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# MULTIPLE SAMPLING AND INTERPOLATION IN BERGMAN SPACES

D. AADI, C. CRUZ, A. HARTMANN, AND K. KELLAY

ABSTRACT. We study multiple sampling and interpolation problems with unbounded multiplicities in the weighted Bergman space, both in the hilbertian case  $p = 2$  and the uniform case  $p = +\infty$ .

## 1. INTRODUCTION

Interpolating and sampling sequences have been studied in a broad variety of settings. We refer to the books [20, 11, 7] for an account on these problems. The particular situation of the Bergman space had been completely solved by Seip in [18] using density conditions, and in more general Hilbert spaces of analytic functions by Berndtsson and Ortega-Cerdà in [4]. Subsequently, the case of multiple interpolation (but not sampling) with uniformly bounded multiplicities had been studied for instance by Krosky and Schuster [12] using also extremal functions (we mention related work by the third named author who considered in [8] finite unions of Bergman interpolating sequences based on extremal functions, the case of multiple interpolation being in a sense a limite case of finite unions). In view of his density characterizations, Seip's results imply that there are no simultaneous sampling and interpolating sequences in the Bergman spaces.

In the Fock space, besides considering the case of simple interpolation and sampling problems, Seip – in particular with Brekke – was interested in the situation of higher multiplicities. Again, the density conditions obtained by these authors imply that there are no simultaneous sampling and interpolating sequences, neither in the simple case nor in the multiple case. Brekke and Seip in [6] also asked whether there could be simultaneous sampling and interpolating sequences when the uniform boundedness condition on the multiplicities is relaxed. In [5] it was shown that at least when the multiplicities tend to infinity, this is not possible (see also [2] for the case of bounded multiplicities in the weighted Fock space).

One difficulty occurring in the case of unbounded multiplicities is the lack of a reasonable definition of densities. In [5], the authors introduce covering and separation conditions related with critical radii suitably related with the multiplicities to circumvent densities. Though those conditions do not characterize multiple interpolation and sampling, they get in a sense closer and closer to a characterization when the multiplicities grow (indeed the difference between necessary and sufficient conditions of the radii remains bounded while the radii tend to infinity).

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Bergman and Fock spaces share many properties, and techniques often translate from one setting to the other. The aim of this paper is to study the situation concerning multiple interpolating and sampling with unbounded multiplicities in the Bergman space. New difficulties and challenges appear in order to adapt the situation from the underlying euclidean metric in the Fock space to pseudohyperbolic metric in the Bergman space. While this might be rather direct for simple interpolation and sampling the situation requires a quite delicate analysis of the critical radii in the pseudohyperbolic metric in particular when the multiplicities are not uniformly bounded. On the technical side, replacing the incomplete  $\Gamma$ -function by the incomplete  $\beta$ -function gives rise to other difficulties.

It is mentionable that generalized interpolation problems (but not sampling problems) have been considered long ago in the Hardy space for which a complete answer is given by the so-called generalized Carleson condition (see [14, 21]). In this situation, the case of interpolating sequences with unbounded multiplicities is completely understood (see also earlier work by Vinogradov-Rukshin [23]).

Without claiming exhaustivity, we finish this first tour on multiple interpolation problems, mentioning work on interpolating sequences with uniformly bounded multiplicity in the Korenblum space, see [13] and for weighted spaces of entire functions see [16, 17].

We now introduce the necessary notation. Let  $\alpha > -1$ , we consider the  $L^2$  weighted Bergman space

$$\mathcal{A}_\alpha^2 = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{\alpha,2}^2 = \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) < +\infty \right\},$$

where  $dA_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha dx dy / \pi$ ,  $z = x + iy$ . The space  $\mathcal{A}_\alpha^2$  is a reproducing kernel Hilbert space with the scalar product

$$\langle f, g \rangle := \int_{\mathbb{D}} f(z) \overline{g(z)} dA_\alpha(z).$$

The standard monomial orthonormal basis for  $\mathcal{A}_\alpha^2$  is given by

$$e_n(z) = \sqrt{\frac{\Gamma(n+2+\alpha)}{n! \Gamma(2+\alpha)}} z^n, \quad n \geq 0, \quad (1)$$

where  $\Gamma(s)$  stands for the usual Gamma function [11, p.4]. Thus, see for instance [11, p.5] the reproducing kernel  $\mathcal{A}_\alpha^2$  is

$$K_w(z) = \sum_{j \geq 0} \overline{e_j(w)} e_j(z) = \frac{1}{(1 - \overline{w}z)^{\alpha+2}},$$

and the normalized Bergman kernel is  $k_w(z) = K_w(z) / \|K_w\|_{\alpha,2}$ . The reproducing kernel gives rise in a standard way to a growth condition in the Bergman space:

$$|f(\lambda)|^2 = |\langle f, K_\lambda \rangle|^2 \leq \left( \frac{1}{1 - |\lambda|^2} \right)^{\alpha+2} \|f\|_{\alpha,2}^2, \quad f \in \mathcal{A}_\alpha^2, \lambda \in \mathbb{D}. \quad (2)$$

We consider the Möbius transform

$$\varphi_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda}z} \quad (3)$$

and the isometric translation operators in  $\mathcal{A}_\alpha^2$  given by  $T_\lambda f(z) = [\varphi'_\lambda(z)]^{\frac{2+\alpha}{2}} f(\varphi_\lambda(z))$ , i.e.,

$$T_\lambda : \mathcal{A}_\alpha^2 \longrightarrow \mathcal{A}_\alpha^2$$

$$f \longrightarrow T_\lambda f := \left[ \frac{|\lambda|^2 - 1}{(1 - \bar{\lambda} \cdot)^2} \right]^{\frac{2+\alpha}{2}} f(\varphi_\lambda(\cdot)).$$

Notice that  $\varphi_\lambda$  and  $T_\lambda$  are involutions, in fact,  $T_\lambda$  is a self-adjoint operator.

In the Bergman space, the underlying metric on the unit disk is the pseudohyperbolic distance which is defined *via* the already mentioned Möbius transform (3):

$$\rho(u, v) = |\varphi_u(v)|, \quad u, v \in \mathbb{D}.$$

We also associate the pseudohyperbolic disk with this distance:  $D(\lambda, r) = \{z \in \mathbb{D} : \rho(\lambda, z) < r\}$  for  $\lambda \in \mathbb{D}$  and  $0 < r < 1$ . In order to be interpolating in the Bergman space, a sequence has to be separated in this metric (see [18]). Also, in order to be sampling (at least in the Hilbertian case under consideration here), it can be deduced that any pseudohyperbolic neighborhood with fixed radius can contain at most a uniformly bounded number of points (Carleson measure condition).

In order to better understand multiple interpolation and sampling problems we shall comment a little bit more on the multiplicity one situation. In this case, given a set of points  $\Lambda \subset \mathbb{D}$ , one is interested in the values of a function in given points  $f(\lambda)$ ,  $\lambda \in \Lambda$ . When  $\Lambda$  is separated in the pseudohyperbolic metric, it can be shown that for  $f \in \mathcal{A}_\alpha^2$ , we have

$$\sum_{\lambda \in \Lambda} \frac{|f(\lambda)|^2}{\|K_\lambda\|^2} = \sum_{\lambda \in \Lambda} (1 - |\lambda|^2)^{2+\alpha} |f(\lambda)|^2 < \infty. \quad (4)$$

Conversely, if the sequence  $\Lambda$  is sufficiently separated, it can be shown that every sequence of values  $(v_\lambda)_{\lambda \in \Lambda}$  the square of which is summable against the weight  $(1 - |\lambda|^2)^{2+\alpha}$  can be interpolated by a function  $f \in \mathcal{A}_\alpha^2$ . Observe that

$$\langle f, T_\lambda e_0 \rangle = T_\lambda f(0) = \left[ \frac{|\lambda|^2 - 1}{(1 - \bar{\lambda} \times 0)^2} \right]^{\frac{2+\alpha}{2}} f(\varphi_\lambda(0)) = (|\lambda|^2 - 1)^{\frac{2+\alpha}{2}} f(\lambda),$$

so that (4) translates to

$$\sum_{\lambda \in \Lambda} |\langle f, T_\lambda e_0 \rangle|^2 < +\infty.$$

Notice that, intuitively, a sequence of interpolation must be sufficiently sparse, and should be a set of zeros of a holomorphic function (at least up to one point). Analogously, any sampling sequence should have sufficiently big density and is in general not a set of zeros (except in spaces which admit complete interpolating sequences). This naive relation points at the connection between our problems and uniqueness questions which will be studied in the next Section 2.

Let us now switch to the multiple case. Instead of studying only the values of a function in the points of the given sequence, it is natural to consider germs of functions in those points, i.e. to consider also derivatives of the function up to a certain order depending on the point (Hermite type interpolation). As long as the

multiplicities are uniformly bounded, the definition of the target space can be based on point evaluations and their derivatives with suitable weights. However, when we allow multiplicities to grow to infinity, the weights and constants have to be chosen in a very precise way. In the Hilbertian case, since  $(e_j)_{j \geq 0}$  is an orthonormal basis and  $T_\lambda$  an isometry, it is natural to consider  $\langle f, T_\lambda e_j \rangle$ , where  $\lambda \in \Lambda$  and  $j$  is bounded by the multiplicity. With this in mind, we can now define sampling and interpolation in the general case. Given a set of points  $\Lambda \subset \mathbb{D}$  with multiplicity  $m_\lambda$ , we call *divisor* a set of pairs  $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ .

**Definition 1.1.** A divisor  $X$  is sampling for  $\mathcal{A}_\alpha^2$  if there exists a constant  $C > 0$ , such that for all  $f \in \mathcal{A}_\alpha^2$

$$\frac{1}{C} \sum_{\lambda \in \Lambda} \sum_{j < m_\lambda} |\langle f, T_\lambda e_j \rangle|^2 \leq \|f\|_{\alpha,2}^2 \leq C \sum_{\lambda \in \Lambda} \sum_{j < m_\lambda} |\langle f, T_\lambda e_j \rangle|^2.$$

Note that  $P_\lambda f = \sum_{j < m_\lambda} \langle f, T_\lambda e_j \rangle T_\lambda e_j$  is an orthogonal projection and it can be shown that

$$\ker P_\lambda = N_{\lambda, m_\lambda}^{2,\alpha} := \{f \in \mathcal{A}_\alpha^2 : f^{(j)}(\lambda) = 0, \quad \forall j < m_\lambda\}, \quad (5)$$

(see also equation (4) in [6, p.114] where this matter is discussed in the Fock space), so that in particular

$$\sum_{j < m_\lambda} |\langle f, T_\lambda e_j \rangle|^2 = \|f\|_{\mathcal{A}_\alpha^2 / N_{\lambda, m_\lambda}^{2,\alpha}}^2.$$

It is useful to recall the weaker notion of uniqueness. A divisor  $X$  is called a *uniqueness divisor* (or simply  $X$  is uniqueness) if every function vanishing up to the order  $m_\lambda - 1$  at  $\lambda$  is necessarily the zero function. Clearly, a sampling divisor is uniqueness, but the converse is in general not true.

The above definition gives rise to a natural definition of the following target space needed for interpolation.

$$\ell^2(X) = \left\{ (v_\lambda^j)_{\lambda \in \Lambda, 0 \leq j < m_\lambda} : \|v\|_2^2 := \sum_{\lambda \in \Lambda} \sum_{j < m_\lambda} |v_\lambda^j|^2 < \infty \right\}.$$

**Definition 1.2.** The divisor  $X$  is interpolating for  $\mathcal{A}_\alpha^2$  if for all sequences  $v \in \ell^2(X)$ , there exists  $f \in \mathcal{A}_\alpha^2$  such that

$$\langle f, T_\lambda e_j \rangle = v_\lambda^j. \quad (\lambda \in \Lambda, j < m_\lambda)$$

Note again that the above interpolation condition is equivalent to interpolation by germs of  $f$  in  $\lambda$  up to the order  $m_\lambda - 1$  (see equation (4) in [6, p.114]). Clearly, if  $f$  interpolates  $v$  in the above way, then  $\sum_{j < m_\lambda} |v_\lambda^j|^2 = \|f\|_{\mathcal{A}_\alpha^2 / N_{\lambda, m_\lambda}^{2,\alpha}}^2$ . The reinterpretation in terms of quotient norms will be useful later when considering the situation in  $\mathcal{A}_\alpha^\infty$ .

In the case of the classical Fock space, multiple interpolation and sampling was related to some critical radius. More precisely, since in the Fock space the underlying metric is euclidean, given a multiplicity  $m_\lambda$  in a point  $\lambda \in \mathbb{C}$  the “influence zone” of  $\lambda$  meaning the knowledge of  $f(\lambda), \dots, f^{m_\lambda-1}(\lambda)$  — or equivalently that of  $\langle f, T_\lambda e_0 \rangle, \dots, \langle f, T_\lambda e_{m_\lambda-1} \rangle$  — was the euclidean disk  $D_e(\lambda, \sqrt{m_\lambda}) = \{z \in \mathbb{C} : |z - \lambda| < \sqrt{m_\lambda}\}$  (see [5]). This corresponds more or less to redistributing the multiplicity in a

regular way in an euclidean disk. It is a priori not so clear how to define this redistribution in the pseudohyperbolic case in particular when we authorize the multiplicity to tend to infinity. The corresponding critical radius appears in the following overlap condition; it will be clear from later discussions where this radius comes from. Our sampling and interpolating conditions are all expressed with respect to this critical value (slightly increasing or decreasing it). We will need an overlap condition that we introduce now.

**Definition 1.3.** *A divisor  $X$  satisfies the finite overlap condition for  $\mathcal{A}_\alpha^2$  if*

$$S_X = \sup_{z \in \mathbb{D}} \sum_{\lambda \in \Lambda} \chi_{D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}}\right)}(z) < \infty.$$

We should mention that this finite overlap condition is intimately related to the Carleson measure condition.

Now we are in a position to state the geometric condition for sampling divisors.

**Theorem 1.4.** *Let  $\alpha > -1$ .*

- (a) *If  $X$  is a sampling divisor for  $\mathcal{A}_\alpha^2$ , then  $X$  satisfies the finite overlap condition and there exists  $0 < C_X < \alpha + 1$  such that*

$$\bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha + 1}}\right) = \mathbb{D}.$$

- (b) *Conversely, suppose the divisor  $X$  satisfies the finite overlap condition. There is a constant  $C > 1$  depending on  $S_X$  such that if for some compact  $K$  of  $\mathbb{D}$  we have*

$$\bigcup_{\lambda \in \Lambda, m_\lambda > C} D\left(\lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha + 1}}\right) = \mathbb{D} \setminus K,$$

*then  $X$  is a sampling divisor for  $\mathcal{A}_\alpha^2$ .*

This theorem tells us that if disks with slightly smaller radii than the critical one already cover the unit disk, then we have a sampling divisor. And if a divisor is sampling then at least disks with slightly bigger radii cover the unit disk (up to a compact set).

One could be tempted to complain about the constant  $C$  appearing in (b) above. Note that the theorem is completely general and applies even in the case of uniformly bounded multiplicities where the result proved in [6] requires density conditions. So there is no hope getting a sufficient condition only from the covering without additional conditions for instance on the critical radius.

In the analogous situation for interpolating divisors the covering condition is replaced by a separation condition of disks with slightly bigger or smaller radii than the critical ones.

**Theorem 1.5.** *Let  $\alpha > -1$ .*

- (a) *If  $X$  is an interpolating divisor for  $\mathcal{A}_\alpha^2$ , then there exists  $C_X > 0$  such that the hyperbolic disks*

$$\left\{ D\left(\lambda, \sqrt{\frac{m_\lambda - C_X}{m_\lambda + \alpha + 1}}\right) \right\}_{\lambda \in \Lambda, m_\lambda > C_X}$$

are pairwise disjoint.

- (b) Conversely, if for some  $C_X$  such that  $(\alpha + 1)(1 - e^{-1}) < C_X < \alpha + 1$ , the hyperbolic disks

$$\left\{ D \left( \lambda, \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha + 1}} \right) \right\}_{\lambda \in \Lambda}$$

are pairwise disjoint, then  $X$  is an interpolating divisor for  $\mathcal{A}_\alpha^2$ .

Notice that the separation condition appearing in the statement (a) implies the finite overlap condition (the overlap is actually void), which is again related to the Carleson measure condition.

Again, we should point out that additional conditions on the constant  $C$  are required in (b) since the theorem is completely general covering the case when the multiplicities are uniformly bounded in which case the result [6] involves again density conditions. So, separation alone for  $C$  arbitrary close to 0 cannot be sufficient for interpolation.

Concerning both Theorems 1.4 and 1.5, we would also like to emphasize the fact that densities, even if they provide characterizations for simple or uniformly bounded multiplicities, are hard to compute in a general situation (particularly in the pseudohyperbolic metric), while our overlapping and separation conditions are much easier to apprehend.

Here is another observation: in case  $X$  is a sampling divisor, the finite overlap condition is necessary. In case  $X$  is an interpolating divisor, we get a separation condition, which obviously also implies the finite overlap (there is actually no overlap and now  $S_X = 1$ ). So in both cases, the area of pseudohyperbolic disks centered at  $\lambda$  and with radius comparable to  $\sqrt{(m_\lambda - C)/(m_\lambda + \alpha + 1)}$  add up to a finite sum, yielding the following Blaschke type condition which seems new:

$$\sum_{\lambda \in \Lambda} (m_\lambda (1 - |\lambda|^2))^2 < +\infty. \quad (6)$$

The result which affirms that in the Fock space (with Gaussian weight) there are no Riesz bases (simultaneously interpolating and sampling) is quite expensive to obtain. First in [18] for simple multiplicity, and later for uniformly bounded multiplicity in [6] this requires the density characterizations of interpolating and sampling sequences. More recently the third and fourth authors discussed this problem in [5] when the multiplicities go to infinity. In this case, there are no characterizations available, but the gap between necessary conditions for multiple sampling and multiple interpolation (given by the corresponding results to Theorems 1.4 and 1.5), together with some geometric lemma, allowed to conclude. The situation in Bergman spaces is dramatically simpler. The main reason is due to the fact that the multiplier algebra for standard Bergman spaces is not trivial and contains bounded analytic functions in the unit disk. As a consequence, any interpolating divisor is a zero divisor (pick a function vanishing in all the points  $\lambda$  up to the order  $m_\lambda - 1$  except for one point  $\lambda_0$  in which we interpolate the value 1, then multiply the interpolating function by  $b_{\lambda_0}^{m_{\lambda_0}}$ ), and can thus not be sampling since sampling divisors are



uniqueness.

In view of the above discussions, the central role of zero and uniqueness sets should be clear. In this connection, we will formulate here a necessary condition for zero divisors which does not seem to follow from those known so far. A fairly precise results on zero sets in  $\mathcal{A}$

**Theorem 1.6.** *Let  $\alpha > 0$ ,  $\varepsilon > 0$ , and  $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$  be a divisor such that*

$$\bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 2 + \varepsilon}}\right) = \mathbb{D} \setminus K \quad (7)$$

*for some compact set  $K \subset \mathbb{D}$ . Then  $X$  is uniqueness divisor for  $\mathcal{A}_\alpha^2$ .*

We recall that a uniqueness divisor is a non zero divisor.

Without going into more details in this introduction, we mention that the same problems can be considered in the uniform norm: let  $\alpha > 0$ , and define

$$\mathcal{A}_\alpha^\infty = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{\alpha, \infty}^2 := \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\frac{\alpha}{2}} |f(z)| < +\infty \right\}.$$

In this setting, the results are completely analogous – replacing essentially  $\alpha + 1$  by  $\alpha$  in the theorems cited above – and will be discussed in Section 4. Note that it follows immediately from (2) that  $\mathcal{A}_\alpha^2 \subset \mathcal{A}_{\alpha+2}^\infty$  which allows to connect some results between both situations. However, in general the results in  $\mathcal{A}_\alpha^\infty$  do not follow immediately from those for  $\mathcal{A}_\alpha^2$  and the proofs have to be rerun. We also point out a curious phenomenon. Indeed the above embedding works for  $\alpha > -1$ , but there is no  $\mathcal{A}_\beta^2$  which embeds into  $\mathcal{A}_\alpha^\infty$  when  $\alpha \in (0, 1]$ .

We would also like to comment on the uniqueness result Theorem 1.6 and its corresponding result Theorem 2.2 below in  $\mathcal{A}_\alpha^\infty$  (which has essentially the same statement just replacing  $\alpha + 2$  by  $\alpha$ ). In [19], Seip gave fairly precise sufficient and necessary conditions exhibiting a small gap between these. His conditions are based on the Korenblum density which is difficult to check in general. The condition appearing in (7) (or in (11) below) yields maybe a more transparent necessary condition for zero divisors (for  $X$  to be a zero divisor it is necessary that the covering condition (7) does not hold for any compact  $K$  and any  $\varepsilon > 0$ ).

The paper is organized as follows: in Section 2, we prove the  $\mathcal{A}_\alpha^\infty$  uniqueness Theorem. Section 3 is devoted to the proofs of Theorem 1.4 and 1.5 respectively. In Section 4 we discuss the uniform case.

Throughout this paper we use the following notations :

- $A \lesssim B$  means that there is an absolute constant  $C$  such that  $A \leq CB$ .
- $A \asymp B$  if both  $A \lesssim B$  and  $B \lesssim A$ .

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## 2. ZEROS AND A JENSEN TYPE FORMULA

In this section we will prove a uniqueness result for  $\mathcal{A}_\alpha^\infty$  (which will also be useful for the case  $p = 2$ ).

Let us introduce the invariant measure on the unit disk  $d\mathcal{V}(z) = (1 - |z|^2)^{-2} dm(z)$ , where  $m$  is the normalized Lebesgue measure such that  $m(\mathbb{D}) = 1$ . It is well-known that  $d\mathcal{V}$  is invariant under Möbius transforms. We mention that a direct calculation shows that

$$\int_{D(\zeta, r)} d\mathcal{V}(z) = \int_{D(0, r)} d\mathcal{V}(z) = \frac{r^2}{1 - r^2}, \quad 0 < r < 1.$$

We observe here that morally speaking, the critical radius  $r$  has to be chosen more or less in such a way that this mass corresponds to the multiplicity. To be more precise, and as we will see below (see (10) below), the critical radius has to be chosen *via* redistributing the mass of the laplacian of the logarithm of an  $\mathcal{A}_\alpha^\infty$ -function (as was done in the euclidean metric appearing in the setting of the Fock space).

We will discuss the situation here in  $\mathcal{A}_\alpha^\infty$  which requires  $\alpha > 0$  contrarily to  $\mathcal{A}_\alpha^2$  where  $\alpha > -1$ .

Now, in the spirit of our observation above (and in particular (10) below), for fixed  $\varepsilon > 0$ , let

$$r_\lambda = r_{\lambda, \alpha, \varepsilon} := \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + \varepsilon}} \quad (8)$$

and set  $D_\lambda = D(\lambda, r_\lambda)$ . (In a sense, the critical radius in  $\mathcal{A}_\alpha^\infty$  corresponds to the situation when  $\varepsilon = 0$ .)

**Lemma 2.1.** *Let  $\alpha > 0$ , if  $X = \{(\lambda_k, m_k)\}_{k \geq 1}$  is a zero divisor for  $\mathcal{A}_\alpha^\infty$ , then for  $\varepsilon > 0$ ,*

$$\int_{D(0, r)} \sum_{k \geq 1} \chi_{D_\lambda}(z) \log \frac{r}{|z|} d\mathcal{V}(z) \leq \frac{\alpha}{2(\alpha + \varepsilon)} \log \frac{1}{1 - r^2} + O(1), \quad r \rightarrow 1.$$

*Proof.* Let  $X = \{(\lambda_k, m_k)\}_{k \geq 1}$  be a zero divisor for  $\mathcal{A}_\alpha^\infty$  then there exists a non zero function  $f \in \mathcal{A}_\alpha^\infty$  such that

$$f^{(k)}(\lambda) = 0, \quad 0 \leq k < m_\lambda, \quad \lambda \in \Lambda.$$

We will obtain our condition by redistributing the mass  $\Delta(\log |f|)$  on the hyperbolic disks  $D_\lambda$ .

The function  $\log |f|$  is subharmonic and not identically  $-\infty$  and we have for all  $z \in \mathbb{D}$

$$\log |f(z)| \leq \log \|f\|_{\alpha, \infty} + \frac{\alpha}{2} \log \frac{1}{1 - |z|^2} =: s(z).$$

We use the same inductive method as in [5]. First we construct a new sub-harmonic function  $h$  in  $\mathbb{D}$  such that

$$\log |f(z)| \leq h(z) \leq s(z).$$

Later, we will obtain our bound using Green's identity. Let  $h_0 = \log |f|$  and recall  $r_{\lambda_k} = \sqrt{\frac{m_k}{m_k + \alpha + \varepsilon}}$ , then

$$h_0(z) = m_1 \log \frac{|\varphi_{\lambda_1}(z)|}{r_{\lambda_1}} + \underbrace{\log |f(z)| - m_1 \log \frac{|\varphi_{\lambda_1}(z)|}{r_{\lambda_1}}}_{=U_0}.$$

Since  $f/\varphi_{\lambda_1}^{m_{\lambda_1}}$  is holomorphic in  $\mathbb{D}$ ,  $U_0$  is a subharmonic function on  $\mathbb{D}$  and harmonic in a small neighborhood of  $\lambda_1$ . In fact, in a small neighborhood  $V$  of  $\lambda_1$  not containing any other zero of  $f$  than  $\lambda_1$ , we have

$$|f(z)| = |\varphi_{\lambda_1}(z)|^{m_1} |g(z)|,$$

where  $g$  is a holomorphic function with no zeros in  $V$ .

Therefore

$$\Delta \log |f(z)| = m_1 \Delta \log |\varphi_{\lambda_1}(z)| + \underbrace{\Delta \log |g|}_{=0}, \quad z \in V.$$

Now we will modify the term  $m_1 \log \frac{|\varphi_{\lambda_1}(z)|}{r_{\lambda_1}}$  to obtain a constant function with respect to the invariant laplacian. Let us consider

$$v_1(z) = \begin{cases} \frac{\alpha + \varepsilon}{2} \log \frac{1 - r_{\lambda_1}^2}{1 - |\varphi_{\lambda_1}(z)|^2}, & z \in D(\lambda_1, r_{\lambda_1}), \\ m_1 \log \frac{|\varphi_{\lambda_1}(z)|}{r_{\lambda_1}}, & \text{otherwise.} \end{cases} \quad (9)$$

Then  $v_1$  is harmonic outside  $D(\lambda_1, r_{\lambda_1})$  and  $v_1 \in C^1(\mathbb{D})$ . Using  $\Delta = 4\partial\bar{\partial}$ , and the fact that  $\log |1 - \bar{\lambda}_1 z|$  is harmonic, we have inside  $D(\lambda_1, r_{\lambda_1})$

$$\begin{aligned} \Delta v_1 &= \Delta \left( \frac{\alpha + \varepsilon}{2} \log \frac{1 - r_{\lambda_1}^2}{1 - |\varphi_{\lambda_1}(z)|^2} \right) = \frac{\alpha + \varepsilon}{2} \Delta \left( \log \frac{|1 - \bar{\lambda}_1 z|^2}{(1 - |z|^2)(1 - |\lambda_1|^2)} \right) \\ &= (\alpha + \varepsilon) \Delta (\log |1 - \bar{\lambda}_1 z|) - \frac{\alpha + \varepsilon}{2} \Delta (\log(1 - |z|^2)) \\ &= \frac{2(\alpha + \varepsilon)}{(1 - |z|^2)^2}. \end{aligned}$$

(We have used that  $\Delta \log(1 - |z|^2) = -\frac{4}{(1 - |z|^2)^2}$ .) Observe that with the definition of the invariant laplacian  $\tilde{\Delta}u = (1 - |z|^2)^2 \Delta u$ , the preceding computation shows that  $\tilde{\Delta}v_1$  is constant. We thus obtain the total mass of the measure  $\Delta v_1$  on  $D(\lambda_1, r_{\lambda_1})$  which is equal to  $2(\alpha + \varepsilon)\mathcal{V}(D(\lambda_1, r_{\lambda_1}))$ :

$$\begin{aligned} \int_{\mathbb{D}} \Delta v_1 dm &= \int_{D(\lambda_1, r_{\lambda_1})} 2(\alpha + \varepsilon) \frac{dm}{(1 - |z|^2)^2} = 2(\alpha + \varepsilon)\mathcal{V}(D(\lambda_1, r_{\lambda_1})) \\ &= 2(\alpha + \varepsilon) \frac{r_{\lambda_1}^2}{1 - r_{\lambda_1}^2}. \end{aligned} \quad (10)$$

By the specific definition of  $r_{\lambda_1}$  this is equal to  $2m_{\lambda_1}$ .

On the other hand, since  $\Delta \log |z| = 2\pi\delta_0$ ,

$$\begin{aligned} \int_{\mathbb{D}} \Delta \left( m_1 \log \frac{|\varphi_{\lambda_1}(z)|}{r_{\lambda_1}} \right) dm(z) &= m_1 \int_{\mathbb{D}} \Delta(\log |\lambda_1 - z|) dm(z) \\ &= 2m_1. \end{aligned}$$

In particular, by the very definition of  $r_{\lambda_1}$ , both total masses coincide so that in terms of Laplacians, we can replace  $m_1 \log \frac{|\varphi_{\lambda_1}(\cdot)|}{r_{\lambda_1}}$  by  $v_1(\cdot)$  in  $D(\lambda_1, r_{\lambda_1})$ . This yields the function  $h_1 = v_1 + U_0$ . Obviously,

$$h_1(z) = \frac{\alpha + \varepsilon}{2} \log \frac{1 - r_{\lambda_1}^2}{1 - |\varphi_{\lambda_1}(z)|^2} \chi_{D(\lambda_1, r_{\lambda_1})}(z) + m_1 \log \frac{|\varphi_{\lambda_1}(z)|}{c_{\lambda_1}} \chi_{\mathbb{D} \setminus D(\lambda_1, r_{\lambda_1})}(z) + U_0.$$

Let us show that

- (a)  $h_1(z) \leq s(z)$ ,  $z \in \mathbb{D}$ ,
- (b)  $h_0(z) \leq h_1(z)$ ,  $z \in \mathbb{D}$ .

We start proving  $h_1 \leq s$ . This is clear for  $z \notin D(\lambda_1, r_{\lambda_1})$ , because  $h_1 = h_0$  in  $\mathbb{D} \setminus D(\lambda_1, r_{\lambda_1})$ . For  $z \in D(\lambda_1, r_{\lambda_1})$ , we consider the function  $w_1 = v_1 + U_0 - s$ , we have

$$\Delta w_1 = \Delta \left( v_1 + U_0 - \frac{\alpha}{2} \log \frac{1}{1 - |z|^2} \right) = \Delta U_0 + \Delta \frac{\varepsilon}{2} \log \frac{1}{1 - |z|^2} \geq 0.$$

Hence  $w_1$  is subharmonic on  $D(\lambda_1, r_{\lambda_1})$ , and for  $\xi \in \partial D(\lambda_1, r_{\lambda_1})$ , since  $v_1(\xi) = 0$ , we have

$$w_1(\xi) = U_0(\xi) - s(\xi) = h_0(\xi) - s(\xi) = \log |f(\xi)| - s(\xi) \leq 0.$$

So in the boundary  $w_1$  is non-positive, so that it is non-positive throughout the disc by the maximum principle.

It remains to see  $h_0 \leq h_1$ . Again outside the hyperbolic disc  $D(\lambda_1, r_{\lambda_1})$  we have  $h_0 = h_1$ . In the disc  $D(\lambda_1, r_{\lambda_1})$  we need to compare the following functions

$$\varphi(z) = \frac{\alpha + \varepsilon}{2} \log \frac{1 - r_{\lambda_1}^2}{1 - |\varphi_{\lambda_1}(z)|^2},$$

and

$$\psi(z) = m_1 \log \frac{|\varphi_{\lambda_1}(z)|}{r_{\lambda_1}}.$$

More precisely, we have to show that  $\psi \leq \varphi$  on  $D(\lambda_1, r_1)$ . For this, we use the auxiliary functions

$$\varphi : x \mapsto \frac{\alpha + \varepsilon}{2} \log \frac{1 - r_{\lambda_1}^2}{1 - x^2},$$

and

$$\psi : x \mapsto m_1 \log \frac{x}{r_{\lambda_1}}.$$

The function  $\psi$  is concave, while  $\varphi$  is convex and  $\psi(r_{\lambda_1}) = \varphi(r_{\lambda_1}) = 0$ . Moreover

$$\varphi'(r_{\lambda_1}) = \psi'(r_{\lambda_1}) = \sqrt{m_1(m_1 + \alpha + \varepsilon)}$$

thus the two functions touch smoothly at  $x = r_{\lambda_1}$ , and  $\psi \leq \varphi$  on  $(0, r_{\lambda_1}]$ . As a consequence  $h_0 \leq h_1$  in  $D(\lambda_1, r_{\lambda_1})$ .

Now we construct  $h_2$  in the same way as before. We have  $h_1 = v_1 + U_0$ , so we can write

$$\begin{aligned} h_1(z) &= m_2 \log \frac{|\varphi_{\lambda_2}(z)|}{r_{\lambda_2}} + \underbrace{v_1 + U_0 - m_2 \log \frac{|\varphi_{\lambda_2}(z)|}{r_{\lambda_2}}}_{U_1} \\ &= m_2 \log \frac{|\varphi_{\lambda_2}(z)|}{r_{\lambda_2}} + U_1, \end{aligned}$$

where, since  $f/(\varphi_{\lambda_1}^{m_1} \varphi_{\lambda_2}^{m_2})$  is holomorphic,  $U_1$  is a subharmonic function that is harmonic in a small neighborhood of  $\lambda_2$ . Again we modify the term  $m_2 \log \frac{|\varphi_{\lambda_2}(z)|}{r_{\lambda_2}}$  in the hyperbolic disc  $D(\lambda_2, r_{\lambda_2})$ , and set

$$v_2(z) = \begin{cases} \frac{\alpha+\varepsilon}{2} \log \frac{1-r_{\lambda_2}^2}{1-|\varphi_{\lambda_2}(z)|^2}, & z \in D(\lambda_2, r_{\lambda_2}) \\ m_2 \log \frac{|\varphi_{\lambda_2}(z)|}{r_{\lambda_2}}, & \text{otherwise.} \end{cases}$$

And as in the first step, set  $h_2 = v_2 + U_1$ .

Iterating this procedure we obtain a sequence of sub-harmonic functions  $(h_n)_n$ , such that for every  $z \in \mathbb{D}$ , the sequence  $(h_n(z))_n$  is increasing and

$$\log |f(z)| \leq h_n(z) \leq s(z).$$

So the pointwise limit  $h$  of the sequence  $(h_n)_n$ , which is still subharmonic on  $\mathbb{D}$ , is comprised between  $\log |f|$  and  $s(z)$ .

Observe that  $h$  has been obtained from  $\log |f|$  by replacing around each zero  $\lambda$  of  $f$  the function  $m \log |\varphi_\lambda|/r_\lambda$  by  $\frac{\alpha+\varepsilon}{2} \log(1-r_\lambda^2)/(1-|\varphi_\lambda|^2)$  and by a harmonic function far from the zeros so that the laplacian of  $\log |f|$  is given by the sum of the laplacians of  $v_k$ .

Since  $h(z) \leq s(z)$ , by Green's formula ([11, Theorem 3.6, p. 59]), for  $0 \leq r < 1$

$$\begin{aligned} \int_{D(0,r)} \Delta h(z) \log \frac{r}{|z|} dm(z) &= -2h(0) + \frac{1}{r\pi} \int_{|z|=r} h(z) d|z| \\ &\leq 2 \left( -h(0) + \log \|f\|_{\alpha,\infty} + \frac{\alpha}{2} \log \frac{1}{1-r^2} \right). \end{aligned}$$

On the other hand,

$$\int_{D(0,r)} \Delta h(z) \log \frac{r}{|z|} dm(z) \geq 2(\alpha + \varepsilon) \int_{D(0,r)} \sum_{k \geq 1} \chi_{D_{\lambda_k}}(z) \log \frac{r}{|z|} d\mathcal{V}(z).$$

Hence,

$$\int_{D(0,r)} \sum_{k \geq 1} \chi_{D_{\lambda_k}}(z) \log \frac{r}{|z|} d\mathcal{V}(z) \leq \frac{\alpha}{2(\alpha + \varepsilon)} \log \frac{1}{1-r^2} + O(1), \quad r \rightarrow 1,$$

as required.  $\square$

We are now in a position to prove the uniqueness result.

**Theorem 2.2.** *Let  $\alpha > 0$ ,  $\varepsilon > 0$ , and  $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$  be a divisor such that*

$$\bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{\alpha + \varepsilon + m_\lambda}}\right) = \mathbb{D} \setminus K \quad (11)$$

*for some compact  $K \subset \mathbb{D}$ . Then  $X$  is a uniqueness divisor for  $\mathcal{A}_\alpha^\infty$ .*

Before giving the proof of this result, we mention that Theorem 1.6 easily follows from this. Indeed, suppose  $X$  is a zero divisor in  $\mathcal{A}_\alpha^2 \subset \mathcal{A}_{\alpha+2}^\infty$ , then (11) does not hold for any compact  $K$  where  $\alpha$  is replaced by  $\alpha + 2$  as required in Theorem 1.6.

*Proof.* By contradiction, suppose  $X$  is a zero divisor for  $\mathcal{A}_\alpha^\infty$ . Recall  $D_\lambda = D\left(\lambda, \sqrt{\frac{m_\lambda}{\alpha + \varepsilon + m_\lambda}}\right)$ . By Lemma 2.1,

$$c + \frac{\alpha}{2(\alpha + \varepsilon)} \log \frac{1}{1 - r^2} \geq \int_{D(0,r)} \sum_{\lambda \in \Lambda} \chi_{D_\lambda}(z) \log \frac{r}{|z|} \frac{dm(z)}{(1 - |z|^2)^2}. \quad (12)$$

Hence

$$\begin{aligned} \int_{D(0,r)} \sum_{\lambda \in \Lambda} \chi_{D_\lambda}(z) \log \frac{r}{|z|} d\mathcal{V}(z) &= \int_{D(0,r)} \log \frac{r}{|z|} d\mathcal{V}(z) \\ &+ \int_{D(0,r) \cap K} \left[ \sum_{\lambda \in \Lambda} \chi_{D_\lambda}(z) - 1 \right] \log \frac{r}{|z|} d\mathcal{V}(z) + \int_{D(0,r) \setminus K} \left[ \sum_{\lambda \in \Lambda} \chi_{D_\lambda}(z) - 1 \right] \log \frac{r}{|z|} d\mathcal{V}(z). \end{aligned}$$

Notice that

- (a) when  $z \in D(0,r) \setminus K$ , then  $\sum_{\lambda \in \Lambda} \chi_{D_\lambda}(z) - 1 \geq 0$ ,
- (b) integration inside  $K$  only contributes at most as an additive (negative) constant,

so that

$$\int_{D(0,r)} \sum_{\lambda \in \Lambda} \chi_{D_\lambda}(z) \log \frac{r}{|z|} d\mathcal{V}(z) \geq \int_{D(0,r)} \log \frac{r}{|z|} d\mathcal{V}(z) + O(1) = \frac{1}{2} \log \frac{1}{1 - r^2} + O(1).$$

But this is in contradiction with (12) which concludes the proof.  $\square$

We should stop here for a little observation. In the case of the Fock space, it was enough to consider the critical radius in order to get the corresponding uniqueness result. This was related to the observation that when we cover the whole plane by disks we will necessarily encounter "big" overlaps of these disks however far we are from the origin, and that in this case  $\log R/|z|$  can be arbitrarily big. The situation changes in the unit disk where  $\log R/|z|$  tends to zero, and much more subtle overlap conditions have to be discussed. For the purpose of our discussions here it is sufficient to play on the parameter  $\varepsilon$ . For that reason we have to impose a covering condition with smaller radii involving  $\varepsilon$  (equivalently the critical radius gives the uniqueness result in all weighted Bergman spaces  $\mathcal{A}_{\alpha'}^\infty$  with weight  $\alpha' > \alpha$ ). It is easy to check that for given  $\varepsilon > 0$  there exist  $C_2 > C_1 > 0$  (actually  $C_2 > \varepsilon > C_1 > 0$ ), such that for every  $m_\lambda > C_2$ ,

$$\frac{m_\lambda - C_2}{m_\lambda + \alpha} \leq \frac{m_\lambda}{m_\lambda + \alpha + \varepsilon} \leq \frac{m_\lambda - C_1}{m_\lambda + \alpha}. \quad (13)$$

We finish this section with the short argument leading to (6). Denoting by  $D_e(u, r) = \{z \in \mathbb{C} : |z - u| < r\}$  a Euclidean disk, we have (see [9, p.4])

$$D(\lambda, t) = D_e\left(\frac{1 - t_\lambda^2}{1 - t_\lambda^2|\lambda|^2}\lambda, \frac{1 - |\lambda|^2}{1 - t_\lambda^2|\lambda|^2}t_\lambda\right) \quad (14)$$

Using the finite overlap, an area argument shows that

$$\sum_{\lambda \in \Lambda} \left(\frac{1 - |\lambda|^2}{1 - t_\lambda^2|\lambda|^2}t_\lambda\right)^2 \lesssim \pi^2, \quad (15)$$

where

$$t_\lambda = \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}}.$$

According to (13), playing on  $C$ , this is comparable to the same expression with  $\varepsilon$  in the denominator, but no constant subtracted in the numerator. This will allow us to treat at the same time the situation with finite overlap (coming from sampling), or the separation condition (coming from interpolation). The value of  $C$  will not be important in our estimates.

The finite overlap implies that the Euclidean radii  $R_\lambda$  tend to zero when  $\lambda$  approaches the boundary  $\partial\mathbb{D}$ :

$$R_\lambda = \frac{1 - |\lambda|^2}{1 - t_\lambda^2|\lambda|^2}t_\lambda \rightarrow 0, \quad \text{as } |\lambda| \rightarrow 1. \quad (16)$$

Since  $t_\lambda = \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}} \sim 1$ , we have from (15)

$$\left(\frac{1 - |\lambda|^2}{1 - t_\lambda^2|\lambda|^2}\right)^2 \rightarrow 0, \quad \text{as } |\lambda| \rightarrow 1.$$

On the other hand

$$\begin{aligned} \frac{1 - |\lambda|^2}{1 - t_\lambda^2|\lambda|^2} &= (1 - |\lambda|^2) \frac{1}{1 - |\lambda|^2 \left(\frac{m_\lambda - C}{m_\lambda + \alpha}\right)} = (1 - |\lambda|^2) \frac{1}{1 - |\lambda|^2 \left(1 - \frac{C + \alpha}{m_\lambda + \alpha}\right)} \\ &= \frac{1}{1 + \left(\frac{(C + \alpha)|\lambda|^2}{(m_\lambda + 1)(1 - |\lambda|^2)}\right)} \rightarrow 0, \quad \text{as } |\lambda| \rightarrow 1. \end{aligned}$$

Necessarily

$$(m_\lambda + 1)(1 - |\lambda|^2) \sim m_\lambda(1 - |\lambda|^2) \rightarrow 0, \quad \text{as } |\lambda| \rightarrow 1,$$

and thus

$$\frac{1 - |\lambda|^2}{1 - t_\lambda^2|\lambda|^2} \simeq m_\lambda(1 - |\lambda|^2),$$

which in view of (15) leads to (6).

### 3. SAMPLING AND INTERPOLATION IN $\mathcal{A}_\alpha^2$

**3.1. Local  $L^2$ -estimates.** We will obtain the results *via* a local control of the  $\mathcal{A}_\alpha^2$ -functions. Let us consider the  $L^2$ -norm in a domain  $\Omega$ , denoted by  $\|\cdot\|_\Omega$ . With this norm, we can obtain a control of the norm of the basis elements in terms of the regularized  $\beta$ -function.

For  $\operatorname{Re} a > 0$  and  $\operatorname{Re} b > 0$ , we define the  $\beta$ -function

$$\beta(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt. \quad (17)$$

Recall that

$$\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad (18)$$

and so by Stirling's formula (see [1])

$$\beta(n+1, \alpha+1) \sim \frac{\Gamma(\alpha+1)}{n^{1+\alpha}}, \quad (19)$$

The following sharp inequality is sometimes useful (see [10])

$$\left(\frac{x}{x+s}\right)^{1-s} \leq \frac{\Gamma(x+s)}{x^s \Gamma(x)} \leq 1, \quad x > 0, \quad 0 < s < 1. \quad (20)$$

For  $x \in (0, 1]$ , we define the *incomplete*  $\beta$ -function

$$\beta(x; a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad (21)$$

and the *regularized incomplete*  $\beta$ -function (or *regularized*  $\beta$ -function for short)

$$I(x; a, b) = \frac{\beta(x; a, b)}{\beta(a, b)}. \quad (22)$$

With these notations in mind we can compute the norm of the orthonormal basis  $\{e_j\}_j$  on smaller disks (we will of course be interested in disks of type  $D_\lambda$ ). For this we need the following result.

**Lemma 3.1.** *Let  $\{e_j\}_j$  be the orthonormal basis for  $\mathcal{A}_\alpha^2$ . Then, for all  $j \geq 0$  and  $0 < r < 1$*

$$\|e_j\|_{D(0,r)}^2 = (\alpha+1)I(r^2; j+1, \alpha+1).$$

*Proof.* Recall first from (1) that

$$e_n(z) = \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}} z^n, \quad n \geq 0.$$

Since  $\Gamma(n) = (n-1)!$ , and with (18) in mind, we can rewrite

$$\beta(n+1, \alpha+1) = \frac{n!\Gamma(1+\alpha)}{\Gamma(n+2+\alpha)} = \frac{n!\Gamma(2+\alpha)}{(1+\alpha)\Gamma(n+2+\alpha)},$$

and

$$e_n(z) = \sqrt{\frac{1}{(1+\alpha)\beta(n+1, \alpha+1)}} z^n, \quad n \geq 0 \quad (23)$$

(which can be found also directly by computing the norm  $\|z^n\|_{\alpha,2}^2$ ).

Now, the lemma follows from the following computation with the obvious substitution  $s = t^2$ ,

$$\begin{aligned} \|z^j\|_{D(0,r)}^2 &= (\alpha+1) \int_{D(0,r)} |z^j|^2 (1-|z|^2)^\alpha dm(z) = 2(\alpha+1) \int_0^r t^{2j+1} (1-t^2)^\alpha dt \\ &= (\alpha+1)\beta(r^2; j+1, \alpha+1). \end{aligned}$$

□



For our later discussions we will thus need estimates on the regularized incomplete  $\beta$ -function.

**Lemma 3.2.** *Let  $\alpha > -1$  and  $r_m = \sqrt{\frac{m}{m+\alpha+1}}$  then*

(a) *For every  $c > 0$ , there is  $\varepsilon = \varepsilon(\alpha, c) > 0$  such that for all  $m \geq c + 1$  and  $n < m$*

$$I\left(\frac{m-c}{m+\alpha+1}; n+1, \alpha+1\right) \geq \varepsilon.$$

(b) *For  $t < r_m$ , there exists  $\varepsilon = \varepsilon(\alpha)$  such that*

$$F_{m,\alpha}(t) = [1-t^2]^{\alpha+2} \sum_{0 \leq j < m} \frac{t^{2j}}{(\alpha+1)\beta(j+1, \alpha+1)} \geq \varepsilon.$$

(c) *Given  $0 < \eta < 1$ , there exists  $a_\alpha(\eta) > 0$  such that for all  $j \geq m \geq a_\alpha(\eta)$*

$$I\left(\frac{m-a(\eta)}{m+\alpha+1}; j+1, \alpha+1\right) \leq \eta I\left(\frac{m}{m+\alpha+1}; j+1, \alpha+1\right).$$

The lemma does not appeal to a specific zero divisor. Later on,  $m$  will correspond to  $m_\lambda$  and  $r_m$  to  $r_\lambda$ . Compare the critical radius  $r_m$  appearing in this lemma with the one given in (8) (as mentioned there, the critical value corresponds to the situation when  $\varepsilon = 0$ ): the term  $\alpha$  appearing there turns into  $\alpha + 1$  here.

*Proof.* (a) Since  $\beta(x, n+1, \alpha+1)$  is increasing in  $x$ , for  $m > n$

$$I\left(\frac{m-c}{m+\alpha+1}; n+1, \alpha+1\right) \geq I\left(\frac{n-c}{n+\alpha+1}; n+1, \alpha+1\right).$$

We need to treat two cases.

First, suppose  $\alpha > 0$ , then

$$\begin{aligned} \beta\left(\frac{n-c}{n+\alpha+1}; n+1, \alpha+1\right) &= \int_0^{\frac{n-c}{n+\alpha+1}} t^n (1-t)^\alpha dt \\ &\geq \left(1 - \frac{n-c}{n+\alpha+1}\right)^\alpha \int_0^{\frac{n-c}{n+\alpha+1}} t^n dt \\ &\geq \left(\frac{\alpha+1+c}{n+\alpha+1}\right)^\alpha \frac{1}{n+1} \left(1 - \frac{\alpha+1+c}{n+\alpha+1}\right)^{n+1}, \end{aligned}$$

which is comparable to  $\frac{1}{(n+1)^{(\alpha+1)}}$  for  $n \geq c+1$  (uniformly in  $n$  and  $m$ ). Now, By (19), this yields inequality (a) at least when  $n \geq c+1$ .

Second, suppose  $\alpha \in (-1, 0]$ . Note that for  $r \in (0, 1), n \geq 0$  we have

$$\begin{aligned} \int_0^r t^n (1-t)^\alpha dt &= \frac{r^{n+1}}{n+1} (1-r)^\alpha + \frac{\alpha}{n+1} \int_0^r t^{n+1} (1-t)^{\alpha-1} dt \\ &\geq \frac{r^{n+1}}{n+1} (1-r)^\alpha + \frac{\alpha}{n+1} \frac{r}{1-r} \int_0^r t^n (1-t)^\alpha dt, \end{aligned}$$

where we have also used the fact that  $\alpha r/(1-r)$  is decreasing in  $r$  since  $\alpha < 0$ . Hence

$$\int_0^r t^n (1-t)^\alpha dt \geq \frac{1}{1 + \frac{-\alpha}{n+1} \frac{r}{1-r}} \frac{r^{n+1}}{n+1} (1-r)^\alpha. \quad (24)$$

Let  $r = \frac{n-c}{n+\alpha+1}$  (the interesting case being  $n \geq c+1$ ). In this situation we have

$$1 + \frac{-\alpha}{n+1} \times \frac{r}{1-r} = 1 + \frac{-\alpha}{\alpha+1+c} \times \frac{n-c}{n+1} \leq 1 + \frac{-\alpha}{\alpha+1+c}.$$

Thus, from (24)

$$\begin{aligned} \int_0^{\frac{n-c}{n+\alpha+1}} t^n (1-t)^\alpha dt &\geq \frac{1}{[1 + \frac{-\alpha}{\alpha+1+c}]} \times \frac{1}{(n+1)} \times \left( \frac{\alpha+1+c}{n+\alpha+1} \right)^\alpha \times \left( \frac{n-c}{n+\alpha+1} \right)^{n+1} \\ &\geq \frac{(\alpha+1+c)^\alpha e^{-(\alpha+1+c)(2+c)}}{[1 + \frac{-\alpha}{\alpha+1+c}]} \times \frac{1}{(n+1)^{\alpha+1}}. \end{aligned}$$

at the second inequality we have used  $\frac{t-1}{t} \leq \log t$ ,  $t > 0$ .

On the other hand, by (20) we get

$$\frac{1}{\beta(n+1, \alpha+1)} = \frac{\Gamma(n+1+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} \geq \frac{(n+1)^{\alpha+1}}{\Gamma(\alpha+1)}.$$

Put

$$\varepsilon = \varepsilon(\alpha, c) := \frac{1}{\Gamma(\alpha+1)} \times \frac{(\alpha+1+c)^\alpha e^{-(\alpha+1+c)(2+c)}}{[1 + \frac{-\alpha}{\alpha+1+c}]}.$$

Hence it follows from the inequalities above

$$I\left(\frac{n-c}{n+\alpha+2}; n+1, \alpha+1\right) = \frac{\int_0^{\frac{n-c}{n+\alpha+2}} t^n (1-t)^\alpha dt}{\beta(n+1, \alpha+1)} \geq \varepsilon, \quad (25)$$

which is the desired result.

Finally, for  $n < c+1$ , independently whether  $\alpha \geq 0$  or  $\alpha \in (-1, 0)$ , the desired estimate follows from the fact that the integration interval  $[0, \frac{m-c}{m+\alpha+1}]$  of the incomplete  $\beta$ -function contains a fixed interval  $[0, \frac{1}{c+\alpha+2}]$  and both powers of  $t$  and  $(1-t)$  appearing in the definition of  $\beta(x; n+1, \alpha+1)$  are controlled. Dividing by  $\beta(n+1, \alpha+1)$  does not change this control since  $n$  is bounded.

(b) Recall that the reproducing kernel of  $\mathcal{A}_\alpha^2$ ,  $K(t, t) = (1-t^2)^{-\alpha-2}$  satisfies also

$$K(t, t) = \sum_{j \geq 0} \frac{t^{2j}}{(\alpha+1)\beta(j+1, \alpha+1)}, \quad 0 \leq t < 1$$

Hence, it suffices to prove that there exists  $\epsilon = \epsilon(\alpha)$  and  $m_0$  such that for  $t < r_m$

$$R_{m,\alpha}(t) = (1-t^2)^{\alpha+2} \sum_{j \geq m} \frac{t^{2j}}{(\alpha+1)\beta(j+1, \alpha+1)} \leq 1 - \epsilon, \quad m \geq m_0,$$

By (19),

$$\frac{1}{(\alpha+1)\beta(j+1, \alpha+1)} = \frac{1}{(\alpha+1)\Gamma(\alpha+1)/j^{\alpha+1}(1+o_j(1))} = \frac{j^{1+\alpha}}{\Gamma(2+\alpha)}(1+o_j(1)),$$

where  $o_j(1)$  tends to zero when  $j$  tends to infinity. So

$$R_{m,\alpha}(t) = \frac{(1-t^2)^{\alpha+2}}{\Gamma(\alpha+2)} \sum_{j \geq m} j^{1+\alpha} t^{2j} (1+o_j(1)) \leq \frac{(1-t^2)^{\alpha+2}}{\Gamma(\alpha+2)} (1+o_m(1)) \sum_{j \geq m} j^{1+\alpha} t^{2j}.$$

We will pass to an integral. For that, note that when  $x \in [j-1, j]$  and  $t \in [0, 1)$ , we have  $t^{2j} \leq t^{2x}$ , and obviously  $j^{1+\alpha} = x^{1+\alpha}(1 + o_j(1))$ . We deduce

$$\begin{aligned} R_{m,\alpha}(t) &\leq (1 + o_m(1)) \frac{(1-t^2)^{\alpha+2}}{\Gamma(\alpha+2)} \int_{m-1}^{\infty} x^{1+\alpha} t^{2x} dx \\ &= (1 + o_m(1)) \frac{(1-t^2)^{\alpha+2}}{\Gamma(\alpha+2)(\log 1/t^2)^{2+\alpha}} \int_{(m-1)(\log 1/t^2)}^{\infty} u^{1+\alpha} e^{-u} du. \end{aligned}$$

Note that the function  $t \mapsto (1-t^2)/\ln(1/t^2)$  is bounded by 1 on  $(0, 1)$ . Also, observe that  $0 < t < r_m = \sqrt{\frac{m}{m+\alpha+1}}$ , and hence

$$\log \frac{1}{t^2} > \log \frac{m+\alpha+1}{m} = \log\left(1 + \frac{\alpha+1}{m+\alpha+1}\right) = \frac{\alpha+1}{m+\alpha+1}(1 + o_m(1)).$$

We deduce

$$R_{m,\alpha}(t) \leq \frac{1 + o_m(1)}{\Gamma(\alpha+2)} \int_{\frac{(\alpha+1)(m-1)}{m+\alpha+1}(1+o_m(1))}^{\infty} u^{1+\alpha} e^{-u} du.$$

Let  $\beta = \frac{(\alpha+1)(m-1)}{m+\alpha+1}(1 + o_m(1))$ , then using the notation  $\Gamma(a, b) := \int_b^{\infty} t^{a-1} e^{-t} dt$  for the incomplete Gamma function, we have

$$R_{m,\alpha}(t) \leq \frac{1 + o_m(1)}{\Gamma(\alpha+2)} \Gamma(\alpha+2, \beta).$$

Now observe that for finite  $m$  the estimate in (b) is trivially true. We can thus assume that  $m$  is big enough so that  $\beta \geq \alpha+1-1 = \alpha$ . So

$$R_{m,\alpha}(t) \leq \frac{1 + o_m(1)}{\Gamma(\alpha+2)} \Gamma(\alpha+2, \alpha).$$

Since  $\alpha$  is fixed, we obviously have  $q = \Gamma(\alpha+2, \alpha)/\Gamma(\alpha+2) < 1$ , and for sufficiently big  $m$  we have  $1 + o_m(1) < (1+q)/(2q)$  implying that  $R_{m,\alpha} \leq (1+q)/2 < 1$ , and we can pick  $\varepsilon = (1-q)/2$ .

(c) Clearly, setting

$$\rho = \frac{m - a(\eta)}{m + \alpha + 1}, \quad r = \frac{m}{m + \alpha + 1},$$

it is enough to show the estimate for the incomplete  $\beta$ -function:

$$\begin{aligned} \beta \left( \frac{m - a(\eta)}{m + \alpha + 1}; j+1, \alpha+1 \right) &= \int_0^{\rho} t^j (1-t)^{\alpha} dt \\ &\leq \eta \int_0^r t^j (1-t)^{\alpha} dt \\ &= \eta \beta \left( \frac{m}{m + \alpha + 1}; j+1, \alpha+1 \right) \end{aligned}$$

First assume  $\alpha > 0$ .

Note that

$$\beta_2 := \int_0^r t^j (1-t)^{\alpha} dt \geq (1-r)^{\alpha} \frac{r^{j+1}}{j+1}, \quad (26)$$

and an integration by part, as well as the fact that  $t/(1-t)$  is increasing in  $t$ , yield

$$\begin{aligned} \int_0^\rho t^j (1-t)^\alpha dt &= \frac{\rho^{j+1}}{j+1} (1-\rho)^\alpha + \frac{\alpha}{j+1} \int_0^\rho t^{j+1} (1-t)^{\alpha-1} dt \\ &\leq \frac{\rho^{j+1}}{j+1} (1-\rho)^\alpha + \frac{\alpha}{j+1} \frac{\rho}{1-\rho} \int_0^\rho t^j (1-t)^\alpha dt. \end{aligned}$$

So, if  $\frac{\alpha}{j+1} \frac{\rho}{1-\rho} < 1$ , we get

$$\beta_1 := \int_0^\rho t^j (1-t)^\alpha dt \leq \frac{1}{1 - \frac{\alpha}{j+1} \frac{\rho}{1-\rho}} \frac{\rho^{j+1}}{j+1} (1-\rho)^\alpha. \quad (27)$$

Given  $0 < \eta < 1$  and  $a > 0$ , it follows from (26) and (27)

$$\frac{\beta_1}{\beta_2} \leq \frac{1}{1 - \frac{\alpha}{j+1} \frac{\rho}{1-\rho}} \left( \frac{1-\rho}{1-r} \right)^\alpha \left( \frac{\rho}{r} \right)^{j+1} \quad (28)$$

Simple computations show

$$\frac{\rho}{r} = \frac{m - a(\eta)}{m} = 1 - \frac{a(\eta)}{m} < 1,$$

and

$$\frac{1-\rho}{1-r} = \frac{\alpha + 1 + a(\eta)}{\alpha + 1} \quad \text{and} \quad \frac{\rho}{1-\rho} = \frac{m - a(\eta)}{\alpha + 1 + a(\eta)}.$$

Now for  $j \geq m \geq a(\eta)$ ,

$$\frac{\alpha}{j+1} \frac{\rho}{1-\rho} = \frac{\alpha}{j+1} \frac{m - a(\eta)}{\alpha + 1 + a(\eta)} \leq \frac{\alpha m}{(m+1)(\alpha + 1) + (m+1)a(\eta)} \leq \frac{\alpha}{\alpha + 1 + a(\eta)},$$

therefore,

$$\frac{1}{1 - \frac{\alpha}{j+1} \frac{\rho}{1-\rho}} \leq \frac{\alpha + 1 + a(\eta)}{1 + a(\eta)}.$$

It remains the term

$$\left( \frac{\rho}{r} \right)^{j+1} = \left( 1 - \frac{a(\eta)}{m} \right)^{j+1} \leq \left( 1 - \frac{a(\eta)}{m} \right)^m \leq e^{-a(\eta)} \quad (29)$$

Hence

$$\frac{\beta_1}{\beta_2} \leq \frac{\alpha + 1 + a(\eta)}{1 + a(\eta)} \times \left( 1 + \frac{a(\eta)}{1 + \alpha} \right)^\alpha \times e^{-a(\eta)}.$$

Since the exponential decrease dominates the polynomial growth, we have

$$\lim_{a \rightarrow +\infty} \frac{\alpha + 1 + a}{1 + a} \times \left( 1 + \frac{a}{1 + \alpha} \right)^{\alpha+1} \times e^{-a} = 0,$$

which proves the claim: for every  $\eta > 0$ , we can find  $a$ , such that  $\beta_1/\beta_2 < \eta$  independently on  $j \geq m \geq a$ , and the same is true for  $I_1/I_2$ .

Second, assume  $\alpha \in (-1, 0]$ . The exact same arguments as in the case  $\alpha > 0$  allow to reverse the inequalities in (26) and (27) (since  $\alpha$  is negative, the expression  $\alpha r/(1-r)$  is decreasing and  $(1-r)^\alpha$  is increasing in  $r$ ). Hence,

$$\beta_2 := \int_0^r t^j (1-t)^\alpha dt \geq \frac{1}{1 + \frac{-\alpha}{j+1} \frac{r}{1-r}} \frac{r^{j+1}}{j+1} (1-r)^\alpha, \quad (30)$$

and

$$\beta_1 := \int_0^\rho t^j (1-t)^\alpha dt \leq (1-\rho)^\alpha \frac{\rho^{j+1}}{j+1}. \quad (31)$$

As in (28) for  $\alpha > 0$ , it follows from (30) and (31)

$$\frac{\beta_1}{\beta_2} \leq \left(1 - \frac{\alpha}{j+1} \frac{r}{1-r}\right) \left(\frac{1-\rho}{1-r}\right)^\alpha \left(\frac{\rho}{r}\right)^{j+1},$$

where the first factor is now in the numerator instead of the denominator. Hence, with  $\rho = \frac{m-a(\eta)}{m+\alpha+1}$  and  $r = \frac{m}{m+\alpha+1}$  in mind, we get as before

$$\frac{\beta_1}{\beta_2} \leq \left[1 + \frac{-\alpha}{j+1} \times \frac{m}{\alpha+1}\right] \times \left(1 + \frac{a(\eta)}{\alpha+1}\right)^\alpha \times \left(1 - \frac{a(\eta)}{m}\right)^{j+1}.$$

Since  $j+1 > m$  and  $\alpha < 0$ , we get

$$1 + \frac{-\alpha}{j+1} \times \frac{m}{\alpha+1} \leq 1 + \frac{-\alpha}{\alpha+1} = \frac{1}{1+\alpha}.$$

Again

$$\left(1 - \frac{a(\eta)}{m}\right)^{j+1} \leq e^{-a},$$

and now

$$\left(1 + \frac{a(\eta)}{\alpha+1}\right)^\alpha < 1,$$

so that

$$\frac{\beta_1}{\beta_2} \leq \frac{1}{1+\alpha} e^{-a},$$

which again goes to 0, proving the claim also in this case.  $\square$

As mentioned earlier, we will need to switch from the orthonormal basis  $\{e_n\}_n$  in  $\mathcal{A}_\alpha^2$  to an orthonormal basis on a smaller disk  $D(0, r)$ . The following lemma recalls this simple fact.

**Lemma 3.3.** *Let  $f = \sum_{n \geq 0} a_n e_n \in \mathcal{A}_\alpha^2$ . Then*

$$(\alpha+1) \int_{D(0,r)} |f(z)|^2 (1-|z|^2)^\alpha dm(z) = \sum_{n \geq 0} I(r^2, n+1, \alpha+1) |a_n|^2,$$

for every  $r \in (0, 1)$

*Proof.* Set

$$f_n = \frac{e_n}{\sqrt{I(r^2, n+1, \alpha+1)}},$$

then, by Lemma (3.1),  $(f_n)$  is orthonormal with respect to the measure  $dA_\alpha$  on  $D(0, r)$ . Writing now

$$f(z) = \sum_{n \geq 0} a_n e_n = \sum_{n \geq 0} (a_n \sqrt{I(r^2, n+1, \alpha+1)}) f_n,$$

we get the required equality.  $\square$

**Lemma 3.4.** *Let  $f \in \mathcal{A}_\alpha^2$ , for every  $c > 0$  there exist constants  $A = A(c) > 0$  such that for every  $m \geq c + 1$ , we have*

$$\sum_{j < m} |\langle f, T_\lambda e_j \rangle|^2 \leq A(\alpha + 1) \int_{D(\lambda, \sqrt{\frac{m-c}{m+\alpha+1}})} |f(z)|^2 (1 - |z|^2)^\alpha dm(z),$$

for any  $\lambda \in \mathbb{D}$ .

*Proof.* Set  $r = r_{m,\alpha,c} := \sqrt{\frac{m-c}{m+\alpha+1}}$ . Taking  $g = T_\lambda f$ , the statement can be rewritten as

$$\sum_{0 \leq j < m} |\langle g, e_j \rangle|^2 \lesssim (\alpha + 1) \int_{D(0,r)} |g(z)|^2 (1 - |z|^2)^\alpha dm(z).$$

Since  $g \in \mathcal{A}_\alpha^2$ ,

$$g(z) = \sum_{j \geq 0} a_j e_j = \sum_{j \geq 0} a_j \frac{z^j}{\sqrt{(\alpha + 1)\beta(j + 1, \alpha + 1)}}$$

and

$$|\langle g, e_j \rangle| = |a_j|, \quad j \geq 0.$$

Using Lemma 3.3 applied to  $g$ , the claim is equivalent to

$$\sum_{0 \leq j < m} |a_j|^2 \lesssim \sum_{j \geq 0} |a_j|^2 I\left(\frac{m-c}{m+\alpha+1}; j+1, \alpha+1\right),$$

where we have substituted back the value of  $r$ . Therefore it is enough to prove

$$I\left(\frac{m-c}{m+\alpha+1}; j+1, \alpha+1\right) \geq \varepsilon(\alpha, c) > 0$$

for  $j < m$  and  $m \geq c + 1$ , but this is given by Lemma 3.2(a).  $\square$

The next lemma relates the finite overlap condition to a kind of Carleson measure type condition.

**Lemma 3.5.** *Let  $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$  be a divisor. Then  $X$  satisfies the finite overlap condition if and only if there exists a constant  $C > 0$  satisfying*

$$\sum_{\lambda \in \Lambda} \sum_{j < m_\lambda} |\langle f, T_\lambda e_j \rangle|^2 \leq C \|f\|_{\alpha,2}^2, \quad f \in \mathcal{A}_\alpha^2.$$

*Proof.* Suppose that the estimate holds. Given  $z \in \mathbb{D}$ , set  $f_z = T_z 1$ , observe that  $|T_z 1| = |k_z|$ , so  $\|f_z\|_{\alpha,2} = 1$  and

$$\begin{aligned} |\langle f_z, T_\lambda e_j \rangle|^2 &= |\langle T_z 1, T_\lambda e_j \rangle|^2 = |\langle e_0, T_z T_\lambda e_j \rangle|^2 = |T_z T_\lambda e_j(0)|^2 \\ &= \frac{1}{(\alpha + 1)\beta(j + 1, \alpha + 1)} [1 - |\varphi_\lambda(z)|^2]^{2+\alpha} |\varphi_\lambda(z)|^{2j}, \end{aligned} \quad (32)$$

where we have used the form (23) of  $e_j$ . Then, by assumption,

$$\begin{aligned} 1 = \|f\|_{\alpha,2}^2 &\gtrsim \sum_{\lambda \in \Lambda} \sum_{j < m_\lambda} \frac{1}{(\alpha + 1)\beta(j + 1, \alpha + 1)} [1 - |\varphi_\lambda(z)|^2]^{2+\alpha} |\varphi_\lambda(z)|^{2j} \\ &\geq \sum_{\lambda \in \Lambda} F_{m_\lambda, \alpha}(|\varphi_\lambda(z)|) \chi_{D(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}})}(z) \geq \epsilon \sum_{\lambda \in \Lambda} \chi_{D(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}})}(z), \end{aligned}$$

where the function  $F$  is defined in Lemma 3.2(b) and the last bound also comes from that result. Since the constants do not depend on  $z$ , we conclude that

$$1 \gtrsim \sum_{\lambda \in \Lambda} \chi_D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}}\right)(z).$$

In the opposite direction, if  $X$  satisfies the finite overlap condition, we just apply Lemma 3.4, which gives

$$\begin{aligned} \sum_{\lambda \in \Lambda} \sum_{j < m_\lambda} |\langle f, T_\lambda e_j \rangle|^2 &\lesssim \sum_{\lambda \in \Lambda} (\alpha + 1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}}\right)} |f(z)|^2 (1 - |z|^2)^\alpha dm(z) \\ &= \sum_{\lambda \in \Lambda} (\alpha + 1) \int_{\mathbb{D}} \chi_D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}}\right)(z) |f(z)|^2 (1 - |z|^2)^\alpha dm(z) \\ &\leq S_X \|f\|_{\alpha, 2}^2, \end{aligned}$$

where  $S_X$  is the overlap constant introduced in Definition (1.3). □

**3.2. Sampling for  $\mathcal{A}_\alpha^2$ .** In order to obtain our geometric conditions we need the following local control of  $\mathcal{A}_\alpha^2$ -functions.

**Lemma 3.6.** *Given  $0 < \eta \leq 1$  there exists  $a(\eta) > 0$  such that if  $f \in \mathcal{A}_\alpha^2$ , and  $m \geq a(\eta)$ , and if*

$$\begin{aligned} \sum_{j < m} |\langle f, e_j \rangle|^2 &\leq \eta/2, \\ (\alpha + 1) \int_{D(0, \sqrt{\frac{m}{m + \alpha + 1}})} |f(z)|^2 (1 - |z|^2)^\alpha dm(z) &\leq 1, \end{aligned}$$

then

$$(\alpha + 1) \int_{D(0, \sqrt{\frac{m - a(\eta)}{m + \alpha + 1}})} |f(z)|^2 (1 - |z|^2)^\alpha dm(z) \leq \eta.$$

As we will see from the proof, we have  $a(\eta) = a_\alpha(\eta/2)$  where  $a_\alpha$  appears in Lemma 3.2(c).

*Proof.* If we write

$$f(z) = \sum_j a_j e_j(z),$$

by the first assumption and (1), using the orthogonality of  $\{e_n\}_n$  with respect to the measure  $dA_\alpha$  on any disk  $D(0, r)$ ,  $0 < r < 1$ , and Lemma 3.3,

$$\begin{aligned} &(\alpha + 1) \int_{D(0, \sqrt{\frac{m - a(\eta)}{m + \alpha + 1}})} |f(z)|^2 (1 - |z|^2)^\alpha dm(z) \\ &= \sum_{j \geq 0} |a_j|^2 I\left(\frac{m - a(\eta)}{m + \alpha + 1}; j + 1, \alpha + 1\right) \\ &\leq \sum_{j < m} |a_j|^2 + \sum_{j \geq m} |a_j|^2 I\left(\frac{m - a(\eta)}{m + \alpha + 1}; j + 1, \alpha + 1\right) \\ &\leq \frac{\eta}{2} + \sum_{j \geq m} |a_j|^2 I\left(\frac{m - a(\eta)}{m + \alpha + 1}; j + 1, \alpha + 1\right). \end{aligned}$$



Now, by Lemma 3.2 (c) for  $j \geq m \geq a_\alpha(\eta/2)$ , another application of Lemma 3.3 and the hypothesis,

$$\begin{aligned} \sum_{j \geq m} |a_j|^2 I\left(\frac{m - a(\eta/2)}{m + \alpha + 1}; j + 1, \alpha + 1\right) &\leq \frac{\eta}{2} \sum_{j \geq m} |a_j|^2 I\left(\frac{m}{m + \alpha + 1}; j + 1, \alpha + 1\right) \\ &\leq (1 + \alpha) \frac{\eta}{2} \int_{D(0, \sqrt{\frac{m}{m + \alpha + 1}})} |f(z)|^2 (1 - |z|^2)^\alpha dm(z) \leq \frac{\eta}{2}, \end{aligned}$$

and the result follows.  $\square$

We are now in a position to prove Theorem 1.4 which we restate here for convenience.

**Theorem.** (a) If  $X$  is a sampling divisor for  $\mathcal{A}_\alpha^2$ , then  $X$  satisfies the finite overlap condition and there exists  $0 < C < \alpha + 1$  such that

$$\bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C}{m_\lambda + \alpha + 1}}\right) = \mathbb{D}.$$

(b) Conversely, let the divisor  $X$  satisfy the finite overlap condition and if there exists  $C = C(S_X) > 0$  such that for some compact  $K$  of  $\mathbb{D}$  we have

$$\bigcup_{\lambda \in \Lambda, m_\lambda > C} D\left(\lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha + 1}}\right) = \mathbb{D} \setminus K,$$

then  $X$  is a sampling divisor for  $\mathcal{A}_\alpha^2$ .

*Proof. Necessary Part (a)*

Let  $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$  be a sampling divisor. By Lemma 3.5, it satisfies the finite overlap condition. Suppose that for every  $0 < C < \alpha + 1$

$$\bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C}{m_\lambda + \alpha + 1}}\right) \neq \mathbb{D}.$$

Then there exists a sequence  $0 < C_j \uparrow (\alpha + 1)$  and  $z_j \in \mathbb{D}$  such that

$$z_j \notin \bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C_j}{m_\lambda + \alpha + 1}}\right).$$

Put  $r_{\lambda, C_j} = \sqrt{\frac{m_\lambda + C_j}{m_\lambda + \alpha + 1}}$  and  $r_\lambda = \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}}$ . Thus

$$z_j \in \mathbb{D} \setminus \left[ \bigcup_{\lambda \in \Lambda} D(\lambda, r_{\lambda, C_j}) \right] \subset \mathbb{D} \setminus \left[ \bigcup_{\lambda \in \Lambda} D(\lambda, r_\lambda) \right].$$

Let

$$\zeta \in \bigcup_{\lambda \in \Lambda} D(\lambda, r_\lambda)$$

that is  $\zeta \in D(\lambda_0, r_{\lambda_0})$  for some  $\lambda_0 \in \Lambda$  and we have for all  $j$

$$\rho(\zeta, \lambda_0) < r_{\lambda_0}, \quad \text{and} \quad \rho(z_j, \lambda_0) > r_{\lambda_0, C_j}.$$

By the triangular inequality for the pseudohyperbolic metric (see [9, p.4]), we get

$$\begin{aligned} \rho(z_j, \zeta) &\geq \frac{\rho(z_j, \lambda_0) - \rho(\zeta, \lambda_0)}{1 - \rho(z_j, \lambda_0)\rho(\zeta, \lambda_0)} \geq \frac{r_{\lambda_0, C_j} - r_{\lambda_0}}{1 - r_{\lambda_0, C_j}r_{\lambda_0}} \\ &\geq \frac{(r_{\lambda_0, C_j})^2 - (r_{\lambda_0})^2}{1 - (r_{\lambda_0, C_j}r_{\lambda_0})^2} \geq \frac{C_j m_{\lambda_0} + C_j(\alpha + 1)}{(2\alpha + 2 - C_j)m_{\lambda_0} + (\alpha + 1)^2} \rightarrow 1, \quad j \rightarrow \infty, \end{aligned}$$

and this latter convergence is uniform in  $m$ . Since we have chosen  $\zeta$  arbitrary in  $\bigcup_{\lambda \in \Lambda} D(\lambda, r_\lambda)$ , there exists a sequence  $(z_j)_j \subset \mathbb{D}$  such that :

$$\rho_j := \text{dist}\left(z_j, \bigcup_{\lambda \in \Lambda} D(\lambda, r_\lambda)\right) \rightarrow 1.$$

Set  $f_j = T_{z_j}1$ , observe that  $|T_{z_j}1| = |k_{z_j}|$ , and applying Lemma 3.4 we obtain

$$\begin{aligned} \sum_{\lambda \in \Lambda} \sum_{k < m_\lambda} |\langle f_j, T_\lambda e_k \rangle|^2 &\lesssim \sum_{\lambda \in \Lambda} (\alpha + 1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}}\right)} \frac{(1 - |z_j|^2)^{2+\alpha}}{|1 - \bar{z}_j \zeta|^{2(2+\alpha)}} (1 - |\zeta|^2)^\alpha dm(\zeta) \\ &\lesssim (\alpha + 1) S_X \int_{\bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}}\right)} [1 - |\varphi_{z_j}(\zeta)|^2]^\alpha |\varphi_{z_j}(\zeta)|^2 dm(\zeta) \\ &\lesssim (\alpha + 1) \int_{|w| \geq \rho_j} (1 - |w|^2)^\alpha dm(w) \rightarrow 0, \end{aligned}$$

since  $\rho_j \rightarrow 1$ . This contradicts the sampling inequality.

*Sufficiency part (b)*

Suppose the divisor is not sampling. Then there exists a sequence  $(f_n)_{n \geq 1}$  such that  $\|f_n\|_{\alpha, 2} = 1$  and

$$\sum_{\lambda \in \Lambda} \sum_{0 \leq j < m_\lambda} |\langle f_n, T_\lambda e_j \rangle|^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Passing to a weakly convergent sub-sequence denoted again by  $(f_n)_{n \geq 1}$  we have two possibilities: either (i)  $(f_n)$  converges weakly to  $f \neq 0$  or (ii)  $(f_n)$  converges weakly to 0.

In the case (i),  $X$  is a zero divisor for a function  $f \in \mathcal{A}_\alpha^2$ . By Theorem 1.6, for every  $\varepsilon > 0$ ,

$$\mathbb{D} \setminus \bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{\alpha + 2 + \varepsilon + m_\lambda}}\right)$$

cannot be compact. Similarly as in (13), given  $\varepsilon > 0$  for every  $C_1 > \varepsilon + 1 > 0$  such that for every  $m \geq 1$ ,

$$\frac{m - C_1}{m + \alpha + 1} \leq \frac{m}{m + \alpha + \varepsilon + 2}$$

Hence

$$D\left(\lambda, \sqrt{\frac{m_\lambda - C_1}{m_\lambda + \alpha + 1}}\right) \subset D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 2 + \varepsilon}}\right),$$

for every  $\lambda \in \Lambda$  with  $m_\lambda > C_2$ . And

$$\mathbb{D} \setminus \bigcup_{\lambda \in \Lambda, m_\lambda > C_1} D\left(\lambda, \sqrt{\frac{m_\lambda - C_1}{m_\lambda + \alpha + 1}}\right)$$

cannot be compact, leading to a contradiction.

In the second case (ii), we define  $\eta = \frac{1}{S_X+1}$  and set the constant  $C = a(\eta)$  where  $a(\eta)$  is given by Lemma 3.6. Denote

$$\Lambda_1 = \{\lambda \in \Lambda : m_\lambda - C > 0\}.$$

In order to reach a contradiction we will assume that the disks  $D\left(\lambda, \sqrt{\frac{m_\lambda - a(\eta)}{m_\lambda + \alpha + 1}}\right)$ ,  $\lambda \in \Lambda_1$ , cover the unit disk up to a compact set, i.e. there is  $R = R(\eta) \in [0, 1)$  such that

$$\mathbb{D} \setminus D(0, R) \subset \bigcup_{\lambda \in \Lambda_1} D\left(\lambda, \sqrt{\frac{m_\lambda - a(\eta)}{m_\lambda + \alpha + 1}}\right).$$

We get for every  $n \geq 1$

$$1 = \int_{\mathbb{D}} |f_n(z)|^2 dA_\alpha(z) \leq \int_{D(0, R)} |f_n(z)|^2 dA_\alpha(z) + \sum_{\lambda \in \Lambda_1} \int_{D\left(\lambda, \sqrt{\frac{m_\lambda - a(\eta)}{m_\lambda + \alpha + 1}}\right)} |f_n(z)|^2 dA_\alpha(z).$$

Denote by  $\Lambda_2$  the set of  $\lambda \in \Lambda_1$  such that

$$\sum_{j < m_\lambda} |\langle f_n, T_\lambda e_j \rangle|^2 \leq \frac{\eta}{2}(\alpha + 1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}}\right)} |f_n(z)|^2 (1 - |z|^2)^\alpha dm(z) =: \frac{\eta'}{2}.$$

By Lemma 3.6 applied for  $\lambda \in \Lambda_2$ ,  $\eta'$  and  $f_n$  with  $\|f_n\|_{\alpha, 2} = 1$ , we obtain

$$\begin{aligned} 1 &\leq (\alpha + 1) \int_{D(0, R)} |f_n(z)|^2 (1 - |z|^2)^\alpha dm(z) \\ &\quad + \sum_{\lambda \in \Lambda_2} (\alpha + 1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda - a(\eta)}{m_\lambda + \alpha + 1}}\right)} |f_n(z)|^2 (1 - |z|^2)^\alpha dm(z) \\ &\quad + \sum_{\lambda \in \Lambda_1 \setminus \Lambda_2} (\alpha + 1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda - a(\eta)}{m_\lambda + \alpha + 1}}\right)} |f_n(z)|^2 (1 - |z|^2)^\alpha dm(z) \\ &\leq (\alpha + 1) \int_{D(0, R)} |f_n(z)|^2 (1 - |z|^2)^\alpha dm(z) \\ &\quad + \eta \sum_{\lambda \in \Lambda_2} (\alpha + 1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}}\right)} |f_n(z)|^2 (1 - |z|^2)^\alpha dm(z) \\ &\quad + \sum_{\lambda \in \Lambda_1 \setminus \Lambda_2} (\alpha + 1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda - a(\eta)}{m_\lambda + \alpha + 1}}\right)} |f_n(z)|^2 (1 - |z|^2)^\alpha dm(z). \end{aligned}$$

For the first term on the right hand side of the last inequality, the weak convergence, *via* dominated convergence, yields

$$(\alpha + 1) \int_{D(0, R)} |f_n(z)|^2 (1 - |z|^2)^\alpha dm(z) = o(1). \quad (n \rightarrow \infty).$$

And by the definition of  $\Lambda_2$

$$\begin{aligned} & \sum_{\lambda \in \Lambda_1 \setminus \Lambda_2} (\alpha + 1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda - a(\eta)}{m_\lambda + \alpha + 1}}\right)} |f_n(z)|^2 (1 - |z|^2)^\alpha dm(z) \\ & \leq \sum_{\lambda \in \Lambda_1 \setminus \Lambda_2} (\alpha + 1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}}\right)} |f_n(z)|^2 (1 - |z|^2)^\alpha dm(z) \\ & \leq \frac{2}{\eta} \sum_{\lambda \in \Lambda_1 \setminus \Lambda_2} \sum_{j < m_\lambda} |\langle f_n, T_\lambda e_j \rangle|^2 = o(1) \quad (n \rightarrow \infty) \end{aligned}$$

Finally,

$$\begin{aligned} 1 & \leq o(1) + \eta \sum_{\lambda \in \Lambda} (\alpha + 1) \int_{D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}}\right)} |f_n(z)|^2 (1 - |z|^2)^\alpha dm(z) \\ & \leq o(1) + \eta S_X (\alpha + 1) \int_{\mathbb{D}} |f_n(z)|^2 (1 - |z|^2)^\alpha dm(z) = o(1) + \frac{S_X}{S_X + 1}, \quad n \rightarrow \infty. \end{aligned}$$

We have reached a contradiction.  $\square$

**3.3. Interpolation for  $\mathcal{A}_\alpha^2$ .** As in [5], we obtain the geometric condition by a  $\bar{\partial}$ -scheme and a local control of the functions in the space. To do so, we adapt the same technique as in [4]. We will need the following version of Hörmander's  $L^2$ -estimates for  $\bar{\partial}$  due to Ohsawa [15]. Recall the definition of the invariant laplacian

$$\tilde{\Delta} = (1 - |z|^2)^2 \Delta$$

and define the invariant convolution of two functions  $f, g$

$$(f \star g)(z) = \int_{\mathbb{D}} g(\varphi_z(\zeta)) f(\zeta) d\mathcal{V}(\zeta).$$

**Theorem 3.7. (Ohsawa)[15]** *Let  $\psi$  be any subharmonic function in the disk such that  $\tilde{\Delta}\psi > \delta > 0$ . Then there is a solution  $u$  to the equation  $\bar{\partial}u = g$  such that*

$$\int_{\mathbb{D}} |u(z)|^2 \frac{e^{-\psi(z)}}{1 - |z|^2} dm(z) \leq C_\delta \int_{\mathbb{D}} |g(z)|^2 e^{-\psi(z)} (1 - |z|^2) dm(z). \quad (33)$$

We need to regularize the weight in such a way that we will not destroy the interpolation after the  $\bar{\partial}$ -surgery. This will be achieved by Lemma 3.8 below. We recall from Lemma 3.2 the critical radius in  $\mathcal{A}_\alpha^2$ ,  $r_\lambda = \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}}$ , and its dilation from Theorem 1.5  $r'_\lambda = \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha + 1}}$ , and denote the associated hyperbolic disks  $D_\lambda = D(\lambda, r_\lambda)$  and  $D'_\lambda = D(\lambda, r'_\lambda)$  respectively.

We will also need the following auxiliary function (see [4, p.119])

$$\xi(\zeta) = \xi_\lambda(\zeta) = \begin{cases} 0, & 0 \leq |\zeta| < r_\lambda, \\ \frac{\log \frac{1}{|\zeta|^2}}{K(m_\lambda, c, \alpha)}, & r_\lambda < |\zeta| < r'_\lambda, \\ 0, & r'_\lambda < |\zeta| < 1, \end{cases}$$

where

$$K = K(m_\lambda, C_X, \alpha) = \int_{r_\lambda < |\zeta| < r'_\lambda} \log \frac{1}{|\zeta|^2} d\mathcal{V}(\zeta).$$

(so that the  $L^1$ -norm of  $\xi$  is one, for some more precise estimates on  $K$  see below). Consider the weight

$$\begin{aligned} w_{\Lambda, \alpha}(z) &:= \sum_{\lambda \in \Lambda} m_\lambda \left[ \log |\varphi_\lambda(z)|^2 - \frac{1}{K} \int_{r_\lambda < |\zeta| < r'_\lambda} \log |\varphi_{\varphi_\lambda(z)}(\zeta)|^2 \log \frac{1}{|\zeta|^2} d\mathcal{V}(\zeta) \right] \chi_{D'_\lambda}(z) \\ &= \sum_{\lambda \in \Lambda} m_\lambda \left[ \log |\varphi_\lambda(z)|^2 - \int_{\mathbb{D}} \log |\varphi_{\varphi_\lambda(z)}(\zeta)|^2 \xi(\zeta) d\mathcal{V}(\zeta) \right] \chi_{D'_\lambda}(z) \end{aligned} \quad (34)$$

**Lemma 3.8.** *Let  $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$  be a divisor and let  $C_X$  be such that  $(\alpha + 1)(1 - e^{-1}) < C_X < \alpha + 1$  the hyperbolic disks*

$$\left\{ D \left( \lambda, \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha + 1}} \right) \right\}_{\lambda \in \Lambda}$$

*are pairwise disjoint. Then the weight  $w_{\Lambda, \alpha}$  above satisfies*

- (a)  $w_{\Lambda, \alpha} \leq 0$ ,
- (b)  $-w_{\Lambda, \alpha} \leq A(\alpha)$  in  $D'_\lambda \setminus D_\lambda$ ,
- (c) and  $\tilde{\Delta} w_{\Lambda, \alpha} \geq -4(\alpha + 1 - \varepsilon)$  for some  $\varepsilon$  depending on  $C_X$ .

*Proof.* To see (a), since  $\log |\varphi_a|$  is sub-harmonic, we have

$$\log |a| \times \log \frac{1}{r^2} \leq \frac{1}{2\pi} \int_0^{2\pi} \log |\varphi_a(re^{i\theta})| \log \frac{1}{r^2} d\theta.$$

Integrating from  $r_\lambda$  to  $r'_\lambda$  with respect to the measure  $\frac{rdr}{(1-r^2)^2}$  and dividing by  $K$  yields the required result.

For (b), observe that the separation condition implies that it is sufficient to consider only one term of the sum. Let us also set  $a = \varphi_\lambda(z)$ , and notice that  $z \in D_{\lambda'} \setminus D_\lambda$  implies in particular  $|a| = |\varphi_\lambda(z)| > r_\lambda$ . Hence

$$\begin{aligned} -w_{\Lambda, \alpha}(z) &= m_\lambda \left[ \frac{1}{K(m_\lambda, C_X, \alpha)} \int_{r_\lambda < |\zeta| < r'_\lambda} \underbrace{\log |\varphi_a(\zeta)|^2 \log \frac{1}{|\zeta|^2}}_{< 0} d\mathcal{V}(\zeta) - \log |a|^2 \right] \\ &\leq m_\lambda \log \frac{1}{|a|^2} < m_\lambda \log \frac{1}{r_\lambda^2} = m_\lambda \log \frac{m_\lambda + \alpha + 1}{m_\lambda} \\ &\leq \alpha + 1. \end{aligned}$$

Let us discuss (c). Setting  $a = \varphi_\lambda(z)$ , we have

$$h(z) := \int_{\mathbb{D}} \log |\varphi_{\varphi_\lambda(z)}(\zeta)|^2 \xi(\zeta) d\mathcal{V}(\zeta) = \int_{\mathbb{D}} \log |\varphi_a(\zeta)|^2 \xi(\zeta) d\mathcal{V}(\zeta) = (\xi \star E)(a),$$

with  $E(u) = \log |u|^2$ . Observe that  $\tilde{\Delta}(h \circ \varphi_\lambda) = (\tilde{\Delta}h) \circ \varphi_\lambda$  (see e.g. [4, p.120]),  $\tilde{\Delta}(\mu \star \log |\cdot|^2) = 4\mu$  for any measure (notice that in [4], the authors define  $\Delta = \partial\bar{\partial}$  so that in our setting we have to introduce an additional factor 4), and hence

$$\tilde{\Delta}h = \tilde{\Delta}[(\xi \star E) \circ \varphi_\lambda] = 4 \times (\xi \circ \varphi_\lambda).$$

Hence, for  $z \in D'_\lambda \setminus D_\lambda$

$$\begin{aligned}\tilde{\Delta}w_{\Lambda,\alpha}(z) &= m_\lambda \left( \underbrace{\tilde{\Delta} \log |\varphi_\lambda(z)|^2}_{4\pi(1-|z|^2)\delta_\lambda} - \tilde{\Delta}h(z) \right) \\ &\geq -4m_\lambda(\xi \circ \varphi_\lambda)(z).\end{aligned}$$

Since  $\xi$  is decreasing, we get

$$\tilde{\Delta}w_{\Lambda,\alpha}(z) \geq -4m_\lambda\xi(r_\lambda) = -4\frac{m_\lambda}{K} \log \frac{m_\lambda + \alpha + 1}{m_\lambda} \geq -4\frac{\alpha + 1}{K}.$$

Let us estimate

$$K = 2 \int_{r_\lambda}^{r'_\lambda} \frac{-r \ln r^2}{(1-r^2)^2} dr$$

The function  $h(r) = \frac{r^2}{1-r^2} \log \frac{1}{r^2}$  is increasing on  $(0, 1)$ , we get

$$\begin{aligned}K &= \int_{(r_\lambda)^2}^{(r'_\lambda)^2} \log \frac{1}{t} \frac{dt}{(1-t)^2} = \left[ \frac{1}{1-t} \log \frac{1}{t} \right]_{r_\lambda^2}^{r_\lambda'^2} + \int_{r_\lambda^2}^{r_\lambda'^2} \frac{1}{t(1-t)} dt \\ &= h(r'_\lambda) - h(r_\lambda) + \log \frac{1-r_\lambda^2}{1-r_\lambda'^2} \\ &\geq \log \frac{\alpha + 1}{\alpha + 1 - C_X}\end{aligned}\tag{35}$$

Since  $C_X > (1 + \alpha)(1 - e^{-1})$ , we have  $K > 1$  as required.  $\square$

We are ready to establish our conditions for interpolating divisors. We recall the statement of the corresponding Theorem 1.5 here for the convenience of the reader.

**Theorem.** *Let  $\alpha > -1$ .*

- (a) *If  $X$  is an interpolating divisor for  $\mathcal{A}_\alpha^2$ , then there exists  $C_X > 0$  such that the hyperbolic disks*

$$\left\{ D \left( \lambda, \sqrt{\frac{m_\lambda - C_X}{m_\lambda + \alpha + 1}} \right) \right\}_{\lambda \in \Lambda, m_\lambda > C_X}$$

*are pairwise disjoint.*

- (b) *Conversely, if for some  $C_X$  such that  $(\alpha + 1)(1 - e^{-1}) < C_X < \alpha + 1$ , the hyperbolic disks*

$$\left\{ D \left( \lambda, \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha + 1}} \right) \right\}_{\lambda \in \Lambda}$$

*are pairwise disjoint, then  $X$  is an interpolating divisor for  $\mathcal{A}_\alpha^2$ .*

*Proof. Sufficiency part.*

The proof is based on a  $\bar{\partial}$ -method which consists, as usual, in constructing first a smooth interpolating function, and then to use Hörmander's solution to the  $\bar{\partial}$ -equation with  $L^2$ -estimates to make the interpolating function holomorphic without destroying the interpolation.

Given  $v = (v_\lambda^j)_{\lambda \in \Lambda, \alpha < m_\lambda} \in \ell^2(X)$ , take polynomials  $p_\lambda$ ,  $\lambda \in \Lambda$ , with  $\deg p_\lambda \leq m_\lambda - 1$ , such that

$$\langle p_\lambda, e_j \rangle = v_\lambda^j, \quad \lambda \in \Lambda, j < m_\lambda.$$

We recall that the above interpolation condition means that we interpolate germs in  $\lambda$  and that it is thus sufficient to guarantee that the interpolating function and its derivatives take the values  $v_\lambda^j$  in  $\lambda$ ,  $0 \leq j \leq m_\lambda - 1$ . Recall from (5) that  $N_{\lambda, m_\lambda}^{2, \alpha}$  denotes the set of functions  $f$  in  $\mathcal{A}_\alpha^2$  vanishing up to the order  $m_\lambda - 1$  in  $\lambda$ . Since  $v \in \ell^2(X)$ , we have

$$\|p_\lambda\|_{\mathcal{A}_\alpha^2/N_{0, m_\lambda}^{2, \alpha}} = \|p_\lambda\|_{\alpha, 2}, \quad \sum_{\lambda \in \Lambda} \|p_\lambda\|_{\mathcal{A}_\alpha^2/N_{0, m_\lambda}^{2, \alpha}}^2 = \|v\|_2^2 < \infty.$$

Let us denote  $Q_\lambda = T_\lambda p_\lambda$ ,  $r_\lambda = \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + 1}}$ ,  $r'_\lambda = \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha + 1}}$ ,  $D_\lambda = D(\lambda, r_\lambda)$  and  $D'_\lambda = D(\lambda, r'_\lambda)$ . Notice that  $\{D'_\lambda\}_{\lambda \in \Lambda}$  are pairwise disjoint by hypothesis. Consider the smooth interpolating function

$$F(z) = \sum_{\lambda \in \Lambda} Q_\lambda(z) \eta(|\varphi_\lambda(z)| - r'_\lambda), \quad (36)$$

where  $\eta = \eta_\lambda$  is a smooth cut-off function on  $\mathbb{R}$ , so that

- (a)  $\text{supp } \eta \subset (-\infty, 0]$ ,
- (b)  $\eta \equiv 1$  on  $(-\infty, r_\lambda - r'_\lambda]$ ,
- (c)  $|\eta'| \lesssim \frac{1}{r'_\lambda - r_\lambda} \simeq 2 \frac{m_\lambda + \alpha + 1}{C_X}$ .

The separation condition implies that in each  $z$ ,  $F(z)$  is given by at most one term.

Notice that  $\text{supp } F \subset \bigcup_{\lambda \in \Lambda} D'_\lambda$ . Also

$$\text{supp } \bar{\partial} F \subset \bigcup_{\lambda \in \Lambda} (D'_\lambda \setminus D_\lambda),$$

since  $\eta(|\varphi_\lambda(z)| - r'_\lambda)$  is constant outside  $\bigcup_{\lambda \in \Lambda} (D'_\lambda \setminus D_\lambda)$ .

A direct calculation shows that

$$|\bar{\partial} \eta(|\varphi_\lambda(z)| - r'_\lambda)| \leq \frac{1}{2} \|\eta'\|_\infty \frac{1 - |\varphi_\lambda(z)|^2}{1 - |z|^2}, \quad (37)$$

so that for  $z \in D'_\lambda \setminus D_\lambda$ , property (c) yields

$$\begin{aligned} |\bar{\partial} F(z)| &\leq |Q_\lambda(z)| \times |\bar{\partial} \eta| \times |\bar{\partial}(|\varphi_\lambda(z)|)| \\ &\lesssim \frac{|Q_\lambda(z)|}{r'_\lambda - r_\lambda} \frac{1 - r_\lambda^2}{1 - |z|^2}. \end{aligned} \quad (38)$$

Since  $T_\lambda$  is an isometry of  $\mathcal{A}_\alpha^2$ ,

$$\|Q_\lambda\|_{\alpha, 2}^2 = \|p_\lambda\|_{\alpha, 2}^2 = \|p_\lambda\|_{\mathcal{A}_\alpha^2/N_{0, m_\lambda}^{2, \alpha}}^2,$$

and therefore  $F$  has the growth of  $\mathcal{A}_\alpha^2$ :

$$\begin{aligned} (\alpha + 1) \int_{\mathbb{D}} |F(z)|^2 (1 - |z|^2)^\alpha dm(z) &\leq \sum_{\lambda \in \Lambda} (\alpha + 1) \int_{D'_\lambda} |Q_\lambda(z)|^2 (1 - |z|^2)^\alpha dm(z) \\ &\leq \sum_{\lambda \in \Lambda} \|Q_\lambda\|_{\mathcal{A}_\alpha^2}^2 = \|v\|_2^2. \end{aligned}$$



Next we construct a holomorphic interpolating function using a  $\bar{\partial}$ -technique. As in the scheme used in [4], we are looking for a holomorphic interpolating function of the form  $f = F - u$ , where  $u$  is a solution to the  $\bar{\partial}$ -problem  $\bar{\partial}u = \bar{\partial}F$  with the conditions

$$\int_{\mathbb{D}} |u(z)|^2 (1 - |z|^2)^\alpha dm(z) < \infty,$$

and

$$\partial^j u(\lambda) = 0, \quad \forall j < m_\lambda.$$

This last condition will ensure that

$$\partial^j f(\lambda) = \partial^j F(\lambda), \quad j < m_\lambda,$$

and we remind that the interpolation condition  $\langle f, T_\lambda e_j \rangle = v_\lambda^j$  translates into an interpolation by germs.

We will apply Ohsawa's Theorem 3.7 with the subharmonic weight

$$\phi(z) = (\alpha + 1) \log \frac{1}{1 - |z|^2} + w_{\Lambda, \alpha}(z),$$

where  $w_{\Lambda, \alpha}$  is the weight in Lemma 3.8. We need to compute  $\tilde{\Delta}\phi$  as in Ohsawa's theorem:

$$\begin{aligned} \tilde{\Delta}\phi &= (1 - |z|^2)^2 \Delta \left( (\alpha + 1) \log \frac{1}{1 - |z|^2} + w_{\Lambda, \alpha} \right) \\ &= 4(\alpha + 1) + (1 - |z|^2)^2 \Delta w_{\Lambda, \alpha} \\ &\geq 4(\alpha + 1) - 4\left(\frac{\alpha + 1}{K}\right) > \varepsilon. \end{aligned}$$

The last inequality is due to Lemma 3.8 (c).

The properties of the weight  $w_{\Lambda, \alpha}$  and Ohsawa's estimate (33) yield

$$\begin{aligned} \int_{\mathbb{D}} |u(z)|^2 (1 - |z|^2)^\alpha dA_\alpha(z) &= (\alpha + 1) \int_{\mathbb{D}} |u(z)|^2 \frac{e^{-(\alpha+1) \log \frac{1}{1-|z|^2}}}{1 - |z|^2} \frac{dm(z)}{\pi} \\ &\leq (\alpha + 1) \int_{\mathbb{D}} |u(z)|^2 \frac{e^{-\phi(z)}}{1 - |z|^2} \frac{dm(z)}{\pi} \quad (\text{Lemma 3.8(a)}) \\ &\lesssim \int_{\mathbb{D}} |\bar{\partial}F(z)|^2 e^{-\phi(z)} (1 - |z|^2) dm(z) \quad (\text{Ohsawa}) \\ &\lesssim \sum_{\lambda \in \Lambda} \int_{D'_\lambda \setminus D_\lambda} \frac{|Q_\lambda(z)|^2}{(r'_\lambda - r_\lambda)^2} \left( \frac{1 - r_\lambda^2}{1 - |z|^2} \right)^2 (1 - |z|^2)^{\alpha+2} dm(z) \quad (\text{Lemma 3.8(b) \& (38)}) \end{aligned}$$

Now

$$\frac{1 - r_\lambda^2}{r'_\lambda - r_\lambda} = \frac{(r'_\lambda + r_\lambda)(1 - r_\lambda^2)}{r_\lambda'^2 - r_\lambda^2} \leq 2 \frac{\alpha + 1}{C_X}, \quad \lambda \in \Lambda, \quad (39)$$

so that

$$\int_{\mathbb{D}} |u(z)|^2 (1 - |z|^2)^\alpha dA_\alpha(z) \lesssim \sum_{\lambda \in \Lambda} \|Q_\lambda\|_{\mathcal{A}_\alpha^2 / N_{0, m_\lambda}^2}^2 < \infty.$$

Hence,  $f = F - u \in \mathcal{A}_\alpha^2$ .

Finally, we want to see that  $\langle f, T_\lambda e_j \rangle = v_\lambda^j$ ,  $j < m_\lambda$ . We have already mentioned that for this we need  $u$  to vanish at order  $m_\lambda$  in each  $\lambda$ , so let us examine the order of the singularity near  $\lambda$ . For each  $z \in D_\lambda$

$$w(z) = m_\lambda \log |\varphi_\lambda(z)|^2 + C_\lambda.$$

and therefore

$$\begin{aligned} +\infty &> \int_{D_\lambda} |u(z)|^2 e^{-w(z)} dm(z) \gtrsim \int_{D_\lambda} |u(z)|^2 e^{-\log |\varphi_\lambda(z)|^{2m_\lambda}} dm(z) \\ &= \int_{D_\lambda} |u(z)|^2 \frac{1}{|\varphi_\lambda(z)|^{2m_\lambda}} dm(z). \end{aligned}$$

This forces  $u$  to vanish at order  $m_\lambda$  on  $\lambda \in \Lambda$ . Therefore

$$\langle f, T_\lambda e_j \rangle = v_\lambda^j, \quad j \leq m_\lambda, \lambda \in \Lambda.$$

*Necessary part.*

Let  $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$  be an interpolating divisor and assume that the discs  $\left\{ D\left(\lambda, \sqrt{\frac{m_\lambda - C_X}{m_\lambda + \alpha + 1}}\right) \right\}_{\lambda, m_\lambda > C_X}$  are not separated for any  $C_X > 0$ . Let  $r_\lambda = \sqrt{\frac{m_\lambda - C_X}{m_\lambda + \alpha + 1}}$  and  $r'_\lambda = \sqrt{\frac{m_\lambda - (C_X - 1)}{m_\lambda + \alpha + 1}}$ . There exists  $\lambda_1, \lambda_2 \in \Lambda$ ,  $\lambda_1 \neq \lambda_2$  and  $m_{\lambda_1}, m_{\lambda_2} > C_X$  such that

$$D(\lambda_1, r_{\lambda_1}) \cap D(\lambda_2, r_{\lambda_2}) \neq \emptyset.$$

And also by the same argument

$$D\left(\lambda_1, \sqrt{\frac{m_{\lambda_1} - (C_X - 1)}{m_{\lambda_1} + \alpha + 1}}\right) \cap D\left(\lambda_2, \sqrt{\frac{m_{\lambda_2} - (C_X - 1)}{m_{\lambda_2} + \alpha + 1}}\right) \neq \emptyset.$$

Let  $\zeta \in \partial D(\lambda_1, r_{\lambda_1})$  and  $\zeta' \in \partial D(\lambda_1, r'_{\lambda_1})$ , we have

$$\begin{aligned} \rho(\zeta, \zeta') &\geq \frac{\rho(\zeta', \lambda_1) - \rho(\lambda_1, \zeta)}{1 - \rho(\zeta', \lambda_1)\rho(\lambda_1, \zeta)} = \rho(r_\lambda, r'_\lambda) = \frac{r'_{\lambda_1} - r_{\lambda_1}}{1 - r_{\lambda_1}r'_{\lambda_1}} \geq \frac{(r'_{\lambda_1})^2 - (r_{\lambda_1})^2}{1 - (r_{\lambda_1}r'_{\lambda_1})^2} \\ &= \frac{m_{\lambda_1} + \alpha + 1}{m_{\lambda_1}(2\alpha + 2C_X + 1) + (\alpha + 2)^2 - C_X^2 + C_X} \geq \frac{1}{2\alpha + 2C_X + 1} =: \delta > 0. \end{aligned}$$

Hence, the estimate of the hyperbolic distance between  $\partial D(\lambda_1, r_{\lambda_1})$  and  $\partial D(\lambda_1, r'_{\lambda_1})$ , is bounded from below by  $\delta$ . Thus, if  $w \in D(\lambda_1, r_{\lambda_1}) \cap D(\lambda_2, r_{\lambda_2}) \subset D(\lambda_1, r'_{\lambda_1}) \cap D(\lambda_2, r'_{\lambda_2})$ , and

$$\varepsilon = \frac{1}{2}\delta,$$

then we have

$$D(w, \varepsilon) \subset \left( \lambda_1, \sqrt{\frac{m_{\lambda_1} - C_X}{m_{\lambda_1} + \alpha + 1}} \right) \cap D\left( \lambda_2, \sqrt{\frac{m_{\lambda_2} - C_X}{m_{\lambda_2} + \alpha + 1}} \right).$$

Since  $X$  is an interpolating divisor, there exists  $f \in \mathcal{A}_\alpha^2$  such that

- (a)  $f \in N_{\lambda_1, m_{\lambda_1}}^2$ ,
- (b)  $f - T_w 1 \in N_{\lambda_2, m_{\lambda_2}}^2$ ,
- (c)  $\|f\|_{\alpha, 2} \leq M_X$ , where  $M_X$  is a fixed (interpolating) constant depending only on  $X$ .

By Lemma 3.6 applied to both  $f$  and  $f - T_w 1$  (for which the sum of the squares of the corresponding Fourier coefficients  $|\langle g, e_j \rangle|$  vanish and the norm on the disks are in particular bounded by  $M_X$ ) we have

$$\begin{aligned} & \int_D \left( \lambda_1, \sqrt{\frac{m_{\lambda_1} - C_X + 1}{m_{\lambda_1} + \alpha + 1}} \right) |f(z)|^2 dA_\alpha(z) \\ & + \int_D \left( \lambda_2, \sqrt{\frac{m_{\lambda_2} - C_X + 1}{m_{\lambda_2} + \alpha + 1}} \right) |(f - T_w 1)(z)|^2 dA_\alpha(z) = o(1) \cdot M_X^2, \quad C_X \rightarrow \infty, \end{aligned}$$

and therefore

$$\begin{aligned} & \int_{D(w, \epsilon)} |f(z)|^2 dA_\alpha(z) + \int_{D(w, \epsilon)} |(f - T_w 1)(z)|^2 dA_\alpha(z) \\ & = o(1) \cdot M_X^2, \quad C_X \rightarrow \infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} (\alpha + 1) \int_{D(w, \epsilon)} |(T_w 1)(z)|^2 dA_\alpha(z) &= (\alpha + 1) \int_{D(0, \epsilon)} (1 - |z|^2)^\alpha dm(z) \\ &= \left(1 - (1 - \epsilon^2)^{\alpha+1}\right) > 0, \end{aligned}$$

which gives a contradiction when  $C_X > C_X(M_X)$ . □

#### 4. THE $\mathcal{A}_\alpha^\infty$ -CASE

Let  $\alpha > 0$ , we now consider

$$\mathcal{A}_\alpha^\infty = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_{\alpha, \infty} := \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\frac{\alpha}{2}} |f(z)| < +\infty \right\}.$$

We shall start recalling the reformulation of interpolation and sampling met in the situation  $p = 2$  in terms of vanishing subspaces.

For each  $\lambda, m_\lambda$  we have already introduced the subspace

$$N_\lambda^2 := N_{\lambda, m_\lambda}^{2, \alpha} = \{f \in \mathcal{A}_\alpha^2 : f^{(j)}(\lambda) = 0, \quad \forall j < m_\lambda\}.$$

Observe that  $\sum_{j < m_\lambda} |\langle f, T_\lambda e_j \rangle|^2 = \|f\|_{\mathcal{A}_\alpha^2/N_\lambda^2}^2$ . Then it becomes clear that  $X$  is a *sampling divisor* for  $\mathcal{A}_\alpha^2$  if, for all  $f \in \mathcal{A}_\alpha^2$ ,

$$\|f\|_{\alpha, 2}^2 \simeq \sum_{\lambda \in \Lambda} \sum_{j < m_\lambda} |\langle f, T_\lambda e_j \rangle|^2 = \sum_{\lambda \in \Lambda} \|f\|_{\mathcal{A}_\alpha^2/N_\lambda^2}^2.$$

Similarly,  $X$  is *interpolating* for  $\mathcal{A}_\alpha^2$ , if for all sequence  $(f_\lambda)_{\lambda \in \Lambda} \subset \mathcal{A}_\alpha^2$  such that

$$\sum_{\lambda \in \Lambda} \|f_\lambda\|_{\mathcal{A}_\alpha^2/N_\lambda^2}^2 < \infty,$$

there exists  $f \in \mathcal{A}_\alpha^2$  so that

$$f - f_\lambda \in N_\lambda^2.$$

In order to consider the corresponding  $L^\infty$  sampling and interpolation problems, we associate to each  $\lambda \in \mathbb{D}$  the subspace

$$N_\lambda^\infty = N_{\lambda, m_\lambda}^{\infty, \alpha} := \{f \in \mathcal{A}_\alpha^\infty : \partial^j f(\lambda) = 0, \forall j < m_\lambda\}.$$

**Definition 4.1.** A divisor is called *sampling* for  $\mathcal{A}_\alpha^\infty$ , if there exists  $L > 0$  such that

$$\|f\|_{\alpha,\infty} \leq L \sup_{\lambda \in \Lambda} \|f\|_{\mathcal{A}_\alpha^\infty / N_\lambda^\infty}.$$

In a similar way we define generalized interpolation.

**Definition 4.2.** The divisor  $X$  is called *interpolating* for  $\mathcal{A}_\alpha^\infty$  if for every sequence  $(f_\lambda)_{\lambda \in \Lambda}$  with

$$\sup_{\lambda \in \Lambda} \|f_\lambda\|_{\mathcal{A}_\alpha^\infty / N_\lambda^\infty} < \infty,$$

there exists a function  $f \in \mathcal{A}_\alpha^\infty$  such that

$$f - f_\lambda \in N_\lambda^\infty, \quad \lambda \in \Lambda.$$

**4.1. Local  $L^\infty$ -estimates.** As in the  $L^2$  case we need a local control of the functions of the space  $\mathcal{A}_\alpha^\infty$  with small quotient norm. Here is the result corresponding to Lemma 3.6 for  $\mathcal{A}_\alpha^\infty$  ( $\alpha > 0$ ).

**Lemma 4.3.** (i) For every  $\eta, \varepsilon \in (0, 1)$ , there exists  $C > 0$  such that if  $f \in \mathcal{A}_\alpha^\infty$  satisfies  $\|f\|_{\alpha,\infty} \leq 1$ ,  $m \geq C$ ,  $\|f\|_{\mathcal{A}_\alpha^\infty / N_{0,m}^{\infty,\alpha}} < \varepsilon$ , then

$$|f(z)| (1 - |z|^2)^{\frac{\alpha}{2}} \leq \eta + \varepsilon, \quad z \in D\left(0, \sqrt{\frac{m-C}{m+\alpha}}\right).$$

(ii) For every  $C > 0$  there exist  $\eta, \varepsilon \in (0, 1)$ , such that if  $f \in \mathcal{A}_\alpha^\infty$  satisfies  $\|f\|_{\alpha,\infty} \leq 1$ ,  $m \geq C$ ,  $\|f\|_{\mathcal{A}_\alpha^\infty / N_{0,m}^{\infty,\alpha}} < \varepsilon$ , then

$$|f(z)| (1 - |z|^2)^{\frac{\alpha}{2}} \leq 1 - \eta, \quad z \in D\left(0, \sqrt{\frac{m-C}{m+\alpha}}\right).$$

The result in (i) is of course of interest when  $\eta + \varepsilon < 1$ , and in particular when  $f \in N_{0,m}^{\infty,\alpha}$  in which case we can pick  $\varepsilon$  arbitrarily small.

Note that the critical radius  $\sqrt{\frac{m}{m+\alpha}}$  is different from the one appearing for  $p = 2$ . We have already met this radius in Theorem 2.2.

*Proof.* Claim (i):

Since  $\|f\|_{\mathcal{A}_\alpha^\infty / N_{0,m}^{\infty,\alpha}} < \varepsilon$ , there exist a function  $g \in N_{0,m}^{\infty,\alpha}$  such that

$$(a) \quad \|f - g\|_{\mathcal{A}_\alpha^\infty} \leq \varepsilon,$$

$$(b) \quad g(z) = z^m h(z), \text{ where } h \text{ is a holomorphic function in the unit disk.}$$

Since  $\|f\|_{\alpha,\infty} \leq 1$  we have the bound

$$|g(z)| \leq |g(z) - f(z)| + |f(z)| \leq (1 + \varepsilon) \frac{1}{(1 - |z|^2)^{\frac{\alpha}{2}}}, \quad z \in \mathbb{D}.$$

and in terms of  $h$  and the functions  $\vartheta_{m,\alpha}(t) = \log \frac{1}{t^m(1 - t^2)^{\frac{\alpha}{2}}}$

$$|h(z)| \leq (1 + \varepsilon) e^{\vartheta_{m,\alpha}(|z|)}, \quad z \in \mathbb{D}.$$

Using the maximum principal we obtain

$$\max_{z \in D\left(0, \sqrt{\frac{m}{m+\alpha}}\right)} |h(z)| = (1 + \varepsilon) e^{\left[\vartheta_{m,\alpha}\left(\sqrt{\frac{m}{m+\alpha}}\right) - \vartheta_{m,\alpha}\left(\sqrt{\frac{m-C}{m+\alpha}}\right) + \vartheta_{m,\alpha}\left(\sqrt{\frac{m-C}{m+\alpha}}\right)\right]}$$

Observe that

$$\vartheta_{m,\alpha} \left( \sqrt{\frac{m-C}{m+\alpha}} \right) - \vartheta_{m,\alpha} \left( \sqrt{\frac{m}{m+\alpha}} \right) = \frac{C}{2} + o(1) - \log \left( \frac{\alpha+2C}{\alpha} \right)^{\frac{\alpha}{2}}. \quad (40)$$

Since the term  $o(1)$  goes to 0 when  $m$  goes to infinity, and  $m \geq C$ , the above expression can be made arbitrarily big. Let  $\delta = \delta(C)$  be the corresponding constant (thus with  $\lim_{C \rightarrow +\infty} \delta(C) = +\infty$ ), we get

$$\max_{z \in D(0, \sqrt{\frac{m}{m+\alpha}})} |h(z)| \leq (1+\varepsilon)e^{-\delta} e^{\vartheta_{m,\alpha}(\sqrt{\frac{m-C}{m+\alpha}})}. \quad (41)$$

Since  $\delta(C) \rightarrow +\infty$  when  $C \rightarrow +\infty$  there exists a  $C$  such that  $(1+\varepsilon)e^{-\delta} \leq \eta$ . Then

$$|h(z)| \leq \eta e^{\vartheta_{m,\alpha}(\sqrt{\frac{m-C}{m+\alpha}})}, \quad z \in \partial D \left( 0, \sqrt{\frac{m}{m+\alpha}} \right).$$

Now, by the maximum principal again, restricting the estimate to the smaller disk  $D \left( 0, \sqrt{\frac{m-C}{m+\alpha}} \right)$ , and using the fact that  $\vartheta_{m,\alpha}$  is decreasing on  $(0, \sqrt{\frac{m}{m+\alpha}})$  we get

$$|h(z)| \leq \eta e^{\vartheta_{m,\alpha}(\sqrt{\frac{m-C}{m+\alpha}})} \leq \eta e^{\vartheta_{m,\alpha}(|z|)}, \quad z \in D \left( 0, \sqrt{\frac{m-C}{m+\alpha}} \right). \quad (42)$$

Finally, for  $z \in D \left( 0, \sqrt{\frac{m-C}{m+\alpha}} \right)$

$$\begin{aligned} |f(z)| (1-|z|^2)^{\frac{\alpha}{2}} &\leq |f(z) - g(z)| (1-|z|^2)^{\frac{\alpha}{2}} + |g(z)| (1-|z|^2)^{\frac{\alpha}{2}} \\ &\leq \varepsilon + |z|^m (1-|z|^2)^{\frac{\alpha}{2}} |h(z)| \\ &= \varepsilon + e^{-\vartheta_{m,\alpha}(|z|)} |h(z)| \\ &\leq \varepsilon + \eta. \end{aligned} \quad (43)$$

Claim (ii):

The proof follows exactly the same lines and ideas. First one should observe that given  $C > 0$ , the difference appearing in (40) is uniformly bounded from below by some  $\delta > 0$  (this is clear when  $m$  is big, say  $m \geq m_0$ , and for  $1 \leq m < m_0$  we just take the smallest of finitely many strictly positive numbers). Then looking at (41), we have to convince ourselves that there are  $\varepsilon, \eta > 0$  such that  $(1+\varepsilon)e^{-\delta} < 1 - \eta - \varepsilon$  which is easily seen to be true. Finally, the same estimates as in (43) lead to

$$|f(z)| (1-|z|^2)^{\frac{\alpha}{2}} \leq \varepsilon + (1 - \eta - \varepsilon) = 1 - \eta.$$

□

**4.2. Sampling for  $\mathcal{A}_\alpha^\infty$ .** Now we are ready to establish our conditions for sampling conditions.

**Theorem 4.4.** *Let  $\alpha > 0$ .*

(a) *If  $X$  is a sampling divisor for  $\mathcal{A}_\alpha^\infty$ , then there exists  $0 < C < \alpha$  such that*

$$\bigcup_{\lambda \in \Lambda} D \left( \lambda, \sqrt{\frac{m_\lambda + C}{m_\lambda + \alpha}} \right) = \mathbb{D}.$$

(b) Conversely, if there exists  $C = C(S_X) > 0$  such that for some compact  $K$  of  $\mathbb{D}$  we have

$$\bigcup_{\lambda \in \Lambda, m_\lambda > C} D\left(\lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}}\right) = \mathbb{D} \setminus K,$$

then  $X$  is a sampling divisor for  $\mathcal{A}_\alpha^\infty$ .

*Proof.*

(a) *Necessary Condition.*

Suppose that for every  $0 < C < \alpha$ , we have

$$\bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C}{m_\lambda + \alpha}}\right) \neq \mathbb{D}.$$

Thus, there exists an increasing sequence of positive numbers  $(C_k)$  tending to  $\alpha$  and a sequence  $(z_k)$  with  $z_k \in \mathbb{D}$  such that :

$$z_k \in \mathbb{D} \setminus \left[ \bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda + C_k}{m_\lambda + \alpha}}\right) \right] \subset \mathbb{D} \setminus \left[ \bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha}}\right) \right].$$

As in the proof of the necessary condition of the sampling theorem in the Hilbertian case, we will show that

$$d_k := \text{dist}\left(z_k, \bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha}}\right)\right) \rightarrow 1.$$

Put  $r_{\lambda, C_k} = \sqrt{\frac{m_\lambda + C_k}{m_\lambda + \alpha}}$  and  $r_\lambda = \sqrt{\frac{m_\lambda}{m_\lambda + \alpha}}$ . Let

$$\zeta \in \bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha}}\right).$$

Then there exists  $\lambda_0 \in \Lambda$  such that  $\zeta \in D(\lambda_0, r_{\lambda_0})$  and  $\rho(z_k, \lambda_0) > r_{\lambda_0, C_k}$ ,  $k \geq 1$ . Since  $\alpha > 0$ , we get as in the hilbertian situation

$$\begin{aligned} \rho(z_k, \zeta) &\geq \frac{\rho(z_k, \lambda_0) - \rho(\zeta, \lambda_0)}{1 - \rho(z_k, \lambda_0)\rho(\zeta, \lambda_0)} \geq \frac{(r_{\lambda_0, C_k})^2 - (r_{\lambda_0})^2}{1 - (r_{\lambda_0})^2(r_{\lambda_0, C_k})^2} \\ &= \frac{C_k(m_{\lambda_0} + \alpha)}{m_{\lambda_0}(2\alpha - C_k) + \alpha^2}. \end{aligned}$$

Observe that this last expression is decreasing in  $m_{\lambda_0}$  so that passing to the limit  $m_{\lambda_0} \rightarrow +\infty$ , we get

$$\rho(z_k, \zeta) \geq \frac{C_k}{2\alpha - C_k}.$$

Thus,  $d_k \geq C_k/(2\alpha - C_k) \rightarrow 1$  when  $k \rightarrow +\infty$ .

Now pick  $f_k(z) = T_{z_k}(1)$ . We will show that we cannot sample uniformly  $f_k$  meaning that  $\sup_{\lambda \in \Lambda} \|f_k\|_{\mathcal{A}_\alpha^\infty / N_{\lambda, m_\lambda}^{\alpha, \infty}} \rightarrow 0$  (while  $\|f_k\|_{\mathcal{A}_\alpha^\infty} = 1$ ). In view of the construction

it is enough to show that when  $u \geq \sqrt{\frac{m + C_k}{m + \alpha}}$ , then  $\|T_u 1\|_{\mathcal{A}_\alpha^\infty / N_{0, m}^{\alpha, \infty}} \rightarrow 0$  uniformly in  $m$  when  $C_k \rightarrow \alpha$ . Recall that

$$T_u 1(z) = \left( \frac{1 - |u|^2}{(1 - \bar{u}z)^2} \right)^{\alpha/2} = (1 - |u|^2)^{\alpha/2} \frac{1}{(1 - \bar{u}z)^\alpha}.$$

Using the standard Taylor series for power functions we get

$$\frac{1}{(1 - \bar{u}z)^\alpha} = \sum_{n \geq 0} \binom{-\alpha}{n} (-\bar{u})^n z^n = \sum_{n=0}^{m-1} \binom{-\alpha}{n} (-\bar{u})^n z^n + z^m h(z) = f_0(z) + z^m h(z).$$

The following estimate is well known

$$\binom{-\alpha}{n} = \frac{(-1)^n}{\Gamma(\alpha) n^{1-\alpha}} (1 + o(1)).$$

Hence

$$|f_0(z)| \leq C \sum_{n=0}^{m-1} \frac{1}{n^{1-\alpha}} u^n |z|^n$$

(we remind that  $u > 0$ ). Here  $C$  is some irrelevant universal constant. Hence

$$\|T_u 1\|_{\mathcal{A}_\alpha^\infty / N_{0,m}^{\alpha,\infty}} \leq (1 - |u|^2)^{\alpha/2} \|f_0\|_{\mathcal{A}^\infty} \leq C(1 - |u|^2)^{\alpha/2} \sup_{|z| < 1} (1 - |z|^2)^{\alpha/2} \sum_{n=0}^{m-1} \frac{1}{n^{1-\alpha}} u^n |z|^n$$

The function  $\varphi_n(x) = (1 - x^2)^{\alpha/2} x^n$  admits a maximum in  $x_n = \sqrt{n/(n + \alpha/2)}$  which, up to a multiplicative constant, behaves like  $1/n^{\alpha/2}$ . Hence

$$\|T_u 1\|_{\mathcal{A}_\alpha^\infty / N_{0,m}^{\alpha,\infty}} \leq C(1 - |u|^2)^{\alpha/2} \sum_{n=0}^{m-1} n^{\alpha/2-1} \leq C(1 - |u|^2)^{\alpha/2} m^{\alpha/2}$$

where in the above inequalities  $C$  are different universal constants. On the other hand

$$(1 - |u|^2)^{\alpha/2} \leq \left(1 - \frac{m + C_k}{m + \alpha}\right)^{\alpha/2} = \left(\frac{\alpha - C_k}{m + \alpha}\right)^{\alpha/2},$$

so that

$$\|T_u 1\|_{\mathcal{A}_\alpha^\infty / N_{0,m}^{\alpha,\infty}} \leq C \left(m \frac{\alpha - C_k}{m + \alpha}\right)^{\alpha/2} \leq C(C_k - \alpha)^{\alpha/2}$$

uniformly in  $m$ . Since  $C_k \rightarrow \alpha$  the above expression goes to 0 (uniformly in  $m$ ), and we reach the desired conclusion.

(b) *Sufficient Condition.*

Suppose that there exists a sequence  $(f_n)_n$  such that  $\|f_n\|_{\alpha,\infty} = 1$ , and

$$\sup_{\lambda \in \Lambda} \|f_n\|_{\mathcal{A}_\alpha^\infty / N_\lambda^\infty} \rightarrow 0, \quad n \rightarrow \infty.$$

Passing to a sub-sequence converging uniformly on compact subsets denoted again by  $(f_n)_n$ , we have two possibilities: either (A) the sequence  $(f_n)_n$  converges to  $f \neq 0$  or (B) the sequence  $(f_n)_n$  converges to 0.

(A): In this case  $X$  is a zero divisor for  $\mathcal{A}_\alpha^\infty$ . Then, by the Uniqueness Theorem 2.2,  $\mathbb{D} \setminus \left[ \bigcup_{\lambda \in \Lambda} D\left(\lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + \varepsilon}}\right) \right]$  cannot be compact for any  $\varepsilon > 0$ . On the other hand, for every  $C > 0$  and for every  $0 < \varepsilon < C$  we have

$$\sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}} < \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + \varepsilon}}. \quad (44)$$

This yields,

$$\mathbb{D} \setminus \left[ \bigcup_{\lambda \in \Lambda} D \left( \lambda, \sqrt{\frac{m_\lambda}{m_\lambda + \alpha + \varepsilon}} \right) \right] \subset \mathbb{D} \setminus \left[ \bigcup_{\lambda \in \Lambda} D \left( \lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}} \right) \right].$$

Therefore, for no  $C > 0$ ,  $\mathbb{D} \setminus \left[ \bigcup_{\lambda \in \Lambda} D \left( \lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}} \right) \right]$  can be compact, contradicting the hypothesis.

(B): In this case, by contradiction we will assume that for some compact set  $K \subset \mathbb{D}$  we have

$$\Omega = \bigcup_{\lambda \in \Lambda} D \left( \lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}} \right) = \mathbb{D} \setminus K.$$

Since by assumption  $(f_n)_n$  converges to 0 on compact subsets, there exists  $n_0 \in \mathbb{N}$  such that

$$|f_n(z)| (1 - |z|^2)^{\frac{\alpha}{2}} < \frac{1}{2}, \quad z \in K, n \geq n_0.$$

Next, for the given  $C$ , Lemma 4.3(ii) implies the existence of  $\eta, \varepsilon > 0$  ensuring a control on  $f$ . Since  $\sup_{\lambda \in \Lambda} \|f_n\|_{\mathcal{A}_\alpha^\infty / N_\lambda^\infty} \rightarrow 0$ , there exists  $n_1$  such that for  $n \geq n_1$ , these quotient norms are strictly smaller than  $\varepsilon$  (uniformly in  $\lambda$ ) as required by the lemma. Since moreover  $\|f_n\|_{\alpha, \infty} = 1$ , Lemma 4.3 implies that

$$|f_n(z)| (1 - |z|^2)^{\frac{\alpha}{2}} < 1 - \eta, \quad z \in \bigcup_{\lambda \in \Lambda, m_\lambda > C} D \left( \lambda, \sqrt{\frac{m_\lambda - C}{m_\lambda + \alpha}} \right).$$

Hence,  $\|f_n\|_{\alpha, \infty} < 1$  for  $n > \max(n_0, n_1)$  and we get a contradiction.  $\square$

**4.3. Interpolation for  $\mathcal{A}_\alpha^\infty$ .** We need the following result by Berndtsson [3, Theorem 4] (see [4, Theorem G]) for the uniform estimates in the  $\bar{\partial}$ -surgery.

**Theorem 4.5.** *Let  $\psi$  be a subharmonic function and*

$$\varphi(z) = \min \left\{ (1 - |z|) \Delta \psi(z), \frac{1}{1 - |z|} \right\}.$$

*Let  $f$  be a function in  $\mathbb{D}$  such that*

$$\sup \frac{|f(z)|}{\varphi(z)} e^{-\psi(z)/2} < \infty.$$

*Let  $u \in L^2(\mathbb{D}, e^{-\psi} dm)$  be the canonical solution to  $\bar{\partial} u = f$ . Then*

$$\sup |u(z)| e^{-\tilde{\psi}(z)/2} \leq \sup \frac{|f(z)|}{\varphi(z)} e^{-\psi(z)/2},$$

*where  $\tilde{\psi}(z) = \sup_{|z - \zeta| < 1/2(1 - |z|)} \psi(\zeta)$ .*

The corresponding result for interpolation in  $\mathcal{A}_\alpha^\infty$  reads as follows.

**Theorem 4.6.** *Let  $\alpha > 0$ .*



- (a) If  $X$  is an interpolating divisor for  $\mathcal{A}_\alpha^\infty$ , then there exists  $C_X > 0$  such that the hyperbolic disks

$$\left\{ D \left( \lambda, \sqrt{\frac{m_\lambda - C_X}{m_\lambda + \alpha}} \right) \right\}_{\lambda \in \Lambda, m_\lambda > C_X}$$

are pairwise disjoint.

- (b) Conversely, if for some  $C_X$  such that  $\alpha(1 - e^{-1}) < C_X < \alpha$ , the hyperbolic disks

$$\left\{ D \left( \lambda, \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha}} \right) \right\}_{\lambda \in \Lambda}$$

are pairwise disjoint, then  $X$  is an interpolating divisor for  $\mathcal{A}_\alpha^\infty$ .

*Proof. Necessary part*

Let  $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$  be an interpolating divisor and assume that the discs

$$\left\{ D \left( \lambda, \sqrt{\frac{m_\lambda - C_X}{m_\lambda + \alpha}} \right) \right\}_{\lambda \in \Lambda, m_\lambda > C_X}$$

are not pairwise disjoint for any  $C_X$ . Arguing as in the proof of Theorem 1.5, we see that there exist  $\lambda, \lambda' \in \Lambda$  and  $w \in \mathbb{D}$  such that

$$D(w, \epsilon) \subset D \left( \lambda, \sqrt{\frac{m_\lambda - C_X + 1}{m_\lambda + \alpha}} \right) \cap D \left( \lambda', \sqrt{\frac{m_{\lambda'} - C_X + 1}{m_{\lambda'} + \alpha}} \right).$$

Since  $X$  is an interpolating divisor, there exists a function  $f \in \mathcal{A}_\alpha^\infty$  such that

- (a)  $f \in N_{\lambda, m_\lambda}^\infty$ ,
- (b)  $f - T_w 1 \in N_{\lambda', m_{\lambda'}}^\infty$ ,
- (c)  $\|f\|_{\alpha, \infty} \leq M_X$ .

Let us denote  $\|\cdot\|_{\infty, U}$  the norm with a supremum taken in the set  $U \subset \mathbb{D}$ . By Lemma 4.3(i) applied to  $f$  and  $f - T_w 1$  we have

$$\|f\|_{\infty, D \left( \lambda, \sqrt{\frac{m_\lambda - C_X + 1}{m_\lambda + \alpha}} \right)} + \|f - T_w 1\|_{\infty, D \left( \lambda', \sqrt{\frac{m_{\lambda'} - C_X + 1}{m_{\lambda'} + \alpha}} \right)} < 2\eta,$$

where we can pick  $\eta < 1/2$  when  $C_X$  is sufficiently big (note that since  $f$  and  $f - T_w 1$  are zero in the corresponding quotient spaces, we can consider  $\epsilon = 0$ ). Therefore

$$\|f\|_{\infty, D(w, \epsilon)} + \|f - T_w 1\|_{\infty, D(w, \epsilon)} < 2\eta$$

However

$$\sup_{z \in D(w, \epsilon)} \|T_w 1\|_{\infty, D(w, \epsilon)} = 1,$$

so  $X$  cannot be interpolating.

*Sufficient part*

Here we use the same scheme as in the  $L^2$ -case: we construct a smooth interpolating function and we modify it to obtain a holomorphic one. However, now we need an  $L^\infty$ -estimate for the solution to the  $\bar{\partial}$ -equation which will be provided by Theorem 4.5.

Let  $(\rho_{\lambda_j})_{j \geq 1}$  be holomorphic data (polynomials) with  $\sup_j \|\rho_{\lambda_j}\|_{\alpha, \infty} \leq 1$ . Given any  $N \geq 1$  we look for functions  $f_N \in \text{Hol}(\mathbb{D})$  and  $M$  independent of  $N$  such that

$$\begin{aligned} f_N - \rho_j &\in N_{\lambda_j}^\infty, \quad j = 1, \dots, N; \\ \|f_N\|_{\alpha, \infty} &\leq M, \quad \forall N \in \mathbb{N}. \end{aligned}$$

Then, by Montel's theorem, the limit  $f = \lim_N f_N$  gives the desired result. For this set,  $m_j = m_{\lambda_j}$ ,

$$D_j = D(\lambda_j, r_j), \quad r_j = \sqrt{\frac{m_j}{m_j + \alpha}} \quad \text{and} \quad D'_j = D(\lambda_j, r'_j), \quad r'_j = \sqrt{\frac{m_j + C_X}{m_j + \alpha}}.$$

Define the smooth interpolating function,

$$F_N(z) = \sum_{j=1}^N \rho_j(z) \eta(|\varphi_{\lambda_j}(z)| - r'_j),$$

where  $\eta = \eta_\lambda$  is a smooth cut-off function on  $\mathbb{R}$ , with

- (a)  $\text{supp } \eta \subset (-\infty, 0]$ ,
- (b)  $\eta \equiv 1$  on  $(-\infty, r_j - r'_j]$ ,
- (c)  $|\eta'| \lesssim \frac{1}{r'_j - r_j} \simeq \frac{m_j + \alpha}{C_X}$ .

By the separation hypothesis we have

$$\text{supp } F_N \subset \bigcup_{j=1}^N D'_j \subset \bigcup_{j \geq 1} D'_j.$$

Furthermore,  $F_N$  has the characteristic growth of  $\mathcal{A}_\alpha^\infty$ , due to the property (b) and the separation hypothesis again, Namely

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\frac{\alpha}{2}} |F_N(z)| = \max_{1 \leq j \leq N} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\frac{\alpha}{2}} |\rho_j(z)| \quad (45)$$

$$\leq \sup_{j \geq 1} \|\rho_j\|_{\mathcal{A}_\alpha^\infty} \leq 1 \quad (46)$$

uniformly in  $N$ . On the other hand,

$$\text{supp } \bar{\partial} F_N \subset \bigcup_{j=1}^N (D'_j \setminus D_j).$$

And

$$\bar{\partial} F_N(z) = \sum_{j \leq N} \rho_j(z) \eta'(|\varphi_{\lambda_j}(z)| - r'_j) \bar{\partial} |\varphi_{\lambda_j}(z)| \chi_{D'_j \setminus D_j}(z).$$

Hence, for  $z \in D'_j \setminus D_j$ , as in (37),

$$\begin{aligned} |\bar{\partial} F_N(z)| &= |\rho_j(z)| |\eta'(|\varphi_{\lambda_j}(z)| - r'_j)| |\bar{\partial} |\varphi_{\lambda_j}(z)|| \\ &\leq \frac{1}{2} |\rho_j(z)| \|\eta'\|_\infty \frac{1 - |\varphi_{\lambda_j}(z)|^2}{1 - |z|^2}. \end{aligned}$$

Therefore, by (b) and (c), we get for  $z \in D'_j \setminus D_j$

$$\begin{aligned} |\bar{\partial}F_N(z)| (1 - |z|^2)^{\frac{\alpha}{2}+1} &\lesssim \|\rho_j\|_{\alpha,\infty} \|\eta'\|_{\infty} (1 - |\varphi_{\lambda_j}(z)|^2) \\ &\lesssim \|\rho_j\|_{\alpha,\infty} \frac{1 - r_j^2}{r'_j - r_j} \\ &\lesssim \frac{2(\alpha + 1)}{C_X} \|\rho_j\|_{\alpha,\infty}, \end{aligned}$$

where we have used a similar estimate as in (39).

This leads to

$$\sup_{z \in \mathbb{D}} \frac{|\bar{\partial}F_N(z)|}{1 - |z|^2} e^{-\frac{\alpha}{2} \log(\frac{1}{1 - |z|^2})} \lesssim \sup_{j \geq 1} \|\rho_j\|_{\mathcal{A}_\alpha^\infty} < \infty, \quad (47)$$

where underlying constants are independent on  $N$ .

Again the holomorphic interpolating function in  $\mathcal{A}_\alpha^\infty$  will be obtained *via* the solution to a  $\bar{\partial}$ -problem:  $f_N = F_N - u_N$ , where  $\bar{\partial}u_N = \bar{\partial}F_N$  with the conditions

$$\sup_{z \in \mathbb{D}} |u_N(z)| (1 - |z|^2)^{\frac{\alpha}{2}} < \infty,$$

and

$$\partial^k u_N(\lambda_j) = 0, \quad \forall k < m_{\lambda_j}.$$

The last condition ensures that  $\partial^k f_N(\lambda_j) = \partial^k F_N(\lambda_j)$ , for  $k < m_{\lambda_j}$ , and then

$$\partial^k (F_N - u_N - \rho_j)(\lambda_j) = 0, \quad k < m_{\lambda_j}, 1 \leq j \leq N. \quad (48)$$

We will use a similar weight function  $w$  as in (34) where now

$$r_\lambda = \sqrt{\frac{m_\lambda}{m_\lambda + \alpha}}, \quad \text{and} \quad r'_\lambda = \sqrt{\frac{m_\lambda + C_X}{m_\lambda + \alpha}}.$$

More precisely, set

$$w := w_{\Lambda, \alpha, N}(z) = \sum_{j=1}^N m_{\lambda_j} [E(\cdot) - E \star \xi_{\lambda_j}(\cdot)] (\varphi_{\lambda_j}(z)) \chi_{D'_{\lambda_j}}(z), \quad z \in \mathbb{D},$$

where  $E(z) = \log |z|^2$ , and for  $\lambda \in \Lambda$

$$\xi_\lambda(\zeta) = \begin{cases} 0 & \text{if } 0 \leq |\zeta| < r_\lambda, \\ \frac{\log \frac{1}{|\zeta|^2}}{K(m_\lambda, C_X, \alpha)} & \text{if } r_\lambda < |\zeta| < r'_\lambda, \\ 0 & \text{if } |\zeta| > r'_\lambda. \end{cases}$$

Again  $K := K(m_\lambda, C_X, \alpha) = \int_{r_\lambda < |\zeta| < r'_\lambda} \log(1/|\zeta|^2) d\mathcal{V}(\zeta)$ . As in (35) we see that  $K \geq \log(\alpha/(\alpha - C_X)) > 1$  when  $C_X > \alpha(1 - e^{-1})$ . Let

$$\psi(z) := \psi_{\Lambda, \alpha}(z) = \alpha \log \frac{1}{1 - |z|^2} + w(z), \quad z \in \mathbb{D}.$$

By the same arguments as in the proof of lemma 3.8, we have

- (a)  $w \leq 0$ ,
- (b)  $-w \leq A(\alpha)$  in  $D'_j \setminus D_j$ ,
- (c)  $\tilde{\Delta}w \geq -4(\alpha - \varepsilon)$  for some  $\varepsilon$  depending on  $C_X$ .

Clearly from (a) and the definition of  $\psi$  we have  $\psi(z) \leq \alpha \log \frac{1}{1-|z|^2}$  for  $z \in \mathbb{D}$ , and with (c) we get that under the condition  $\alpha(1 - e^{-1}) < C_X < \alpha$ , for every  $z \in \mathbb{D}$ ,

$$(1 - |z|^2)\Delta\psi(z) = \frac{\tilde{\Delta}\psi(z)}{1 - |z|^2} \gtrsim \frac{\varepsilon(C_X)}{1 - |z|^2}.$$

Thus,

$$\varphi(z) := \min\{(1 - |z|)\Delta\psi(z), \frac{1}{1 - |z|}\} \asymp \frac{1}{1 - |z|^2}, \quad z \in \mathbb{D}.$$

Now applying Theorem 4.5, we see that there exists  $u_N \in L^2(\mathbb{D}, e^{-\psi} dm)$ , a canonical solution of the  $\bar{\partial}$ -equation  $\bar{\partial}u_N = \bar{\partial}F_N$  satisfying

$$\sup_{z \in \mathbb{D}} |u_N(z)| e^{-\frac{1}{2}\widetilde{\psi}(z)} \leq \sup_{z \in \mathbb{D}} \frac{|\bar{\partial}F_N(z)|}{\varphi(z)} e^{-\frac{1}{2}\psi(z)},$$

where  $\widetilde{\psi}(z) := \sup_{|z-\zeta| < 1/2(1-|z|)} \psi(\zeta)$ . By (47) and (b)

$$\begin{aligned} \sup_{z \in \mathbb{D}} \frac{|\bar{\partial}F_N(z)|}{\varphi(z)} e^{-\frac{1}{2}\psi(z)} &\lesssim \sup_{z \in \mathbb{D}} \frac{|\bar{\partial}F_N(z)|}{\frac{1}{1-|z|^2}} e^{-\frac{\alpha}{2} \log(\frac{1}{1-|z|^2}) - w(z)} \\ &\lesssim \sup_{j=1, \dots, N} \sup_{z \in D'_j \setminus D_j} \frac{|\bar{\partial}F_N(z)|}{\frac{1}{1-|z|^2}} e^{-\alpha \log \frac{1}{1-|z|^2}} e^{A(\alpha)} \\ &\lesssim \sup_{j \geq 1} \|\rho_j\|_{A_\alpha^\infty} \lesssim 1. \end{aligned}$$

Thus, uniformly in  $N$

$$\sup_{z \in \mathbb{D}} |u_N(z)| e^{-\frac{1}{2}\widetilde{\psi}(z)} < \infty.$$

On the other hand, by (a)

$$\begin{aligned} \widetilde{\psi}(z) &:= \sup_{|z-\zeta| < 1/2(1-|z|)} \psi(\zeta) = \sup_{|z-\zeta| < 1/2(1-|z|)} \left( \alpha \log \frac{1}{1-|\zeta|^2} + w(\zeta) \right) \\ &\leq \alpha \log \frac{1}{1-|z|^2} + \log \frac{3}{2}. \end{aligned}$$

We obtain finally

$$\sup_{z \in \mathbb{D}} |u_N(z)| e^{-\frac{\alpha}{2} \log \frac{1}{1-|z|^2}} \leq \frac{2}{3} \sup_{z \in \mathbb{D}} |u_N(z)| e^{-\frac{1}{2}\widetilde{\psi}(z)} \lesssim 1. \quad (49)$$

Hence, by (45) and (48),  $f_N = F_N - u_N \in \mathcal{A}_\alpha^\infty$ . This completes the proof.  $\square$

## REFERENCES

- [1] M. Abramowitz and I. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. National Bureau of Standards Applied Mathematics Series, No. 55 U. S. Government Printing Office, Washington, D.C., 1964 xiv+1046 pp.
- [2] Escudero, Luis Alberto; Haimi, Antti; Romero, Jos  l Luis, *Multiple sampling and interpolation in weighted Fock spaces of entire functions*. Complex Anal. Oper. Theory 15 (2021), no. 2, Paper No. 35, 32 pp.
- [3] B. Berndtsson, *Weighted estimates for  $\bar{\partial}$  in domains in  $\mathbb{C}$* . Duke Math. J. 66 (1992), no. 2, 239–255.
- [4] B. Berndtsson and J. Ortega-Cerd  , *On interpolation and sampling in Hilbert spaces of analytic functions*, J. Reine Angew. Math. 464 (1995), 109–128.
- [5] A. Borichev, A. Hartmann, K. Kellay, and X. Massaneda, *Geometric conditions for multiple sampling and interpolation in the Fock space*, Adv. Math. 304 (2017), 1262 – 1295.

- [6] S. Brekke and K. Seip, *Density theorems for sampling and interpolation in the Bargmann-Fock space III*, Math. Scand. 73 (1993) 112 - 126.
- [7] P.L. Duren and A. Schuster, *Bergman Spaces*, Mathematical surveys and monographs, American Mathematical Society, 2004.
- [8] A. Hartmann, *Generalized Interpolation in Bergman Spaces and Extremal Functions*. Mathematische Nachrichten, 2001, vol. 224, no 1, p. 123-144.
- [9] J. Garnett, *Bounded Analytic Functions*, Graduate Texts in Mathematics, Springer New York, 2006.
- [10] W. Gautschi. *Some elementary inequalities relating to the gamma and incomplete gamma function*. J. Math. Phys 38.1 (1959): 77-81.
- [11] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, Graduate Texts in Mathematics, Springer New York, 2000.
- [12] M. Krosky, and A. Schuster, *Multiple interpolation and extremal functions in the Bergman spaces*, J. Anal. Math. 85 (2001), 141-156.
- [13] X. Massaneda, *Density conditions for interpolation in  $A^{-\infty}$* , J. Analyse Math. 79 (1999), no. 1, 299-314.
- [14] N. K. Nikolski, *Bases of invariant subspaces and operator interpolation*, Tr. Mat. Inst. Akad. Nauk SSSR 130 (1978) 50-123 (Russian); English translation: Proc. Steklov Inst. Math. (4) (1979) 55-132.
- [15] T. Ohsawa, *On the extension of  $L^2$ -holomorphic functions IV: a new density concept*, Geometry and analysis on complex manifolds, 157-170, World Sci. Publ., River Edge, NJ, 1994.
- [16] M. Ounaies, *Geometric conditions for interpolation in weighted spaces of entire functions*, J. Geom. Anal 17 (2007), no. 8, 701-716.
- [17] M. Ounaies, *Interpolation by entire functions with growth conditions*, Michigan Math. J. (2007).
- [18] K. Seip, *Beurling type density theorems in the unit disk*, Inv. Math. 113 (1993), no. 1, 21-40.
- [19] K. Seip, *On Korenblum's density condition for the zero sequences of  $A^{-\alpha}$* , J. Anal. Math. 67 (1995), 307-322.
- [20] K. Seip, *Interpolation and sampling in spaces of analytic functions*. University Lecture Series, 33. American Mathematical Society, Providence, RI, 2004.
- [21] V.I. Vasyunin, *Unconditionally convergent spectral decompositions and interpolation problems*, Tr. Mat. Inst. Akad. Nauk SSSR 130 (1978) 5-49 (Russian); English translation: Proc. Steklov Inst. Math. 4 (1979) 1-53.
- [22] R. Rochberg, *Interpolation by functions in Bergman spaces*, Michigan Math. J. 29 (1982), no. 2, 229-236.
- [23] S.A. Vinogradov, S.E. Rukshin, *On the free interpolation of germs of analytic functions in Hardy spaces*, Investigations on linear operators and the theory of functions, X. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 107 (1982), 36-45, 229.

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