{\mu}_0 = \hat{\mu} - \beta \hat{\mu}_1$ . In this way,  $\hat{\mu}_0$  and  $\hat{\mu}_1$  are both probability measures.

**Lemma 6.** *The measures  $\hat{\mu}_0$  and  $\hat{\mu}_1$  satisfy the following properties:*

- $\hat{\mu}_1$  and  $\hat{\mu}_0$  are  $F$ -invariant,

- $\lambda^+(x) \geq \delta$  for  $\mu_1$ -a.e.  $x$  and  $\hat{\mu}_1 = \hat{\mu}_1^+$ .

*Proof.* The neutral blocks in the complement set of  $E^M(x)$  have length larger than  $M$ . Therefore for any continuous function  $\varphi : \mathbb{P}TM \rightarrow \mathbb{R}$  and for any  $k$ , we have

$$\left| \int \varphi d\hat{\xi}_k^M - \int \varphi \circ F d\hat{\xi}_k^M \right| \leq \frac{2 \sup_{\hat{x}} |\varphi(\hat{x})|}{M}.$$

Letting  $k$ , then  $M$  go to infinity, we get  $\int \varphi d\hat{\mu}_1 = \int \varphi \circ F d\hat{\mu}_1$ , i.e.  $\hat{\mu}_1$  is  $F$ -invariant.

We let  $K_M$  be the compact subset of  $\mathbb{P}TM$  given by  $K_M = \{\hat{x} \in \mathbb{P}TM, \exists 1 \leq m \leq M \text{ } \phi_m(\hat{x}) \geq m\delta\}$ . Let  $\hat{x} \in \mathbf{G}_k$ , where  $\mathbf{G}_k$  is the set where the empirical measures are converging to  $\hat{\xi}_k^M$  (see Subsection 2.1). Observe that

$$(2.7) \quad \lim_n \mu_{\hat{x},n}^M(K_M) = \hat{\xi}_k^M(K_M) = \hat{\xi}_k^M(\mathbb{P}TM).$$

Indeed for any  $k \in E^M(\hat{x})$  there is  $1 \leq m \leq M$  with  $F^m(F^k \hat{x}) \in G \subset H_\delta$ . Moreover, as already mentioned,  $\delta$ -hyperbolic points w.r.t.  $\psi$  are  $\delta$ -hyperbolic w.r.t.  $\phi$ . Therefore  $\phi_m(F^k \hat{x}) \geq m\delta$ . Consequently we have  $\lim_n \mu_{\hat{x},n}^M(K_M) = \lim_n \mu_{\hat{x},n}^M(\mathbb{P}TM) = \hat{\xi}_k^M(\mathbb{P}TM)$ . The set  $K_M$  being compact in  $\mathbb{P}TM$ , we get  $\hat{\xi}_k^M(K_M) \geq \lim_n \mu_{\hat{x},n}^M(K_M)$  and (2.7) follows.

Also we have  $\hat{\xi}_\infty^M(K_M) \geq \limsup_k \hat{\xi}_k^M(K_M) = \limsup_k \hat{\xi}_k^M(\mathbb{P}TM) = \beta_M^\infty$ . Therefore we have  $\hat{\mu}_1(\bigcup_M K_M) = 1$  as  $\hat{\xi}_\infty^M$  goes increasingly in  $M$  to  $\beta\hat{\mu}_1$ . The  $F$ -invariant set  $\bigcap_{k \in \mathbb{Z}} F^{-k}(\bigcup_M K_M)$  has also full  $\hat{\mu}_1$ -measure and for all  $\hat{x} = (x, v)$  in this set we have  $\limsup_n \frac{1}{n} \log \|d_x f^n(v)\| \geq \delta$ . Consequently the measure  $\hat{\mu}_1$  is supported on the unstable bundle  $\mathcal{E}_+(x)$  and  $\lambda^+(x) \geq \delta$  for  $\mu_1$ -a.e.  $x$ .  $\square$

**Remark 6.** In Theorem C of [10], the measure  $\beta\hat{\mu}_1^+$  is obtained as the limit when  $\delta$  goes to zero of the component associated to the set  $G^\delta := \{x, \forall l > 0 \text{ } \phi_l(\hat{x}) \geq \delta l\} \supset \pi^{-1}G_\kappa \cap H_\delta$ .

We pursue now the proof of Proposition 5. Let  $q, M \in \mathbb{N}^*$ . Fix a sequence  $(\nu_k)_k$  of positive numbers with  $\nu_k \xrightarrow{k} 0$ . We consider a partition  $Q$  satisfying  $\text{diam}(Q) < \epsilon_q$  with  $\epsilon_q$  as in Lemma 5. The sequence  $(f_k)_k$  being  $C^r$  bounded, one can choose  $\epsilon_q$  independently of  $f_k$ ,  $k \in \mathbb{N}$ .

By a standard argument of countability we may assume that for all  $m \in \mathbb{N}^*$  the boundary of  $Q^m$  has zero-measure for  $\hat{\mu}_1$  and all the measures  $\hat{\xi}_k^M$ ,  $M \in \mathbb{N}^*$  and  $k \in \mathbb{N} \cup \{\infty\}$ . Combining Lemma 5 and Lemma 3 we get with  $\gamma_{q,Q,M}(f) = \gamma_{q,M}(f) + H\left(\frac{2}{M}\right) + \frac{12 \log \#Q}{M}$  :

$$\begin{aligned} h(\nu_k) &\leq \beta_M^k \frac{1}{m} H_{\hat{\xi}_k^M}(Q^m) + (1 - \beta_M^k) C(f_k) \\ &\quad + \left( \log 2 + \frac{1}{r-1} \right) \left( \int \frac{\log^+ \|df_k^q\|}{q} d\hat{\xi}_k^M - \int \phi d\hat{\xi}_k^M \right) \\ &\quad + \gamma_{q,Q,M}(f_k) + \nu_k. \end{aligned}$$

By letting  $k$ , then  $M$  go to infinity, we obtain for all  $m$ :

$$\begin{aligned} \limsup_k h(\nu_k) &\leq \beta \frac{1}{m} H_{\hat{\mu}_1^+}(Q^m) + (1 - \beta) C(f) \\ &\quad + \left( \log 2 + \frac{1}{r-1} \right) \left( \int \frac{\log^+ \|df^q\|}{q} d\mu_1 - \int \phi d\hat{\mu}_1^+ \right) \\ &\quad + \limsup_M \sup_k \gamma_{q,Q,M}(f_k). \end{aligned}$$

By letting  $m$  go to infinity, we get:

$$\begin{aligned} \limsup_k h(\nu_k) &\leq \beta h(\hat{\mu}_1^+) + (1 - \beta)C(f) \\ &\quad + \left( \log 2 + \frac{1}{r-1} \right) \left( \int \frac{\log^+ \|df^q\|}{q} d\mu_1 - \int \phi d\hat{\mu}_1^+ \right) \\ &\quad + \limsup_M \sup_k \gamma_{q,M}(f_k). \end{aligned}$$

But  $h(\hat{\mu}_1^+) = h(\mu_1)$  (see e.g. Corollary 4.2 in [10]) and  $\int \phi d\hat{\mu}_1^+ = \lambda^+(\mu_1) = \lim_q \int \frac{\log^+ \|df^q\|}{q} d\mu_1$ . Therefore by letting  $q$  go to infinity we finally obtain with the asymptotic property (2.5) of  $\gamma_{q,M}$ :

$$\limsup_k h(\nu_k) \leq \beta h(\mu_1) + (1 - \beta)C(f).$$

### 3. SEMI-LOCAL REPARAMETRIZATION LEMMA

In this section we prove the semi-local *Reparametrization Lemma* stated in Proposition 4.

**3.1. Strongly bounded curves.** To simplify the exposition (by avoiding irrelevant technical details involving the exponential map) we assume that  $\mathbf{M}$  is the two-torus  $\mathbb{T}^2$  with the usual Riemannian structure inherited from  $\mathbb{R}^2$ . Borrowing from [2] we first make the following definitions.

A  $C^r$  embedded curve  $\sigma : [-1, 1] \rightarrow \mathbf{M}$  is said *bounded* when  $\max_{k=2, \dots, r} \|d^k \sigma\|_\infty \leq \frac{\|d\sigma\|_\infty}{6}$ .

**Lemma 7.** *Assume  $\sigma$  is a bounded curve. Then for any  $x \in \sigma([-1, 1])$ , the curve  $\sigma$  contains the graph of a  $\kappa$ -admissible map at  $x$  with  $\kappa = \frac{\|d\sigma\|_\infty}{6}$ .*

*Proof.* Let  $x = \sigma(s)$ ,  $s \in [-1, 1]$ . One checks easily (see Lemma 7 in [4] for further details) that for all  $t \in [-1, 1]$  the angle  $\angle \sigma'(s), \sigma'(t) < \frac{\pi}{6} \leq 1$  and therefore  $\int_0^1 \sigma'(t) \cdot \frac{\sigma'(s)}{\|\sigma'(s)\|} dt \geq \frac{\|d\sigma\|_\infty}{6}$ . Therefore, as  $\sigma'(s) \in \mathcal{E}_+(x)$ , the image of  $\sigma$  contains the graph of an  $\frac{\|d\sigma\|_\infty}{6}$ -admissible map at  $x$ .  $\square$

A  $C^r$  bounded curve  $\sigma : [-1, 1] \rightarrow \mathbf{M}$  is said *strongly  $\epsilon$ -bounded* for  $\epsilon > 0$  if  $\|d\sigma\|_\infty \leq \epsilon$ . For  $n \in \mathbb{N}^*$  and  $\epsilon > 0$  a curve is said *strongly  $(n, \epsilon)$ -bounded* when  $f^k \circ \sigma$  is strongly  $\epsilon$ -bounded for all  $k = 0, \dots, n-1$ .

We consider a  $C^r$  smooth diffeomorphism  $g : \mathbf{M} \circlearrowleft$  with  $\mathbb{N} \ni r \geq 2$ . For  $\hat{x} = (x, v) \in \mathbb{P}T\mathbf{M}$  with  $\pi(\hat{x}) = x$ , we let  $k_g(x) \geq k'_g(\hat{x})$  be the following integers:

$$k_g(x) := [\log \|d_x g\|],$$

$$k'_g(\hat{x}) := [\log \|d_x g(v)\|] = [\phi_g(\hat{x})].$$

In the next lemma, we reparametrize the image by  $g$  of a bounded curve. The proof of this lemma is mostly contained in the proof of the Reparametrization Lemma [2], but we reproduce it for the sake of completeness.

**Lemma 8.** Let  $\frac{R_{inj}}{2} > \epsilon = \epsilon_g > 0$  satisfying  $\|d^s g_{2\epsilon}^x\|_\infty \leq 3\epsilon \|d_x g\|$  for all  $s = 1, \dots, r$  and all  $x \in \mathbf{M}$ , where  $g_{2\epsilon}^x = g \circ \exp_x(2\epsilon) = g(x + 2\epsilon) : \{w_x \in T_x \mathbf{M}, \|w_x\| \leq 1\} \rightarrow \mathbf{M}$ . We assume  $\sigma : [-1, 1] \rightarrow \mathbf{M}$  is a strongly  $\epsilon$ -bounded  $C^r$  curve and we let  $\hat{\sigma} : [-1, 1] \rightarrow \mathbb{P}TM$  be the associated induced map.

Then for some universal constant  $C_r > 0$  depending only on  $r$  and for any pair of integers  $(k, k')$  there is a family  $\Theta$  of affine maps from  $[-1, 1]$  to itself satisfying:

- $\hat{\sigma}^{-1}(\{\hat{x} \in \mathbb{P}TM, k_g(x) = k \text{ and } k'_g(\hat{x}) = k'\}) \subset \bigcup_{\theta \in \Theta} \theta([-1, 1])$ ,
- $\forall \theta \in \Theta$ , the curve  $g \circ \sigma \circ \theta$  is bounded,
- $\forall \theta \in \Theta$ ,  $|\theta'| \leq e^{\frac{k'-k-1}{r-1}}/4$ ,
- $\#\Theta \leq C_r e^{\frac{k-k'}{r-1}}$ .

*Proof. First step : Taylor polynomial approximation.* One computes for an affine map  $\theta : [-1, 1] \rightarrow \mathbf{M}$  with contraction rate  $b$  precised later and with  $y = \sigma(t)$ ,  $k_g(y) = k$ ,  $k'_g(y) = k'$ ,  $t \in \theta([-1, 1])$ :

$$\begin{aligned} \|d^r(g \circ \sigma \circ \theta)\|_\infty &\leq b^r \|d^r(g_{2\epsilon}^y \circ \sigma_{2\epsilon}^y)\|_\infty, \text{ with } \sigma_{2\epsilon}^y := (2\epsilon)^{-1} \exp_y^{-1} \circ \sigma = 2\epsilon^{-1}(\sigma(\cdot) - y), \\ &\leq b^r \left\| d^{r-1} \left( d_{\sigma_{2\epsilon}^y} g_{2\epsilon}^y \circ d_{\sigma_{2\epsilon}^y} \right) \right\|_\infty, \\ &\leq b^r 2^r \max_{s=0, \dots, r-1} \left\| d^s \left( d_{\sigma_{2\epsilon}^y} g_{2\epsilon}^y \right) \right\|_\infty \max_{k=1, \dots, r} \|d^k \sigma_{2\epsilon}^y\|_\infty. \end{aligned}$$

By assumption on  $\epsilon$ , we have  $\|d^s g_{2\epsilon}^y\|_\infty \leq 3\epsilon \|d_y g\|$  for any  $r \geq s \geq 1$ . Moreover  $\max_{k=1, \dots, r} \|d^k \sigma_{2\epsilon}^y\|_\infty \leq 1$  as  $\sigma$  is strongly  $\epsilon$ -bounded. Therefore by Faà di Bruno's formula, we get for some ¶ constants  $C_r > 0$  depending only on  $r$ :

$$\max_{s=0, \dots, r-1} \|d^s (d_{\sigma_{2\epsilon}^y} g_{2\epsilon}^y)\|_\infty \leq \epsilon C_r \|d_y g\|,$$

then ,

$$\begin{aligned} \|d^r(g \circ \sigma \circ \theta)\|_\infty &\leq \epsilon C_r b^r \|d_y g\| \max_{k=1, \dots, r} \|d^k \sigma_{2\epsilon}^y\|_\infty, \\ &\leq C_r b^r \|d_y g\| \|d\sigma\|_\infty, \\ &\leq (C_r b^{r-1} \|d_y g\|) \|d(\sigma \circ \theta)\|_\infty, \\ &\leq (C_r b^{r-1} e^k) \|d(\sigma \circ \theta)\|_\infty, \text{ because } k(y) = k, \\ &\leq e^{k'-4} \|d(\sigma \circ \theta)\|_\infty, \text{ by taking } b = \left( C_r e^{k-k'+4} \right)^{-\frac{1}{r-1}}. \end{aligned}$$

Therefore the Taylor polynomial  $P$  at 0 of degree  $r-1$  of  $d(g \circ \sigma \circ \theta)$  satisfies on  $[-1, 1]$ :

$$\|P - d(g \circ \sigma \circ \theta)\|_\infty \leq e^{k'-4} \|d(\sigma \circ \theta)\|_\infty.$$

We may cover  $[-1, 1]$  by at most  $b^{-1} + 1$  such affine maps  $\theta$ .

*Second step : Bezout theorem.* Let  $a = e^{k'} \|d(\sigma \circ \theta)\|_\infty$ . Note that for  $s \in [-1, 1]$  with  $k(\sigma \circ \theta(s)) = k$  and  $k'(\sigma \circ \theta(s)) = k'$  we have  $\|d(g \circ \sigma \circ \theta)(s)\| \in [ae^{-2}, ae^2]$ , therefore  $\|P(s)\| \in [ae^{-3}, ae^3]$ . Moreover if we have now  $\|P(s)\| \in [ae^{-3}, ae^3]$  for some  $s \in [-1, 1]$  we get also  $\|d(g \circ \sigma \circ \theta)(s)\| \in [ae^{-4}, ae^4]$ .

¶ Although these constants may differ at each step, they are all denoted by  $C_r$ .

By Bezout theorem the semi-algebraic set  $\{s \in [-1, 1], \|P(s)\| \in [e^{-3}a, e^3a]\}$  is the disjoint union of closed intervals  $(J_i)_{i \in I}$  with  $\#I$  depending only on  $r$ . Let  $\theta_i$  be the composition of  $\theta$  with an affine reparametrization from  $[-1, 1]$  onto  $J_i$ .

*Third step : Landau-Kolmogorov inequality.* By the Landau-Kolmogorov inequality on the interval (see Lemma 6 in [2]), we have for some constants  $C_r \in \mathbb{N}^*$  and for all  $1 \leq s \leq r$ :

$$\begin{aligned} \|d^s(g \circ \sigma \circ \theta_i)\|_\infty &\leq C_r (\|d^r(g \circ \sigma \circ \theta_i)\|_\infty + \|d(g \circ \sigma \circ \theta_i)\|_\infty), \\ &\leq C_r \frac{|J_i|}{2} \left( \|d^r(g \circ \sigma \circ \theta)\|_\infty + \sup_{t \in J_i} \|d(g \circ \sigma \circ \theta)(t)\| \right), \\ &\leq C_r a \frac{|J_i|}{2}. \end{aligned}$$

We cut again each  $J_i$  into  $1000C_r$  intervals  $\tilde{J}_i$  of the same length with

$$\theta(\tilde{J}_i) \cap \sigma^{-1} \{x, k_g(x) = k \text{ and } k'_g(x) = k'\} \neq \emptyset.$$

Let  $\tilde{\theta}_i$  be the affine reparametrization from  $[-1, 1]$  onto  $\theta(\tilde{J}_i)$ . We check that  $g \circ \sigma \circ \tilde{\theta}_i$  is bounded:

$$\begin{aligned} \forall s = 2, \dots, r, \|d^s(g \circ \sigma \circ \tilde{\theta}_i)\|_\infty &\leq (1000C_r)^{-2} \|d^s(g \circ \sigma \circ \theta_i)\|_\infty, \\ &\leq \frac{1}{6} (1000C_r)^{-1} \frac{|J_i|}{2} a_n e^{-4}, \\ &\leq \frac{1}{6} (1000C_r)^{-1} \frac{|J_i|}{2} \min_{s \in J_i} \|d(g \circ \sigma \circ \theta)(s)\|, \\ &\leq \frac{1}{6} (1000C_r)^{-1} \frac{|J_i|}{2} \min_{s \in \tilde{J}_i} \|d(g \circ \sigma \circ \theta)(s)\|, \\ &\leq \frac{1}{6} \|d(g \circ \sigma \circ \tilde{\theta}_i)\|_\infty. \end{aligned}$$

This concludes the proof with  $\Theta$  being the family of all  $\tilde{\theta}_i$ 's.  $\square$

We recall now a useful property of bounded curve (see Lemma 7 in [4] for a proof).

**Lemma 9.** *Let  $\sigma : [-1, 1] \rightarrow \mathbf{M}$  be a  $C^r$  bounded curve and let  $B$  be a ball of radius less than  $\epsilon$ . Then there exists an affine map  $\theta : [-1, 1] \circlearrowleft$  such that :*

- $\sigma \circ \theta$  is strongly  $3\epsilon$ -bounded,
- $\theta([-1, 1]) \supset \sigma^{-1}B$ .

**3.2. Choice of the parameters  $\kappa$  and  $\epsilon_q$ .** For a diffeomorphism  $f : \mathbf{M} \circlearrowleft$  the scale  $\epsilon_f$  in Lemma 8 may be chosen such that  $\epsilon_{fk} \leq \epsilon_{fl} \leq \max(1, \|df\|_\infty)^{-k}$  for any  $q \geq k \geq l \geq 1$ . We take  $\kappa = \frac{\epsilon_f}{36}$  and we choose  $\epsilon_q < \frac{\epsilon_f^q}{3}$  such that for any  $\hat{x}, \hat{y} \in \mathbb{P}T\mathbf{M}$  which are  $\epsilon_q$ -close and for any  $0 \leq l \leq q$ :

$$(3.1) \quad \begin{aligned} |k_{fl}(x) - k_{fl}(y)| &\leq 1, \\ |k'_{fl}(\hat{x}) - k'_{fl}(\hat{y})| &\leq 1. \end{aligned}$$

Without loss of generality we can assume the local unstable curve  $D$  (defined in Subsection 2.3) is reparametrized by a  $C^r$  strongly  $\epsilon_q$ -bounded map  $\sigma : [-1, 1] \rightarrow D$ .

Let  $F_n$  be an atom of the partition  $Q^{E_n^M}$  and let  $E_n^M = E_n^M(x)$  for any  $\hat{x} \in F_n$ . Recall that the diameter of  $Q$  is less than  $\epsilon_q$ . It follows from (3.1) that for any  $\hat{x} \in F_n$  we have with  $\hat{\zeta}_{F_n}^M = \int \mu_{\hat{x},n}^M d\zeta_{F_n}(x)$ :

$$\sum_{l \in E_n^M} \left| k_{f^l}(f^l x) - k'_{f^l}(F^l \hat{x}) \right| \leq 10 \#E_n^M + \int \log^+ \|d_y f^q\| d\zeta_{F_n}^M(y) - \int \phi_q d\hat{\zeta}_{F_n}^M.$$

Therefore we may fix some  $0 \leq c < q$ , such that for any  $x \in F_n$

$$\begin{aligned} \sum_{l \in (c+q\mathbb{N}) \cap E_n^M} \left| k_{f^l}(f^l x) - k'_{f^l}(F^l \hat{x}) \right| &\leq 10 \frac{n}{q} + \frac{1}{q} \left( \int \log^+ \|d_y f^q\| d\zeta_{F_n}^M(y) - \int \phi_q d\hat{\zeta}_{F_n}^M \right), \\ &\leq 10 \frac{n}{q} + 2A_f \frac{qn}{M} + \frac{1}{q} \int \log^+ \|d_y f^q\| d\zeta_{F_n}^M(y) - \int \phi d\hat{\zeta}_{F_n}^M. \end{aligned}$$

**3.3. Combinatorial aspects.** We put  $\partial_l E_n^M := \{a \in E_n^M \text{ with } a-1 \notin E_n^M\}$ . Then we let  $\mathcal{A}_n := \{0 = a_1 < a_2 < \dots < a_m\}$  be the union of  $\partial_l E_n^M$ ,  $[0, n] \setminus E_n^M$  and  $(c+q\mathbb{N}) \cap [0, n]$ . We also let  $b_i = a_{i+1} - a_i$  for  $i = 1, \dots, m-1$  and  $b_m = n - a_m$ .

For a sequence  $\mathbf{k} = (k_l, k'_l)_{l \in \mathcal{A}_n}$  of integers, a positive integer  $m_n$  and a subset  $\bar{E}$  of  $[0, n]$ , we let  $F_n^{\mathbf{k}, \bar{E}, m_n}$  be the subset of points  $\hat{x} \in F_n$  satisfying:

- $\bar{E} = E_n(x) \setminus E_n^M(x)$ ,
- $k_{a_i} = k_{f^{b_i}}(f^{a_i} x)$  and  $k'_{a_i} = k'_{f^{b_i}}(F^{a_i} \hat{x})$  for  $i = 1, \dots, m$ ,
- $m_n(x) = m_n$ .

**Lemma 10.**

$$\# \left\{ (\mathbf{k}, \bar{E}, m_n), F_n^{\mathbf{k}, \bar{E}, m_n} \neq \emptyset \right\} \leq n e^{2nA_f H(A_f^{-1})} 3^{n(1/q+1/M)} e^{nH(1/M)}.$$

*Proof.* Firstly observe that if  $a_i \notin E_n^M$  then  $b_i = 1$ . In particular  $\sum_{i, a_i \notin E_n^M} k_{a_i} \leq (n - \#E_n^M) \log^+ \|df\|_\infty \leq (n - \#E_n^M)(A_f - 1)$ . The number of such sequences  $(k_{a_i})_{i, a_i \notin E_n^M}$  is therefore bounded above by  $\binom{r_n A_f}{r_n}$  with  $r_n = n - \#E_n^M$  and its logarithm is dominated by  $r_n A_f H(A_f^{-1}) + 1 \leq n A_f H(A_f^{-1}) + 1$ . Similarly the number of sequence  $(k'_{a_i})_{i, a_i \notin E_n^M}$  is less than  $n A_f H(A_f^{-1}) + 1$ .

Then from the choice of  $\epsilon_q$  in (3.1) there are at most three possible values of  $k_{a_i}(x)$  for  $a_i \in E_n^M$  and  $x \in F_n$ .

Finally as  $\#\bar{E} \leq n/M$ , the number of admissible sets  $\bar{E}$  is less than  $\binom{n}{[n/M]}$  and thus its logarithm is bounded above by  $nH(1/M) + 1$ . Clearly we can also fix the value of  $m_n$  up to a factor  $n$ . □

**3.4. The induction.** We fix  $\mathbf{k}$ ,  $m_n$  and  $\bar{E}$  and we reparametrize appropriately the set  $F_n^{\mathbf{k}, \bar{E}, m_n}$ .

**Lemma 11.** *With the above notations there are families  $(\Theta_i)_{i \leq m}$  of affine maps from  $[-1, 1]$  into itself such that :*

- $\forall \theta \in \Theta_i \forall j \leq i$  the curve  $f^{a_i} \circ \sigma \circ \theta$  is strongly  $\epsilon_{f^{b_i}}$ -bounded,
- $\hat{\sigma}^{-1} \left( F_n^{\mathbf{k}, \bar{E}, m_n} \right) \subset \bigcup_{\theta \in \Theta_i} \theta([-1, 1])$ ,

- $\forall \theta_i \in \Theta_i \forall j < i, \exists \theta_j^i \in \Theta_j, \frac{|\theta_i'|}{|(\theta_j^i)'|} \leq \prod_{j \leq l < i} e^{\frac{k_{a_l}' - k_{a_l} - 1}{r-1}} / 4,$
- $\#\Theta_i \leq C \max(1, \|df\|_\infty)^{\#\bar{E} \cap [1, a_i]} \prod_{j < i} C_r e^{\frac{k_{a_j}' - k_{a_j}'}{r-1}}.$

*Proof.* We argue by induction on  $i \leq m$ . By changing the constant  $C$ , it is enough to consider  $i$  with  $a_i > N$ . Recall that the integer  $N$  was chosen in such a way that for any  $x \in F$  there is  $0 \leq k \leq N$  with  $F^k \hat{x} \in H_\delta$ . We assume the family  $\Theta_i$  for  $i < m$  already built and we will define  $\Theta_{i+1}$ . Let  $\theta_i \in \Theta_i$ . We apply Lemma 8 to the strongly  $\epsilon_{f^{b_i}}$ -bounded curve  $f^{a_i} \circ \sigma \circ \theta_i$  with  $g = f^{b_i}$ . Let  $\Theta$  be the family of affine reparametrizations of  $[-1, 1]$  satisfying the conclusions of Lemma 8, in particular  $f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta$  is bounded,  $|\theta'| \leq e^{\frac{k_{a_i}' - k_{a_i} - 1}{r-1}} / 4$  for all  $\theta \in \Theta$  and  $\#\Theta \leq C_r e^{\frac{k_{a_i}' - k_{a_i}'}{r-1}}$ . We distinguish three cases:

- $a_{i+1} \in E_n^M$ . The diameter of  $F^{a_{i+1}} F_n$  is less than  $\epsilon_q \leq \frac{\epsilon_{f^{b_{i+1}}}}{3}$ . By Lemma 9 there is an affine map  $\psi : [-1, 1] \circlearrowleft$  such that  $f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta \circ \psi$  is strongly  $\epsilon_{f^{b_{i+1}}}$ -bounded and its image contains the intersection of the bounded curve  $f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta$  with  $f^{a_{i+1}} F_n$ . We let then  $\theta_{i+1} = \theta_i \circ \theta \circ \psi \in \Theta_{i+1}$ .
- $a_{i+1} \in E \setminus E_n^M$ . Observe that  $b_{i+1} = 1$ , therefore  $\epsilon_{f^{b_i}} \leq \epsilon_{f^{b_{i+1}}}$ . Then the length of the curve  $f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta$  is less than  $3\|df\|_\infty \epsilon_{f^{b_i}}$ , thus may be covered by  $[3\|df\|_\infty] + 1$  balls of radius less than  $\epsilon_{f^{b_{i+1}}}$ . We then use Lemma 9 as in the previous case to reparametrize the intersection of this curve with each ball by a strongly  $\epsilon_{f^{b_{i+1}}}$ -bounded curve. We define in this way the associated parametrizations of  $\Theta_{i+1}$ .
- $a_{i+1} \notin E$  and  $a_{i+1} \notin E_n^M$ . We claim that  $\|d(f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta)\| \leq \epsilon_f / 6$ . Take  $\hat{x} \in F_n^{\mathbf{k}, \bar{E}, m_n}$  with  $x = \pi(\hat{x}) = \sigma \circ \theta_i \circ \theta(s)$ . Let  $K_x = \max\{k < a_{i+1}, F^k \hat{x} \in H_\delta\} \geq N$ . Observe that  $[K_x, a_{i+1}] \cap E_n^M = \emptyset$ , therefore for  $K_x \leq a_l < a_{i+1}$ , we have  $b_l = 1$ , then  $a_l = a_{i+1} - i - 1 + l$ . We argue by contradiction by assuming :

$$(3.2) \quad \|d(f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta)\| \geq \epsilon_f / 6 = 6\kappa$$

By Lemma 7, the point  $f^{a_{i+1}} x$  belongs to  $G_\kappa$ . We will show  $F^{a_{i+1}} \hat{x} \in H_\delta$ . Therefore we will get  $F^{a_{i+1}} \hat{x} \in G = \pi^{-1} G_\kappa \cap H_\delta$  contradicting  $a_{i+1} \notin E$ . To prove  $F^{a_{i+1}} \hat{x} \in H_\delta$  it is enough to show  $\sum_{j \leq l < a_{i+1}} \psi(F^l \hat{x}) \geq (a_{i+1} - j)\delta$  for any  $K_x \leq j < a_{i+1}$  because  $F^{K_x}(\hat{x})$  belongs to  $H_\delta$ . For any  $K_x \leq j < a_{i+1}$  we have :

$$(3.3) \quad \begin{aligned} \|d(f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta)\|_\infty &\leq 2\|d_s(f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta)\|, \text{ because } f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta \text{ is bounded,} \\ &\leq 2\|d_{f^j x} f^{a_{i+1}-j}(\hat{x})\| \times \|d_s(f^{a_{\bar{j}}} \circ \sigma \circ \theta_{\bar{j}}^i)\| \times \frac{|\theta_i'| \times |\theta'|}{|(\theta_{\bar{j}}^i)'|}, \text{ with } a_{\bar{j}} = j, \\ &\leq \frac{\epsilon_f}{3} \|d_{f^j x} f^{a_{i+1}-j}(\hat{x})\| \prod_{\bar{j} \leq l \leq i} e^{\frac{k_{a_l}' - k_{a_l} - 1}{r-1}} / 4 \text{ by induction hypothesis,} \\ &\frac{1}{2} \leq \|d_{f^j x} f^{a_{i+1}-j}(\hat{x})\| \prod_{\bar{j} \leq l \leq i} e^{\frac{k_{a_l}' - k_{a_l} - 1}{r-1}} / 4 \text{ by assumption (3.2).} \end{aligned}$$

Recall again that for  $\bar{j} \leq l \leq i$ , we have  $b_l = 1$ , thus

$$|k_{a_l} - \log \|d_{f^{a_l} x} f\| \leq 1$$

and

$$k'_{a_l} \leq \phi(F^{a_l} \hat{x}).$$

Therefore we get for any  $K_x \leq j < a_{i+1}$  from (3.3):

$$\begin{aligned} 2^{a_{i+1}-j} &\leq e^{\frac{r}{r-1} \sum_{j \leq l < a_{i+1}} \phi(F^l \hat{x})} e^{-\frac{1}{r-1} \sum_{j \leq l < a_{i+1}} \log^+ \|d_{f^l x} f\|}, \\ (a_{i+1} - j) \log 2 &\leq \frac{r}{r-1} \sum_{j \leq l < a_{i+1}} \psi(F^l \hat{x}), \text{ by definition of } \psi, \\ (a_{i+1} - j) \delta &\leq \sum_{j \leq l < a_{i+1}} \psi(F^l \hat{x}), \text{ as } \delta \text{ was chosen less than } \frac{r-1}{r} \log 2. \end{aligned}$$

□

**Lemma 12.**

$$\sum_{i, m_n > a_i \notin E_n^M} \frac{k_{a_i} - k'_{a_i}}{r-1} \leq (n - \#E_n^M) \left( \frac{\log^+ \|df\|_\infty}{r} + \frac{1}{r-1} \right).$$

*Proof.* The intersection of  $[0, m_n[$  with the complement set of  $E_n^M$  is the disjoint union of neutral blocks and possibly an interval of integers of the form  $[l, m_n[$ . In any case  $F^j \hat{x}$  belongs to  $H_\delta$  for such an interval  $[i, j[$  for any  $x \in F_n^{\mathbf{k}, \bar{E}, m_n}$ . In particular, we have

$$\sum_{l, a_l \in [i, j[} k'_{a_l} - \frac{k_{a_l}}{r} \geq (\delta - 1)(j - i)$$

therefore

$$\begin{aligned} \sum_{i, m_n > a_i \notin E_n^M} k'_{a_i} - \frac{k_{a_i}}{r} &\geq -(n - \#E_n^M), \\ \sum_{i, m_n > a_i \notin E_n^M} \frac{k_{a_i} - k'_{a_i}}{r-1} &\leq \frac{n - \#E_n^M}{r-1} + \frac{\sum_{i, m_n > a_i \notin E_n^M} k_{a_i}}{r}, \\ &\leq (n - \#E_n^M) \left( \frac{\log^+ \|df\|_\infty}{r} + \frac{1}{r-1} \right). \end{aligned}$$

□

**3.5. Conclusion.** We let  $\Psi_n$  be the family of  $\mathcal{C}^r$  curves  $\sigma \circ \theta$  for  $\theta \in \Theta_m = \Theta_m(\mathbf{k}, \bar{E}, m_n)$  with  $\Theta_m$  as in Lemma 11 over all admissible parameters  $\mathbf{k}, \bar{E}, m_n$ . For  $\theta \in \Theta_m$  the curve  $f^{a_i} \circ \sigma \circ \theta$  is strongly  $\epsilon_{f^{b_i}}$ -bounded for any  $i = 1, \dots, m$ , in particular

$$\forall i = 1, \dots, m, \|d(f^{a_i} \circ \sigma \circ \theta)\|_\infty \leq \epsilon_{f^{b_i}} \leq \max(1, \|df\|_\infty)^{-b_i},$$

therefore

$$\forall j = 0, \dots, n, \|d(f^j \circ \sigma \circ \theta)\|_\infty \leq 1.$$

By combining the previous estimates, we get moreover:

$$\begin{aligned}
\#\Psi_n &\leq \#\left\{(\mathbf{k}, \bar{E}, m_n), F_n^{\mathbf{k}, \bar{E}, m_n} \neq \emptyset\right\} \times \sup_{\mathbf{k}, \bar{E}, m_n} \#\Theta_n(\mathbf{k}, \bar{E}, m_n), \\
&\leq ne^{2(n-\#\mathcal{E}_n^M)A_f H(A_f)} 3^{n(1/q+1/M)} e^{nH(1/M)} \sup_{\mathbf{k}, \bar{E}, m_n} \#\Theta_n(\mathbf{k}, \bar{E}, m_n), \text{ by Lemma 10,} \\
&\leq ne^{2(n-\#\mathcal{E}_n^M)A_f H(A_f)} 3^{n(1/q+1/M)} e^{nH(1/M)} \max(1, \|df\|_\infty)^{\#\bar{E}} \prod_{j \leq m} C_r e^{\frac{k_{a_j} - k'_{a_j}}{r-1}}, \text{ by Lemma 11.}
\end{aligned}$$

Then we decompose the product into four terms :

- $\sum_{i, m_n > a_i \notin E_n^M} \frac{k_{a_i} - k'_{a_i}}{r-1} \leq (n - \#\mathcal{E}_n^M) \left( \frac{\log^+ \|df\|_\infty}{r} + \frac{1}{r-1} \right)$  by Lemma 12,
- $\sum_{i, m_n \leq a_i} \frac{k_{a_i} - k'_{a_i}}{r-1} \leq (n - m_n) \frac{A_f}{r-1}$ ,
- $\sum_{i, a_i \in E_n^M \cap (c+q\mathbb{N})} \frac{k_{a_i} - k'_{a_i}}{r-1} \leq 10 \frac{n}{q} + 2A_f \frac{qn}{M} + \frac{1}{r-1} \left( \int \frac{\log^+ \|d_y f^q\|}{q} d\zeta_{F_n}^M(y) - \int \phi d\hat{\zeta}_{F_n}^M \right)$ ,
- $\sum_{i, a_i \in E_n^M \setminus (c+q\mathbb{N})} \frac{k_{a_i} - k'_{a_i}}{r-1} \leq 2A_f \frac{qn}{M}$ .

By letting

$$B_r = \frac{1}{r-1} + \log C_r,$$

$$\gamma_{q,M}(f) := 2 \left( \frac{1}{q} + \frac{1}{M} \right) \log C_r + H(1/M) + \frac{10 + \log 3}{q} + \frac{4qA_f + \log 3}{M},$$

$$\tau_n = \sup_{x \in \mathbb{F}} \left( 1 - \frac{m_n(x)}{n} \right) \frac{A_f}{r-1} + \frac{\log(nC)}{n},$$

we get with  $C(f) := 2A_f H(A_f^{-1}) + \frac{\log^+ \|df\|_\infty}{r} + B_r$ :

$$\begin{aligned}
\frac{1}{n} \log \#\Psi_{F_n} &\leq \left( 1 - \frac{\#\mathcal{E}_n^M}{n} \right) C(f) \\
&\quad + \left( \log 2 + \frac{1}{r-1} \right) \left( \int \frac{\log^+ \|d_x f^q\|}{q} d\zeta_{F_n}^M(x) - \int \phi d\hat{\zeta}_{F_n}^M \right) \\
&\quad + \gamma_{q,M}(f) + \tau_n,
\end{aligned}$$

This concludes the proof of Proposition 4.

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