MAXIMAL MEASURE AND ENTROPIC CONTINUITY OF LYAPUNOV EXPONENTS FOR C^r SURFACE DIFFEOMORPHISMS WITH LARGE ENTROPY

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MAXIMAL MEASURE AND ENTROPIC CONTINUITY OF LYAPUNOV EXPONENTS FOR $\mathcal{C}^r$ SURFACE DIFFEOMORPHISMS WITH LARGE ENTROPY

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Abstract. We prove a finite smooth version of the entropic continuity of Lyapunov exponents of Buzzi-Crovisier-Sarig for C^∞ surface diffeomorphisms [10]. As a consequence we show that any C^r, $r > 1$, smooth surface diffeomorphism f with $h_{top}(f) > \frac{1}{r} \limsup_n \frac{1}{n} \log^+ \|df^n\|$ admits a measure of maximal entropy. We also prove the $\mathcal{C}^r$ continuity of the topological entropy at f.

Introduction

The entropy of a dynamical system quantifies the dynamical complexity by counting distinct orbits. There are topological and measure theoretical versions which are related by a variational principle : the topological entropy of a continuous map on a compact space is equal to the supremum of the entropy of the invariant (probability) measures. An invariant measure is said of maximal entropy when its entropy is equal to the topological entropy, i.e. this measure realizes the supremum in the variational principle. In general a topological system does not admit measures of maximal entropy. But such measures exist for systems satisfying some expansiveness properties. In particular Newhouse [14] has proved their existence for C^∞ systems by using Yomdin's theory. In the present paper we show the existence of a measure of maximal entropy for $\mathcal{C}^r$ smooth surface diffeomorphisms with $1 < r < +\infty$ with large entropy.

Other important dynamical quantities for smooth systems are given by the Lyapunov exponents which estimate the exponential growth of the derivative. For $\mathcal{C}^\infty$ surface diffeomorphisms, J. Buzzi, S. Crovisier and O. Sarig proved recently a property of continuity in the entropy of the Lyapunov exponents with many statistical applications [10]. More precisely, for a $\mathcal{C}^\infty$ surface diffeomorphism f, if ν_k is a sequence of ergodic measures with $\lim_k h(\nu_k) = h_{top}(f)$, then the Lyapunov exponents of ν_k are going to the (average) Lyapunov exponents of μ. We prove a $\mathcal{C}^r$ version of their result for $1 < r < +\infty$.

1. Statements

We define now some notations to state our main result. For a diffeomorphism f on a compact Riemannian surface $(M, \|\cdot\|)$ we let $F : \mathbb{P}TM \circlearrowleft$ be the induced map on the projective tangent bundle $\mathbb{P}TM = T^1M/\pm 1$ and

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$\phi : \mathbb{P}TM \to \mathbb{R}$ be the continuous observable on $\mathbb{P}TM$ given by $(x,v) \mapsto \log \|d_x f(v)\|$. For $k \in \mathbb{N}^*$ we define more generally $\phi_k : (x,v) \mapsto \log \|d_x f^k(v)\|$. Then we let $\lambda^+(x)$ and $\lambda^-(x)$ be the pointwise Lyapunov exponents given by $\lambda^+(x) = \limsup_{n\to+\infty} \frac{1}{n} \log \|d_x f^n\|$ and $\lambda^-(x) = \liminf_{n\to-\infty} \frac{1}{n} \log \|d_x f^n\|$, for any $x \in M$ and $\lambda^+(\mu) = \int \lambda^+(x)\, d\mu(x)$ for any f-invariant measure μ.

Also we put $\lambda^+(f) := \lim_n \frac{1}{n} \log \|df^n\|$. Observe that $f \mapsto \lambda^+(f)$ is upper semi-continuous in the $\mathcal{C}^1$ topology. For a f-invariant measure μ with $\lambda^+(x) > 0 \geq \lambda^-(x)$ for μ a.e. x, there are by Oseledets* theorem one-dimensional invariant vector spaces $\mathcal{E}_+(x)$ and $\mathcal{E}_-(x)$, resp. called the unstable and stable Oseledets bundle, such that

$$\forall \text{ a.e. } x \ \forall v \in \mathcal{E}_\pm(x) \setminus \{0\}, \ \lim_{n\to\pm\infty} \frac{1}{n} \log \|d_x f^n(v)\| = \lambda^\pm(x).$$

Then we let $\hat{\mu}^+$ be the F-invariant measure given by the lift of μ on $\mathbb{P}TM$ supported on the one-dimensional Oseledets unstable bundle. When writing $\hat{\mu}^+$ we assume implicitly that the push-forward measure μ on M satisfies $\lambda^+(x) > 0 \geq \lambda^-(x)$ for μ a.e. x.

Theorem (Buzzi-Crovisier-Sarig, Theorem C in [10])**.** *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of $\mathcal{C}^r$, with $r > 1$, surface diffeomorphisms converging to f in the C^r topology. Assume there is a sequence $(\hat{\nu}_k^+)_k$ of ergodic F_k-invariant measures converging to $\hat{\mu}$.*

Then there are F-invariant measures $\hat{\mu}_0$ and $\hat{\mu}_1^+$ with $\hat{\mu} = (1-\beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$, $\beta > 0$, such that:

- *$\lambda^+(x) > 0 \geq \lambda^-(x)$ for μ_1-a.e. x,*
- *$\int \phi\, d\hat{\mu}_0 = 0$,*
- *$\limsup_{k\to+\infty} h(\nu_k) \leq \beta h(\mu_1) + \frac{\lambda^+(f)+\lambda^+(f^{-1})}{r-1}$.*

Observe that $\lambda^+(\nu_k) = \int \phi\, d\hat{\nu}_k^+ \overset{k}{\to} \int \phi\, d\hat{\mu} = \beta\lambda^+(\mu_1)$, therefore $\beta = \lim_k \frac{\lambda^+(\nu_k)}{\lambda^+(\mu_1)}$. In particular when f is $\mathcal{C}^\infty$ and $h(\nu_k)$ goes to the topological entropy of f, then β is equal to 1 and therefore $\lambda^+(\nu_k)$ goes to $\lambda^+(\mu)$. We state an improved version of Buzzi-Crovisier-Sarig Theorem, which allows to prove the same entropy continuity of Lyapunov exponents for $\mathcal{C}^r$ surface diffeomorphisms with large enough entropy.

A sequence of $\mathcal{C}^r$, with $r > 1$, surface diffeomorphisms $(f_k)_k$ on M is said to converge $\mathcal{C}^r$ weakly to f, when f_k goes to f in the $\mathcal{C}^1$ topology and the sequence $(f_k)_k$ is $\mathcal{C}^r$ bounded (in particular f is $\mathcal{C}^{r-1}$ and for $r \leq 2$ the derivative of f is $(r-1)$-Hölder).

Main Theorem. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of $\mathcal{C}^r$, with $r > 1$, surface diffeomorphisms converging $\mathcal{C}^r$ weakly to f. Assume there is a sequence $(\hat{\nu}_k^+)_k$ of ergodic F_k-invariant measures converging to $\hat{\mu}$ with $\int \phi\, d\hat{\mu} > \alpha > \frac{\lambda^+(f)}{r}$.*

Then there are F-invariant measures $\hat{\mu}_0$ and $\hat{\mu}_1^+$ with $\hat{\mu} = (1-\beta)\hat{\mu}_0 + \beta\hat{\mu}_1^+$, $\beta > 0$, such that

*We refer to [15] for background on Lyapunov exponents and Pesin theory.

- $\lambda^+(x) \geq \alpha > 0 \geq \lambda^-(x)$ *for* μ_1*-a.e.* x,
- $\int \phi \, d\hat{\mu}_0 = \alpha$,
- $\limsup_{k\to+\infty} h(\nu_k) \leq \beta h(\mu_1) + (1-\beta)\alpha$.

The Main Theorem implies Buzzi-Crovisier-Sarig statement. Indeed, either $\lim_k \lambda^+(\nu_k) = \int \phi \, d\hat{\mu} \leq \frac{\lambda^+(f)}{r}$ and we get by Ruelle inequality, $\limsup_k h(\nu_k) \leq \frac{\lambda^+(f)}{r}$ or we are for some $\alpha \in \left]\frac{\lambda^+(f)}{r}, \min\left(\int \phi \, d\hat{\mu}, \frac{\lambda^+(f)}{r-1}\right)\right[$ in the settings of the Main Theorem. By applying the Main Theorem, there is then a decomposition $\hat{\mu} = (1-\beta_\alpha)\hat{\mu}_{0,\alpha} + \beta_\alpha \hat{\mu}^+_{1,\alpha}$ satisfying $\limsup_{k\to+\infty} h(\nu_k) \leq \beta_\alpha h(\mu_{1,\alpha}) + (1-\beta_\alpha)\alpha$. But it follows from the proofs that $\beta_\alpha \mu_{1,\alpha}$ is a component of $\beta\mu_1$ with β and μ_1 being as in Buzzi-Crovisier-Sarig's statement (see Remark 4). In particular $\beta_\alpha h(\mu_{1,\alpha}) \leq \beta h(\mu_1)$, therefore $\limsup_{k\to+\infty} h(\nu_k) \leq \beta h(\mu_1) + \frac{\lambda^+(f)+\lambda^+(f^{-1})}{r-1}$.

Corollary 1 (Existence of maximal measures)**.** *Let f be a $\mathcal{C}^r$, with $r > 1$, surface diffeomorphism satisfying $h_{top}(f) > \frac{\lambda^+(f)}{r}$.*

Then f admits a measure of maximal entropy. More precisely, if $(\nu_k)_k$ is a sequence of ergodic measures converging to μ with $\lim_k h(\nu_k) = h_{top}(f)$, then

- $h(\mu) = h_{top}(f)$,
- $\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)$.

It was proved in [9] that any $\mathcal{C}^r$ surface diffeomorphism satisfying $h_{top}(f) > \frac{\lambda^+(f)}{r}$ admits at most finitely many ergodic measures of maximal entropy. In the other hand, J. Buzzi has built examples of $\mathcal{C}^r$ surface diffeomorphisms for any $+\infty > r > 1$ with $\frac{h_{top}(f)}{\lambda^+(f)}$ arbitrarily close to $1/r$ without measure of maximal entropy [7]. Such results were already known for interval maps [3, 6, 8].

Proof. We consider the constant sequence of diffeomorphisms equal to f. By using the notations of the Main Theorem with $h_{top}(f) > \alpha > \frac{\lambda^+(f)}{r}$, we have

$$\begin{aligned} h_{top}(f) &= \lim_{k\to+\infty} h(\nu_k), \\ &\leq \beta h(\mu_1) + (1-\beta)\alpha, \\ &\leq \beta h_{top}(f) + (1-\beta)\alpha, \\ (1-\beta)h_{top}(f) &\leq (1-\beta)\alpha. \end{aligned}$$

But $h_{top}(f) > \alpha$, therefore $\beta = 1$, i.e. $\mu_1 = \mu$ and $\lim_k \lambda^+(\nu_k) = \lambda^+(\mu)$. Moreover $h_{top}(f) = \lim_{k\to+\infty} h(\nu_k) \leq \beta h(\mu_1) + (1-\beta)\alpha = h(\mu)$. Consequently μ is a measure of maximal entropy.

□

Corollary 2 (Continuity of topological entropy)**.** *Let $(f_k)_k$ be a sequence of $\mathcal{C}^r$, with $r > 1$, surface diffeomorphisms converging $\mathcal{C}^r$ weakly to f with $h_{top}(f) \geq \frac{\lambda^+(f)}{r}$. Then*

$$h_{top}(f) = \lim_k h_{top}(f_k).$$

Moreover if $h_{top}(f) > \frac{\lambda^+(f)}{r}$ and ν_k is a maximal measure of f_k for large enough k, then any limit measure of $(\nu_k)_k$ for the weak-$*$ topology is a maximal measure of f.

Proof. By Katok's horseshoes theorem [13], the topological entropy is lower semi-continuous for the $\mathcal{C}^1$ topology on the set of $\mathcal{C}^r$ surface diffeomorphisms. Therefore it is enough to show the upper semi-continuity.

Let $(\nu_k)_{k\in K}$, $K \subset \mathbb{N}$ with $\sharp K = \infty$, be a sequence of ergodic f_k-invariant measures with $\lim_{k\in K} h(\nu_k) = \limsup_{k\in\mathbb{N}} h_{top}(f_k)$. If $\limsup_{k\in K} \lambda^+(\nu_k) \leq \frac{\lambda^+(f)}{r}$ then we get by Ruelle's inequality

$$\limsup_{k\in\mathbb{N}} h_{top}(f_k) = \lim_{k\in K} h(\nu_k) \leq \frac{\lambda^+(f)}{r} \leq h_{top}(f).$$

By extracting a subsequence we can therefore assume $\lim_k \lambda^+(\nu_k) > \frac{\lambda^+(f)}{r}$ and that $(\nu_k)_k$ is converging to a measure μ in the weak-$*$ topology (this case always holds when $h_{top}(f) > \frac{\lambda^+(f)}{r}$). We can then apply the Main Theorem for some $\alpha \in]\frac{\lambda^+(f)}{r}, \min(\lim_k \lambda^+(\nu_k), h_{top}(f))[$ to get for some f-invariant measure μ_1 and $\beta \in [0,1]$:

$$\begin{aligned}\limsup_k h_{top}(f_k) &= \lim_k h(\nu_k),\\ &\leq \beta h(\mu_1) + (1-\beta)\alpha,\\ &\leq h_{top}(f).\end{aligned}$$

In fact, by the lower semi-continuity, we have necessarily $\beta = 1$ and $h(\mu_1) = h_{top}(f)$. □

The corresponding result was proved for interval maps in [5] by using a different method. We also refer to [5] for counterexamples of the upper semi-continuity property for interval maps f with $h_{top}(f) < \frac{\lambda^+(f)}{r}$. Finally, in [7], the author built, for any $r > 1$, a $\mathcal{C}^r$ surface diffeomorphism f with $\limsup_{g\xrightarrow{\mathcal{C}^r} f} h_{top}(g) = \frac{\lambda^+(f)}{r} > h_{top}(f) = 0$. We recall also that upper semi-continuity of the topological entropy in the $\mathcal{C}^\infty$ topology was established in any dimension by Y. Yomdin [16].

2. Main steps of the proof

We follow the strategy of the proof of [10]. We point out below the main differences:

- *hyperbolic and neutral empirical component.* For $\lambda^+(\nu_k) > \alpha > 0$ we split the orbit of a ν_k-typical point x into the *hyperbolic* part and its complement the *neutral* part. The hyperbolic part is given by the integers k lying between to M-close consecutive α-hyperbolic times for the sequence $(\phi(F^k\hat{x}))_k$ with $\hat{x} = (x, \mathcal{E}_+(x))$. We then consider the associated M-empirical measures and their limit in k, then in M. In this way we get an invariant component of $\hat{\mu}$ (called the hyperbolic

component). This decomposition follows exactly the approach in [10]. But here $\alpha > 0$ is fixed, whereas α is taken close to zero in [10].

- *Entropy computations.* To compute the asymptotic entropy of the ν_k's in terms of the entropy of its neutral and hyperbolic empirical component, we use the static entropy w.r.t. partitions and its conditional version. Instead the authors in [10] used Katok's like formulas and combinatorial arguments.
- $\mathcal{C}^r$ *Reparametrizations.* Finally to estimate the entropy of the neutral component and the local entropy of the hyperbolic component we use here reparametrization methods from [4] and [2] respectively rather than Yomdin's reparametrizations of the projective action F as done in [10]. This is the principal difference with [10].

2.1. **Hyperbolic times and associated empirical measures.** In this section we mostly follow the presentation of the neutral decomposition of Section 6 in [10]. We work with the map F induced on the projective tangent bundle and with the continuous observable ϕ, but the same construction may be applied to any continuous observable of a topological system.

We let H be the α-hyperbolic set and $E(\hat{x})) \subset \mathbb{Z}$ for $\hat{x} \in \mathbb{P}TM$ be the associated set of α-hyperbolic times at $\hat{x}$:

$$H = \left\{\hat{x} \in \mathbb{P}TM,\ \forall l > 0\ \phi_l(F^{-l}\hat{x}) \geq \alpha l\right\}\},$$

$$E(\hat{x}) = \{n \in \mathbb{Z},\ F^n\hat{x} \in H\}\,.$$

When $a < b$ are two consecutive times in $E(\hat{x})$, then $[a, b[$ is a *neutral block* as defined in [10], i.e. $\phi_k(F^a\hat{x}) < k\alpha$ for all $1 \leq k < \cdots < b - a$. From $\phi_{b-a-1}(F^a\hat{x}) < (b-a-1)\alpha$ and $\phi_{b-a}(F^a\hat{x}) \geq (b-a)\alpha$ we get:

$$|\phi_{b-a}(F^a\hat{x}) - (b-a)\alpha| \leq \|\phi\|_\infty. \tag{2.1}$$

For all M we let

$$E^M(\hat{x}) = \bigcup_{a<b\in E(\hat{x}),\ |a-b|\leq M} [a, b[.$$

The complement of $E^M(\hat{x})$ is made of neutral blocks of length larger than M. We consider the associated empirical measures :

$$\forall n,\ \hat{\mu}^M_{\hat{x},n} = \frac{1}{n} \sum_{k\in E^M(\hat{x})\cap[0,n[} \delta_{F^k\hat{x}}.$$

Let ν be an ergodic measure with $\lambda^+(\nu) > \alpha > 0$. By applying the Ergodic Maximal Inequality (see e.g. Theorem 1.1 in [1]) to the measure preserving system $(F^{-1}, \hat{\nu}^+)$ with the observable $\psi = \alpha - \phi \circ F^{-1}$, we get with $A = \{\hat{x},\ \exists k \geq 1\ \sum_{l=0}^{k-1} \psi(F^{-l}\hat{x}) > 0\}$:

$$\int_A \psi\, d\hat{\nu}^+ \geq 0.$$

But $A = \mathbb{P}TM \setminus H$, therefore $\int_H (\alpha - \phi \circ F^{-1})\, d\hat{\nu}^+ \leq \int (\alpha - \phi \circ F^{-1})\, d\hat{\nu}^+ = \alpha - \lambda^+(\nu)$. In particular we have $\hat{\nu}^+(H) > \frac{\lambda^+(\nu)-\alpha}{\|\phi\|_\infty} > 0$. Let χ^M be the indicator

function of $\{\hat{x}, 0 \in E^M(\hat{x})\}$. By Birkhoff ergodic theorem, there is a set $\mathtt{G}$ of full ν-measure such that the empirical measures $\left(\hat{\mu}^M_{\hat{x},n}\right)_n$ with $\hat{x} = (x, \mathcal{E}_+(x))$ are converging for any $x \in \mathtt{G}$ and any $M \in \mathbb{N}^*$ to $\hat{\xi}^M := \chi^M \hat{\nu}^+$ in the weak-$*$ topology. We also let $\hat{\eta}^M = \hat{\nu}^+ - \hat{\xi}^M$. Moreover we put $\beta_M = \int \chi^M \, d\hat{\nu}^+$, then $\hat{\xi}^M = \beta_M \cdot \underline{\hat{\xi}}^M$ and $\hat{\eta}^M = (1-\beta_M) \cdot \underline{\hat{\eta}}^M$ with $\underline{\hat{\xi}}^M$, $\underline{\hat{\eta}}^M$ being thus probability measures. Finally in the next sections we will write ξ^M, η^M, $\mu^M_{x,n}$, ... for the push-forward measures on M of $\hat{\xi}^M$, $\hat{\eta}^M$, $\hat{\mu}^M_{\hat{x},n}$,...

Lemma 1. *The empirical measures $\hat{\xi}^M$ and $\hat{\eta}^M$ satisfy the following properties:*

(1) $\hat{\xi}^M \geq \hat{\xi}^N$ for $M \geq N$.
(2) $\beta_M > 0$ for M large enough,
(3) $\left|\int \phi \, d\hat{\eta}^M - \alpha\right| \leq \frac{\|\phi\|_\infty}{M}$,
(4) for $\underline{\hat{\xi}}^M(K_M) = 1$ with $K_M := \{\hat{x},\ \exists 0 \leq m \leq M$ with $\phi_m(\hat{x}) \geq m\alpha\}$.

Proof. The first item follows obviously from the definitions. The second item follows from $\hat{\nu}^+(H) = \lim_n \frac{1}{n} \sharp E(\hat{x}) \cap [1,n] > 0$ for $\hat{\nu}^+$ a.e. $\hat{x}$ and the inequality $\limsup_n \frac{1}{n} \sup_{\hat{x}} \sharp \left(E(\hat{x}) \setminus E^M(\hat{x})\right) \cap [1,n] < 1/M$. To prove the third item, it is enough to sum Inequality (2.1) over all neutral blocks $[a,b[$ in the complement set of $E^M(\hat{x})$. The last item follows immediately from the definition of the empirical measure $\hat{\xi}^M$ and the compactness of K_M.

□

For $\hat{x} \in \mathbb{P}TM$ we let $m_n(\hat{x}) = \max E(\hat{x}) \cap [0,n[$.

Lemma 2.

$$\frac{m_n(\hat{x})}{n} \xrightarrow{n} 1 \text{ for } \hat{\nu}^+ \text{ a.e. } \hat{x}.$$

Proof. For $\hat{\nu}^+$ a.e. $\hat{x}$, we have $\epsilon_n(\hat{x}) := \frac{1}{n}\phi_n(\hat{x}) - \lambda^+(\nu) \xrightarrow{n} 0$ and $m_n(\hat{x}) \xrightarrow{n} +\infty$. Then by definition of $m_n(\hat{x})$, the interval of integers $[m_n(\hat{x}), n]$ lies in a neutral block starting at $m_n(\hat{x})$, therefore $\phi_{n-m_n(\hat{x})}(F^{m_n(\hat{x})}\hat{x}) \leq \alpha(n - m_n(\hat{x}))$. Consequently

$$\begin{aligned}
\phi_n(\hat{x}) - \phi_{m_n(\hat{x})}(\hat{x}) &= \phi_{n-m_n(\hat{x})}(F^{m_n(\hat{x})}\hat{x}),\\
(n - m_n(\hat{x}))\lambda^+(\nu) + n\epsilon_n(\hat{x}) - m_n(\hat{x})\epsilon_{m_n(\hat{x})}(\hat{x}) &\leq \alpha(n - m_n(\hat{x})),\\
\frac{1}{n}(\lambda^+(\nu) - \alpha)(n - m_n(\hat{x})) &\leq |\epsilon_n(\hat{x})| + |\epsilon_{m_n(\hat{x})}(\hat{x})| \xrightarrow{n} 0.
\end{aligned}$$

□

2.2. **Entropy of conditional measures.** A measurable partition ς is *subordinated* to the Pesin unstable local lamination W^u of ν if the atom of ς containing x is a neighborhood of x inside the curve $W^u(x)$ and $f^{-1}\varsigma \succ \varsigma$. By Rokhlin's disintegration theorem, there is for ν-a.e. x a probability measure ν_x on $\varsigma(x)$, called the condiational measures on unstable manifolds, satisfying

$\nu = \int \nu_x \, d\nu(x)$. Ledrappier-Young [12] proved the existence of such subordinated measurable partitions and that for ν-a.e. x, we have with $B_n(x,\rho)$ being the Bowen ball $B_n(x,\rho) = \bigcap_{0\le k<n} f^{-k}B(f^k x,\rho)$:

$$\lim_{\rho\to 0}\liminf_n -\frac{1}{n}\log \nu_x\left(B_n(x,\rho)\right) = h(\nu). \tag{2.2}$$

Fix an error term δ depending[†] on ν. There is $\rho>0$ and a measurable set $\mathtt{F}\subset \mathtt{G}$ with $\nu(\mathtt{F})>0$ such that the empirical measures $\left(\hat{\mu}^M_{\hat{x},n}\right)_n$ are converging uniformly in $x\in\mathtt{F}$ to $\hat{\xi}^M$, i.e. for all continuous function $\psi : \mathbb{P}TM \to \mathbb{R}$

$$\frac{1}{n}\sum_{k\in E^M(\hat{x})\cap[1,n[} \psi(F^k\hat{x}) \xrightarrow{n} \int \psi \, d\xi^M \text{ uniformly in } x\in\mathtt{F} \tag{2.3}$$

and such that

$$\forall x\in\mathtt{F}, \liminf_n -\frac{1}{n}\log \nu_x\left(B_n(x,\rho)\right) \ge h(\nu)-\delta.$$

We fix $x_*\in\mathtt{F}$ with $\nu_{x_*}(\mathtt{F})>0$ and we let ζ be the probability measure induced by ν_{x_*} on $\mathtt{F}$. We let D be the $\mathcal{C}^r$ curve given by the Pesin local unstable manifold $W^u(x_*)$ at x_*. For a finite measurable partition P and a Borel probability measure μ we let $H_\mu(P)$ be the static entropy, $H(\mu) = -\sum_{A\in P}\mu(A)\log\mu(A)$. For a partition P of M, we let $P^n = \bigvee_{k=0}^{n-1} f^{-k}P$ be the n-iterated partition, $n\in\mathbb{N}$. We also let P_x be the atom of P containing the point $x\in M$.

Lemma 3. *For any (finite measurable) partition P with diameter less than ρ, we have*

$$\liminf_n \frac{1}{n}H_\zeta(P^n) \ge h(\nu)-\delta.$$

Proof.

$$\begin{aligned}
\liminf_n \frac{1}{n}H_\zeta(P^n) &= \liminf_n \int -\frac{1}{n}\log\zeta(P^n_x)\,d\zeta(x), \text{ by definition of } H_\zeta,\\
&\ge \int \liminf_n -\frac{1}{n}\log\zeta(P^n_x)\,d\zeta(x), \text{ by Fatou's Lemma},\\
&\ge \int \liminf_n -\frac{1}{n}\log\nu_x(P^n_x)\,d\zeta(x), \text{ by definition of } \zeta,\\
&\ge \int \liminf_n -\frac{1}{n}\log\nu_x(B_n(x,\rho))\,d\zeta(x), \text{ as } \operatorname{diam}(P)<\rho,\\
&\ge h(\nu)-\delta, \text{ by the choice of } \mathtt{F}.
\end{aligned}$$

□

[†]In the proof of the Main Theorem we will take $\delta = \delta(\nu_k) \xrightarrow{k} 0$ for the converging sequence of ergodic measures $(\nu_k)_k$.

2.3. **Entropy splitting of the neutral and the hyperbolic component.** In this section we split the entropy contribution of the neutral and hyperbolic components. We consider a partition P with $\nu(\partial P)=0$ and with diameter less than ρ (with ρ as in Lemma 3). Fix M. For each $n\in\mathbb{N}$ and $x\in \mathtt{F}$ we let $E_n^M(x)=E^M(\hat{x})\cap[0,n[$. We also let $\mathtt{E}_n^M$ be the partition of $\mathtt{F}$ with atoms $A_E:=\{x\in D,\, E_n^M(x)=E\}$ for $E\subset[0,n[$. Given a partition Q of M, we also let $Q^{\mathtt{E}_n^M}$ be the partition of $\mathtt{F}$ finer than $\mathtt{E}_n^M$ with atoms $\left\{x\in\mathtt{F},\, E_n^M(x)=E \text{ and } \forall k\in E,\ f^kx\in Q_k\right\}$ for $E\subset[0,n[$ and $(Q_k)_{k\in E}\in Q^E$. For a measure η and a subset A of M with $\eta(A)>0$ we denote by $\eta_A=\frac{\eta(A\cap\cdot)}{\eta(A)}$ the induced probability measure on A. Moreover, for sets A,B we let $A\Delta B$ denote the symmetric difference of A and B, i.e. $A\Delta B=(A\backslash B)\cup(B\backslash A)$.

Lemma 4. *For any partition Q with $\xi^M(\partial Q)=0$ we have*

$$h(\nu)\le \beta_M\frac{1}{m}H_{\underline{\xi}^M}(Q^m)+\limsup_n\frac{1}{n}H_\zeta(P^n|Q^{\mathtt{E}_n^M})+\frac{2}{M}+\frac{12\log\sharp Q}{M}+\delta. \tag{2.4}$$

Proof. As the complement of $E_n^M(x)$ is the disjoint union of neutral blocks with length larger than M, there are at most $A_n^M=\sum_{k=0}^{[2n/M]+1}\binom{n}{k}$ possible values for $E_n^M(x)$ so that

$$\begin{aligned}\frac{1}{n}H_\zeta(P^n)&=\frac{1}{n}H_\zeta(P^n|\mathtt{E}_n^M)+H_\zeta(\mathtt{E}_n^M),\\&\le\frac{1}{n}H_\zeta(P^n|\mathtt{E}_n^M)+\log A_n^M,\\ \liminf_n\frac{1}{n}H_\zeta(P^n)&\le\limsup_n\frac{1}{n}H_\zeta(P^n|\mathtt{E}_n^M)+\frac{2}{M}.\end{aligned}$$

Then

$$\frac{1}{n}H_\zeta(P^n|\mathtt{E}_n^M)\le\frac{1}{n}H_\zeta(Q^{\mathtt{E}_n^M}|\mathtt{E}_n^M)+\frac{1}{n}H_\zeta(P^n|Q^{\mathtt{E}_n^M}).$$

For $E\subset[0,n[$ we let $\zeta_{E,n}=\frac{n}{\sharp E}\int\mu_{x,n}^M\,d\zeta_{A_E}$. Observe that $\zeta_{E,n}$ and $\sharp E/n$ are converging to $\underline{\xi}^M$ and β_M respectively uniformly in E when n goes to infinity (as $\zeta(\mathtt{F})=1$). By Lemma 6 in [2] we have for all $n>m\in\mathbb{N}^*$:

$$\begin{aligned}H_\zeta\left(Q^{\mathtt{E}_n^M}|\mathtt{E}_n^M\right)&=\sum_E\zeta(A_E)H_{\zeta_{A_E}}(Q^E),\\&\le\sum_E\zeta(A_E)\sharp E\left(\frac{1}{m}H_{\zeta_{E,n}}(Q^m)+6m\log\sharp Q\frac{\sharp(E+1)\Delta E}{\sharp E}\right).\end{aligned}$$

Then as $\xi^M(\partial Q)=0$ we get by taking the limit in n:

$$\begin{aligned}\limsup_n\frac{1}{n}H_\zeta\left(Q^{\mathtt{E}_n^M}|\mathtt{E}_n^M\right)\le&\beta_M\frac{1}{m}H_{\underline{\xi}^M}(Q^m)+\frac{12\log\sharp Q}{M},\\ h(\nu)-\delta\le\liminf_n\frac{1}{n}H_\zeta(P^n)\le&\beta_M\frac{1}{m}H_{\underline{\xi}^M}(Q^m)+\limsup_n\frac{1}{n}H_\zeta(P^n|Q^{\mathtt{E}_n^M})\\&+\frac{2}{M}+\frac{12\log\sharp Q}{M}.\end{aligned}$$

□

2.4. **Bounding the entropy of the neutral component.** In this section we assume the ergodic measure ν satisfies $\lambda^+(\nu) > \alpha > \frac{\lambda^+(f)}{r}$. We bound the term $\limsup_n \frac{1}{n} H_\zeta(P^n | Q^{\mathtt{E}_n^M})$ in the right member of Lemma 4, which corresponds to the local entropy contribution plus the entropy in the neutral part. In the next statement, for any $\epsilon > 0$ we let $\|df^p\|_\epsilon : x \mapsto \|d_x f^p\|_\epsilon := \sup_{y \in B(x,\epsilon)} \|d_y f^p\|$.

Lemma 5. *For all $p, M \in \mathbb{N}^*$, there are $\epsilon_p > 0$ (depending only on p, $\alpha - \frac{\lambda^+(f)}{r}$ and the $\mathcal{C}^r$ norm of f) and $\gamma_{p,M} > 0$ with*

$$\limsup_p \limsup_M \gamma_{p,M} = 0$$

such that for any partition Q with diameter less than $\epsilon < \epsilon_p$, we have:

$$\begin{aligned}\limsup_n \frac{1}{n} H_\zeta(P^n | Q^{\mathtt{E}_n^M}) \leq & (1 - \beta_M)\alpha \\ & + \left(\log 2 + \frac{1}{r-1}\right)\left(\int \frac{\log^+ \|df^p\|_\epsilon}{p} d\xi^M - \int \phi \, d\hat{\xi}^M\right) \\ & + \gamma_{p,M}.\end{aligned}$$

We will prove Lemma 5 assuming a *semi-local Reparametrization Lemma* (Proposition 3). Let $r > 1$. Following [2] a curve $\sigma : [-\epsilon, \epsilon] \to M$ is said (resp. strongly) (ϵ, r)-bounded when the image of σ is contained in the ball centered at $\sigma(0)$ with radius equal to the radius of injectivity of the Riemannian surface and $\tilde{\sigma} := \exp_{\sigma(0)}^{-1} \circ \sigma$ satisfies $\max_{k=1,\cdots,r} \|d^k \tilde{\sigma}\| \leq \|d\tilde{\sigma}\|$ (resp. ≤ 1). The curve σ is said strongly (ϵ, r, n)-bounded for $n \in \mathbb{N}^*$ if $f^k \circ \sigma$ is (ϵ, r)-bounded for all $k = 0, \cdots, n-1$. When σ is strongly (ϵ, r, n)-bounded, then the distortion is uniformly bounded along σ, i.e. for all $\hat{x} = (x, v_x)$, $\hat{y} = (y, v_y)$ in $\mathbb{P}TM$ tangent to σ and for all $k = 0, \cdots, n-1$ we have $\left|\phi_{n-k}(F^k \hat{x}) - \phi_{n-k}(F^k \hat{y})\right| \leq \log 2$ for $\epsilon < 1/100$ by Lemma 8 in [2].

Lemma 6. *For any $0 < \alpha' < \alpha$, there exists $C > 0$ such that for any strongly (ϵ, r, n)-bounded subcurve σ of D intersecting $\mathtt{F}$, the length of $f^k \circ \sigma$ is less than $Ce^{-\alpha'(n-k)}$.*

Proof. Let $x \in \mathtt{F} \cap \mathrm{Im}(\sigma)$ and $\hat{x} = (x, \mathcal{E}_+(x))$. For $0 \leq k \leq m_n(x)$, we have $\phi_{m_n(\hat{x})-k}(F^k \hat{x}) \geq \alpha(m_n(\hat{x}) - k)$ as $m_n(\hat{x})$ belongs to $E(\hat{x})$. Therefore the length of $f^k \circ \sigma$ is less than $8\epsilon e^{\alpha(k - m_n(x))}$ by the aforementioned bounded distorsion property. Finally, observe that $\frac{m_n(\hat{x})}{n}$ is converging uniformly to 1 on $\mathtt{F}$ by Lemma 2 and the definition of $\mathtt{F}$. □

Proposition 3. *For all $p \in \mathbb{N}^*$ there are $\epsilon_p > 0$ (depending only on p, $\alpha - \frac{\lambda^+(f)}{r}$ and the $\mathcal{C}^r$ norm of f) and $\gamma_{p,M} > 0$ with*

$$\limsup_p \limsup_M \gamma_{p,M} = 0$$

such that for any partition Q with diameter less than $\epsilon < \epsilon_p$, the following property holds for n large enough.

The intersection of any atom F_n of the partition $Q^{\mathtt{E}_n^M}$ with $D \cap \mathtt{F}$ may be covered by a family Ψ_{F_n} of strongly (ϵ, r, n)-bounded curves satisfying

$$\frac{1}{n}\log \sharp \Psi_{F_n} \leq \left(1 - \frac{\sharp E_n}{n}\right)\alpha + \left(\log 2 + \frac{1}{r-1}\right)\left(\int \frac{\log^+ \|d_x f^p\|_\epsilon}{p}\, d\zeta_{F_n}^M(x) - \int \phi\, d\hat{\zeta}_{F_n}^M\right) + \gamma_{p,M}.$$

where $E_n = E_n^M(x)$ for $x \in F_n$, $\hat{\zeta}_{F_n}^M = \frac{1}{n}\int \hat{\mu}_{\hat{x},n}\, d\zeta_{F_n}(x)$ and $\zeta_{F_n}^M$ its push-forward on M.

The proof of Proposition 3 is given in the last section. Proposition 3 is very similar to the Reparametrization Lemma in [4]. Here we reparametrize $D \cap \mathtt{F} \cap F_n$ for F_n an atom of $Q^{\mathtt{E}_n}$ instead of Q^n in [4]. The arguments are the same as in [4] for the hyperbolic part, whereas in the neutral part we use the bounded geometry property at hyperbolic times established in [2].

Proof of Lemma 5. First observe that

$$H_\zeta(P^n | Q^{\mathtt{E}_n}) \leq \sum_{F_n \in Q^{\mathtt{E}_n}} \zeta(F_n) \log \sharp\{A^n \in P^n,\ A^n \cap D \cap \mathtt{F} \cap F_n \neq \emptyset\}.$$

As $\nu(\partial P) = 0$, for all $\gamma > 0$, there is $\chi > 0$ and a continuous function $\psi : X \to \mathbb{R}^+$ equal to 1 on the χ-neighborhood ∂P^χ of ∂P satisfying $\int \psi\, d\nu < \gamma$. Then we have uniformly in $x \in \mathtt{F}$:

$$\limsup_n \frac{1}{n}\sharp\{0 \leq k < n,\ f^k x \in \partial P^\chi\} \leq \lim_n \frac{1}{n}\sum_{k=0}^{n-1} \psi(f^k x) < \gamma. \tag{2.5}$$

Assume that for arbitrarily large n there is $F_n \in Q^{\mathtt{E}_n^M}$ and $\sigma \in \Psi_{F_n}$ with $\sharp\{A^n \in P^n,\ A^n \cap \mathrm{Im}(\sigma) \neq \emptyset\} > \sharp P^{\gamma n}$. Then we have $\sharp\{0 \leq k < n,\ \partial P \cap \mathrm{Im}(f^k \circ \sigma) \neq \emptyset\} > \gamma n$. By Lemma 6, the length of $f^k \circ \sigma$ is less than χ for all $0 \leq k \leq n - C$ for some constant C independent of n. In particular we get $\sharp\{0 \leq k < n,\ f^k x \in \partial P^\chi\} > \gamma n - C$ for any $x \in \mathrm{Im}(\sigma)$, which contradicts (2.5). Therefore we have

$$\limsup_n \sup_{F_n,\, \sigma \in \Psi_{F_n}} \frac{1}{n}\log\{A^n \in P^n,\ A^n \cap \mathrm{Im}(\sigma) \neq \emptyset\} = 0.$$

Together with Proposition 3 this concludes the proof of Lemma 5.

□

2.5. **Proof of the Main Theorem.** Let $(\hat{\nu}_k^+)_k$ be a sequence of ergodic f_k-invariant measures converging to $\hat{\mu}$ with $\int \phi\, d\hat{\mu} = \lim_k \lambda^+(\nu_k) > \alpha > \frac{\lambda^+(f)}{r}$. Without loss of generality we can assume $\liminf_k h(\nu_k) > 0$. For μ a.e. x, we have $\lambda^-(x) \leq 0$. If not, some ergodic component $\tilde{\mu}$ of μ would have two positive Lyapunov exponents and therefore should be the periodic measure at a source. But then ν_k would be equal to $\tilde{\mu}$ for large k contradicting $h(\nu_k) > 0$.

There is a subsequence in k such that $\hat{\xi}_k^M \to \hat{\xi}_\infty^M$ for all M. Let us also denote by β_M^∞ the limit of β_M^k. Then consider a subsequence in M such that $\hat{\xi}_\infty^M$ is converging to $\beta\hat{\mu}_1 (= \beta_\alpha \hat{\mu}_{1,\alpha})$ with $\beta = \lim_M \beta_M^\infty$. We also let $(1-\beta)\hat{\mu}_0 = \hat{\mu} - \beta\hat{\mu}_1$. In this way, $\hat{\mu}_0$ and $\hat{\mu}_1$ are both probability measures.

Lemma 7. *The measures $\hat{\mu}_0$ and $\hat{\mu}_1$ satisfy the following properties:*

- *$\hat{\mu}_1$ is F-invariant,*
- *$\lambda^+(x) \geq \alpha$ for μ_1-a.e. x and $\hat{\mu}_1 = \hat{\mu}_1^+$,*
- *$\int \phi \, d\hat{\mu}_0 = \alpha$.*

Proof. For any continuous function $\psi : \mathbb{P}TM \to \mathbb{R}$ and for any k, we have

$$\left| \int \psi \, d\hat{\xi}_k^M - \int \psi \circ F \, d\hat{\xi}_k^M \right| \leq \frac{2 \sup_{\hat{x}} |\psi(\hat{x})|}{M}.$$

Letting k, then M go to infinity, we get $\int \psi \, d\hat{\mu}_1 = \int \psi \circ F \, d\hat{\mu}_1$, i.e. $\hat{\mu}_1$ is F-invariant. In particular $\hat{\mu}_1$ is supported on the Oseledets bundles. But the set K_M being compact, we have $\hat{\xi}_\infty^M(K_M) \geq \limsup_k \hat{\xi}_k^M(K_M) = \beta_M^\infty$ by Lemma 1 (4). Therefore we have $\hat{\mu}_1(\bigcup_M K_M) = 1$ as $\hat{\xi}_\infty^M$ goes increasingly in M to $\hat{\mu}_1$ by Lemma 1 (1). The F-invariant set $\bigcap_k F^{-k}(\bigcup_M K_M)$ has also full $\hat{\mu}_1$-measure and for all $\hat{x} = (x, v)$ in this set we have $\limsup_n \frac{1}{n} \log \|d_x f^n(v)\| \geq \alpha$. Consequently the measure $\hat{\mu}_1$ is supported on the unstable bunddle and $\lambda^+(x) \geq \alpha$ for μ_1-a.e. x. To check finally $\int \phi \, d\hat{\mu}_0 = \alpha$, it is enough to let M go to infinity in the third item of Lemma 1.

□

Remark 4. *The measure $\hat{\mu}_{1,\alpha}^+$ increases when α decreases to zero. In Theorem C of* [10]*, the measure $\hat{\mu}_1^+$ is obtained as the limit of $\hat{\mu}_{1,\alpha}^+$ when α goes to zero.*

We pursue now the proof of the Main Theorem. Fix a sequence $(\delta_k)_k$ of positive numbers with $\delta_k \overset{k}{\to} 0$. Let $p, M \in \mathbb{N}^*$. We consider a partition Q satisfying $\operatorname{diam}(Q) < \epsilon < \epsilon_p$ with ϵ_p as in Lemma 5. As $\alpha > \frac{\lambda^+(f)}{r} \geq \limsup_k \frac{\lambda^+(f_k)}{r}$, we may extract a subsequence such that $\alpha - \frac{\lambda^+(f_k)}{r}$ is bounded away from zero. The sequence $(f_k)_k$ being morevoer $\mathcal{C}^r$ bounded, one can choose ϵ_p independently of f_k, $k \in \mathbb{N}$.

We may also assume ∂Q has zero-measure for μ_1 and all the measures ξ_k^M, $M \in \mathbb{N}^*$ and $k \in \mathbb{N} \cup \{\infty\}$. Combining Lemma 5 and Lemma 4 we get for $\epsilon < \epsilon_p$ and for some constant[‡] $\gamma_{p,Q,M}$ with $\limsup_p \sup_Q \limsup_M \gamma_{p,Q,M} = 0$:

$$\begin{aligned} h(\nu_k) \leq & \beta_M^k \frac{1}{m} H_{\underline{\xi_k}^M}(Q^m) + (1 - \beta_M^k)\alpha \\ & + \left(\log 2 + \frac{1}{r-1}\right) \left(\int \frac{\log^+ \|df_k^p\|_\epsilon}{p} d\xi_k^M - \int \phi \, d\hat{\xi_k}^M \right) \\ & + \gamma_{p,Q,M} + \delta_k. \end{aligned}$$

[‡] $\gamma_{p,Q,M} = \gamma_{p,M} + \frac{2}{M} + \frac{12 \log \sharp Q}{M\vec{\beta}}$

By letting k, then M go to infinity, we obtain for all m:

$$\begin{aligned}\limsup_k h(\nu_k) \leq & \beta\frac{1}{m}H_{\mu_1}(Q^m) + (1-\beta)\alpha \\ & + \left(\log 2 + \frac{1}{r-1}\right)\left(\int \frac{\log^+ \|df^p\|_\epsilon}{p} d\mu_1 - \int \phi\, d\hat{\mu}_1^+\right) \\ & + \limsup_M \gamma_{p,Q,M}.\end{aligned}$$

By letting m go to infinity, then $\epsilon < \epsilon_p$ go to 0, we get:

$$\begin{aligned}\limsup_k h(\nu_k) \leq & \beta h(\mu_1) + (1-\beta)\alpha \\ & + \left(\log 2 + \frac{1}{r-1}\right)\left(\int \frac{\log^+ \|df^p\|}{p} d\mu_1 - \int \phi\, d\hat{\mu}_1^+\right) \\ & + \sup_Q \limsup_M \gamma_{p,Q,M}.\end{aligned}$$

But $\int \phi\, d\hat{\mu}_1^+ = \lambda^+(\mu_1) = \lim_p \int \frac{\log^+ \|df^p\|}{p} d\mu_1$. Therefore by letting p go to infinity we finally obtain :

$$\limsup_k h(\nu_k) \leq \beta h(\mu_1) + (1-\beta)\alpha.$$

3. Dynamical Reparametrizations with bounded distorsion

In this section we establish a $\mathcal{C}^r$, $+\infty > r > 1$, version of the bounded geometry property at hyperbolic times established in [2]. Then we prove the semi-local *Reparametrization Lemma* stated in Proposition 3. We first recall some terminology from [2].

3.1. **Control of $\mathcal{C}^r$ norms with Landau-Kolmogorov inequality.** We let exp and $R = R_{inj}$ be respectively the exponential map and the radius of injectivity of the smooth Riemanian manifold $(M, \|\cdot\|)$. For $x \in M$ we let for a given diffeomorphism f on M

$$f^x = \exp_{fx}^{-1} \circ f \circ \exp_x : B(R) \subset T_xM \sim \mathbb{R}^2 \to T_{fx}M \sim \mathbb{R}^2.$$

The $\mathcal{C}^r$ norm of f is defined as $\|f\|_r = \sup_{x\in X} \|f_\epsilon^x\|_r$ with

$$\|f_\epsilon^x\|_r = \max\left(\max_{\mathbb{N}\ni s\leq r} \|d^s f^x\|_0, \|d^{[r]} f\|_{r-[r]}\right),$$

where $\|g\|_s$, $s > 0$, denotes the s-Hölder norm of a function g and $\|g\|_0$ its supremum norm. A sequence $(f_k)_k$ of C^r diffeomorphisms on M is said $\mathcal{C}^r$ bounded when $\sup_k \|f_k\|_r < \infty$.

There is $\epsilon > 0$ depending only on f so small that for any $x \in M$ the map $f_\epsilon^x = f^x(\epsilon\cdot)$ satisfies $\|f_\epsilon^x\|_r = \|df_\epsilon^x\|$. For any $1 \geq b > 0$ we let $\psi_b : [-\epsilon, \epsilon] \circlearrowleft$, be the linear map $x \mapsto bx$. For any (ϵ, r)-bounded curve $\sigma : [-\epsilon, \epsilon] \to M$ with $\sigma(0) = x$ we let $\sigma_\epsilon = \epsilon^{-1} \exp_x^{-1} \circ \sigma$.

Lemma 8. *Let σ be a strongly (ϵ, r)-bounded curve with $\sigma(0) = x$ and v_x the projective vector tangent to σ at x. There exists a constant $C = C(r) > 1$ depending only on r and $(M, \|\cdot\|)$ such that the map $f \circ \sigma \circ \psi_a : [-\epsilon, \epsilon] \to M$ is (ϵ, r)-bounded with*

$$a = a(x, f) = C^{-1} \left(\|d_x f\|_\epsilon / \|d_x f(v_x)\| \right)^{-1/r-1}.$$

Proof. We assume $r \in \mathbb{N}N$. The easy adpatations to the general case are left to the reader. We compute for some constants $C_1 > \cdots > C_4$ depending only on r and $(M, \|\cdot\|)$ and with $b = (\|d_x f\|_\epsilon / \|d_x f(v_x)\|)^{-1/r-1}$:

$$\begin{aligned}
d^r(\exp_{fx}^{-1} \circ f \circ \sigma \circ \psi_b) &= d^r(\exp_{fx}^{-1} \circ f \circ \sigma) b^r, \\
\|d^r(\exp_{fx}^{-1} \circ f \circ \sigma \circ \psi_b)\| &\leq \frac{C_4}{2} \|f_\epsilon^x\|_r \|\sigma_\epsilon\|_r b^r, \text{ by Fa di Bruno's formula,} \\
&\leq \frac{C_4}{2} \|f_\epsilon^x\|_r \epsilon^{-1} \| \exp_x^{-1} \circ \sigma\|_r b^r, \\
&\leq \frac{C_4}{2} \|df_\epsilon^x\| \epsilon^{-1} \| \exp_x^{-1} \circ \sigma\|_r b^r, \text{ by the choice of } \epsilon, \\
&\leq \frac{C_4}{2} \|df_\epsilon^x\| \epsilon^{-1} \|d(\exp_x^{-1} \circ \sigma)\| b^r, \text{ since } \sigma \text{ is } (\epsilon, r)\text{-bounded,} \\
&\leq \frac{C_3}{2} b^{r-1} \|d_x f\|_\epsilon \|d\sigma\| b, \\
&\leq C_3 \|d_x f(v_x)\| \frac{\|d\sigma\|}{2} b, \text{ by definition of } b, \\
&\leq C_3 \|d_x f(v_x)\| \|\sigma'(0)\| b, \\
&\leq C_2 \|d(\exp_{fx}^{-1} \circ f \circ \sigma \circ \psi_b)\|.
\end{aligned}$$

Then by the Landau-Kolmogorov inequality (see [2]), we have :

$$\| \exp_{fx}^{-1} \circ f \circ \sigma \circ \psi_b\|_r \leq C_1 \|d(\exp_{fx}^{-1} \circ f \circ \sigma \circ \psi_b)\|.$$

By taking $a = b/C_1$ we get the desired result. □

3.2. **Bounded geometry at hyperbolic times k.** Let $\sigma : I \to M$, where I is an open bounded interval of $\mathbb{R}$, be a $\mathcal{C}^r$ embedded smooth curve with $\overline{D} \subset \mathrm{Im}(\sigma)$. For $x \in \mathrm{Im}(\sigma)$ and $n \in \mathbb{N}$ we let $I_n^x = I_n^x(\epsilon, \sigma, r)$ be the largest subinterval of I such that $\sigma \circ \psi_n^x$ is strongly (ϵ, n, r)-bounded, where ψ_n^x denotes the affine reparametrization $\psi_n^x = \psi_{n,r}^x : [-\epsilon, \epsilon] \to I_n^x$ with $\sigma \circ \psi_n^x(0) = x$. For $y \in \mathrm{Im}(\sigma)$ we let $D_k(y) (= D_{k,\sigma}^\epsilon(y)) := \mathrm{Im}(f^k \circ \sigma \circ \psi_k^y)$ and $B_k(y) := \mathrm{Im}(\sigma \circ \psi_k^y)$.

A (ϵ, r)-bounded curve σ with $\sigma(0) = x$ is said **ample** when $\|d\left(\exp_x^{-1} \circ \sigma\right)\| > 1/2$. Then the length of σ is larger $\frac{\epsilon}{2}$.

Recall that for $x \in D$ we let $\hat{x} = (x, v_x) \in \mathbb{P}TM$ with v_x being the line tangent to D at x.

Lemma 9. *Let $\alpha > \frac{\lambda^+(f)}{r}$. There exist $\epsilon > 0$ and $1 > \tau > 0$ depending only on α and f, and N depending on f, α and D such that for any $x \in D$ and any $n \in E(\hat{x})$ larger than N, the length of $D_k(x)$ is larger than $\tau\epsilon$.*

Proof. Assume n is a α-hyperbolic time for x with $\alpha > \frac{\lambda^+(f)}{r}$. Let C be the universal constant in Lemma 8. We may choose p so large that $b := \alpha - \frac{\log \|df^p\| + (r-1)\log C}{pr} > 0$. Finally take $\epsilon > 0$ with $\|(f^p)^x_\epsilon\|_r = \|d(f^p)^x_\epsilon\|$ for all $x \in M$.

Let m be the largest integer m' less than or equal to n such that $f^{m'} \circ \sigma \circ \psi^x_m$ is ample, i.e. $\left\| d\left(\exp^{-1}_{f^{m'}x} \circ f^{m'} \circ \sigma \circ \psi^x_{m'}\right)\right\| > 1/2$. If there is no such m', we let $m = 0$. For $L \in \mathbb{N}$ we let $A_L := \min_{0 \le l < L, x \in M} a(x, f^l)$. Let $K_0 = K_0(\alpha, f)$ and $K_1 = K_1(\alpha, f, D)$ be integers fixed later on.

- If $m > n - K_0 p$ and $m > 0$ then $\sigma \circ \psi^x_m(A_{Kp}\cdot)$ is strongly (ϵ, n, r)-bounded by Lemma 8. Indeed observe that for $j = m+1, \cdots, n$ we have $\|d(\exp^{-1}_{f^j x} \circ f^j \circ \sigma \circ \psi^x_m(A_{Kp}\cdot))\| \le 1/2$, otherwise this would contradict the definition of m. Moreover the length of $D_n(x)$ is larger than $\tau\epsilon$ with
$$\tau = \frac{A_{K_0 p}}{2} \min_{k < K_0 p} \|df^{-k}\|^{-1}.$$

- Consider now the case $k := n - m \ge N := K_0 p$ and $m > 0$. Write $n - m = qp + r$ with $0 \le r < p$. By Lemma 8 we have then with C as in Lemma 8:
$$\begin{aligned}
\frac{|\psi^x_n|}{|\psi^x_m|} &\ge \left(\frac{(C^{r-1}\|df^p\|)^{k/p}}{e^{\phi_k(F^m \hat{x})}}\right)^{-1/r-1}, \\
&\ge \left(\frac{(C^{r-1}\|df^p\|)^{1/p}}{e^{\alpha}}\right)^{-k/r-1}, \text{ as } n \text{ is a } \alpha\text{-hyperbolic time}, \\
&\ge e^{-k\alpha}\left(\frac{(C^{r-1}\|df^p\|)^{1/pr}}{e^{\alpha}}\right)^{-kr/r-1}, \\
&\ge e^{-k\alpha} e^{\frac{K_0 pbr}{r-1}}, \text{as } b = \alpha - \frac{\log\|df^p\| + (r-1)\log C}{pr} > 0. \qquad (3.1)
\end{aligned}$$

It follows that for others universal constant B:

$$\begin{aligned}
\|d(\exp^{-1}_{f^n x} \circ f^n \circ \sigma \circ \psi^x_n)\| &\ge B\|d_0(f^n \circ \sigma \circ \psi^x_n)\|, \\
&\ge Be^{k\alpha}\|d_0(f^m \circ \sigma \circ \psi^x_n)\|, \text{ as } n \text{ is a } \alpha\text{-hyperbolic time}, \\
&\ge Be^{k\alpha}\|d_0(f^m \circ \sigma \circ \psi^x_m)\| \times \frac{|\psi^x_n|}{|\psi^x_m|}, \text{ as } \psi^x_m, \psi^x_n \text{ are affine}, \\
&\ge Be^{\frac{K_0 pbr}{r-1}}\|d(f^m \circ \sigma \circ \psi^x_m)\|, \text{by (3.1)}, \qquad (3.2) \\
\|d(\exp^{-1}_{f^n x} \circ f^n \circ \sigma \circ \psi^x_n)\| &\ge Be^{\frac{K_0 pbr}{r-1}}\|d(\exp^{-1}_{f^m x} \circ f^m \circ \sigma \circ \psi^x_m)\|, \\
&\ge Be^{\frac{K_0 pbr}{r-1}}/2, \\
&\ge 1/2, \text{ by choosing } K_0 = K_0(\alpha, f) \text{ appropriately}.
\end{aligned}$$

Therefore the (ϵ, n, r)-bounded curve $\exp^{-1}_{f^n x} \circ f^n \circ \sigma \circ \psi^x_n$ is ample, contradicting the definition of m. Consequently, only this case can not occur.

- Finally assume $m = 0$ and $n > K_1$. We can follow the same lines as in the previous case till (3.2):

$$\|d(\exp^{-1}_{f^n x} \circ f^n \circ \sigma \circ \psi^x_n)\| \geq Be^{\frac{K_0 pbr}{r-1}} \|d(\sigma \circ \psi^x_0)\|,$$
$$\geq 1/2, \text{ by choosing } K_1 = K_1(\alpha, f, D) \text{ appropriately.}$$

Again the (ϵ, n, r)-bounded curve $\exp^{-1}_{f^n x} \circ f^n \circ \sigma \circ \psi^x_n$ is ample, contradicting the definition of m.

This concludes the proof by taking $N = pK_1(\alpha, f, D)$.

□

3.3. **Proof of Proposition 3.** Fix $p \in \mathbb{N}^*$ and $0 \leq q < p$. Without loss of generality we can take $M \gg p$. Let Q be a partition with diameter less than some $\epsilon < \epsilon_p$ with ϵ_p being fixed later on in the proof of Lemma 11. We consider $F_n \in Q^{\mathtt{E}^M_n}$ with $n > N$ (as defined in Lemma 9). We let E_n be the set $E^M_n(x)$ for any $x \in F_n$. Put $E^{p,q}_n = E_n \cap (q + p\mathbb{N})$. For a sequence of positive integers $\mathcal{K} = (k_i)_{i \in E^{p,q}_n}$, we let

$$\mathcal{H}(\mathcal{K}) := \left\{ y \in \mathtt{F},\ \forall i\ k_i = \left[\frac{\log^+ \|d_{f^i y} f^p\|}{\phi_p(F^i \hat{y})} \right] + 1 \right\}.$$

We first estimate the number of sequences $\mathcal{K}$ with $\mathcal{H}(\mathcal{K}) \cap F_n \neq \emptyset$, then reparametrize $\mathcal{H}(\mathcal{K}) \cap F_n$ for each $\mathcal{K}$. Let $\|df^{\pm}\| = \max(\|df\|, \|df^{-1}\|)$.

Lemma 10. *There is a sequence $(\varepsilon_n)_n$ with $\lim_n \varepsilon_n = 0$ depending only on $\mathtt{F}$ such that for any $y \in F_n$ we have with $\hat{\zeta}^M_{p,q,F_n} = \frac{1}{n} \int \sum_{k \in E^{p,q}_n} \delta_{F^k(x, v_x)}\, d\zeta_{F_n}(x)$ and ζ^M_{p,q,F_n} its push-forward on M:*

$$\frac{1}{n} \log_2 \sharp \{\mathcal{K},\ \mathcal{H}(\mathcal{K}) \cap F_n \neq \emptyset\} \leq \int \log^+ \|df^p\|_\epsilon d\zeta^M_{p,q,F_n} - \int \phi\, d\hat{\zeta}^M_{F_n} + \frac{4(p+1)\log \|df^{\pm}\|}{M} + \frac{1}{p} + \varepsilon_n. \tag{3.3}$$

Proof. First observe $E_n \Delta (\{0, \cdots, p-1\} + E^{q,p}_n) \subset \{-p+1, \cdots, p-1\} + E_n \Delta (E_n + 1)$. As $\sharp E_n \Delta (E_n + 1) \leq 2n/M$ we have also:

$$\sharp E_n \Delta (\{0, \cdots, p-1\} + E^{q,p}_n) \leq 4pn/M.$$

For all $y \in F_n$ we get:

- $|\sum_{i \in E^{q,p}_n} \phi_p(F^i \hat{y}) - \sum_{i \in E_n} \phi(F^i \hat{y})| \leq n \frac{4p \log \|df^{\pm}\|}{M}$, as observed above,
- $|\sum_{i \notin E_n} \phi(F^i \hat{y}) - \alpha(n - \sharp E_n)| \leq n \frac{\log \|df^{\pm}\|}{M}$, by Lemma 1 (3),
- $\varepsilon_n := 2 \sup_{y \in \mathtt{F}} |\sum_{0 \leq i \leq n} \phi(F^i \hat{y}) - n\lambda^+(\nu)| \xrightarrow{n} 0$ by (2.3).

By combining the two last inequalities we get for all $y, z \in F_n$:

$$\left| \sum_{i \in E_n} \phi(F^i \hat{y}) - \phi(F^i \hat{z}) \right| \leq n \left(\varepsilon_n + 2 \frac{\log \|df^{\pm}\|}{M} \right),$$

then by integrating $\hat{z}$ with respect to $\hat{\zeta}_{F_n}$

$$\left| \sum_{i \in E_n} \phi(F^i \hat{y}) - n \int \phi \, d\hat{\zeta}^M_{F_n} \right| \leq n \left(\varepsilon_n + 2 \frac{\log \|df^{\pm}\|}{M} \right).$$

Together the inequality of the first item we get

$$\left| \sum_{i \in E_n^{q,p}} \phi_p(F^i \hat{y}) - n \int \phi \, d\hat{\zeta}^M_{F_n} \right| \leq n \left(\varepsilon_n + \frac{4(p+1) \log \|df^{\pm}\|}{M} \right). \tag{3.4}$$

For any $y, z \in F_n$ and any $i \in E_n^{q,p}$, the points $f^i y$ and $f^i z$ lie in the same atom of Q, thus they are ϵ-close, in particular $\log^+ \|d_{f^i y} f^p\| \leq \log^+ \|d_{f^i z} f^p\|_\epsilon$. Therefore if y belongs moreover to $\mathcal{H}(\mathcal{K})$ we get together with (3.4):

(3.5)

$$\frac{1}{n} \sum_i k_i \leq A_n := \int \log^+ \|df^p\|_\epsilon d\zeta^M_{p,q,F_n} - \int \phi \, d\hat{\zeta}^M_{F_n} + \frac{4(p+1) \log \|df^{\pm}\|}{M} + \frac{1}{p} + \varepsilon_n.$$

Finally the cardinality of sequences of positive integers $(k_1, \cdots, k_N)$ with $\sum_{i=1,\cdots,N} k_i \leq S$ is less than $\binom{N}{S} \leq 2^S$.

□

Lemma 11. *One can choose ϵ_p with the following property for any $\epsilon < \epsilon_p$. For any F_n as above there is a family $\Psi_n = \Psi_n^q$ of affine reparametrizations $\psi : [-\epsilon, \epsilon] \circlearrowleft$, such that for some constant C depending only on r and f (not on p, M or n):*

- $\mathrm{Im}(\sigma) \cap \mathcal{H}(\mathcal{K}) \cap F_n \subset \bigcup_{\psi_n \in \Psi_n} \mathrm{Im}(\sigma \circ \psi_n)$,
- $\forall \psi_n \in \Psi_n \ \forall \ q + p\mathbb{N} \ni k \leq n$, $f^k \circ \sigma \circ \psi_n$ *is strongly (ϵ, r)-bounded,*
-

$$\log \sharp \Psi_n \leq \frac{\sum_i k_i}{r-1} + \alpha(n - \sharp E_n) + C \left(\frac{n}{p} + n - m_n + p\sharp(E_n + 1)\Delta E_n \right). \tag{3.6}$$

Proof. First let us observe by arguing as in Lemma 8 that if σ is strongly (ϵ, r)-bounded, then there is a family $\mathcal{F}$ of affine reparametrizations ϕ of $[-\epsilon, \epsilon]$ such that $f \circ \sigma \circ \phi$ is strongly (ϵ, r)-bounded with $\sharp\mathcal{F}$ depending only on f and r.

We argue by induction on n. Let $F_n \in Q^{\mathtt{E}_n^M}$ and let E_n be the associated atom of $\mathtt{E}_n^M$. Without loss of generality we can assume n belongs to E_n according to our first observation, i.e. $m_n = n$. The term $C(n - m_n)$ in (3.6) follows from this reduction). Let n' be the largest integer in $q + p\mathbb{N}$ less than or equal to n. We have the following alternative:

(1) $[n' - p, n'] \subset E_n$, then we let $m = n' - p$,
(2) there is $n' > m \in E_n$ with $]m, n'[\cap E_n = \emptyset$,

(3) there is $q \in [n-2p, n[\cap (E_n+1)\Delta E_n$, then we let $m = q$ or $q-1$ depending whether q belongs to E_n or E_n+1.

In the first case (1) we argue as in the Reparamerization Lemma in [4]. By induction hypothesis there is a family Ψ_m satisfying the conclusion of Lemma 11 for m with F_m being the atom of $Q^{E_n^M}$ containing F_n. Let $\psi_m \in \Psi_m$ with $\mathrm{Im}(\sigma \circ \psi_m) \cap \mathcal{H}(\mathcal{K}) \cap F_n \neq \emptyset$. Let $\epsilon_p > 0$ so small that $\|(f^p)^x_\epsilon\|_r = \|d(f^p)^x_\epsilon\|$ for all $x \in M$ and for all $\epsilon < \epsilon_p$. By using Lemma 8 for f^p as in [4, Proposition 2], there is a family Θ of affine reparametrizations of $[-\epsilon, \epsilon]$ such that:

- $f^{n'} \circ \sigma \circ \psi_m \circ \theta$ is strongly (ϵ, r)-bounded,
- $\mathrm{Im}(\sigma \circ \psi_m) \cap \mathcal{H}(\mathcal{K}) \cap F_n \subset \bigcup_{\theta \in \Theta} \mathrm{Im}(\sigma \circ \psi_m \circ \theta)$,
- $\sharp\Theta \leq C e^{\frac{k_{n'}}{r-1}}$.

This concludes the proof in case (1) by letting Ψ_n be the union of affine reparametrizations $\psi_m \circ \theta$ over all $\psi_m \in \Psi_m$ and $\theta \in \Theta = \Theta(\psi_m)$. The contribution of case (1) in (3.6) is given by $\frac{\sum_i k_i}{r-1} + C\frac{n}{p}$.

In the second case (2) we argue as in Lemma 13 in [2]. For any $x \in F_n$, the integers m and n are hyperbolic times. By Lemma 9, there is $\tilde{\epsilon}$ and τ depending only on f and α such that for $\epsilon' < \tilde{\epsilon}$ the sizes of $D_m(x) = D_m^{\epsilon'}(x)$ and $D_n(x) = D_n^{\epsilon'}(x)$ are bounded from below by some $\epsilon = \tau\epsilon' < \epsilon'$. But as already observed, the distorsion is bounded on these curves and $[m, n'[$ is contained in a neutral block, consequently:

$$|\phi_{n-m}(F^m \hat{x}) - \alpha(n-m)| \leq p \log \|df^{\pm}\|.$$

Therefore the length of $f^{-(n-m)} D_n(x) \subset D_m(x)$ is larger than $\|df^{\pm}\|^{-p} e^{-\alpha(n-m)}\epsilon/2$.

We can choose ϵ_p with $\epsilon_p < \tau\tilde{\epsilon}/2$. We assume the diameter of Q is less than $\epsilon < \epsilon_p$. Write $F_n = \bigcap_{k \in E_n} f^{-k} Q_k$ for $(Q_k)_k \in Q^{E_n}$. For two points $x, y \in F_n \cap f^{-n} Q_n$, either $D_n(x)$ and $D_n(y)$ are disjoint or $D_n(x) \cap Q_n = D_n(y) \cap Q_n$ by Lemma 11 in [2]. Let Ψ_m be the family of affine reparametrizations at step m with respect to the atom F_m of $Q^{\mathrm{E}_n^M}$ containing F_n. Let $\psi_m \in \Psi_m$ with $\mathrm{Im}(\sigma \circ \psi_m) \cap \mathcal{H}(\mathcal{K}) \cap F_n \neq \emptyset$. There is a minimal collection $(x_i)_{i \in I(\psi_m)}$ of points in $\mathrm{Im}(\sigma \circ \psi_m) \cap \mathcal{H}(\mathcal{K})$ such that the pairwise disjoint curves $(E_n(x_i))_i$ cover $\mathrm{Im}(\sigma \circ \psi_m) \cap \mathcal{H}(\mathcal{K}) \cap F_n$. For any i, we have $D_m(x_i) \cap \mathrm{Im}(f^m \circ \sigma \circ \psi_m) \neq \emptyset$. From the above lower bound on the length of $D_m(x_i)$, we get $\sharp I(\psi_m) \leq 4\|df^{\pm}\|^{-1} e^{-\alpha(n-m)}$. Moreover each strongly (ϵ', r, n)-bounded curve $\psi_n^{x_i} = E_n(x_i)$ may be obviously covered by a family $\mathcal{F}_i$ of (ϵ, r, n)-bounded curves with $\sharp\mathcal{F}_i \leq \tau^{-1} + 1$. We conclude by letting Ψ_n be the union of $\mathcal{F}_i$ over $i \in I(\psi_m)$ and $\psi_m \in \Psi_m$. Case (2) corresponds to the term $\alpha(n - \sharp E_n) + Cp\sharp(E_n+1)\Delta E_n$ in (3.6).

The last case follows easily from our starting observation and leads to another error term of the form $Cp\sharp(E_n+1)\Delta E_n$ in (3.6).

□

By increasing the cardinality of Ψ_n with a factor depending only on p and f we may assume $\sigma \circ \psi_n$ is (ϵ, r, n)-bounded for any $\psi_n \in \Psi_n$. We recall that

$\frac{m_n(\hat{x})}{n}$ is converging uniformly to 1 on $\mathtt{F}$. Together with (3.3), (3.6), (3.5), we get for some $\gamma_{p,M}$ with $\limsup_p \limsup_M \gamma_{p,M} = 0$:

$$\begin{aligned}\frac{1}{n}\log \sharp\Psi_n^q \leq &\left(1-\frac{\sharp E_n}{n}\right)\alpha\\ &+\left(\log 2+\frac{1}{r-1}\right)\left(\int \log^+\|d_xf^p\|_{\epsilon_p} d\zeta^M_{p,q,F_n}(x)-\int \phi\, d\hat{\zeta}^M_{F_n}\right)\\ &+\gamma_{p,M}.\end{aligned}$$

We finally let Ψ_{F_n} be the collection $\Psi_n^{q'}$ with $\sharp\Psi_n^{q'} = \min_{0\leq q<p}\sharp\Psi_n^q$. We conclude:

$$\begin{aligned}\frac{1}{n}\log\sharp\Psi_{F_n} &\leq \frac{1}{p}\sum_{0\leq q<p}\frac{1}{n}\log\sharp\Psi_n^q,\\ &\leq \left(1-\frac{\sharp E_n}{n}\right)\alpha\\ &\quad+\left(\log 2+\frac{1}{r-1}\right)\left(\frac{1}{p}\sum_{0\leq q<p}\int \log^+\|d_xf^p\|_\epsilon d\zeta^M_{p,q,F_n}(x)-\int\phi\, d\hat{\zeta}^M_{F_n}\right)\\ &\quad+\gamma_{p,M},\\ &\leq \left(1-\frac{\sharp E_n}{n}\right)\alpha\\ &\quad+\left(\log 2+\frac{1}{r-1}\right)\left(\int \frac{\log^+\|d_xf^p\|_\epsilon}{p}\, d\zeta^M_{F_n}(x)-\int\phi\, d\hat{\zeta}^M_{F_n}\right)\\ &\quad+\gamma_{p,M}.\end{aligned}$$

REFERENCES

[1] J. Brown, *Ergodic theory and topological dynamics*, Pure and applied mathematics, 1976.

[2] Burguet, David, *SRB measure for $\mathcal{C}^\infty$ surface diffeomorphisms*, arXiv:2111.06651

[3] Burguet, David, *Existence of measures of maximal entropy for $\mathcal{C}^r$ interval maps*, Proc. Amer. Math. Soc. 142 (2014), p; 957-968

[4] Burguet, David, *Symbolic extensions in intermediate smoothness on surfaces*, Ann. Sci. Éc. Norm. Supér. (4), 45 (2012), no. 2, 337-362

[5] Burguet, David, *Jumps of entropy for C^r interval maps*, Fund. Math., 231, (2015), no.3, p.299-317.

[6] Buzzi, Jérôme and Ruette, Sylvie, *Large entropy implies existence of a maximal entropy measure for interval maps*, Discrete Contin. Dyn. Syst. A, 14, (2006), p.673-688,

[7] Buzzi, Jérôme, *$\mathcal{C}^r$ surface diffeomorphisms with no maximal entropy measure*, Ergodic Theory Dynam. Systems 34, 2014, p 1770-1793.

[8] Buzzi, Jérôme, *Représentation markovienne des applications réulieres de lintervalle*, PhD thesis, Université Paris-Sud, Orsay, 1995.

[9] J. Buzzi, S. Crovisier, and O. Sarig, *Measures of maximal entropy for surface diffeomorphisms*, to appear in Annals of Math.

[10] J. Buzzi, S. Crovisier, and O. Sarig, *Continuity properties of Lyapunov exponents for surface diffeomorphisms*, arXiv:2103.02400.

[11] J. Buzzi, S. Crovisier, and O. Sarig, *In preparation.*
[12] F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesins entropy formula*, Ann. of Math., 122(1985), 505-539.
[13] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Publications Mathmatiques de lIHES,51(1980),137-174.
[14] S. Newhouse. *Continuity properties of entropy*, Ann. of Math. (2), 129(2):215235, 1989.
[15] Y. Pesin and L. Barreira, *Lyapunov Exponents and Smooth Ergodic Theory*, University Lecture Series, v. 23, AMS, Providence, 2001
[16] Yosef Yomdin, *Volume growth and entropy*, Israel J. Math., 57(3), p. 285-300, 1987.

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