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# Robust DOF control for uncertain polynomial fuzzy systems in finite frequency domain

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## ABSTRACT

This study addresses the issue of robust dynamic output feedback control (DOF) for polynomial Takagi–Sugeno (T–S) fuzzy systems in the Finite Frequency (FF) domain. Sufficient conditions for designing the robust DOF control are derived in terms of the sum of squares (SOS). The proposed strategy is built in the FF domain to reduce conservatism generated by the techniques established in the whole frequency domain. In addition, there are no transformation matrices or equality constraints under these conditions, which simplifies the numerical solution. To show the validity of the suggested technique, several numerical examples are presented.

## 1. Introduction

Many issues of analysis and control design of a large class of nonlinear systems have been effectively solved using Takagi–Sugeno (T–S) fuzzy models [1]. It has been demonstrated that T–S fuzzy models can Transform higher-order nonlinear systems into a weighted combination of a set of linear systems, by using fuzzy IF–THEN rules. In both continuous and discrete-time, several methods for T–S systems have addressed in the literature. The stability and stabilization issues of nonlinear systems represented by T–S fuzzy systems have been investigated in the literature. In [2], authors proposed quadratic Lyapunov function (LF) to derive stability conditions represented in terms of LMI for T–S fuzzy control systems that unfortunately tend to give conservative conditions. Furthermore, various relaxed stabilization conditions were also suggested to derive less conservative results in [3]. The robust controller for T–S fuzzy systems with uncertainties is explored in [4,5]. Hence, It is widely understood that the existence of uncertainty may decrease the performance of many control systems and can even cause instability, that is why robust stability of uncertain T–S fuzzy systems is a critical concern.

The majority of the previous references have concentrated on state feedback control issues. Unfortunately, state variables are not always fully measurable throughout many industrial applications. Hence, the output feedback control and particularly the dynamic output feedback control is for remarkable interest. Some results on the DOF case exist in the literature [6–8] using the descriptor representation approach and LMI-based design conditions [6]. In [7], authors proposed an unified systematic framework for designing DOF controllers for nonlinear systems described by T–S models. These existing LMI based results, however, do not take into account uncertainties. [9], has studied the robust DOF control for a class of discrete-time nonlinear fuzzy systems with parametric uncertainties where the control is designed via an LMI based approach.

In general, polynomial systems are an extension of T–S systems that allow the study of a larger class of nonlinear systems [10,11]. Indeed, polynomial T–S fuzzy systems can overcome the quadratic approach and allows reducing the conservatism of the existing LMI results. To provide more relaxed stability, in [12,13], some results on static output feedback control of continuous-time T–S

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fuzzy systems based on a polynomial Lyapunov function are proposed. As well known, solving problems by SOS-based conditions of polynomial systems is less conservative than using LMI-based conditions for T–S systems. At this stage, it is worth noting that the DOF control for polynomial systems is not widely studied as its linear counterpart. In [14], the problem of designing DOF controller for discrete polynomial fuzzy model-based was investigated. Moreover, all the DOF control designs previously cited deal with the entire frequency range (EF) which may bring some conservatism. As a result, it is essential and more practical to develop DOF control in the finite frequency domain (FF). Some approaches have been proposed to deal with the GKYP lemma using LMI techniques. One can refer to [15], where the problem of fault detection filtering design has been provided for discrete-time T–S fuzzy systems in the FF domain. Moreover, [16], discussed the problem of SOF  $H_\infty$  control of continuous T–S systems in FF domain. A new descriptor method in FF was recently developed in [17] to facilitate the output feedback controller design. The aforementioned results show that FF conditions are less conservative than the entire frequency domain and encompass the standard ones as a particular case. Nevertheless, relatively few research efforts have been devoted to polynomial fuzzy systems in the FF domain. Problem of fault detection was addressed in [18] for polynomial fuzzy system in the FF domain. To the best of our knowledge, there has not been any research done yet on the robust DOF control design for polynomial fuzzy systems in the FF domain and this issue remains an open and challenging point. The present research is motivated by the issue of how to reduce conservatism even more and construct polynomial fuzzy discrete-time systems.

We present a new technique for robust DOF control for polynomial T–S fuzzy systems in the FF domain in this study. Using polynomial Lyapunov function and FF bounded lemma sufficient conditions were derived in terms of SOS, to guarantee the stability of the polynomial fuzzy system. This paper presents an SOS-based technique to solve the stabilization of discrete DOF polynomial systems. Compared to the existing approaches, the polynomial nonlinearities can be precisely manipulated and a wide class of other nonlinearities can be treated by adding auxiliary variables and constraints. The suggested method does not need any transformation matrices nor equality constraints that are difficult to satisfy. The proposed methodology not only ensures the closed-loop system's stability but also leads to less conservative outcomes. The remainder of this paper is organized as follows. The T–S system description and some preliminary results are stated in Section 2. The proposed approach is presented in Section 3 while in Section 4 some numerical examples are given to illustrate the effectiveness of the proposed method. Finally, a conclusion takes place in Section 5.

Notation:

- $\mathbb{R}$ : the set of real numbers;
- $I$ : identity matrix (of size specified by the context);
- $\rho(\cdot)$ : eigenvalue of a matrix;
- $*$ : represents a term induced by symmetry.
- $M^\perp$  orthogonal matrix of  $M$
- $M^{-1}$  inverse matrix of  $M$ .
- $\text{sym}(M)$ : is defined as  $\text{sym}(M) = M + M^T$ .
- $A^T$  represents the transpose matrix of  $A$ .
- $P > 0$  means that  $P$  is real symmetric and positive definite.

## 2. Problem formulation

Consider a polynomial discrete-time system described by the following T–S fuzzy model, in which the  $i$ th rule is described as follows:

Plant Rule  $i$ : IF  $\varsigma_1(x(k))$  is  $M_{i1}$  AND ... AND  $\varsigma_p(x(k))$  is  $M_{ip}$  THEN

$$\begin{cases} x(k+1) = \bar{A}_i(x(k))\tilde{x}(x(k)) + \bar{B}_i(x(k))u(k) \\ y(k) = \bar{C}_i(x(k))\tilde{x}(x(k)) \end{cases} \quad (1)$$

where  $x(k) \in \mathbb{R}^n$  is the state vector,  $\tilde{x}(x(k))$  is a vector of monomials in  $x(k)$ ,  $u(k) \in \mathbb{R}^{m_1}$  is the control input,  $y(k) \in \mathbb{R}^{m_2}$  is the measurement output.  $i = 1, 2, \dots, r$  and  $r$  is the number of IF–THEN rules,  $\varsigma_1(x(k)) = [\varsigma_1(x(k)) \ \varsigma_2(x(k)) \ \dots \ \varsigma_p(x(k))]$  are known premise variables,  $M_{ij}$  are fuzzy sets,  $\tilde{x}(x(k)) \in \mathbb{R}^N$  is an  $N \times 1$  vector of monomial in  $x(k)$ .  $\bar{A}_i(x(k))$ ,  $\bar{B}_i(x(k))$ , and  $\bar{C}_i(x(k))$  are polynomial matrices in  $x(k)$  which are composed of two parts as:

$$\begin{aligned} \bar{A}_i(x(k)) &= A_i(x(k)) + \Delta A_i(k), \\ \bar{B}_i(x(k)) &= B_i(x(k)) + \Delta B_i(k) \\ \bar{C}_i(x(k)) &= C_i(x(k)) + \Delta C_i(k) \end{aligned} \quad (2)$$

where  $A_i(x(k))$ ,  $B_i(x(k))$  and  $C_i(x(k))$  are polynomial matrices.  $\Delta A_i(k)$ ,  $\Delta B_i(k)$  and  $\Delta C_i(k)$  are unknown matrices that account for time-varying parameter uncertainties and are assumed to be structured under the form:

$$\begin{aligned} \Delta A_i(k) &= X_{\Delta i}(k)\Delta(k)Y_{\Delta i}, \quad \Delta B_i(k) = X_{\Delta i}(k)\Delta(k)Y_{B i}, \\ \Delta C_i(k) &= X_{C i}(k)\Delta(k)Y_{\Delta i}, \end{aligned} \quad (3)$$

and  $\Delta(k)$  are unknown time-varying matrix functions satisfying:

$$\Delta(k)^T \Delta(k) \leq I$$

The defuzzification process of the polynomial T-S system (1) are inferred as follows:

$$\begin{cases} x(k+1) = A(h)\tilde{x}(x(k)) + B(h)u(k) \\ y(k) = C(h)\tilde{x}(x(k)) \end{cases} \quad (4)$$

where

$$\begin{aligned} A(h) &:= \sum_{i=1}^r h_i(\zeta(x(k))) \bar{A}_i(x(k)), \quad B(h) := \sum_{i=1}^r h_i(\zeta(x(k))) \bar{B}_i(x(k)) \\ C(h) &:= \sum_{i=1}^r h_i(\zeta(x(k))) \bar{C}_i(x(k)) \\ h_i(\zeta(x(k))) &= \frac{w_i(\zeta(x(k)))}{\sum_{i=1}^r w_i(\zeta(x(k)))}, \quad w_i(\zeta(x(k))) = \prod_{j=1}^s M_{ij}(\zeta_j(x(k))) \end{aligned}$$

where  $M_{ij}(\zeta_j(x(k)))$  is the grade of membership of  $\zeta_j(x(k))$  in  $M_{ij}$  and  $w_i(\zeta(x(k)))$  represents the weight of the  $i$ th rule. It should be noted from the properties of membership functions that  $h_i(\zeta(x(k))) \geq 0$ , for  $i = 1, 2, \dots, r$  and  $\sum_{i=1}^r h_i(\zeta(x(k))) = 1$  for all  $k$ .

In this paper, it is assumed that  $\tilde{x}(x(k)) = 0$  if  $x(k) = 0$ . Moreover, we defined the polynomial transformation matrix from  $x(k)$  to  $\tilde{x}(x(k))$  by  $T(x(k))$  as follows  $\tilde{x}(x(k+1)) = T(x(k+1))x(k+1)$ . Thus, the polynomial T-S system (4) can be rewritten as:

$$\begin{cases} \tilde{x}(x(k+1)) = T(x(k+1))[A(h)\tilde{x}(x(k)) + B(h)u(k)] \\ y(k) = C(h)\tilde{x}(x(k)) \end{cases} \quad (5)$$

Recall that the premise variables of the fuzzy system (5) cannot be used to design the controller since they are assumed to be non-measurable. It means that the well-known parallel distributed compensation control cannot be applied in this case. Hence, we adopt the following DOF controller

$$K : \begin{cases} x_d(k+1) = A_d(x(k))x_d(k) + B_d(x(k))y(k) \\ u(k) = C_d(x(k))x_d(k) + D_d(x(k))y(k) \end{cases} \quad (6)$$

where  $x_d(k)$  represents the state vector of the DOF controller, and  $A_d(x(k))$ ,  $B_d(x(k))$ ,  $C_d(x(k))$ , and  $D_d(x(k))$  are the sought polynomial matrices. The overall closed-loop system with DOF controller  $K$  can be represented by

$$\xi(k+1) = \mathbb{A}\xi(k) \quad (7)$$

where

$$\begin{aligned} \xi(k+1) &= \begin{bmatrix} \tilde{x}(x(k+1)) \\ x_d(k+1) \end{bmatrix}, \quad T(x^+) = T(x(k+1)), \\ \mathbb{A} &= \begin{bmatrix} T(x^+)[A(h) + B(h)D_d(x(k))C(h)] & T(x^+)B(h)C_d(x(k)) \\ B_d(x(k))C(h) & A_d(x(k)) \end{bmatrix} \end{aligned}$$

The following Lemmas will be used intensively in the sequel.

**Lemma 1 ([19]).** Let  $\mathbb{A} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$  System (7) is asymptotically stable if and only if the following condition holds:  $\rho(A_4 + A_3(e^{j\theta}I - A_1)^{-1}A_2) < 1$  for all  $\theta \in [-\pi, \pi]$ , and  $\rho(A_1) < 1$ .

**Lemma 2 ([20]).** Let the matrices  $\theta$ ,  $F$ ,  $\phi$  and  $\Psi$  be given, and denote  $N_\theta$  is the null space of  $T_\theta F$ , where  $T_\theta = [I \quad -e^{j\theta}]$ . The inequality

$$N_\theta^T \theta N_\theta, \quad \text{with } \theta \in [\theta_1, \theta_2] \quad (8)$$

holds if and only if there exist  $Q > 0$  and a symmetric matrix  $P$ , such that

$$F^T(\phi \otimes P + \Sigma_\theta \otimes Q)F + \theta < 0 \quad (9)$$

where  $\phi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $\Sigma_\theta$  is in Table 1, with  $\theta_c = \frac{\theta_1 + \theta_2}{2}$ ,  $\theta_w = \frac{\theta_2 - \theta_1}{2}$ , and  $\theta_1, \theta_2$  satisfying  $-\pi \leq \theta_1 \leq \theta_2 \leq \pi$ , where  $\theta_1, \theta_2, \theta_l, \theta_h$  are known scalars.

**Definition 1 ([21]).** A multivariate polynomial  $f(x)$ , for  $x \in \mathbb{R}^N$  is a Sum of Squares (SOS) if there exist polynomials  $f_i(x)$ ,  $i = 1, \dots, n$  such that

$$f(x) = \sum_{i=1}^n f_i^2(x) \quad (10)$$

This implies  $f(x) \geq 0$  for any  $x \in \mathbb{R}^n$ .

**Table 1**  
 $\Sigma_\theta$  in different frequency ranges.

Frequency range $\Omega$	Low-frequency $ \theta  \leq \theta_l$	Middle-frequency $\theta_l \leq \theta \leq \theta_2$	High-frequency $\theta_h \leq  \theta $
$\Sigma_\theta = \begin{bmatrix} 0 & \Sigma_1 \\ \Sigma_1^T & \Sigma_2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -2\cos(\theta_l) \end{bmatrix}$	$\begin{bmatrix} 0 & e^{j\theta_c} \\ e^{(-j\theta_c)} & -2\cos(\theta_o) \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 2\cos(\theta_h) \end{bmatrix}$

**Proposition 1** ([22]). Let  $f(x)$  be a polynomial in  $x \in \mathbb{R}^n$  of degree  $2d$ . Let  $Z(x)$  be a column vector whose entries are all monomials in  $x$  with degree no greater than  $d$ . Then,  $f(x)$  is said to be SOS if and only if there exists a positive semi-definite matrix  $Q$  such that

$$f(x) = Z^T(x)QZ(x) \quad (11)$$

**Lemma 3** ([23]). Given a symmetric matrix  $\Sigma \in \mathbb{R}^{p \times p}$  and two matrices  $X, Z$  of column dimension  $p$ , there exists a matrix  $Y$  such that the LMI

$$\Sigma + \text{sym}\{X^T Y Z\} < 0 \quad (12)$$

holds if and only if the following two projection inequalities with respect to  $Y$  are satisfied:

$$X^\perp{}^T \Sigma X^\perp < 0, \quad Z^\perp{}^T \Sigma Z^\perp < 0. \quad (13)$$

where  $X^\perp$  and  $Z^\perp$  are arbitrary matrices whose columns form a basis of the null spaces of  $X$  and  $Z$ , respectively.

**Lemma 4** ([24]). Let  $T, Q, U$ , and  $W$  be some given matrices and  $\alpha$  a scalar. The following inequality is fulfilled

$$T + W^T Q^T + QW < 0 \quad (14)$$

if the following condition holds:

$$\begin{bmatrix} T & * \\ \alpha Q^T + UW & -\alpha U - \alpha U^T \end{bmatrix} < 0$$

**Lemma 5** ([9]). Given matrices  $T, X$ , and  $Y$  of appropriate dimensions and with  $T$  symmetrical

$$T + X\Delta Y + Y^T \Delta^T X^T < 0 \quad (15)$$

where  $\Delta = \Delta(k)(I - J\Delta(k))^{-1}$ ,  $\Delta(k)^T \Delta(k) < R$ , and  $R^{-1} - JJJ^T > 0$ . The inequality in (15) holds if and only if for matrix  $N$  and scalar  $\epsilon$

$$\begin{bmatrix} T & * & * \\ NY & -N - N^T + \epsilon R & * \\ X^T & J^T N^T & -\epsilon I \end{bmatrix} < 0 \quad (16)$$

### 3. Main result

The objective is to design a DOF controller in finite frequency domain, that stabilizes the polynomial fuzzy system. For brevity, in the following analysis,  $x$  is used instead of  $x(k)$  and  $\tilde{x}(x)$  instead of  $\tilde{x}(x(k))$ . Moreover,  $\mathbf{K} = \{\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_m\}$  denotes the row indices of  $B_i(x)$  whose corresponding row is zero and  $\hat{x} = (x_{\mathbf{k}_1}, x_{\mathbf{k}_2}, \dots, x_{\mathbf{k}_m})$ .

**Theorem 1.** The closed-loop system (7) is asymptotically stable if there exist symmetric matrices  $\bar{P}(\hat{x}) > 0$ ,  $P_1(\hat{x})$ ,  $P_2(\hat{x}) > 0$  and  $Q(x) > 0$  such that the following conditions are satisfied:

$$F^T(\phi \otimes P_1(\hat{x}) + \Sigma_\theta \otimes Q(x))F + \theta < 0 \quad (17)$$

$$A_1^T \bar{P}(\hat{x}) A_1 - \bar{P}(\hat{x}) < 0 \quad (18)$$

where  $F = \begin{bmatrix} A_1 & A_2 \\ I & 0 \end{bmatrix}$ ,  $\phi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $\theta = \begin{bmatrix} A_3 & A_4 \\ 0 & I \end{bmatrix}^T (\phi \otimes P_2(\hat{x})) \begin{bmatrix} A_3 & A_4 \\ 0 & I \end{bmatrix}$ ,  $\Sigma_\theta$  and  $\Omega$  are in Table 1

**Proof.** By Lemma 1, we have  $\rho(A_1) < 1$  if and only if there exist  $\bar{P}(\hat{x}) > 0$  such that  $A_1^T \bar{P}(\hat{x}) A_1 - \bar{P}(\hat{x}) < 0$ , LMI in (18) is satisfied. Denote

$$S(e^{j\theta}) = A_4 + A_3(e^{j\theta} I - A_1)^{-1} A_2 = A_3 G(e^{j\theta}) + A_4 \quad (19)$$

by Lemma 2, inequality (17) is equivalent to:

$$\begin{bmatrix} G(e^{j\theta}) \\ I \end{bmatrix}^T \theta \begin{bmatrix} G(e^{j\theta}) \\ I \end{bmatrix} < 0, \quad \forall \theta \in \Omega \quad (20)$$

or in a more compact form

$$S(e^{j\theta})^T P_2(\hat{x}) S(e^{j\theta}) - P_2(\hat{x}) < 0, \quad P_2(\hat{x}) > 0, \quad \forall \theta \in \Omega \quad (21)$$

So  $\rho(S(e^{j\theta})) < 1$  is finally guaranteed for all  $\theta \in \Omega$ . Combining  $\rho(A_1) < 1$ , and  $\rho(S(e^{j\theta})) < 1$ , we conclude that system (7) is asymptotically stable based on Lemma 1.

The proof is completed.

**Theorem 2.** The closed-loop fuzzy system (7) is robustly stable if there exist polynomial matrix  $F_l(x)$ ,  $G_l(x)$ ,  $l = 1, 2, 3$  symmetric polynomial matrices  $\bar{P}(\hat{x}) > 0$ ,  $Q(x) > 0$  and  $P_s(\hat{x})$ ,  $s = 1, 2$ .  $\varepsilon_l(x)$  are non-negative polynomials such that the following conditions are satisfied:

$$v_1^T (\bar{P}(\hat{x}) - \varepsilon_1(x)I) v_1 \text{ is SOS} \quad (22)$$

$$v_1^T (Q(x) - \varepsilon_1(x)I) v_1 \text{ is SOS} \quad (23)$$

$$-v_2^T (\varphi(x) + \varepsilon_2(x)I) v_2 \text{ is SOS} \quad (24)$$

$$-v_3^T (\bar{\varphi}(x) + \varepsilon_3(x)I) v_3 \text{ is SOS} \quad (25)$$

where

$$\varphi(x) = \begin{bmatrix} \varphi_{11}(x) & \varphi_{12}(x) & \varphi_{13}(x) & \varphi_{14}(x) \\ * & \varphi_{22}(x) & \varphi_{23}(x) & F_1(x)A_2 \\ * & * & \varphi_{33}(x) & \varphi_{34}(x) \\ * & * & * & -P_2(\hat{x}) \end{bmatrix} < 0 \quad (26)$$

$$\bar{\varphi}(x) = \begin{bmatrix} \bar{P}(\hat{x}) - G_1(x) - G_1^T(x) & G_1(x)A_1 \\ * & -\bar{P}(\hat{x}) \end{bmatrix} < 0 \quad (27)$$

$$\varphi_{11}(x) = P_1(\hat{x}) - G_1(x) - G_1^T(x)$$

$$\varphi_{12}(x) = G_1(x)A_1 + \lambda G_2(x)A_3 + \Sigma_1 Q(x) - F_1^T(x)$$

$$\varphi_{13}(x) = -\lambda G_2(x) - G_3^T(x)$$

$$\varphi_{14}(x) = G_1(x)A_2 + \lambda G_2(x)A_4$$

$$\varphi_{22}(x) = \Sigma_3 Q(x) - P_1(\hat{x}) + \text{sym}\{F_1(x)A_1\}$$

$$\varphi_{23}(x) = A_1^T G_3^T(x) + A_3^T G_2^T(x)$$

$$\varphi_{33}(x) = P_2(\hat{x}) - G_2(x) - G_2^T(x)$$

$$\varphi_{34}(x) = G_3(x)A_2 + G_2(x)A_4$$

with  $\Sigma_1$  and  $\Sigma_2$  are given in Table 1 and  $v_i$   $i = 1, \dots, 7$  are independent vectors of  $x(k)$ .

**Proof.** we can verify that (27) is equivalent to,

$$\begin{bmatrix} \bar{P}(\hat{x}) & 0 \\ * & -\bar{P}(\hat{x}) \end{bmatrix} + \text{sym} \left( \begin{bmatrix} G_1(x) \\ 0 \end{bmatrix} \begin{bmatrix} -I & A_1 \end{bmatrix} \right) < 0 \quad (28)$$

By Lemma 3 with

$$\Sigma_a = \begin{bmatrix} \bar{P}(\hat{x}) & 0 \\ * & -\bar{P}(\hat{x}) \end{bmatrix}, \quad X_a = I, \quad Y_a = \begin{bmatrix} G_1(x) \\ 0 \end{bmatrix}, \quad Z_a = \begin{bmatrix} -I & A_1 \end{bmatrix}$$

the inequality (28) can guarantee

$$\begin{bmatrix} A_1^T & I \end{bmatrix} \begin{bmatrix} \bar{P}(\hat{x}) & 0 \\ * & -\bar{P}(\hat{x}) \end{bmatrix} \begin{bmatrix} A_1 \\ I \end{bmatrix} < 0 \quad (29)$$

this implies that the closed-loop system is robustly stable.

Let

$$\Sigma_b = \begin{bmatrix} P_1(\hat{x}) & \Sigma_1 Q(x) & 0 & 0 \\ \Sigma_1^T Q(x) & \Sigma_2 Q(x) - P_1(\hat{x}) & 0 & 0 \\ 0 & 0 & P_2(\hat{x}) & 0 \\ 0 & 0 & 0 & -P_2(\hat{x}) \end{bmatrix}, \quad X_b = I,$$

$$Y_b = \begin{bmatrix} G_1(x) & \lambda G_2(x) \\ F_1(x) & 0 \\ G_3(x) & G_2(x) \\ 0 & 0 \end{bmatrix}, Z_b = \begin{bmatrix} -I & A_1 & 0 & A_2 \\ 0 & A_3 & -I & A_4 \end{bmatrix}$$

with  $\Sigma_1$  and  $\Sigma_2$  as in Table 1. (26) is equivalent to

$$\Sigma + \text{sym}\{X^T Y Z\} < 0 \quad (30)$$

Choosing  $Z^\perp = \begin{bmatrix} A_1 & A_2 \\ I & 0 \\ A_3 & A_4 \\ 0 & I \end{bmatrix}$  and applying Lemma 3, we obtain from (30) that (17) holds.

The proof is completed.

**Remark 1.** Because of the presence of bi-linear variables  $G_2(x)D_d(x)C(h)$  and  $G_1(x)T(x^+)B(h)D_d(x)C(h)$ , Theorem 1 is not a convex issue. Many convex approaches exist in the literature to solve this problem. In [12], an equality constraint was utilized for the SOF control. An iterative SOS technique has been proposed in [25], to solve numerically the SOF control for polynomial systems. It is worth noting that the iterative techniques are dependent on the initial values. However, how to choose the initial values is still an open problem. Thus, to overcome such drawbacks, an SOS design method is provided in this work without imposing any constraint on the system matrices.

**Theorem 3.** The closed-loop fuzzy system (7) is robustly stable if there exist a known scalar  $\alpha$ , symmetric positive definite polynomial matrices  $\tilde{P}(\hat{x})$ ,  $Q(x)$ , symmetric polynomial matrices  $P_s(\hat{x})$   $s = 1, 2$ , matrices  $F_l(x)$ ,  $G_l(x)$   $l = 1, 2, 3$ ,  $U(x)$ ,  $\hat{A}_d(x)$ ,  $\hat{B}_d(x)$ ,  $\hat{C}_d(x)$ ,  $\hat{D}_d(x)$  for  $i = 1, 2, \dots, r$  such that the following conditions are satisfied

SOS condition (22)

SOS condition (23)

$$-v_4^T(\Psi_{ii}(x) + \varepsilon_i(x)I)v_4 \text{ is SOS } i = 1, 2, \dots, r \quad (31)$$

$$-v_4^T(\Psi_{ij}(x) + \Psi_{ji}(x) + \varepsilon_{ij}(x)I)v_4 \text{ is SOS } i = 1, \dots, r-1; j = i+1, \dots, r \quad (32)$$

$$-v_5^T(\tilde{\Psi}_{ii}(x) + \varepsilon_i(x)I)v_5 \text{ is SOS } i, s = 1, 2, \dots, r \quad (33)$$

$$-v_5^T(\tilde{\Psi}_{ij}(x) + \tilde{\Psi}_{ji}(x) + \varepsilon_{ij}(x)I)v_5 \text{ is SOS } i = 1, \dots, r-1; j = i+1, \dots, r \quad (34)$$

where

$$\Psi_{ij}(x) = \begin{bmatrix} \Psi_{11}(x) & \Psi_{12ij}(x) & \Psi_{13}(x) & \Psi_{14}(x) & \Psi_{15i}(x) \\ * & \Psi_{22}(x) & \Psi_{23ij}(x) & k_1 \hat{C}_d(x) & \Psi_{25j}(x) \\ * & * & \Psi_{33}(x) & \Psi_{34}(x) & \Psi_{35i}(x) \\ * & * & * & -P_2(\hat{x}) & \hat{C}_d^T(x) \\ * & * & * & * & R(x) \end{bmatrix} < 0 \quad (35)$$

$$\tilde{\Psi}_{ij}(x) = \begin{bmatrix} \tilde{\Psi}_{11}(x) & \tilde{\Psi}_{12ij}(x) & \tilde{\Psi}_{13i}(x) \\ * & -\tilde{P}(\hat{x}) & \tilde{C}_j^T(x)\hat{D}_d^T(x) \\ * & * & R(x) \end{bmatrix} < 0 \quad (36)$$

$$\begin{aligned}
\Psi_{11}(x) &= P_1(\hat{x}) - G_1(x) - G_1^T(x) \\
\Psi_{12ij}(x) &= G_1(x)T(x^+)\bar{A}_i(x) + \Sigma_1 Q(x) + (k\hat{D}_d(x) + \lambda\hat{B}_d(x))\bar{C}_j(x) - F_1^T(x) \\
\Psi_{13}(x) &= -\lambda G_2(x) - G_3^T(x) \\
\Psi_{14}(x) &= k\hat{C}_d(x) + \lambda\hat{A}_d(x) \\
\Psi_{15i}(x) &= \alpha(G_1(x)T(x^+)\bar{B}_i(x) - kU(x)) \\
\Psi_{22}(x) &= \Sigma_2 Q(x) - P_1(\hat{x}) + \text{sym}\{F_1(x)T(x^+)\bar{A}_i(x) + k_1\hat{D}_d(x)\bar{C}_j(x)\} \\
\Psi_{23ij}(x) &= \bar{A}_i^T(x)T(x^+)G_3^T(x) + \bar{C}_j^T(x)(\lambda_3\hat{D}_d^T(x)k^T + \hat{B}_d^T(x)) \\
\Psi_{25j}(x) &= \alpha(F_1(x)T(x^+)\bar{B}_i(x) - k_1U(x)) + \bar{C}_j^T(x)\hat{D}_d^T(x) \\
\Psi_{33}(x) &= P_2(\hat{x}) - G_2(x) - G_2^T(x) \\
\Psi_{34}(x) &= \lambda_3 k\hat{C}_d(x) + \hat{A}_d(x) \\
\Psi_{35i}(x) &= \alpha(G_3(x)T(x^+)\bar{B}_i(x) - \lambda_3 kU(x)) \\
R(x) &= -\alpha(U(x) + U^T(x)) \\
\bar{\Psi}_{11}(x) &= \bar{P}(\hat{x}) - G_1(x) - G_1^T(x) \\
\bar{\Psi}_{12ij}(x) &= G_1(x)T(x^+)\bar{A}_i(x) + k\hat{D}_d(x)\bar{C}_j(x) \\
\bar{\Psi}_{13i}(x) &= \alpha(G_1(x)T(x^+)\bar{B}_i(x) - kU(x))
\end{aligned}$$

**Proof.** According to the SOS conditions (33)–(34), we write

$$\begin{aligned}
&\sum_{i=1}^r h_i^2(\zeta(x(k)))\bar{\Psi}_{ii} + \sum_{i=1}^{r-1} \sum_{j=i+1}^r h_i(\zeta(x(k)))h_j(\zeta(x(k))) (\bar{\Psi}_{ij} + \bar{\Psi}_{ji}) \\
&= \sum_{i=1}^r \sum_{j=1}^r h_i(\zeta(x(k)))h_j(\zeta(x(k)))\bar{\Psi}_{ij} < 0
\end{aligned}$$

We can verify that (36) implies

$$\begin{aligned}
&\sum_{i=1}^r \sum_{j=1}^r h_i(\zeta(x(k)))h_j(\zeta(x(k))) \left\{ \begin{bmatrix} \bar{\Psi}_{11}(x) & \bar{\Psi}_{12ij}(x) \\ * & -\bar{P}(\hat{x}) \end{bmatrix} \right. \\
&+ \text{sym} \left\{ \begin{bmatrix} G_1(x)T(x^+)\bar{B}_i(x) - kU(x) \\ 0 \end{bmatrix} U^{-1}(x) \begin{bmatrix} 0 & \hat{D}_d(x)\bar{C}_j(x) \end{bmatrix} \right\} \left. \right\} < 0
\end{aligned} \tag{37}$$

by Lemma 4, letting

$$\bar{W}_x = U^{-1}(x) \begin{bmatrix} 0 & \hat{D}_d(x)\bar{C}_j(x) \end{bmatrix}, \quad \bar{Q}_x = \begin{bmatrix} G_1(x)T(x^+)\bar{B}_i(x) - kU(x) \\ 0 \end{bmatrix}, \quad \bar{T}_x = \begin{bmatrix} \bar{\Psi}_{11}(x) & \bar{\Psi}_{12ij}(x) \\ * & -\bar{P}(\hat{x}) \end{bmatrix}$$

and defining  $\hat{D}_d(x) = U(x)D_d$ , we can guarantee that (37) is equivalent to

$$\sum_{i=1}^r \sum_{j=1}^r h_i(\zeta(x(k)))h_j(\zeta(x(k))) \begin{bmatrix} \bar{\Psi}_{11}(x) & G_1(x)A_1 \\ * & -\bar{P}(\hat{x}) \end{bmatrix} < 0$$

which is equivalent to (27).

Let

$$\begin{aligned}
T_{ij} &= \begin{bmatrix} \Psi_{11}(x) & \Psi_{12ij}(x) & \Psi_{13}(x) & \Psi_{14}(x) \\ * & \Psi_{22}(x) & \Psi_{23ij}(x) & k_1\hat{C}_d(x) \\ * & * & \Psi_{33}(x) & \Psi_{34}(x) \\ * & * & * & -P_2(\hat{x}) \end{bmatrix} \quad Q_i = \begin{bmatrix} G_1(x)T(x^+)\bar{B}_i(x) - kU(x) \\ F_1(x)T(x^+)\bar{B}_i(x) - k_1U(x) \\ G_3(x)T(x^+)\bar{B}_i(x) - \lambda_3 kU(x) \\ 0 \end{bmatrix}, \\
W_j &= U^{-1}(x) \begin{bmatrix} 0 & \hat{D}_d(x)\bar{C}_j(x) & 0 & \hat{C}_d(x) \end{bmatrix},
\end{aligned}$$

applying Lemma 4, the inequality in (35) leads to

$$T_{ij} + \text{sym} \left\{ \begin{bmatrix} G_1(x)T(x^+)\bar{B}_i(x) - kU(x) \\ F_1(x)T(x^+)\bar{B}_i(x) - k_1U(x) \\ G_3(x)T(x^+)\bar{B}_i(x) - \lambda_3 kU(x) \\ 0 \end{bmatrix} U^{-1}(x) \begin{bmatrix} 0 & \hat{D}_d(x)\bar{C}_j(x) & 0 & \hat{C}_d(x) \end{bmatrix} \right\} < 0 \tag{38}$$



By defining  $G_2(x)B_d(x) = \hat{B}_d(x)$ ,  $G_2(x)A_d(x) = \hat{A}_d(x)$ ,  $U(x)D_d(x) = \hat{D}_d(x)$ ,  $U(x)C_d(x) = \hat{C}_d$ , we can verify that (38) is equivalent to

$$\sum_{i=1}^r \sum_{j=1}^r h_i(\zeta(x(k)))h_j(\zeta(x(k))) \left\{ \begin{bmatrix} \Psi_{11}(x) & \Psi_{12ij}(x) & \Psi_{13}(x) & \Psi_{14}(x) \\ * & \Psi_{22}(x) & \Psi_{23ij}(x) & k_1 \hat{C}_d(x) \\ * & * & \Psi_{33}(x) & \Psi_{34}(x) \\ * & * & * & -P_2(\hat{x}) \end{bmatrix} + \begin{bmatrix} 0 & \mathbb{T}_{12}(x) & 0 & \mathbb{T}_{14}(x) \\ * & \mathbb{T}_{22}(x) & \mathbb{T}_{23}(x) & \mathbb{T}_{24}(x) \\ * & * & 0 & \mathbb{T}_{34}(x) \\ * & * & * & 0 \end{bmatrix} \right\} < 0 \quad (39)$$

$$\mathbb{T}_{12}(x) = (G_1(x)T(x^+) \bar{B}_i(x) - kU(x))U^{-1}(x) \hat{D}_d(x) \bar{C}_j(x)$$

$$\mathbb{T}_{14}(x) = (G_1(x)T(x^+) \bar{B}_i(x) - kU(x))U^{-1}(x) \hat{C}_d(x)$$

$$\mathbb{T}_{22}(x) = \text{sym}\{(F_1(x)T(x^+) \bar{B}_i(x) - k_1 U(x))U^{-1}(x) \hat{D}_d(x) \bar{C}_j(x)\}$$

$$\mathbb{T}_{23}(x) = \bar{C}_j^T(x) \hat{D}_d^T(x) U^{-T}(x) (G_3(x)T(x^+) \bar{B}_i(x) - \lambda_3 k U(x))^T$$

$$\mathbb{T}_{24}(x) = (F_1(x)T(x^+) \bar{B}_i(x) - k_1 U(x))U^{-1}(x) \hat{C}_d(x)$$

$$\mathbb{T}_{34}(x) = (G_3(x)T(x^+) \bar{B}_i(x) - \lambda_3 k U(x))U^{-1}(x) \hat{C}_d(x)$$

From (39), the condition (26) is obtained. The proof is completed.

**Remark 2.** We can notice that the proposed results contain some slack variables, the introduction of these parameters is not necessary to derive our results, but it can provide more degrees of freedom and more flexibility in the resolution space.

**Theorem 4.** Let  $\alpha$ ,  $\lambda$  and  $k$ , be some given scalars. The polynomial fuzzy system (7) is robustly stable if there exist a symmetric positive definite matrices  $\bar{P}(\hat{x})$ , and  $\bar{Q}(x)$ , symmetric matrices  $P_s(\hat{x})$   $s = 1, 2$  and matrices  $F_l(x)$ ,  $G_l(x)$   $l = 1, 2, 3$ ,  $U(x)$  and  $\hat{A}_d(x)$ ,  $\hat{B}_d(x)$ ,  $\hat{C}_d(x)$ ,  $\hat{D}_d(x)$  and  $N_i(x)$ , and positive scalars  $\epsilon_i$  for  $i = 1, 2, \dots, r$ , given scalars  $\alpha > 0$  such that the following conditions are satisfied

SOS condition (22)

SOS condition (23)

$$-v_6^T(\Xi_{ii}(x) + \epsilon_i(x)I)v_6 \text{ is SOS } i, s = 1, 2, \dots, r \quad (40)$$

$$-v_6^T(\Xi_{ij}(x) + \Xi_{ji}(x) + \epsilon_{ij}(x)I)v_6 \text{ is SOS } i \neq j = 1, 2, \dots, r \quad (41)$$

$$-v_7^T(\tilde{\Xi}_{ii}(x) + \epsilon_i(x)I)v_7 \text{ is SOS } i, s = 1, 2, \dots, r \quad (42)$$

$$-v_7^T(\tilde{\Xi}_{ij}(x) + \tilde{\Xi}_{ji}(x) + \epsilon_{ij}(x)I)v_7 \text{ is SOS } i \neq j = 1, 2, \dots, r \quad (43)$$

where

$$\Xi_{ij}(x) = \begin{bmatrix} Y_{ij}(x) & Y_i^T(x)N_i^T(x) & X_i(x) \\ * & \epsilon_i R_1(x) - N_i(x) - N_i^T(x) & N_i(x)J_i(x) \\ * & * & -\epsilon_i I \end{bmatrix} < 0 \quad (44)$$

$$\tilde{\Xi}_{ij}(x) = \begin{bmatrix} \bar{Y}_{ij}(x) & \bar{Y}_i^T(x)N_i^T(x) & \bar{X}_i(x) \\ * & \epsilon_i R_1(x) - N_i(x) - N_i^T(x) & N_i(x)J_i(x) \\ * & * & -\epsilon_i I \end{bmatrix} < 0 \quad (45)$$

$$Y_{ij}(x) = \begin{bmatrix} \Psi_{11}(x) & Y_{12ij}(x) & \Psi_{13}(x) & \Psi_{14}(x) & Y_{15i}(x) \\ * & Y_{22i}(x) & Y_{23ij}(x) & k_1 \hat{C}_d(x) & Y_{25j}(x) \\ * & * & \Psi_{33}(x) & \Psi_{34}(x) & Y_{35i}(x) \\ * & * & * & -P_2(\hat{x}) & \hat{C}_d^T(x) \\ * & * & * & * & R(x) \end{bmatrix} < 0 \quad (46)$$

$$\begin{aligned}
Y_{12ij}(x) &= G_1(x)T(x^+)A_i(x) + \Sigma_1 Q(x) + (k\hat{D}_d(x) + \lambda\hat{B}_d(x))C_j(x) - F_1^T(x) \\
Y_{15i}(x) &= \alpha(G_1(x)T(x^+)B_i(x) - kU(x)) \\
Y_{22i}(x) &= \Sigma_2 Q(x) - P_1(\hat{x}) + \text{sym}\{F_1(x)T(x^+)A_i(x) + k_1\hat{D}_d(x)C_j(x)\} \\
Y_{23ij}(x) &= A_i^T(x)T(x^+)G_3^T(x) + C_j^T(x)(\lambda_3\hat{D}_d^T k^T + \hat{B}_d^T(x)) \\
Y_{25jj}(x) &= \alpha(F_1(x)T(x^+)B_i(x) - k_1U(x)) + C_j^T(x)\hat{D}_d^T(x) \\
Y_{35i}(x) &= \alpha(G_3(x)T(x^+)B_i(x) - \lambda_3 kU(x))
\end{aligned}$$

$$\bar{Y}_{ij}(x) = \begin{bmatrix} \bar{\Psi}_{11}(x) & \bar{Y}_{12i}(x) & \bar{Y}_{13i}(x) \\ * & -\bar{P}(\hat{x}) & \bar{Y}_{23ij}(x) \\ * & * & R(x) \end{bmatrix} < 0 \quad (47)$$

$$\begin{aligned}
\bar{Y}_{12ij}(x) &= G_1(x)T(x^+)A_i(x) + k\hat{D}_d(x)C_j(x) \\
\bar{Y}_{13i}(x) &= \alpha(G_1(x)T(x^+)B_i(x) - kU(x)) \\
\bar{Y}_{23ij}(x) &= C_j^T(x)\hat{D}_d^T(x)
\end{aligned}$$

**Proof.** The SOS conditions (40)–(41) can be expressed as:

$$\begin{aligned}
\sum_{i=1}^r h_i^2(\zeta(x(k)))\Xi_{ii}(x) + \sum_{i=1}^{r-1} \sum_{j=i+1}^r h_i(\zeta(x(k)))h_j(\zeta(x(k)))(\Xi_{ij}(x) + \Xi_{ij}(x)) \\
= \sum_{i=1}^r \sum_{j=1}^r h_i(\zeta(x(k)))h_j(\zeta(x(k)))(\Xi_{ij}(x) + \Xi_{ij}(x)) < 0
\end{aligned}$$

which is verified if  $\Xi_{ij}(x) < 0$

using Lemma 5, we obtain

$$Y_{ij} + \Delta Y_{ij} = Y_{ij} + \text{sym}\{X_i \Delta Y_i\} < 0 \quad (48)$$

where for  $X_{ci} = 0$ ,

$$X_i = \begin{bmatrix} G_1(x)T(x^+)X_{\Delta i} \\ F_1(x)T(x^+)X_{\Delta i} \\ G_3(x)T(x^+)X_{\Delta i} \\ 0 \\ 0 \end{bmatrix}, \quad Y_i = \begin{bmatrix} 0 & Y_{\Delta i} & 0 & 0 & \alpha Y_{Bi} \end{bmatrix}$$

and for  $Y_{Bi} = 0$ ,

$$X_i = \begin{bmatrix} G_1(x)T(x^+)X_{\Delta i} + (k\hat{D}_d(x) + \lambda\hat{B}_d(x))X_{ci} \\ F_1(x)T(x^+)X_{\Delta i} + k\hat{D}_d(x)X_{ci} \\ G_3(x)T(x^+)X_{\Delta i} + (\lambda_3 k\hat{D}_d(x) + \hat{B}_d(x))X_{ci} \\ 0 \\ \hat{D}_d(x)X_{ci} \end{bmatrix}, \quad Y_i = \begin{bmatrix} 0 & Y_{\Delta i} & 0 & 0 & 0 \end{bmatrix}$$

$$\Delta Y_{ij} = \begin{bmatrix} 0 & \Delta Y_{12i} & 0 & 0 & \Delta Y_{15i} \\ * & \Delta Y_{22i} & \Delta Y_{23i} & 0 & \Delta Y_{25i} \\ * & * & 0 & 0 & \Delta Y_{35i} \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix} < 0 \quad (49)$$

where

$$\begin{aligned}\Delta Y_{12i} &= G_1(x)T(x^+)X_{\Delta i}\Delta(k)Y_{\Delta i} + (k\hat{D}_d(x) + \lambda\hat{B}_d(x))X_{C_i}\Delta(k)Y_{\Delta i} \\ \Delta Y_{15i} &= \alpha(G_1(x)T(x^+)X_{\Delta i}\Delta(k)Y_{B_i}) \\ \Delta Y_{22i} &= \text{sym}\{F_1(x)X_{\Delta i}\Delta(k)Y_{\Delta i} + k_1\hat{D}_d(x)X_{C_i}\Delta(k)Y_{\Delta i}\} \\ \Delta Y_{23i} &= Y_{\Delta i}^T\Delta^T(k)X_{\Delta i}^T T(x^+)^T G_3^T(x) + Y_{\Delta i}^T\Delta^T(k)X_{C_i}^T(\lambda_3\hat{D}_d(x)^T k^T + \hat{B}_d(x)^T) \\ \Delta Y_{25i} &= \alpha(F_1(x)T(x^+)X_{\Delta i}\Delta(k)Y_{B_i}) + Y_{C_j}^T\Delta^T(k)X_{C_j}^T\hat{D}_d^T(x) \\ \Delta Y_{35i} &= \alpha(G_3(x)T(x^+)X_{\Delta i}\Delta(k)Y_{B_i})\end{aligned}$$

Inequality (48) reduces to (35). By the same way, we can show that (47) equivalent to (36).

The proof is completed.

**Corollary 1.** *The polynomial fuzzy system (7) is robustly stable if there exist a symmetric positive definite matrices  $P_s(\hat{x})$   $s = 1, 2$ , matrices  $F_l(x)$ ,  $G_l(x)$   $l = 1, 2, 3$ ,  $U(x)$  and  $N_i(x)$  for  $i = 1, 2, \dots, r$  and positive scalars  $\epsilon_i$  such that the following conditions are satisfied*

*SOS condition (22)*

*SOS condition (23)*

$$-v_6^T(A_{ii}(x) + \epsilon_i(x)I)v_6 \text{ is SOS } i, s = 1, 2, \dots, r \quad (50)$$

$$-v_6^T(A_{ij}(x) + \Lambda_{ji}(x) + \epsilon_{ij}(x)I)v_6 \text{ is SOS } i \neq j = 1, 2, \dots, r \quad (51)$$

$$A_{ij}(x) = \begin{bmatrix} \bar{A}_{ij}(x) & Y_i^T(x)N_i^T(x) & X_i(x) \\ * & \epsilon_i R_1(x) - N_i(x) - N_i^T(x) & N_i(x)J_i(x) \\ * & * & -\epsilon_i I \end{bmatrix} < 0 \quad (52)$$

$$\bar{A}_{ij}(x) = \begin{bmatrix} \Psi_{11}(x) & \bar{A}_{12ij}(x) & \Psi_{13}(x) & \Psi_{14}(x) & Y_{15i}(x) \\ * & \bar{A}_{22i} & Y_{23ij} & k_1\hat{C}_d & Y_{25j} \\ * & * & \Psi_{33}(x) & \Psi_{34}(x) & Y_{35i}(x) \\ * & * & * & -P_2(\hat{x}) & \hat{C}_d^T(x) \\ * & * & * & * & R(x) \end{bmatrix} < 0 \quad (53)$$

$$\bar{A}_{12ij}(x) = G_1(x)T(x^+)A_i(x) + (k\hat{D}_d(x) + \lambda\hat{B}_d(x))C_j(x) - F_1^T(x)$$

$$\bar{A}_{22i}(x) = -P_1(\hat{x}) + \text{sym}\{F_1(x)T(x^+)A_i(x) + k_1\hat{D}_d(x)C_j(x)\}$$

**Proof.** Corollary 1 follows directly from conditions (40)–(41) in Theorem 4 by letting  $Q(x) = 0$  and  $P_s(\hat{x}) > 0$ .

**Remark 3.** In [9], the robust DOF control of discrete-time was investigated, for nonlinear fuzzy systems with parametric uncertainties via LMIs, however these conditions are only in the EF. On the other hand, in [14] authors have developed some results to design an  $H_\infty$  DOF control for discrete polynomial fuzzy model. In comparison with the above-mentioned design methods, this study proposes a robust DOF control for polynomial fuzzy systems. Based on a GKYP and some lemmas, a novel control law is derived by using the SOS approach. In addition, the introduction of slack variables provides more flexibility and more degrees of freedom in the resolution space. This can help in reducing the conservatism. As discussed in [16,18,26], the FF conditions are more general and contain the entire frequency conditions as special cases, which means that our work covers the solutions set of [9,14] while ensuring less conservativeness.

**Remark 4.** Pre- and post multiplying (52) by

$$H = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} < 0 \quad (54)$$

and  $H^T$ , respectively, and letting  $F_1(x) = 0$  and  $P_2(\hat{x}) = P_3$  then Corollary 1 reduces to Theorem 1 in [9] in the case  $P_2 = 0$ . That is, Theorem 1 in [9] is a special case of Corollary 1 in this paper. Thus, when compared to existing results in the literature, our technique of polynomial fuzzy models in FF can give less conservative results.

**Table 2**  
 $\beta$ 's values, Example 1.

Methods	$\beta$ for Case A $\lambda = 0.08$	$\beta$ for Case B $\lambda = 0.28$
Theorem 1 in [9]	0.49	0.199
Corollary 1 for $2d = 0$	0.48	0.185
Theorem 4 MF $\frac{\pi}{6} \leq  \theta  \leq \frac{\pi}{3}$ $2d = 0$	0.81	0.67
Theorem 4 LF $ \theta  \leq \frac{\pi}{6}$ $2d = 0$	0.62	0.56
Theorem 4 HF $ \theta  \geq \frac{\pi}{3}$ $2d = 0$	0.90	0.72

#### 4. Simulation examples

In this section, we provide two numerical examples in order to highlight the effectiveness and the advantages of the proposed results. The first one is used here to compare the proposed results in terms of conservatism when the system is not polynomial. The second example shows that higher order polynomial Lyapunov functions achieve more relaxed stability results. From the simulation results, it can be seen that the proposed approach has clear advantages over LMI-based ones.

**Example 1.** To demonstrate the validity of the studied method, let us consider the following uncertain fuzzy plant model, which is represented by a two-rule [9]

$$A_1 = \begin{bmatrix} 1 & -\beta \\ -1 & -0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & \beta \\ -1 & -0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 5 + \beta \\ 2\beta \end{bmatrix}, \quad B_2 = \begin{bmatrix} 5 - \beta \\ -2\beta \end{bmatrix},$$

$$X_{d1} = G_{d2} = \begin{bmatrix} 0.4 \\ -0.4 \end{bmatrix}, \quad C_1 = C_2 = [0.5 \quad 0.2], \quad Y_{d1} = [-0.2 \quad \beta],$$

$$Y_{d2} = [-0.2 \quad -\beta], \quad J_1 = J_2 = 1.2$$

where the membership functions are taken as the same as those used in [9]

$$h_1(\zeta(x(k))) = \frac{x_1(k) + \beta}{2\beta}, \quad h_2(\zeta(x(k))) = 1 - h_1(\zeta(x(k)))$$

Let us consider the following two case as taken from [9]:

case A:  $X_{c1} = X_{c2} = 0$  and  $Y_{B1} = Y_{B2} = 0.3$

case B:  $Y_{B1} = Y_{B2} = 0$  and  $X_{c1} = X_{c2} = -0.4$

Choose the following control parameters  $\alpha = 2$ . Then, using the proposed methods for the degree ( $2d = 0$ ) of the polynomial Lyapunov function, the maximum  $\beta_{max}$  which guarantee the robust stability of DOF T-S fuzzy system are listed in Table 2.

Table 2 shows the values of  $\beta_{max}$  obtained with the EF approaches existing in [9], that designed by Corollary 1 in this paper, and the FF approach presented for different frequency ranges, where LF, MF and HF denote low-frequency, middle-frequency and high-frequency ranges, respectively. We can see that Theorem 4 with ( $2d = 0$ ) provides larger value of the parameter  $\beta_{max}$  than the results in [9]. And it is also easy to see that the FF controller synthesis results achieve better performances than the full frequency ones.

Solving the SOS conditions in Theorem 4, with  $\beta = 0.8$   $\lambda = 0.08$  and  $\alpha = 1$ , we obtain the following DOF gain matrices:

$$A_d = \begin{bmatrix} -0.6830 & 0.3151 \\ -0.1613 & 0.0663 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0.0748 \\ 0.1663 \end{bmatrix},$$

$$C_d = [-0.8520 \quad -0.1626], \quad D_d = -0.4619.$$

Fig. 1 shows the state responses of the close-loop fuzzy system (7) from the initial condition  $x(0) = [-0.4 \quad 1]^T$  and  $x_d(0) = [0 \quad 0]^T$ . According to the simulation results, we can see that the designed robust DOF controller presented in this paper is effective.

**Remark 5.** Note that with the T-S polynomial fuzzy system, as the number of variables and/or the degree of the polynomial increases, the conservatism of the result decreases, but the computational complexity increases. Furthermore, the SOS approach has a significant benefit in this work because it is a generalization of existing T-S fuzzy system approaches and is more efficient for describing nonlinear control systems.

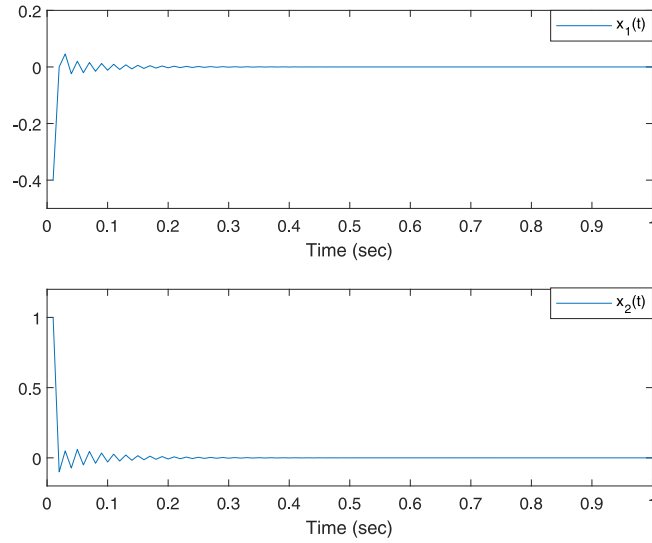


Fig. 1. System response with the polynomial DOF.

**Example 2.** Consider a two-rules polynomial fuzzy model [27] in the form of (7). The corresponding parameter matrices are given as follows:

$$A_1 = \begin{bmatrix} 0.5 & 0 \\ -x_2^2(k) & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 0 \\ -x_2^2(k) & -0.2172 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

The membership functions are formulated as [27]

$$h_1(k) = \frac{\sin(x_2(k)) + 0.2172x_2(k)}{1.2172x_2(k)} \quad (55)$$

$$h_2(k) = 1 - h_1(k)$$

Chosen  $\lambda = 0.1$ ,  $\alpha = 0.2$ , solving the SOS conditions in Theorem 4 in the case  $\Delta(k) = 0$ , within FF domain  $\frac{\pi}{6} \leq |\theta| \leq \frac{\pi}{3}$  the corresponding control gain matrices are as follows

$$A_d = \begin{bmatrix} -4.573 \times 10^{-5}x_2^2(k) & 5.46 \times 10^{-6}x_2^2(k) \\ -3.525x_2^2(k) & 0.421x_2^2(k) \end{bmatrix}, \quad B_d = \begin{bmatrix} 1.879 \times 10^{-5}x_2^2(k) \\ 1.566x_2^2(k) \end{bmatrix},$$

$$C_d = \begin{bmatrix} -0.2556x_2^2(k) & -0.09437x_2^2(k) \end{bmatrix}, \quad D_d = \begin{bmatrix} -1.837x_2^2(k) \end{bmatrix},$$

The proposed polynomial DOF controller (6) is employed to control the polynomial fuzzy system subject to the initial condition. Fig. 2 shows the transient response of state variables for the polynomial fuzzy system without parametric uncertainties, where the initial values of the state variables are  $x(0) = [0.5 \ 0.5]^T$  and  $x_d(0) = [0 \ 0]^T$ . We can see that the system is asymptotically stable.

## 5. Conclusion

The problem of DOF controller design for uncertain polynomial T-S fuzzy systems in the FF field (in discrete-time setting) has been considered in this paper. Based on the Polynomial Lyapunov function the robust DOF control is established such that the closed-loop polynomial T-S system is robustly stable. Specifically, employing the GKYP lemma, less conservative results are obtained in terms of SOS. Some examples are provided to show the validity and efficacy of the suggested method. Our future study will include the application of the theoretical results obtained in this work to some real applications, such as vehicle active suspension systems.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

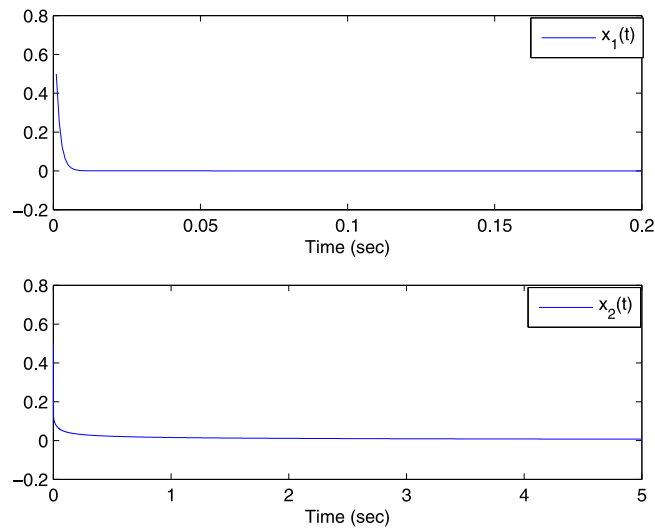


Fig. 2. Time response of the closed-loop system states.

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