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► **To cite this version:**

David Burguet. MAXIMAL MEASURE AND ENTROPIC CONTINUITY OF LYAPUNOV EXPONENTS FOR C^r SURFACE DIFFEOMORPHISMS WITH LARGE ENTROPY. 2022. hal-03568570v3

HAL Id: hal-03568570

<https://hal.science/hal-03568570v3>

Preprint submitted on 20 Sep 2022 (v3), last revised 29 Mar 2023 (v4)

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MAXIMAL MEASURE AND ENTROPIC CONTINUITY OF LYAPUNOV EXPONENTS FOR C^r SURFACE DIFFEOMORPHISMS WITH LARGE ENTROPY

DAVID BURGNET

ABSTRACT. We prove a finite smooth version of the entropic continuity of Lyapunov exponents proved recently by Buzzi, Crovisier and Sarig for C^∞ surface diffeomorphisms [10]. As a consequence we show that any C^r , $r > 1$, smooth surface diffeomorphism f with $h_{top}(f) > \frac{1}{r} \limsup_n \frac{1}{n} \log^+ \|d f^n\|_\infty$ admits a measure of maximal entropy. We also prove the C^r continuity of the topological entropy at f .

INTRODUCTION

The entropy of a dynamical system quantifies the dynamical complexity by counting distinct orbits. There are topological and measure theoretical versions which are related by a variational principle : the topological entropy of a continuous map on a compact space is equal to the supremum of the entropy of the invariant (probability) measures. An invariant measure is said to be of maximal entropy (or a maximal measure) when its entropy is equal to the topological entropy, i.e. this measure realizes the supremum in the variational principle. In general a topological system may not admit a measure of maximal entropy. But such a measure exists for dynamical systems satisfying some expansiveness properties. In particular Newhouse [15] has proved their existence for C^∞ systems by using Yomdin's theory. In the present paper we show the existence of a measure of maximal entropy for C^r , $1 < r < +\infty$, smooth surface diffeomorphisms with large entropy.

Other important dynamical quantities for smooth systems are given by the Lyapunov exponents which estimate the exponential growth of the derivative. For C^∞ surface diffeomorphisms, J. Buzzi, S. Crovisier and O. Sarig proved recently a property of continuity in the entropy of the Lyapunov exponents with many statistical applications [10]. More precisely, they showed that for a C^∞ surface diffeomorphism f , if ν_k is a converging sequence of ergodic measures with $\lim_k h(\nu_k) = h_{top}(f)$, then the Lyapunov exponents of ν_k are going to the (average) Lyapunov exponents of the limit (which is a measure of maximal entropy). We prove a C^r version of this fact for $1 < r < +\infty$.

1. STATEMENTS

We define now some notations to state our main results. For a C^r , $r \geq 1$, diffeomorphism f on a compact Riemannian surface $(\mathbf{M}, \|\cdot\|)$ we let $F : \mathbb{P}T\mathbf{M} \rightarrow \mathbb{P}T\mathbf{M}$ be the induced map on the projective tangent bundle $\mathbb{P}T\mathbf{M} = T^1\mathbf{M}/\pm 1$ and we denote by $\phi, \psi : \mathbb{P}T\mathbf{M} \rightarrow \mathbb{R}$ the continuous observables on $\mathbb{P}T\mathbf{M}$ given respectively by $\phi : (x, v) \mapsto \log \|d_x f(v)\|$ and $\psi : (x, v) \mapsto \log \|d_x f(v)\| - \frac{1}{r} \log^+ \|d_x f\|$ with $\|d_x f\| = \sup_{v \in T_x \mathbf{M} \setminus \{0\}} \frac{\|d_x f(v)\|}{\|v\|}$. For $k \in$

Date: September 2022.

2010 Mathematics Subject Classification. 37 A35, 37C40, 37 D25.

\mathbb{N}^* we define more generally $\phi_k : (x, v) \mapsto \log \|d_x f^k(v)\|$ and $\psi_k : (x, v) \mapsto \phi_k(x, v) - \frac{1}{r} \sum_{l=0}^{k-1} \log^+ \|d_{f^l x} f\|$. Then we let $\lambda^+(x)$ and $\lambda^-(x)$ be the pointwise Lyapunov exponents given by $\lambda^+(x) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \|d_x f^n\|$ and $\lambda^-(x) = \liminf_{n \rightarrow -\infty} \frac{1}{n} \log \|d_x f^n\|$ for any $x \in \mathbf{M}$ and $\lambda^+(\mu) = \int \lambda^+(x) d\mu(x)$, $\lambda^-(\mu) = \int \lambda^-(x) d\mu(x)$, for any f -invariant measure μ .

Also we put $\lambda^+(f) := \lim_n \frac{1}{n} \log^+ \|df^n\|_\infty$ with $\|df^n\|_\infty = \sup_{x \in \mathbf{M}} \|d_x f^n\|$. The function $f \mapsto \lambda^+(f)$ is upper semi-continuous in the \mathcal{C}^1 topology on the set of \mathcal{C}^1 diffeomorphisms on \mathbf{M} . For an f -invariant measure μ with $\lambda^+(x) > 0 \geq \lambda^-(x)$ for μ a.e. x , there are by Oseledets* theorem one-dimensional invariant vector spaces $\mathcal{E}_+(x)$ and $\mathcal{E}_-(x)$, resp. called the unstable and stable Oseledets bundle, such that

$$\forall \mu \text{ a.e. } x \forall v \in \mathcal{E}_\pm(x) \setminus \{0\}, \quad \lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|d_x f^n(v)\| = \lambda^\pm(x).$$

Then we let $\hat{\mu}^+$ be the F -invariant measure given by the lift of μ on $\mathbb{P}\mathbf{T}\mathbf{M}$ with $\hat{\mu}^+(\mathcal{E}_+) = 1$. When writing $\hat{\mu}^+$ we assume implicitly that the push-forward measure μ on \mathbf{M} satisfies $\lambda^+(x) > 0 \geq \lambda^-(x)$ for μ a.e. x .

A sequence of \mathcal{C}^r , with $r > 1$, surface diffeomorphisms $(f_k)_k$ on \mathbf{M} is said to converge \mathcal{C}^r weakly to a diffeomorphism f , when f_k goes to f in the \mathcal{C}^1 topology and the sequence $(f_k)_k$ is \mathcal{C}^r bounded. In particular f is \mathcal{C}^{r-1} .

Theorem (Buzzi-Crovisier-Sarig, Theorem C [10]). *Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of \mathcal{C}^r , with $r > 1$, surface diffeomorphisms converging \mathcal{C}^r weakly to a diffeomorphism f . Let $(F_k)_{k \in \mathbb{N}}$ and F be the lifts of $(f_k)_{k \in \mathbb{N}}$ and f to $\mathbb{P}\mathbf{T}\mathbf{M}$. Assume there is a sequence $(\hat{\nu}_k^+)_{k \in \mathbb{N}}$ of ergodic F_k -invariant measures converging to $\hat{\mu}$.*

Then there are $\beta \in [0, 1]$ and F -invariant measures $\hat{\mu}_0$ and $\hat{\mu}_1^+$ with $\hat{\mu} = (1 - \beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$, such that:

$$\limsup_{k \rightarrow +\infty} h(\nu_k) \leq \beta h(\mu_1) + \frac{\lambda^+(f) + \lambda^+(f^{-1})}{r - 1}.$$

In particular when f ($= f_k$ for all k) is \mathcal{C}^∞ and $h(\nu_k)$ goes to the topological entropy of f , then β is equal to 1 and therefore $\lambda^+(\nu_k)$ goes to $\lambda^+(\mu)$:

Corollary (Entropic continuity of Lyapunov exponents [10]). *Let f be a \mathcal{C}^∞ surface diffeomorphism with $h_{top}(f) > 0$.*

Then if $(\nu_k)_k$ is a sequence of ergodic measures converging to μ with $\lim_k h(\nu_k) = h_{top}(f)$, then

- $h(\mu) = h_{top}(f)^\dagger$,
- $\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)$.

We state an improved version of Buzzi-Crovisier-Sarig Theorem, which allows to prove the same entropy continuity of Lyapunov exponents for \mathcal{C}^r , $1 < r < +\infty$, surface diffeomorphisms with large enough entropy (see Corollary 1).

Main Theorem. *Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of \mathcal{C}^r , with $r > 1$, surface diffeomorphisms converging \mathcal{C}^r weakly to a diffeomorphism f . Let $(F_k)_{k \in \mathbb{N}}$ and F be the lifts of $(f_k)_{k \in \mathbb{N}}$ and f*

*We refer to [16] for background on Lyapunov exponents and Pesin theory.

†This follows from the upper semi-continuity of the entropy function h on the set of f -invariant probability measures for a \mathcal{C}^∞ diffeomorphism f (in any dimension), which was first proved by Newhouse in [15].

to $\mathbb{P}TM$. Assume there is a sequence $(\hat{\nu}_k^+)_k$ of ergodic F_k -invariant measures converging to $\hat{\mu}$.

Then for any $\alpha > \frac{\lambda^+(f)}{r}$, there are $\beta = \beta_\alpha \in [0, 1]$ and F -invariant measures $\hat{\mu}_0 = \hat{\mu}_{0,\alpha}$ and $\hat{\mu}_1^+ = \hat{\mu}_{1,\alpha}^+$ with $\hat{\mu} = (1 - \beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$, such that:

$$\limsup_{k \rightarrow +\infty} h(\nu_k) \leq \beta h(\mu_1) + (1 - \beta)\alpha.$$

The Main Theorem implies Buzzi-Crovisier-Sarig statement. Indeed, either $\lim_k \lambda^+(\nu_k) = \int \phi d\hat{\mu} \leq \frac{\lambda^+(f)}{r}$ and we get by Ruelle inequality, $\limsup_k h(\nu_k) \leq \frac{\lambda^+(f)}{r}$ or there exists $\alpha \in \left] \frac{\lambda^+(f)}{r}, \min \left(\int \phi d\hat{\mu}, \frac{\lambda^+(f)}{r-1} \right) \right[$. By applying our Main Theorem with respect to α , there is a decomposition $\hat{\mu} = (1 - \beta_\alpha)\hat{\mu}_{0,\alpha} + \beta_\alpha\hat{\mu}_{1,\alpha}^+$ satisfying $\limsup_{k \rightarrow +\infty} h(\nu_k) \leq \beta_\alpha h(\mu_{1,\alpha}) + (1 - \beta_\alpha)\alpha$. But it follows from the proofs that $\beta_\alpha\mu_{1,\alpha}$ is a component of $\beta\mu_1$ with β and μ_1 being as in Buzzi-Crovisier-Sarig's statement (see Remark 6). In particular $\beta_\alpha h(\mu_{1,\alpha}) \leq \beta h(\mu_1)$, therefore $\limsup_{k \rightarrow +\infty} h(\nu_k) \leq \beta h(\mu_1) + \frac{\lambda^+(f) + \lambda^+(f-1)}{r-1}$. In Theorem C [10], the authors also proved $\int \phi d\hat{\mu}_0 = 0$ whenever $\beta \neq 1$. Therefore we get here $(1 - \beta_\alpha) \int \phi d\hat{\mu}_{0,\alpha} \geq (1 - \beta) \int \phi d\hat{\mu}_0 = 0$, then $\int \phi d\hat{\mu}_{0,\alpha} \geq 0$. But maybe we could have $\int \phi d\hat{\mu}_{0,\alpha} > 0$.

Corollary 1 (Existence of maximal measures and entropic continuity of Lyapunov exponents). *Let f be a C^r , with $r > 1$, surface diffeomorphism satisfying $h_{top}(f) > \frac{\lambda^+(f)}{r}$.*

Then f admits a measure of maximal entropy. More precisely, if $(\nu_k)_k$ is a sequence of ergodic measures converging to μ with $\lim_k h(\nu_k) = h_{top}(f)$, then

- $h(\mu) = h_{top}(f)$,
- $\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)$.

It was proved in [9] that any C^r surface diffeomorphism satisfying $h_{top}(f) > \frac{\lambda^+(f)}{r}$ admits at most finitely many ergodic measures of maximal entropy. On the other hand, J. Buzzi has built examples of C^r surface diffeomorphisms for any $+\infty > r > 1$ with $\frac{h_{top}(f)}{\lambda^+(f)}$ arbitrarily close to $1/r$ without a measure of maximal entropy [7]. Such results were already known for interval maps [3, 6, 8].

Proof. We consider the constant sequence of diffeomorphisms equal to f . By taking a subsequence, we can assume that $(\hat{\nu}_k^+)_k$ is converging to a lift $\hat{\mu}$ of μ . By using the notations of the Main Theorem with $h_{top}(f) > \alpha > \frac{\lambda^+(f)}{r}$, we have

$$\begin{aligned} h_{top}(f) &= \lim_{k \rightarrow +\infty} h(\nu_k), \\ &\leq \beta h(\mu_1) + (1 - \beta)\alpha, \\ &\leq \beta h_{top}(f) + (1 - \beta)\alpha, \\ (1 - \beta)h_{top}(f) &\leq (1 - \beta)\alpha. \end{aligned}$$

But $h_{top}(f) > \alpha$, therefore $\beta = 1$, i.e. $\hat{\mu}_1^+ = \hat{\mu}$ and $\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)$. Moreover $h_{top}(f) = \lim_{k \rightarrow +\infty} h(\nu_k) \leq \beta h(\mu_1) + (1 - \beta)\alpha = h(\mu)$. Consequently μ is a measure of maximal entropy of f . □

Corollary 2 (Continuity of topological entropy and maximal measures). *Let $(f_k)_k$ be a sequence of C^r , with $r > 1$, surface diffeomorphisms converging C^r weakly to a diffeomorphism*

f with $h_{top}(f) \geq \frac{\lambda^+(f)}{r}$.

Then

$$h_{top}(f) = \lim_k h_{top}(f_k).$$

Moreover if $h_{top}(f) > \frac{\lambda^+(f)}{r}$ and ν_k is a maximal measure of f_k for large k , then any limit measure of $(\nu_k)_k$ for the weak-* topology is a maximal measure of f .

Proof. By Katok's horseshoes theorem [14], the topological entropy is lower semi-continuous for the \mathcal{C}^1 topology on the set of \mathcal{C}^r surface diffeomorphisms. Therefore it is enough to show the upper semi-continuity.

By the variational principle there is a sequence of probability measures $(\nu_k)_{k \in K}$, $K \subset \mathbb{N}$ with $\#K = \infty$, such that :

- ν_k is an ergodic f_k -invariant measure for each k ,
- $\lim_{k \in K} h(\nu_k) = \limsup_{k \in \mathbb{N}} h_{top}(f_k)$.

By extracting a subsequence we can assume $(\nu_k^+)_k$ is converging to a F -invariant measure $\hat{\mu}$ in the weak-* topology. We can then apply the Main Theorem for any $\alpha > \frac{\lambda^+(f)}{r}$ to get for some f -invariant measures μ_1, μ_0 and $\beta \in [0, 1]$ (depending on α) with $\mu = (1 - \beta)\mu_0 + \beta\mu_1$:

$$(1.1) \quad \begin{aligned} \limsup_k h_{top}(f_k) &= \lim_k h(\nu_k), \\ &\leq \beta h(\mu_1) + (1 - \beta)\alpha, \\ &\leq \beta h_{top}(f) + (1 - \beta)\alpha, \\ &\leq \max(h_{top}(f), \alpha). \end{aligned}$$

By letting α go to $\frac{\lambda^+(f)}{r}$ we get

$$\limsup_k h_{top}(f_k) \leq h_{top}(f).$$

If $h_{top}(f) > \frac{\lambda^+(f)}{r}$, we can fix $\alpha \in \left] \frac{\lambda^+(f)}{r}, h_{top}(f) \right[$ and the inequalities (1.1) may be then rewritten as follows :

$$\begin{aligned} \limsup_k h_{top}(f_k) &\leq \beta h(\mu_1) + (1 - \beta)\alpha, \\ &\leq h_{top}(f). \end{aligned}$$

By the lower semi-continuity of the topological entropy, we have $h_{top}(f) \leq \limsup_k h_{top}(f_k)$ and therefore these inequalities are equalities, which implies $\beta = 1$, then $\mu_1 = \mu$, and $h(\mu) = h_{top}(f)$. \square

The corresponding result was proved for interval maps in [5] by using a different method. We also refer to [5] for counterexamples of the upper semi-continuity property for interval maps f with $h_{top}(f) < \frac{\lambda^+(f)}{r}$. Finally, in [7], the author built, for any $r > 1$, a \mathcal{C}^r surface diffeomorphism f with $\limsup_{g \xrightarrow{\mathcal{C}^r} f} h_{top}(g) = \frac{\lambda^+(f)}{r} > h_{top}(f) = 0$. We recall also that upper semi-continuity of the topological entropy in the \mathcal{C}^∞ topology was established in any dimension by Y. Yomdin in [18].

Newhouse proved that for a \mathcal{C}^∞ system (\mathbf{M}, f) , the entropy function $h : \mathcal{M}(\mathbf{M}, f) \rightarrow \mathbb{R}^+$ is an upper semi-continuous function on the set $\mathcal{M}(\mathbf{M}, f)$ of f -invariant probability measure.

It follows from our Main Theorem, that the entropy function is upper semi-continuous at ergodic measures with entropy larger than $\frac{\lambda^+(f)}{r}$ for a C^r , $r > 1$, surface diffeomorphism f .

Corollary 3 (Upper semi-continuity of the entropy function at ergodic measures with large entropy). *Let $f : \mathbf{M} \circlearrowleft$ be a C^r , $r > 1$, surface diffeomorphism.*

Then for any ergodic measure μ with $h(\mu) \geq \frac{\lambda^+(f)}{r}$, we have

$$\limsup_{\nu \rightarrow \mu} h(\nu) \leq h(\mu).$$

Proof. By continuity of the ergodic decomposition at ergodic measures and by harmonicity of the entropy function, we have for any ergodic measure μ (see e.g. Lemma 8.2.13 in [12]):

$$\limsup_{\nu \text{ ergodic}, \nu \rightarrow \mu} h(\nu) = \limsup_{\nu \rightarrow \mu} h(\mu).$$

Let $(\nu_k)_{k \in \mathbb{N}}$ be a sequence of ergodic f -invariant measures with $\lim_k h(\nu_k) = \limsup_{\nu \rightarrow \mu} h(\nu)$. By extracting a subsequence we can assume that the sequence $(\hat{\nu}_k^+)_{k \in \mathbb{N}}$ is converging to some lift $\hat{\mu}$ of μ . Take α with $\alpha > \frac{\lambda^+(f)}{r}$. Then, in the decomposition $\hat{\mu} = (1 - \beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$ given by the Main Theorem, we have $\mu_1 = \mu_0$ by ergodicity of μ . Therefore

$$\lim_k h(\nu_k) \leq \beta h(\mu) + (1 - \beta)\alpha.$$

By letting α go to $\frac{\lambda^+(f)}{r}$ we get

$$\begin{aligned} \lim_k h(\nu_k) &\leq \beta h(\mu) + (1 - \beta) \frac{\lambda^+(f)}{r}, \\ &\leq h(\mu). \end{aligned}$$

□

2. MAIN STEPS OF THE PROOF

We follow the strategy of the proof of [10]. We point out below the main differences:

- *Geometric and neutral empirical component.* For $\lambda^+(\nu_k) > \frac{\lambda^+(f)}{r}$ we split the orbit of a ν_k -typical point x into two parts. We consider the empirical measures from x at times lying between to M -close consecutive times where the unstable manifold has a "bounded geometry". We take their limit in k , then in M . In this way we get an invariant component of $\hat{\mu}$. In [10] the authors consider rather such empirical measures for α -hyperbolic times and then take the limit when α go to zero.
- *Entropy computations.* To compute the asymptotic entropy of the ν_k 's, we use the static entropy w.r.t. partitions and its conditional version. Instead the authors in [10] used Katok's like formulas.
- *C^r Reparametrizations.* Finally we use here reparametrization methods from [4] and [2] respectively rather than Yomdin's reparametrizations of the projective action F as done in [10]. This is the principal difference with [10].

2.1. Empirical measures. Let (X, T) be a topological system. For a fixed Borel measurable subset G of X we let $E(x) = E_G(x)$ be the set of times of visits in G from x :

$$E(x) = \{n \in \mathbb{Z}, T^n x \in G\}.$$

When $a < b$ are two consecutive times in $E(x)$, then $[a, b[$ is called a *neutral block* (by following the terminology of [9]). For all M we let then

$$E^M(x) = \bigcup_{a < b \in E(x), |a-b| \leq M} [a, b[.$$

The complement of $E^M(x)$ is made of disjoint neutral blocks of length larger than M . We consider the associated empirical measures :

$$\forall n, \mu_{x,n}^M = \frac{1}{n} \sum_{k \in E^M(x) \cap [0, n[} \delta_{T^k x}.$$

Let ν be an ergodic measure. We denote by χ^M the indicator function of $\{x, 0 \in E^M(x)\}$. By the Birkhoff ergodic theorem, there is a set \mathbf{G} of full ν -measure such that the empirical measures $(\mu_{x,n}^M)_n$ are converging for any $x \in \mathbf{G}$ and any $M \in \mathbb{N}^*$ to $\xi^M := \chi^M \nu$ in the weak-* topology. We also let $\eta^M = \nu - \xi^M$. Moreover we put $\beta_M = \int \chi^M d\nu$, then $\xi^M = \beta_M \cdot \underline{\xi}^M$ when $\beta_M \neq 0$ and $\eta^M = (1 - \beta_M) \cdot \underline{\eta}^M$ when $\beta_M \neq 1$ with $\underline{\xi}^M, \underline{\eta}^M$ being thus probability measures. Following partially [10], the measures ξ^M and η^M are respectively called here the *geometric and neutral components* of ν . In general these measures are not T -invariant. From the definition one easily checks that $\xi^M \geq \xi^N$ for $M \geq N$.

2.2. Pesin unstable manifolds. We consider a smooth compact riemannian manifold $(\mathbf{M}, \|\cdot\|)$. Let \exp_x be the exponential map at x and let R_{inj} be the radius of injectivity of $(\mathbf{M}, \|\cdot\|)$. We consider the distance d on \mathbf{M} induced by the Riemannian structure. Let $f : \mathbf{M} \circlearrowleft$ be a \mathcal{C}^r , $r > 1$, surface diffeomorphism. We denote by \mathcal{R} the set of Lyapunov regular points with $\lambda^+(x) > 0 > \lambda^-(x)$. For $x \in \mathbf{M}$ we let $W^u(x)$ denote the unstable manifold at x :

$$W^u(x) := \left\{ y \in \mathbf{M}, \lim_n \frac{1}{n} \log d(f^n x, f^n y) < 0 \right\}.$$

By Pesin unstable manifold theorem, the set $W^u(x)$ for $x \in \mathcal{R}$ is a \mathcal{C}^r submanifold tangent to $\mathcal{E}_+(x)$ at x .

For $x \in \mathcal{R}$, we let \hat{x} be the vector in $\mathbb{P}T\mathbf{M}$ associated to the unstable Oseledets bundle $\mathcal{E}_+(x)$. For $\delta > 0$ the point x is said *δ -hyperbolic* with respect to ϕ (resp. ψ) when we have $\phi_l(F^{-l}\hat{x}) \geq \delta l$ (resp. $\psi_l(F^{-l}\hat{x}) \geq \delta l$) for all $l > 0$. Note that if x is δ -hyperbolic with respect to ψ then it is δ -hyperbolic with respect to ϕ .

Let ν be an ergodic measure with $\lambda^+(\nu) - \frac{\log^+ \|df\|_\infty}{r} > \delta > 0 > \lambda^-(\nu)$. By applying the Ergodic Maximal Inequality (see e.g. Theorem 1.1 in [1]) to the measure preserving system $(F^{-1}, \hat{\nu}^+)$ with the observable $\psi^\delta = \delta - \psi \circ F^{-1}$, we get with $A_\delta = \{\hat{x} \in \mathbb{P}T\mathbf{M}, \exists k \geq 0 \text{ s.t. } \sum_{l=0}^k \psi^\delta(F^{-l}\hat{x}) > 0\}$:

$$\int_{A_\delta} \psi^\delta d\hat{\nu}^+ \geq 0.$$

But the set $H_\delta := \{\hat{x} \in \mathbb{P}T\mathbf{M}, \forall l > 0 \psi_l(F^{-l}\hat{x}) \geq \delta l\}$ of δ -hyperbolic points w.r.t. ψ is just the complement set $\mathbb{P}T\mathbf{M} \setminus A_\delta$ of A_δ . Therefore $\int_{H_\delta} (\delta - \psi \circ F^{-1}) d\hat{\nu}^+ \leq \int (\delta - \psi \circ F^{-1}) d\hat{\nu}^+ = \delta - \lambda^+(\nu) + \frac{1}{r} \int \frac{\log^+ \|df\|}{r} d\nu < 0$. In particular we have $\hat{\nu}^+(H_\delta) > 0$.

A point $x \in \mathcal{R}$ is said to have κ -bounded geometry for $\kappa > 0$ when $\exp_x^{-1} W^u(x)$ contains the graph of an κ -admissible map at x , which is defined as a 1-Lipschitz map $f : I \rightarrow \mathcal{E}_+(x)^\perp \subset T_x \mathbf{M}$, with I being an interval of $\mathcal{E}_+(x)$ containing 0 with length κ . We let G_κ be the subset of points in \mathcal{R} with κ -bounded geometry.

Lemma 1. *The set G_κ is Borel measurable.*

Proof. For $x \in \mathcal{R}$ we have $W^u(x) = \bigcup_{n \in \mathbb{N}} f^n W_{loc}^u(f^{-n}x)$ with W_{loc}^u being the Pesin unstable local manifold at x . The sequence $(f^{-n} W_{loc}^u(f^n x))_n$ is increasing in n for the inclusion. Therefore, if we let G_κ^n be the subset of points x in G_κ , such that $\exp_x^{-1} f^n W_{loc}^u(f^{-n}x)$ contains the graph of a κ -admissible map, then we have

$$G_\kappa = \bigcup_n G_\kappa^n.$$

There are closed subsets, $(\mathcal{R}_l)_{l \in \mathbb{N}}$, called the Pesin blocks, such that $\mathcal{R} = \bigcup_l \mathcal{R}_l$ and $x \mapsto W_{loc}^u(x)$ is continuous on \mathcal{R}_l for each l (see e.g. [16]). Let $(x_p)_p$ be sequence in $G_\kappa^n \cap \mathcal{R}_l$ which converges to $x \in \mathcal{R}_l$. By extracting a subsequence we can assume that the associated sequence of κ -admissible maps f_p at x_p is converging pointwisely to a κ -admissible map at x , when p goes to infinity. In particular $G_\kappa^n \cap \mathcal{R}_l$ is a closed set and therefore $G_\kappa = \bigcup_{l,n} (G_\kappa^n \cap \mathcal{R}_l)$ is Borel measurable. \square

2.3. Entropy of conditional measures. We consider an ergodic hyperbolic measure ν , i.e an ergodic measure with $\nu(\mathcal{R}) = 1$. A measurable partition ζ is *subordinated* to the Pesin unstable local lamination W_{loc}^u of ν if the atom $\zeta(x)$ of ζ containing x is a neighborhood of x inside the curve $W_{loc}^u(x)$ and $f^{-1}\zeta \succ \zeta$. By Rokhlin's disintegration theorem, there are a measurable set Z of full ν -measure and probability measures ν_x on $\zeta(x)$ for $x \in Z$, called the *conditional measures* on unstable manifolds, satisfying $\nu = \int \nu_x d\nu(x)$. Moreover $\nu_y = \nu_x$ for $x, y \in Z$ in the same atom of ζ . Ledrappier and Young [13] proved the existence of such subordinated measurable partitions and showed that for ν -a.e. x , we have with $B_n(x, \rho)$ being the Bowen ball $B_n(x, \rho) := \bigcap_{0 \leq k < n} f^{-k} B(f^k x, \rho)$ (where $B(f^k x, \rho)$ denotes the ball for d at $f^k x$ with radius ρ):

$$(2.1) \quad \lim_{\rho \rightarrow 0} \liminf_n -\frac{1}{n} \log \nu_x(B_n(x, \rho)) = h(\nu).$$

Fix an error term $\iota > 0$ depending[‡] on ν . There is $\rho > 0$ and a measurable set $\mathbf{F} \subset Z \cap \mathcal{R}$ with $\nu(\mathbf{F}) > 0$ such that

$$\forall x \in \mathbf{F}, \liminf_n -\frac{1}{n} \log \nu_x(B_n(x, \rho)) \geq h(\nu) - \iota.$$

We fix $x_* \in \mathbf{F}$ with $\nu_{x_*}(\mathbf{F}) > 0$ and we let $\zeta = \frac{\nu_{x_*}(\cdot)}{\nu_{x_*}(\mathbf{F})}$ be the probability measure induced by ν_{x_*} on \mathbf{F} . Observe that $\nu_x = \nu_{x_*}$ for ζ a.e. x . We let D be the C^r curve given by the Pesin local unstable manifold $W_{loc}^u(x_*)$ at x_* . For a finite measurable partition P and a Borel probability measure μ we let $H_\mu(P)$ be the static entropy, $H_\mu(P) = -\sum_{A \in P} \mu(A) \log \mu(A)$. Moreover we let $P^n = \bigvee_{k=0}^{n-1} f^{-k} P$ be the n -iterated partition, $n \in \mathbb{N}$. We also denote by P_x^n the atom of P^n containing the point $x \in \mathbf{M}$.

[‡]In the proof of the Main Theorem we will take $\iota = \iota(\nu_k) \xrightarrow{k} 0$ for the converging sequence of ergodic measures $(\nu_k)_k$.

Lemma 2. *For any (finite measurable) partition P with diameter less than ρ , we have*

$$\liminf_n \frac{1}{n} H_\zeta(P^n) \geq h(\nu) - \iota.$$

Proof.

$$\begin{aligned} \liminf_n \frac{1}{n} H_\zeta(P^n) &= \liminf_n \int -\frac{1}{n} \log \zeta(P_x^n) d\zeta(x), \text{ by the definition of } H_\zeta, \\ &\geq \int \liminf_n -\frac{1}{n} \log \zeta(P_x^n) d\zeta(x), \text{ by Fatou's Lemma,} \\ &\geq \int \liminf_n -\frac{1}{n} \log \nu_{x^*}(P_x^n) d\zeta(x), \text{ by the definition of } \zeta, \\ &\geq \int \liminf_n -\frac{1}{n} \log \nu_x(P_x^n) d\zeta(x), \text{ as } \nu_x = \nu_{x^*} \text{ for } \zeta \text{ a.e. } x, \\ &\geq \int \liminf_n -\frac{1}{n} \log \nu_x(B_n(x, \rho)) d\zeta(x), \text{ as } \text{diam}(P) < \rho, \\ &\geq h(\nu) - \iota, \text{ by the choice of } \mathbf{F}. \end{aligned}$$

□

2.4. Entropy splitting of the neutral and the geometric component. The natural projection from $\mathbb{P}\mathbf{T}\mathbf{M}$ to \mathbf{M} is denoted by π . We consider a distance \hat{d} on the projective tangent bundle $\mathbb{P}\mathbf{T}\mathbf{M}$, such that $\hat{d}(\hat{x}, \hat{y}) \geq d(\pi\hat{x}, \pi\hat{y})$ for all $\hat{x}, \hat{y} \in \mathbb{P}\mathbf{T}\mathbf{M}$. In this section we split the entropy contribution of the neutral and geometric components $\hat{\eta}^M$ and $\hat{\xi}^M$ of the ergodic F -invariant measure $\hat{\nu}^+$ associated to $G = H_\delta \cap \pi^{-1}G_\kappa \subset \mathbb{P}\mathbf{T}\mathbf{M}$, where the parameters δ and κ will be fixed later on. We also consider their projections η^M and ξ^M on \mathbf{M} . Let \mathbf{F} and P as in the previous subsection. Without loss of generality we can assume

- $\{\hat{x}, x \in \mathbf{F}\} \subset \mathbf{G}$ with \mathbf{G} being the set of full $\hat{\nu}^+$ -measure of points \hat{x} such that the empirical measures $\mu_{\hat{x}, n}^M$ are converging to $\hat{\xi}^M$ for any M (see Subsection 2.1),
- the boundary of P has zero ν -measure,
- for any $M \in \mathbb{N}$ and for any continuous function $\varphi : \mathbb{P}\mathbf{T}\mathbf{M} \rightarrow \mathbb{R}$,

$$(2.2) \quad \frac{1}{n} \sum_{k \in E^M(x) \cap [1, n[} \varphi(F^k \hat{x}) \xrightarrow{n} \int \varphi d\hat{\xi}^M \text{ uniformly in } x \in \mathbf{F}.$$

- for any continuous function $\vartheta : \mathbf{M} \rightarrow \mathbb{R}$,

$$(2.3) \quad \frac{1}{n} \sum_{k \in [1, n[} \vartheta(f^k x) \xrightarrow{n} \int \vartheta d\nu \text{ uniformly in } x \in \mathbf{F}.$$

Let us detail the proof of the third item. If $\mathcal{F} = (\varphi_k)_{k \in \mathbb{N}}$ is a dense countable family in the set $\mathcal{C}^0(\mathbb{P}\mathbf{T}\mathbf{M}, \mathbb{R})$ of real continuous functions on $\mathbb{P}\mathbf{T}\mathbf{M}$ endowed with the supremum norm $\|\cdot\|_\infty$, then for all k, M , by Egorov's theorem applied to the pointwise converging sequence $(f_n : \mathbf{F} \rightarrow \mathbb{R})_n = \left(x \mapsto \int \varphi_k d\mu_{\hat{x}, n}^M\right)_n$, there is a subset \mathbf{F}_k^M of \mathbf{F} with $\nu(\mathbf{F}_k^M) > \nu(\mathbf{F}) \left(1 - \frac{1}{2^{k+M+3}}\right)$ such that $\int \varphi_k d\mu_{\hat{x}, n}^M$ converges to $\int \varphi_k d\hat{\xi}^M$ uniformly in $x \in \mathbf{F}_k^M$. Let $\mathbf{F}' = \bigcap_{k, M} \mathbf{F}_k^M$. We have $\nu(\mathbf{F}') \geq \frac{\nu(\mathbf{F})}{2}$. Then, if $\varphi \in \mathcal{C}^0(\mathbb{P}\mathbf{T}\mathbf{M}, \mathbb{R})$, we may find for any $\epsilon > 0$ a function $\varphi_k \in \mathcal{F}$

with $\|\varphi - \varphi_k\|_\infty < \epsilon$. Let $M \in \mathbb{N}$. Take $N = N_\epsilon^{k,M}$ such that $|\int \varphi_k d\mu_{\hat{x},n}^M - \int \varphi_k d\xi^M| < \epsilon$ for $n > N$ and for all $x \in \mathbf{F}_k^M$. In particular for all $x \in \mathbf{F}'$ we have for $n > N$

$$\begin{aligned} \left| \int \varphi d\mu_{\hat{x},n}^M - \int \varphi d\xi^M \right| &\leq \left| \int \varphi_k d\mu_{\hat{x},n}^M - \int \varphi d\mu_{\hat{x},n}^M \right| + \left| \int \varphi_k d\mu_{\hat{x},n}^M - \int \varphi_k d\xi^M \right| \\ &\quad + \left| \int \varphi_k d\xi^M - \int \varphi d\xi^M \right|, \\ &\leq 2\|\varphi - \varphi_k\|_\infty + \left| \int \varphi_k d\mu_{\hat{x},n}^M - \int \varphi_k d\xi^M \right|, \\ &< 3\epsilon. \end{aligned}$$

This proves (2.2) by taking \mathbf{F}' in the place of \mathbf{F} . One proves similarly (2.3).

Fix now M . For each $n \in \mathbb{N}$ and $x \in \mathbf{F}$ we let $E_n(x) = E(\hat{x}) \cap [0, n[$ and $E_n^M(x) = E^M(\hat{x}) \cap [0, n[$. We also let \mathbf{E}_n^M be the partition of \mathbf{F} with atoms $A_E := \{x \in D, E_n^M(x) = E\}$ for $E \subset [0, n[$. Given a partition Q of $\mathbb{P}\mathbf{T}\mathbf{M}$, we also let $Q^{\mathbf{E}_n^M}$ be the partition of $\hat{\mathbf{F}} := \{\hat{x}, x \in \mathbf{F} \cap D\}$ finer than $\pi^{-1}\mathbf{E}_n^M$ with atoms $\{\hat{x} \in \hat{\mathbf{F}}, E_n^M(x) = E$ and $\forall k \in E, F^k \hat{x} \in Q_k\}$ for $E \subset [0, n[$ and $(Q_k)_{k \in E} \in Q^E$. We let ∂Q be the boundary of the partition Q , which is the union of the boundaries of its atoms. For a measure η and a subset A of \mathbf{M} with $\eta(A) > 0$ we denote by $\eta_A = \frac{\eta(A \cap \cdot)}{\eta(A)}$ the induced probability measure on A . Moreover, for two sets A, B we let $A \Delta B$ denote the symmetric difference of A and B , i.e. $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Finally, let $H :]0, 1[\rightarrow \mathbb{R}^+$ be the map $t \mapsto -t \log t - (1-t) \log(1-t)$. Recall that $\hat{\zeta}^+$ is the lift of ζ on $\mathbb{P}\mathbf{T}\mathbf{M}$ to the unstable Oseledets bundle (with ζ as in Subsection 2.3).

Lemma 3. *For any finite partition Q and any $m \in \mathbb{N}^*$ with $\hat{\xi}^M(\partial Q^m) = 0$ we have*

$$(2.4) \quad h(\nu) \leq \beta_M \frac{1}{m} H_{\hat{\xi}^M}(Q^m) + \limsup_n \frac{1}{n} H_{\hat{\zeta}^+}(\pi^{-1} P^n | Q^{\mathbf{E}_n^M}) + H(2/M) + \frac{12 \log \#Q}{M} + \iota.$$

Before the proof of Lemma 3, we first recall a technical lemma from [2].

Lemma 4 (Lemma 6 in [2]). *Let (X, T) be a topological system. Let μ be a Borel probability measure on X and let E be a finite subset of \mathbb{N} . For any finite partition Q of X , we have with $\mu^E := \frac{1}{\#E} \sum_{k \in E} T_*^k \mu$ and $Q^E := \bigvee_{k \in E} T^{-k} Q$:*

$$\frac{1}{\#E} H_\mu(Q^E) \leq \frac{1}{m} H_{\mu_E}(Q^m) + 6m \frac{\#(E+1)\Delta E}{\#E} \log \#Q.$$

Proof of Lemma 3. As the complement of $E_n^M(x)$ is the disjoint union of neutral blocks with length larger than M , there are at most $A_n^M = \sum_{k=0}^{[2n/M]+1} \binom{n}{k}$ possible values for $E_n^M(x)$ so that

$$\begin{aligned} \frac{1}{n} H_\zeta(P^n) &= \frac{1}{n} H_\zeta(P^n | \mathbf{E}_n^M) + H_\zeta(\mathbf{E}_n^M), \\ &\leq \frac{1}{n} H_\zeta(P^n | \mathbf{E}_n^M) + \log A_n^M, \\ \liminf_n \frac{1}{n} H_\zeta(P^n) &\leq \limsup_n \frac{1}{n} H_\zeta(P^n | \mathbf{E}_n^M) + H(2/M) \text{ by using Stirling's formula.} \end{aligned}$$

Moreover

$$\begin{aligned} \frac{1}{n}H_\zeta(P^n|\mathbf{E}_n^M) &= \frac{1}{n}H_{\hat{\zeta}_+}(\pi^{-1}P^n|\pi^{-1}\mathbf{E}_n^M), \\ &\leq \frac{1}{n}H_{\hat{\zeta}_+}(Q^{\mathbf{E}_n^M}|\pi^{-1}\mathbf{E}_n^M) + \frac{1}{n}H_{\hat{\zeta}_+}(\pi^{-1}P^n|Q^{\mathbf{E}_n^M}). \end{aligned}$$

For $E \subset [0, n[$ we let $\hat{\zeta}_{E,n}^+ = \frac{n}{\#E} \int \mu_{\hat{x},n}^M d\zeta_{A_E}(x)$, which may be also written as $\left(\hat{\zeta}_{\pi^{-1}A_E}^+\right)^E$ by using the notations of Lemma 4. By Lemma 4 applied to the system $(\mathbb{P}\mathbf{T}\mathbf{M}, F)$ and the measures $\mu := \hat{\zeta}_{\pi^{-1}A_E}^+$ for $A_E \in \mathbf{E}_n^M$ we have for all $n > m \in \mathbb{N}^*$:

$$\begin{aligned} H_{\hat{\zeta}_+}(Q^{\mathbf{E}_n^M}|\pi^{-1}\mathbf{E}_n^M) &= \sum_E \zeta(A_E)H_{\hat{\zeta}_{\pi^{-1}A_E}^+}(Q^E), \\ &\leq \sum_E \zeta(A_E)\#E \left(\frac{1}{m}H_{\hat{\zeta}_{E,n}^+}(Q^m) + 6m \frac{\#(E+1)\Delta E}{\#E} \log \#Q \right). \end{aligned}$$

Recall again that if $E = E_n^M(x)$ for some x then the complement set of E in $[1, n[$ is made of neutral blocks of length larger than M , therefore $\#(E+1)\Delta E \leq \frac{2M}{n}$. Moreover it follows from $\xi^M(\partial Q^m) = 0$ and (2.2), that $\mu_{\hat{x},n}^M(A^m)$ for $A^m \in Q^m$ and $\#E_n^M(x)/n$ are converging to $\hat{\xi}^M(A^m)$ and β_M respectively uniformly in $x \in \mathbf{F}$ when n goes to infinity. Then we get by taking the limit in n :

$$\begin{aligned} \limsup_n \frac{1}{n}H_{\hat{\zeta}_+}(Q^{\mathbf{E}_n^M}|\pi^{-1}\mathbf{E}_n^M) &\leq \beta_M \frac{1}{m}H_{\hat{\xi}^M}(Q^m) + \frac{12m \log \#Q}{M}, \\ h(\nu) - \iota &\leq \liminf_n \frac{1}{n}H_\zeta(P^n) \leq \beta_M \frac{1}{m}H_{\hat{\xi}^M}(Q^m) + \limsup_n \frac{1}{n}H_{\hat{\zeta}_+}(\pi^{-1}P^n|Q^{\mathbf{E}_n^M}) \\ &\quad + H(2/M) + \frac{12m \log \#Q}{M}. \end{aligned}$$

□

2.5. Bounding the entropy of the neutral component. For a \mathcal{C}^1 diffeomorphism f on \mathbf{M} we put $C(f) := 2A_f H(A_f^{-1}) + \frac{\log^+ \|df\|_\infty}{r} + B_r$ with $A_f = \log^+ \|df\|_\infty + \log^+ \|df^{-1}\|_\infty + 1$ and a universal constant B_r depending only r precised later on. Clearly $f \mapsto C(f)$ is continuous in the \mathcal{C}^1 topology and $\frac{\lambda^+(f)}{r} = \lim_{\mathbb{N} \ni p \rightarrow +\infty} \frac{C(f^p)}{p}$ whenever $\lambda^+(f) > 0$ (indeed $A_{f^p} \xrightarrow{p} +\infty$, therefore $H(A_{f^p}^{-1}) \xrightarrow{p} 0$). In particular, if $\frac{\lambda^+(f)}{r} < \alpha$ and $f_k \xrightarrow{k} f$ in the \mathcal{C}^1 topology, then there is p with $\lim_k \frac{C(f_k^p)}{p} < \alpha$.

In this section we consider the empirical measures associated to an ergodic hyperbolic measure ν with $\lambda^+(\nu) > \frac{\log \|df\|_\infty}{r} + \delta$, $\delta > 0$. Without loss of generality we can assume $\delta < \frac{r-1}{r} \log 2$. Then as observed in Subsection 2.2 we have $\hat{\nu}^+(H_\delta) > 0$. For $x \in \mathcal{R}$ we let $m_n(x) = \max\{k < n, F^k \hat{x} \in H_\delta\}$. By a standard application of the ergodic theorem we have

$$\frac{m_n(x)}{n} \xrightarrow{n} 1 \text{ for } \nu \text{ a.e. } x.$$

By taking a smaller subset \mathbf{F} , we can assume the above convergence of m_n is uniform on \mathbf{F} and that $\sup_{x \in \mathbf{F}} \min\{k \leq n, F^k \hat{x} \in H_\delta\} \leq N$ for some positive integer N .

We bound the term $\limsup_n \frac{1}{n} H_{\hat{\zeta}_+}(\pi^{-1} P^n | Q_n^{E_n^M})$ in the right member of (2.4) Lemma 3, which corresponds to the local entropy contribution plus the entropy in the neutral part.

Lemma 5. *There is $\kappa > 0$ such that the empirical measures associated to $G := \pi^{-1} G_\kappa \cap H_\delta$ satisfy the following properties. For all $q, M \in \mathbb{N}^*$, there are $\epsilon_q > 0$ (depending only on $\|d^k(f^q)\|_\infty$, $2 \leq q \leq r$ §) and $\gamma_{q,M}(f) > 0$ with*

$$(2.5) \quad \forall K > 0 \limsup_q \limsup_M \left(\sup_f \{ \gamma_{q,M}(f) \mid \|df\|_\infty \vee \|df^{-1}\|_\infty < K \} \right) = 0$$

such that for any partition Q of $\mathbb{P}TM$ with diameter less than ϵ_q , we have:

$$\begin{aligned} \limsup_n \frac{1}{n} H_{\hat{\zeta}_+}(\pi^{-1} P^n | Q_n^{E_n^M}) &\leq (1 - \beta_M) C(f) \\ &\quad + \left(\log 2 + \frac{1}{r-1} \right) \left(\int \frac{\log^+ \|df^q\|}{q} d\zeta^M - \int \phi d\hat{\zeta}^M \right) \\ &\quad + \gamma_{q,M}(f). \end{aligned}$$

The proof of Lemma 5 appears after the statement of Proposition 4, which is a *semi-local Reparametrization Lemma*.

Proposition 4. *There is $\kappa > 0$ such that the empirical measures associated to $G := \pi^{-1} G_\kappa \cap H_\delta$ satisfy the following properties. For all $q \in \mathbb{N}^*$ there are $\epsilon_q > 0$ (depending only on $\|d^k(f^q)\|_\infty$, $2 \leq q \leq r$) and $\gamma_{q,M}(f) > 0$ with*

$$\forall K > 0 \limsup_q \limsup_M \left(\sup_f \{ \gamma_{q,M}(f) \mid \|df\|_\infty \vee \|df^{-1}\|_\infty < K \} \right) = 0$$

such that for any partition Q with diameter less than $\epsilon < \epsilon_q$, the following property holds for n large enough.

Any atom F_n of the partition $Q_n^{E_n^M}$ may be covered by a family Ψ_{F_n} of C^r curves $\psi : [-1, 1] \rightarrow \mathbf{M}$ satisfying $\|d(f^k \circ \psi)\|_\infty \leq 1$ for any $k = 0, \dots, n-1$, such that

$$\begin{aligned} \frac{1}{n} \log \#\Psi_{F_n} &\leq \left(1 - \frac{\#E_n^M}{n} \right) C(f) \\ &\quad + \left(\log 2 + \frac{1}{r-1} \right) \left(\int \frac{\log^+ \|d_x f^q\|_\epsilon}{q} d\zeta_{F_n}^M(x) - \int \phi d\hat{\zeta}_{F_n}^M \right) \\ &\quad + \gamma_{q,M}(f) + \tau_n, \end{aligned}$$

where $\lim_n \tau_n = 0$, $E_n^M = E_n^M(x)$ for $x \in F_n$, $\hat{\zeta}_{F_n}^M = \int \mu_{\hat{x},n}^M d\zeta_{F_n}^M(x)$ and $\zeta_{F_n}^M = \pi_* \hat{\zeta}_{F_n}^M$ its push-forward on \mathbf{M} .

The proof of Proposition 4 is given in the last section. Proposition 4 is very similar to the Reparametrization Lemma in [4]. Here we reparametrize an atom F_n of $Q_n^{E_n^M}$ instead of Q^n in [4].

§Here

$$\|d^k(f^q)\|_\infty = \sup_{\alpha \in \mathbb{N}^2, |\alpha|=k} \sup_{x,y} \left\| \partial_y^\alpha \left(\exp_{f(x)}^{-1} \circ f \circ \exp_x \right) (\cdot) \right\|_\infty$$

Proof of Lemma 5 assuming Proposition 4. We take $\kappa > 0$ and $\epsilon_q > 0$ as in Proposition 4. Observe that

$$H_{\hat{\zeta}^+}(\pi^{-1}P^n|Q^{\mathbb{E}_n^M}) \leq \sum_{F_n \in Q^{\mathbb{E}_n^M}} \hat{\zeta}^+(F_n) \log \#\{A^n \in P^n, \pi^{-1}(A^n) \cap \hat{F} \cap F_n \neq \emptyset\}.$$

As $\nu(\partial P) = 0$, for all $\gamma > 0$, there is $\chi > 0$ and a continuous function $\vartheta : \mathbf{M} \rightarrow \mathbb{R}^+$ equal to 1 on the χ -neighborhood ∂P^χ of ∂P satisfying $\int \vartheta d\nu < \gamma$. Then we have uniformly in $x \in \mathbf{F}$ by (2.3):

$$(2.6) \quad \limsup_n \frac{1}{n} \#\{0 \leq k < n, f^k x \in \partial P^\chi\} \leq \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \vartheta(f^k x) = \int \vartheta d\nu < \gamma.$$

Assume that for arbitrarily large n there is $F_n \in Q^{\mathbb{E}_n^M}$ and $\psi \in \Psi_{F_n}$ with $\#\{A^n \in P^n, A^n \cap \psi([-1, 1]) \cap \mathbf{F} \neq \emptyset\} > ([\chi^{-1}] + 1) \#P^{\gamma n}$. We reparametrize ψ on \mathbf{F} by $[\chi^{-1}] + 1$ affine contractions θ so that the length of $f^k \circ \psi \circ \theta$ is less than χ for all $0 \leq k < n$ and $(\psi \circ \theta)([-1, 1]) \cap \mathbf{F} \neq \emptyset$. Then we have $\#\{0 \leq k < n, \partial P \cap f^k \circ \psi \circ \theta([-1, 1]) \neq \emptyset\} > \gamma n$ for some θ . In particular we get $\#\{0 \leq k < n, f^k x \in \partial P^\chi\} > \gamma n$ for any $x \in \psi \circ \theta([-1, 1])$, which contradicts (2.6). Therefore we have

$$\limsup_n \sup_{F_n, \psi \in \Psi_{F_n}} \frac{1}{n} \log \#\{A^n \in P^n, A^n \cap \psi([-1, 1]) \cap \mathbf{F} \neq \emptyset\} = 0.$$

Together with Proposition 4 we get

$$\begin{aligned} \limsup_n \frac{1}{n} H_{\hat{\zeta}^+}(\pi^{-1}P^n|Q^{\mathbb{E}_n^M}) &\leq \limsup_n \sum_{F_n \in Q^{\mathbb{E}_n^M}} \hat{\zeta}^+(F_n) \frac{1}{n} \log \#\Psi_{F_n}, \\ &\leq \limsup_n \sum_{F_n \in Q^{\mathbb{E}_n^M}} \hat{\zeta}^+(F_n) \left(1 - \frac{\#E_n^M}{n}\right) C(f) + \\ &\quad + \limsup_n \sum_{F_n \in Q^{\mathbb{E}_n^M}} \hat{\zeta}^+(F_n) \left(\log 2 + \frac{1}{r-1}\right) \left(\int \frac{\log^+ \|df^q\|}{q} d\zeta_{F_n}^M - \int \phi d\hat{\zeta}_{F_n}^M\right) \\ &\quad + \gamma_{q,M}(f), \\ &\leq (1 - \beta_M)C(f) + \left(\log 2 + \frac{1}{r-1}\right) \left(\int \frac{\log^+ \|df^q\|}{q} d\xi^M - \int \phi d\hat{\xi}^M\right) + \gamma_{q,M}(f). \end{aligned}$$

This concludes the proof of Lemma 5. □

2.6. Proof of the Main Theorem. We first reduce the Main Theorem to the following statement.

Proposition 5. *Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of \mathcal{C}^r , with $r > 1$, surface diffeomorphisms converging \mathcal{C}^r weakly to a diffeomorphism f . Assume there is a sequence $(\hat{\nu}_k^+)_{k \in \mathbb{N}}$ of ergodic F_k -invariant measures converging to $\hat{\mu}$ with $\lim_k \lambda^+(\nu_k) > \frac{\log^+ \|df\|_\infty}{r}$.*

Then, there are F -invariant measures $\hat{\mu}_0$ and $\hat{\mu}_1^+$ with $\hat{\mu} = (1 - \beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$, $\beta \in [0, 1]$, such that:

$$\limsup_{k \rightarrow +\infty} h(\nu_k) \leq \beta h(\mu_1) + (1 - \beta)C(f).$$

Proof of the Main Theorem assuming Proposition 5. Let $(\hat{\nu}_k^+)_k$ be a sequence of ergodic F_k -invariant measures converging to $\hat{\mu}$.

As previously mentioned, for any $\alpha > \lambda^+(f)/r$ there is $p \in \mathbb{N}^*$ with $\alpha > \frac{C(f^p)}{p}$. We can also assume $\frac{\log \|df^p\|_\infty}{pr} < \alpha$. Let $\hat{\nu}_k^{+,p}$ be an ergodic component of $\hat{\nu}_k^+$ for F_k^p and let us denote by ν_k^p its push forward on \mathbf{M} . We have $h_{f_k^p}(\nu_k^p) = ph_{f_k}(\nu_k)$ for all k . By taking a subsequence we can assume that $(\hat{\nu}_k^{+,p})_k$ is converging. Its limit $\hat{\mu}^p$ satisfies $\frac{1}{p} \sum_{0 \leq l < p} F_*^l \hat{\mu}^p = \hat{\mu}$. If $\lim_k \lambda^+(\nu_k^p) < \frac{\log^+ \|df^p\|_\infty}{r} < p\alpha$, then by Ruelle's inequality we get

$$\begin{aligned} \limsup_{k \rightarrow +\infty} h_{f_k}(\nu_k) &= \limsup_{k \rightarrow +\infty} \frac{1}{p} h_{f_k^p}(\nu_k^p), \\ &\leq \lim_{k \rightarrow +\infty} \frac{1}{p} \lambda^+(\nu_k^p), \\ &\leq \alpha. \end{aligned}$$

This proves the Main Theorem with $\beta = 1$.

We consider then the case $\lim_k \lambda^+(\nu_k^p) > \frac{\log^+ \|df^p\|_\infty}{r}$. By applying Proposition 4 to the p -power systems, we get F^p -invariant measure $\hat{\mu}_0^p$ and $\hat{\mu}_1^{+,p}$ with $\hat{\mu}^p = (1 - \beta)\hat{\mu}_0^p + \beta\hat{\mu}_1^{+,p}$, $\beta \in [0, 1]$, such that we have with $\mu_1^p = \pi_* \hat{\mu}_1^{+,p}$:

$$\limsup_{k \rightarrow +\infty} h_{f_k^p}(\nu_k^p) \leq \beta h_{f^p}(\mu_1^p) + (1 - \beta)C(f^p).$$

But $h_{f^p}(\mu_1^p) = ph_f(\mu_1)$ with $\mu_1 = \frac{1}{p} \sum_{0 \leq l < p} f^l \mu_1^p$. One easily checks that $\hat{\mu}_1^+ = \frac{1}{p} \sum_{0 \leq l < p} F_*^l \hat{\mu}_1^{+,p}$. Moreover we have :

$$\begin{aligned} \limsup_{k \rightarrow +\infty} h_{f_k}(\nu_k) &= \limsup_{k \rightarrow +\infty} \frac{1}{p} h_{f_k^p}(\nu_k^p), \\ &\leq \beta \frac{1}{p} h_{f^p}(\mu_1^p) + (1 - \beta) \frac{C(f^p)}{p}, \\ &\leq \beta h_f(\mu_1) + (1 - \beta)\alpha. \end{aligned}$$

□

We show now Proposition 5 by using Lemma 5. Without loss of generality we can assume $\liminf_k h(\nu_k) > 0$. For μ a.e. x , we have $\lambda^-(x) \leq 0$. If not, some ergodic component $\tilde{\mu}$ of μ would have two positive Lyapunov exponents and therefore should be the periodic measure at a source S (see e.g. Proposition 4.4 in [17]). But then for large k the probability ν_k would give positive measure to the basin of attraction of the sink S for f^{-1} and therefore ν_k would be equal to $\tilde{\mu}$ contradicting $\liminf_k h(\nu_k) > 0$.

Let $\delta > 0$ with $\lim_k \lambda^+(\nu_k) > \frac{\log \|df\|_\infty}{r} + \delta$. Then take κ as in Lemma 5. We consider the empirical measures associated to $G = \pi^{-1}G_\kappa \cap H_\delta$. By a diagonal argument, there is a subsequence in k such that the geometric component $\hat{\xi}_k^M$ of $\hat{\nu}_k^+$ is converging to some $\hat{\xi}_\infty^M$ for all $M \in \mathbb{N}$. Let us also denote by β_M^∞ the limit in k of β_M^k . Then consider a subsequence in M such that $\hat{\xi}_\infty^M$ is converging to $\beta \hat{\mu}_1$ with $\beta = \lim_M \beta_M^\infty$. We also let $(1 - \beta)\hat{\mu}_0 = \hat{\mu} - \beta \hat{\mu}_1$. In this way, $\hat{\mu}_0$ and $\hat{\mu}_1$ are both probability measures.

Lemma 6. *The measures $\hat{\mu}_0$ and $\hat{\mu}_1$ satisfy the following properties:*

- $\hat{\mu}_1$ and $\hat{\mu}_0$ are F -invariant,

- $\lambda^+(x) \geq \delta$ for μ_1 -a.e. x and $\hat{\mu}_1 = \hat{\mu}_1^+$.

Proof. The neutral blocks in the complement set of $E^M(x)$ have length larger than M . Therefore for any continuous function $\varphi : \mathbb{P}TM \rightarrow \mathbb{R}$ and for any k , we have

$$\left| \int \varphi d\hat{\xi}_k^M - \int \varphi \circ F d\hat{\xi}_k^M \right| \leq \frac{2 \sup_{\hat{x}} |\varphi(\hat{x})|}{M}.$$

Letting k , then M go to infinity, we get $\int \varphi d\hat{\mu}_1 = \int \varphi \circ F d\hat{\mu}_1$, i.e. $\hat{\mu}_1$ is F -invariant.

We let K_M be the compact subset of $\mathbb{P}TM$ given by $K_M = \{\hat{x} \in \mathbb{P}TM, \exists 1 \leq m \leq M \text{ } \phi_m(\hat{x}) \geq m\delta\}$. Let $\hat{x} \in \mathbf{G}_k$, where \mathbf{G}_k is the set where the empirical measures are converging to $\hat{\xi}_k^M$ (see Subsection 2.1). Observe that

$$(2.7) \quad \lim_n \mu_{\hat{x},n}^M(K_M) = \hat{\xi}_k^M(K_M) = \hat{\xi}_k^M(\mathbb{P}TM).$$

Indeed for any $k \in E^M(\hat{x})$ there is $1 \leq m \leq M$ with $F^m(F^k \hat{x}) \in G \subset H_\delta$. Moreover, as already mentioned, δ -hyperbolic points w.r.t. ψ are δ -hyperbolic w.r.t. ϕ . Therefore $\phi_m(F^k \hat{x}) \geq m\delta$. Consequently we have $\lim_n \mu_{\hat{x},n}^M(K_M) = \lim_n \mu_{\hat{x},n}^M(\mathbb{P}TM) = \xi_k^M(\mathbb{P}TM)$. The set K_M being compact in $\mathbb{P}TM$, we get $\xi_k^M(K_M) \geq \lim_n \mu_{\hat{x},n}^M(K_M)$ and (2.7) follows.

Also we have $\hat{\xi}_\infty^M(K_M) \geq \limsup_k \hat{\xi}_k^M(K_M) = \limsup_k \hat{\xi}_k^M(\mathbb{P}TM) = \beta_M^\infty$. Therefore we have $\hat{\mu}_1(\bigcup_M K_M) = 1$ as $\hat{\xi}_\infty^M$ goes increasingly in M to $\beta\hat{\mu}_1$. The F -invariant set $\bigcap_{k \in \mathbb{Z}} F^{-k}(\bigcup_M K_M)$ has also full $\hat{\mu}_1$ -measure and for all $\hat{x} = (x, v)$ in this set we have $\limsup_n \frac{1}{n} \log \|d_x f^n(v)\| \geq \delta$. Consequently the measure $\hat{\mu}_1$ is supported on the unstable bundle $\mathcal{E}_+(x)$ and $\lambda^+(x) \geq \delta$ for μ_1 -a.e. x . \square

Remark 6. In Theorem C of [10], the measure $\beta\hat{\mu}_1^+$ is obtained as the limit when δ goes to zero of the component associated to the set $G^\delta := \{x, \forall l > 0 \text{ } \phi_l(\hat{x}) \geq \delta l\} \supset \pi^{-1}G_\kappa \cap H_\delta$.

We pursue now the proof of Proposition 5. Let $q, M \in \mathbb{N}^*$. Fix a sequence $(\nu_k)_k$ of positive numbers with $\nu_k \xrightarrow{k} 0$. We consider a partition Q satisfying $\text{diam}(Q) < \epsilon_q$ with ϵ_q as in Lemma 5. The sequence $(f_k)_k$ being C^r bounded, one can choose ϵ_q independently of $f_k, k \in \mathbb{N}$.

By a standard argument of countability we may assume that for all $m \in \mathbb{N}^*$ the boundary of Q^m has zero-measure for $\hat{\mu}_1$ and all the measures $\hat{\xi}_k^M, M \in \mathbb{N}^*$ and $k \in \mathbb{N} \cup \{\infty\}$. Combining Lemma 5 and Lemma 3 we get with $\gamma_{q,Q,M}(f) = \gamma_{q,M}(f) + H\left(\frac{2}{M}\right) + \frac{12 \log \#Q}{M}$:

$$\begin{aligned} h(\nu_k) &\leq \beta_M^k \frac{1}{m} H_{\hat{\xi}_k^M}(Q^m) + (1 - \beta_M^k) C(f_k) \\ &\quad + \left(\log 2 + \frac{1}{r-1} \right) \left(\int \frac{\log^+ \|df_k^q\|}{q} d\hat{\xi}_k^M - \int \phi d\hat{\xi}_k^M \right) \\ &\quad + \gamma_{q,Q,M}(f_k) + \nu_k. \end{aligned}$$

By letting k , then M go to infinity, we obtain for all m :

$$\begin{aligned} \limsup_k h(\nu_k) &\leq \beta \frac{1}{m} H_{\hat{\mu}_1^+}(Q^m) + (1 - \beta) C(f) \\ &\quad + \left(\log 2 + \frac{1}{r-1} \right) \left(\int \frac{\log^+ \|df^q\|}{q} d\mu_1 - \int \phi d\hat{\mu}_1^+ \right) \\ &\quad + \limsup_M \sup_k \gamma_{q,Q,M}(f_k). \end{aligned}$$

By letting m go to infinity, we get:

$$\begin{aligned} \limsup_k h(\nu_k) &\leq \beta h(\hat{\mu}_1^+) + (1 - \beta)C(f) \\ &\quad + \left(\log 2 + \frac{1}{r-1} \right) \left(\int \frac{\log^+ \|df^q\|}{q} d\mu_1 - \int \phi d\hat{\mu}_1^+ \right) \\ &\quad + \limsup_M \sup_k \gamma_{q,M}(f_k). \end{aligned}$$

But $h(\hat{\mu}_1^+) = h(\mu_1)$ (see e.g. Corollary 4.2 in [10]) and $\int \phi d\hat{\mu}_1^+ = \lambda^+(\mu_1) = \lim_q \int \frac{\log^+ \|df^q\|}{q} d\mu_1$. Therefore by letting q go to infinity we finally obtain with the asymptotic property (2.5) of $\gamma_{q,M}$:

$$\limsup_k h(\nu_k) \leq \beta h(\mu_1) + (1 - \beta)C(f).$$

3. SEMI-LOCAL REPARAMETRIZATION LEMMA

In this section we prove the semi-local *Reparametrization Lemma* stated in Proposition 4.

3.1. Strongly bounded curves. To simplify the exposition (by avoiding irrelevant technical details involving the exponential map) we assume that \mathbf{M} is the two-torus \mathbb{T}^2 with the usual Riemannian structure inherited from \mathbb{R}^2 . Borrowing from [2] we first make the following definitions.

A C^r embedded curve $\sigma : [-1, 1] \rightarrow \mathbf{M}$ is said *bounded* when $\max_{k=2, \dots, r} \|d^k \sigma\|_\infty \leq \frac{\|d\sigma\|_\infty}{6}$.

Lemma 7. *Assume σ is a bounded curve. Then for any $x \in \sigma([-1, 1])$, the curve σ contains the graph of a κ -admissible map at x with $\kappa = \frac{\|d\sigma\|_\infty}{6}$.*

Proof. Let $x = \sigma(s)$, $s \in [-1, 1]$. One checks easily (see Lemma 7 in [4] for further details) that for all $t \in [-1, 1]$ the angle $\angle \sigma'(s), \sigma'(t) < \frac{\pi}{6} \leq 1$ and therefore $\int_0^1 \sigma'(t) \cdot \frac{\sigma'(s)}{\|\sigma'(s)\|} dt \geq \frac{\|d\sigma\|_\infty}{6}$. Therefore, as $\sigma'(s) \in \mathcal{E}_+(x)$, the image of σ contains the graph of an $\frac{\|d\sigma\|_\infty}{6}$ -admissible map at x . \square

A C^r bounded curve $\sigma : [-1, 1] \rightarrow \mathbf{M}$ is said *strongly ϵ -bounded* for $\epsilon > 0$ if $\|d\sigma\|_\infty \leq \epsilon$. For $n \in \mathbb{N}^*$ and $\epsilon > 0$ a curve is said *strongly (n, ϵ) -bounded* when $f^k \circ \sigma$ is strongly ϵ -bounded for all $k = 0, \dots, n-1$.

We consider a C^r smooth diffeomorphism $g : \mathbf{M} \circlearrowleft$ with $\mathbb{N} \ni r \geq 2$. For $\hat{x} = (x, v) \in \mathbb{P}T\mathbf{M}$ with $\pi(\hat{x}) = x$, we let $k_g(x) \geq k'_g(\hat{x})$ be the following integers:

$$k_g(x) := \lceil \log \|d_x g\| \rceil,$$

$$k'_g(\hat{x}) := \lceil \log \|d_x g(v)\| \rceil = \lceil \phi_g(\hat{x}) \rceil.$$

In the next lemma, we reparametrize the image by g of a bounded curve. The proof of this lemma is mostly contained in the proof of the Reparametrization Lemma [2], but we reproduce it for the sake of completeness.

Lemma 8. Let $\frac{R_{inj}}{2} > \epsilon = \epsilon_g > 0$ satisfying $\|d^s g_{2\epsilon}^x\|_\infty \leq 3\epsilon \|d_x g\|$ for all $s = 1, \dots, r$ and all $x \in \mathbf{M}$, where $g_{2\epsilon}^x = g \circ \exp_x(2\epsilon) = g(x + 2\epsilon) : \{w_x \in T_x \mathbf{M}, \|w_x\| \leq 1\} \rightarrow \mathbf{M}$. We assume $\sigma : [-1, 1] \rightarrow \mathbf{M}$ is a strongly ϵ -bounded C^r curve and we let $\hat{\sigma} : [-1, 1] \rightarrow \mathbb{P}TM$ be the associated induced map.

Then for some universal constant $C_r > 0$ depending only on r and for any pair of integers (k, k') there is a family Θ of affine maps from $[-1, 1]$ to itself satisfying:

- $\hat{\sigma}^{-1}(\{\hat{x} \in \mathbb{P}TM, k_g(x) = k \text{ and } k'_g(\hat{x}) = k'\}) \subset \bigcup_{\theta \in \Theta} \theta([-1, 1])$,
- $\forall \theta \in \Theta$, the curve $g \circ \sigma \circ \theta$ is bounded,
- $\forall \theta \in \Theta$, $|\theta'| \leq e^{\frac{k'-k-1}{r-1}}/4$,
- $\#\Theta \leq C_r e^{\frac{k-k'}{r-1}}$.

Proof. First step : Taylor polynomial approximation. One computes for an affine map $\theta : [-1, 1] \rightarrow \mathbf{M}$ with contraction rate b precised later and with $y = \sigma(t)$, $k_g(y) = k$, $k'_g(y) = k'$, $t \in \theta([-1, 1])$:

$$\begin{aligned} \|d^r(g \circ \sigma \circ \theta)\|_\infty &\leq b^r \|d^r(g_{2\epsilon}^y \circ \sigma_{2\epsilon}^y)\|_\infty, \text{ with } \sigma_{2\epsilon}^y := (2\epsilon)^{-1} \exp_y^{-1} \circ \sigma = 2\epsilon^{-1}(\sigma(\cdot) - y), \\ &\leq b^r \left\| d^{r-1} \left(d_{\sigma_{2\epsilon}^y} g_{2\epsilon}^y \circ d_{\sigma_{2\epsilon}^y} \right) \right\|_\infty, \\ &\leq b^r 2^r \max_{s=0, \dots, r-1} \left\| d^s \left(d_{\sigma_{2\epsilon}^y} g_{2\epsilon}^y \right) \right\|_\infty \max_{k=1, \dots, r} \|d^k \sigma_{2\epsilon}^y\|_\infty. \end{aligned}$$

By assumption on ϵ , we have $\|d^s g_{2\epsilon}^y\|_\infty \leq 3\epsilon \|d_y g\|$ for any $r \geq s \geq 1$. Moreover $\max_{k=1, \dots, r} \|d^k \sigma_{2\epsilon}^y\|_\infty \leq 1$ as σ is strongly ϵ -bounded. Therefore by Faà di Bruno's formula, we get for some ¶ constants $C_r > 0$ depending only on r :

$$\max_{s=0, \dots, r-1} \|d^s (d_{\sigma_{2\epsilon}^y} g_{2\epsilon}^y)\|_\infty \leq \epsilon C_r \|d_y g\|,$$

then ,

$$\begin{aligned} \|d^r(g \circ \sigma \circ \theta)\|_\infty &\leq \epsilon C_r b^r \|d_y g\| \max_{k=1, \dots, r} \|d^k \sigma_{2\epsilon}^y\|_\infty, \\ &\leq C_r b^r \|d_y g\| \|d\sigma\|_\infty, \\ &\leq (C_r b^{r-1} \|d_y g\|) \|d(\sigma \circ \theta)\|_\infty, \\ &\leq (C_r b^{r-1} e^k) \|d(\sigma \circ \theta)\|_\infty, \text{ because } k(y) = k, \\ &\leq e^{k'-4} \|d(\sigma \circ \theta)\|_\infty, \text{ by taking } b = \left(C_r e^{k-k'+4} \right)^{-\frac{1}{r-1}}. \end{aligned}$$

Therefore the Taylor polynomial P at 0 of degree $r-1$ of $d(g \circ \sigma \circ \theta)$ satisfies on $[-1, 1]$:

$$\|P - d(g \circ \sigma \circ \theta)\|_\infty \leq e^{k'-4} \|d(\sigma \circ \theta)\|_\infty.$$

We may cover $[-1, 1]$ by at most $b^{-1} + 1$ such affine maps θ .

Second step : Bezout theorem. Let $a = e^{k'} \|d(\sigma \circ \theta)\|_\infty$. Note that for $s \in [-1, 1]$ with $k(\sigma \circ \theta(s)) = k$ and $k'(\sigma \circ \theta(s)) = k'$ we have $\|d(g \circ \sigma \circ \theta)(s)\| \in [ae^{-2}, ae^2]$, therefore $\|P(s)\| \in [ae^{-3}, ae^3]$. Moreover if we have now $\|P(s)\| \in [ae^{-3}, ae^3]$ for some $s \in [-1, 1]$ we get also $\|d(g \circ \sigma \circ \theta)(s)\| \in [ae^{-4}, ae^4]$.

¶ Although these constants may differ at each step, they are all denoted by C_r .

By Bezout theorem the semi-algebraic set $\{s \in [-1, 1], \|P(s)\| \in [e^{-3}a, e^3a]\}$ is the disjoint union of closed intervals $(J_i)_{i \in I}$ with $\#I$ depending only on r . Let θ_i be the composition of θ with an affine reparametrization from $[-1, 1]$ onto J_i .

Third step : Landau-Kolmogorov inequality. By the Landau-Kolmogorov inequality on the interval (see Lemma 6 in [2]), we have for some constants $C_r \in \mathbb{N}^*$ and for all $1 \leq s \leq r$:

$$\begin{aligned} \|d^s(g \circ \sigma \circ \theta_i)\|_\infty &\leq C_r (\|d^r(g \circ \sigma \circ \theta_i)\|_\infty + \|d(g \circ \sigma \circ \theta_i)\|_\infty), \\ &\leq C_r \frac{|J_i|}{2} \left(\|d^r(g \circ \sigma \circ \theta)\|_\infty + \sup_{t \in J_i} \|d(g \circ \sigma \circ \theta)(t)\| \right), \\ &\leq C_r a \frac{|J_i|}{2}. \end{aligned}$$

We cut again each J_i into $1000C_r$ intervals \tilde{J}_i of the same length with

$$\theta(\tilde{J}_i) \cap \sigma^{-1} \{x, k_g(x) = k \text{ and } k'_g(x) = k'\} \neq \emptyset.$$

Let $\tilde{\theta}_i$ be the affine reparametrization from $[-1, 1]$ onto $\theta(\tilde{J}_i)$. We check that $g \circ \sigma \circ \tilde{\theta}_i$ is bounded:

$$\begin{aligned} \forall s = 2, \dots, r, \|d^s(g \circ \sigma \circ \tilde{\theta}_i)\|_\infty &\leq (1000C_r)^{-2} \|d^s(g \circ \sigma \circ \theta_i)\|_\infty, \\ &\leq \frac{1}{6} (1000C_r)^{-1} \frac{|J_i|}{2} a_n e^{-4}, \\ &\leq \frac{1}{6} (1000C_r)^{-1} \frac{|J_i|}{2} \min_{s \in J_i} \|d(g \circ \sigma \circ \theta)(s)\|, \\ &\leq \frac{1}{6} (1000C_r)^{-1} \frac{|J_i|}{2} \min_{s \in \tilde{J}_i} \|d(g \circ \sigma \circ \theta)(s)\|, \\ &\leq \frac{1}{6} \|d(g \circ \sigma \circ \tilde{\theta}_i)\|_\infty. \end{aligned}$$

This concludes the proof with Θ being the family of all $\tilde{\theta}_i$'s. \square

We recall now a useful property of bounded curve (see Lemma 7 in [4] for a proof).

Lemma 9. *Let $\sigma : [-1, 1] \rightarrow \mathbf{M}$ be a C^r bounded curve and let B be a ball of radius less than ϵ . Then there exists an affine map $\theta : [-1, 1] \circlearrowleft$ such that :*

- $\sigma \circ \theta$ is strongly 3ϵ -bounded,
- $\theta([-1, 1]) \supset \sigma^{-1}B$.

3.2. Choice of the parameters κ and ϵ_q . For a diffeomorphism $f : \mathbf{M} \circlearrowleft$ the scale ϵ_f in Lemma 8 may be chosen such that $\epsilon_{fk} \leq \epsilon_{fl} \leq \max(1, \|df\|_\infty)^{-k}$ for any $q \geq k \geq l \geq 1$. We take $\kappa = \frac{\epsilon_f}{36}$ and we choose $\epsilon_q < \frac{\epsilon_f^q}{3}$ such that for any $\hat{x}, \hat{y} \in \mathbb{P}T\mathbf{M}$ which are ϵ_q -close and for any $0 \leq l \leq q$:

$$(3.1) \quad \begin{aligned} |k_{fl}(x) - k_{fl}(y)| &\leq 1, \\ |k'_{fl}(\hat{x}) - k'_{fl}(\hat{y})| &\leq 1. \end{aligned}$$

Without loss of generality we can assume the local unstable curve D (defined in Subsection 2.3) is reparametrized by a C^r strongly ϵ_q -bounded map $\sigma : [-1, 1] \rightarrow D$.

Let F_n be an atom of the partition $Q^{E_n^M}$ and let $E_n^M = E_n^M(x)$ for any $\hat{x} \in F_n$. Recall that the diameter of Q is less than ϵ_q . It follows from (3.1) that for any $\hat{x} \in F_n$ we have with $\hat{\zeta}_{F_n}^M = \int \mu_{\hat{x},n}^M d\zeta_{F_n}(x)$:

$$\sum_{l \in E_n^M} \left| k_{f^l}(f^l x) - k'_{f^l}(F^l \hat{x}) \right| \leq 10 \#E_n^M + \int \log^+ \|d_y f^q\| d\zeta_{F_n}^M(y) - \int \phi_q d\hat{\zeta}_{F_n}^M.$$

Therefore we may fix some $0 \leq c < q$, such that for any $x \in F_n$

$$\begin{aligned} \sum_{l \in (c+q\mathbb{N}) \cap E_n^M} \left| k_{f^l}(f^l x) - k'_{f^l}(F^l \hat{x}) \right| &\leq 10 \frac{n}{q} + \frac{1}{q} \left(\int \log^+ \|d_y f^q\| d\zeta_{F_n}^M(y) - \int \phi_q d\hat{\zeta}_{F_n}^M \right), \\ &\leq 10 \frac{n}{q} + 2A_f \frac{qn}{M} + \frac{1}{q} \int \log^+ \|d_y f^q\| d\zeta_{F_n}^M(y) - \int \phi d\hat{\zeta}_{F_n}^M. \end{aligned}$$

3.3. Combinatorial aspects. We put $\partial_l E_n^M := \{a \in E_n^M \text{ with } a-1 \notin E_n^M\}$. Then we let $\mathcal{A}_n := \{0 = a_1 < a_2 < \dots < a_m\}$ be the union of $\partial_l E_n^M$, $[0, n] \setminus E_n^M$ and $(c+q\mathbb{N}) \cap [0, n]$. We also let $b_i = a_{i+1} - a_i$ for $i = 1, \dots, m-1$ and $b_m = n - a_m$.

For a sequence $\mathbf{k} = (k_l, k'_l)_{l \in \mathcal{A}_n}$ of integers, a positive integer m_n and a subset \bar{E} of $[0, n]$, we let $F_n^{\mathbf{k}, \bar{E}, m_n}$ be the subset of points $\hat{x} \in F_n$ satisfying:

- $\bar{E} = E_n(x) \setminus E_n^M(x)$,
- $k_{a_i} = k_{f^{b_i}}(f^{a_i} x)$ and $k'_{a_i} = k'_{f^{b_i}}(F^{a_i} \hat{x})$ for $i = 1, \dots, m$,
- $m_n(x) = m_n$.

Lemma 10.

$$\# \left\{ (\mathbf{k}, \bar{E}, m_n), F_n^{\mathbf{k}, \bar{E}, m_n} \neq \emptyset \right\} \leq n e^{2nA_f H(A_f^{-1})} 3^{n(1/q+1/M)} e^{nH(1/M)}.$$

Proof. Firstly observe that if $a_i \notin E_n^M$ then $b_i = 1$. In particular $\sum_{i, a_i \notin E_n^M} k_{a_i} \leq (n - \#E_n^M) \log^+ \|df\|_\infty \leq (n - \#E_n^M)(A_f - 1)$. The number of such sequences $(k_{a_i})_{i, a_i \notin E_n^M}$ is therefore bounded above by $\binom{r_n A_f}{r_n}$ with $r_n = n - \#E_n^M$ and its logarithm is dominated by $r_n A_f H(A_f^{-1}) + 1 \leq n A_f H(A_f^{-1}) + 1$. Similarly the number of sequence $(k'_{a_i})_{i, a_i \notin E_n^M}$ is less than $n A_f H(A_f^{-1}) + 1$.

Then from the choice of ϵ_q in (3.1) there are at most three possible values of $k_{a_i}(x)$ for $a_i \in E_n^M$ and $x \in F_n$.

Finally as $\#\bar{E} \leq n/M$, the number of admissible sets \bar{E} is less than $\binom{n}{[n/M]}$ and thus its logarithm is bounded above by $nH(1/M) + 1$. Clearly we can also fix the value of m_n up to a factor n . □

3.4. The induction. We fix \mathbf{k} , m_n and \bar{E} and we reparametrize appropriately the set $F_n^{\mathbf{k}, \bar{E}, m_n}$.

Lemma 11. *With the above notations there are families $(\Theta_i)_{i \leq m}$ of affine maps from $[-1, 1]$ into itself such that :*

- $\forall \theta \in \Theta_i \forall j \leq i$ the curve $f^{a_i} \circ \sigma \circ \theta$ is strongly $\epsilon_{f^{b_i}}$ -bounded,
- $\hat{\sigma}^{-1} \left(F_n^{\mathbf{k}, \bar{E}, m_n} \right) \subset \bigcup_{\theta \in \Theta_i} \theta([-1, 1])$,

- $\forall \theta_i \in \Theta_i \forall j < i, \exists \theta_j^i \in \Theta_j, \frac{|\theta_i'|}{|(\theta_j^i)'|} \leq \prod_{j \leq l < i} e^{\frac{k_{a_l}' - k_{a_l} - 1}{r-1}} / 4,$
- $\#\Theta_i \leq C \max(1, \|df\|_\infty)^{\#\bar{E} \cap [1, a_i]} \prod_{j < i} C_r e^{\frac{k_{a_j} - k_{a_j}'}{r-1}}.$

Proof. We argue by induction on $i \leq m$. By changing the constant C , it is enough to consider i with $a_i > N$. Recall that the integer N was chosen in such a way that for any $x \in F$ there is $0 \leq k \leq N$ with $F^k \hat{x} \in H_\delta$. We assume the family Θ_i for $i < m$ already built and we will define Θ_{i+1} . Let $\theta_i \in \Theta_i$. We apply Lemma 8 to the strongly $\epsilon_{f^{b_i}}$ -bounded curve $f^{a_i} \circ \sigma \circ \theta_i$ with $g = f^{b_i}$. Let Θ be the family of affine reparametrizations of $[-1, 1]$ satisfying the conclusions of Lemma 8, in particular $f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta$ is bounded, $|\theta'| \leq e^{\frac{k_{a_i}' - k_{a_i} - 1}{r-1}} / 4$ for all $\theta \in \Theta$ and $\#\Theta \leq C_r e^{\frac{k_{a_i} - k_{a_i}'}{r-1}}$. We distinguish three cases:

- $\underline{a_{i+1} \in E_n^M}$. The diameter of $F^{a_{i+1}} F_n$ is less than $\epsilon_q \leq \frac{\epsilon_{f^{b_{i+1}}}}{3}$. By Lemma 9 there is an affine map $\psi : [-1, 1] \circlearrowleft$ such that $f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta \circ \psi$ is strongly $\epsilon_{f^{b_{i+1}}}$ -bounded and its image contains the intersection of the bounded curve $f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta$ with $f^{a_{i+1}} F_n$. We let then $\theta_{i+1} = \theta_i \circ \theta \circ \psi \in \Theta_{i+1}$.
- $\underline{a_{i+1} \in E \setminus E_n^M}$. Observe that $b_{i+1} = 1$, therefore $\epsilon_{f^{b_i}} \leq \epsilon_{f^{b_{i+1}}}$. Then the length of the curve $f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta$ is less than $3\|df\|_\infty \epsilon_{f^{b_i}}$, thus may be covered by $[3\|df\|_\infty] + 1$ balls of radius less than $\epsilon_{f^{b_{i+1}}}$. We then use Lemma 9 as in the previous case to reparametrize the intersection of this curve with each ball by a strongly $\epsilon_{f^{b_{i+1}}}$ -bounded curve. We define in this way the associated parametrizations of Θ_{i+1} .
- $\underline{a_{i+1} \notin E \text{ and } a_{i+1} \notin E_n^M}$. We claim that $\|d(f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta)\| \leq \epsilon_f / 6$. Take $\hat{x} \in F_n^{\mathbf{k}, \bar{E}, m_n}$ with $x = \pi(\hat{x}) = \sigma \circ \theta_i \circ \theta(s)$. Let $K_x = \max\{k < a_{i+1}, F^k \hat{x} \in H_\delta\} \geq N$. Observe that $[K_x, a_{i+1}] \cap E_n^M = \emptyset$, therefore for $K_x \leq a_l < a_{i+1}$, we have $b_l = 1$, then $a_l = a_{i+1} - i - 1 + l$. We argue by contradiction by assuming :

$$(3.2) \quad \|d(f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta)\| \geq \epsilon_f / 6 = 6\kappa$$

By Lemma 7, the point $f^{a_{i+1}} x$ belongs to G_κ . We will show $F^{a_{i+1}} \hat{x} \in H_\delta$. Therefore we will get $F^{a_{i+1}} \hat{x} \in G = \pi^{-1} G_\kappa \cap H_\delta$ contradicting $a_{i+1} \notin E$. To prove $F^{a_{i+1}} \hat{x} \in H_\delta$ it is enough to show $\sum_{j \leq l < a_{i+1}} \psi(F^l \hat{x}) \geq (a_{i+1} - j)\delta$ for any $K_x \leq j < a_{i+1}$ because $F^{K_x}(\hat{x})$ belongs to H_δ . For any $K_x \leq j < a_{i+1}$ we have :

$$(3.3) \quad \begin{aligned} \|d(f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta)\|_\infty &\leq 2\|d_s(f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta)\|, \text{ because } f^{a_{i+1}} \circ \sigma \circ \theta_i \circ \theta \text{ is bounded,} \\ &\leq 2\|d_{f^j x} f^{a_{i+1}-j}(\hat{x})\| \times \|d_s(f^{a_{\bar{j}}} \circ \sigma \circ \theta_{\bar{j}}^i)\| \times \frac{|\theta_i'| \times |\theta'|}{|(\theta_{\bar{j}}^i)'|}, \text{ with } a_{\bar{j}} = j, \\ &\leq \frac{\epsilon_f}{3} \|d_{f^j x} f^{a_{i+1}-j}(\hat{x})\| \prod_{\bar{j} \leq l \leq i} e^{\frac{k_{a_l}' - k_{a_l} - 1}{r-1}} / 4 \text{ by induction hypothesis,} \\ &\frac{1}{2} \leq \|d_{f^j x} f^{a_{i+1}-j}(\hat{x})\| \prod_{\bar{j} \leq l \leq i} e^{\frac{k_{a_l}' - k_{a_l} - 1}{r-1}} / 4 \text{ by assumption (3.2).} \end{aligned}$$

Recall again that for $\bar{j} \leq l \leq i$, we have $b_l = 1$, thus

$$|k_{a_l} - \log \|d_{f^{a_l} x} f\| \leq 1$$

and

$$k'_{a_l} \leq \phi(F^{a_l} \hat{x}).$$

Therefore we get for any $K_x \leq j < a_{i+1}$ from (3.3):

$$\begin{aligned} 2^{a_{i+1}-j} &\leq e^{\frac{r}{r-1} \sum_{j \leq l < a_{i+1}} \phi(F^l \hat{x})} e^{-\frac{1}{r-1} \sum_{j \leq l < a_{i+1}} \log^+ \|d_{f^l x} f\|}, \\ (a_{i+1} - j) \log 2 &\leq \frac{r}{r-1} \sum_{j \leq l < a_{i+1}} \psi(F^l \hat{x}), \text{ by definition of } \psi, \\ (a_{i+1} - j) \delta &\leq \sum_{j \leq l < a_{i+1}} \psi(F^l \hat{x}), \text{ as } \delta \text{ was chosen less than } \frac{r-1}{r} \log 2. \end{aligned}$$

□

Lemma 12.

$$\sum_{i, m_n > a_i \notin E_n^M} \frac{k_{a_i} - k'_{a_i}}{r-1} \leq (n - \#E_n^M) \left(\frac{\log^+ \|df\|_\infty}{r} + \frac{1}{r-1} \right).$$

Proof. The intersection of $[0, m_n[$ with the complement set of E_n^M is the disjoint union of neutral blocks and possibly an interval of integers of the form $[l, m_n[$. In any case $F^j \hat{x}$ belongs to H_δ for such an interval $[i, j[$ for any $x \in F_n^{\mathbf{k}, \bar{E}, m_n}$. In particular, we have

$$\sum_{l, a_l \in [i, j[} k'_{a_l} - \frac{k_{a_l}}{r} \geq (\delta - 1)(j - i)$$

therefore

$$\begin{aligned} \sum_{i, m_n > a_i \notin E_n^M} k'_{a_i} - \frac{k_{a_i}}{r} &\geq -(n - \#E_n^M), \\ \sum_{i, m_n > a_i \notin E_n^M} \frac{k_{a_i} - k'_{a_i}}{r-1} &\leq \frac{n - \#E_n^M}{r-1} + \frac{\sum_{i, m_n > a_i \notin E_n^M} k_{a_i}}{r}, \\ &\leq (n - \#E_n^M) \left(\frac{\log^+ \|df\|_\infty}{r} + \frac{1}{r-1} \right). \end{aligned}$$

□

3.5. Conclusion. We let Ψ_n be the family of \mathcal{C}^r curves $\sigma \circ \theta$ for $\theta \in \Theta_m = \Theta_m(\mathbf{k}, \bar{E}, m_n)$ with Θ_m as in Lemma 11 over all admissible parameters \mathbf{k}, \bar{E}, m_n . For $\theta \in \Theta_m$ the curve $f^{a_i} \circ \sigma \circ \theta$ is strongly $\epsilon_{f^{b_i}}$ -bounded for any $i = 1, \dots, m$, in particular

$$\forall i = 1, \dots, m, \|d(f^{a_i} \circ \sigma \circ \theta)\|_\infty \leq \epsilon_{f^{b_i}} \leq \max(1, \|df\|_\infty)^{-b_i},$$

therefore

$$\forall j = 0, \dots, n, \|d(f^j \circ \sigma \circ \theta)\|_\infty \leq 1.$$

By combining the previous estimates, we get moreover:

$$\begin{aligned}
\#\Psi_n &\leq \#\left\{(\mathbf{k}, \bar{E}, m_n), F_n^{\mathbf{k}, \bar{E}, m_n} \neq \emptyset\right\} \times \sup_{\mathbf{k}, \bar{E}, m_n} \#\Theta_n(\mathbf{k}, \bar{E}, m_n), \\
&\leq ne^{2(n-\#E_n^M)A_f H(A_f)} 3^{n(1/q+1/M)} e^{nH(1/M)} \sup_{\mathbf{k}, \bar{E}, m_n} \#\Theta_n(\mathbf{k}, \bar{E}, m_n), \text{ by Lemma 10,} \\
&\leq ne^{2(n-\#E_n^M)A_f H(A_f)} 3^{n(1/q+1/M)} e^{nH(1/M)} \max(1, \|df\|_\infty)^{\#\bar{E}} \prod_{j \leq m} C_r e^{\frac{k_{a_j} - k'_{a_j}}{r-1}}, \text{ by Lemma 11.}
\end{aligned}$$

Then we decompose the product into four terms :

- $\sum_{i, m_n > a_i \notin E_n^M} \frac{k_{a_i} - k'_{a_i}}{r-1} \leq (n - \#E_n^M) \left(\frac{\log^+ \|df\|_\infty}{r} + \frac{1}{r-1} \right)$ by Lemma 12,
- $\sum_{i, m_n \leq a_i} \frac{k_{a_i} - k'_{a_i}}{r-1} \leq (n - m_n) \frac{A_f}{r-1}$,
- $\sum_{i, a_i \in E_n^M \cap (c+q\mathbb{N})} \frac{k_{a_i} - k'_{a_i}}{r-1} \leq 10 \frac{n}{q} + 2A_f \frac{qn}{M} + \frac{1}{r-1} \left(\int \frac{\log^+ \|d_y f^q\|}{q} d\zeta_{F_n}^M(y) - \int \phi d\hat{\zeta}_{F_n}^M \right)$,
- $\sum_{i, a_i \in E_n^M \setminus (c+q\mathbb{N})} \frac{k_{a_i} - k'_{a_i}}{r-1} \leq 2A_f \frac{qn}{M}$.

By letting

$$B_r = \frac{1}{r-1} + \log C_r,$$

$$\gamma_{q,M}(f) := 2 \left(\frac{1}{q} + \frac{1}{M} \right) \log C_r + H(1/M) + \frac{10 + \log 3}{q} + \frac{4qA_f + \log 3}{M},$$

$$\tau_n = \sup_{x \in \mathbb{F}} \left(1 - \frac{m_n(x)}{n} \right) \frac{A_f}{r-1} + \frac{\log(nC)}{n},$$

we get with $C(f) := 2A_f H(A_f^{-1}) + \frac{\log^+ \|df\|_\infty}{r} + B_r$:

$$\begin{aligned}
\frac{1}{n} \log \#\Psi_{F_n} &\leq \left(1 - \frac{\#E_n^M}{n} \right) C(f) \\
&\quad + \left(\log 2 + \frac{1}{r-1} \right) \left(\int \frac{\log^+ \|d_x f^q\|}{q} d\zeta_{F_n}^M(x) - \int \phi d\hat{\zeta}_{F_n}^M \right) \\
&\quad + \gamma_{q,M}(f) + \tau_n,
\end{aligned}$$

This concludes the proof of Proposition 4.

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