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On the preferred extensions of argumentation frameworks: bijections with naive extensions

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Abstract

This paper deals with the problem of finding the preferred extensions of an argumentation framework by means of a bijection with the naive extensions of another framework. First we consider the case where an argumentation framework is naive-realizable: its naive and preferred extensions are equal. Recognizing naive-realizable argumentation frameworks is hard, but we show that it is tractable for frameworks with bounded in-degree. Next, we give a bijection between the preferred extensions of an argumentation framework being admissible-closed (the intersection of two admissible sets is admissible) and the naive extensions of another framework on the same set of arguments. On the other hand, we prove that identifying admissible-closed argumentation frameworks is coNP-complete. At last, we introduce the notion of irreducible self-defending sets as those that are not the union of others. It turns out there exists a bijection between the preferred extensions of an argumentation framework and the naive extensions of a framework on its irreducible self-defending sets. Consequently, the preferred extensions of argumentation frameworks with some lattice properties can be listed with polynomial delay and polynomial space.

Keywords: argumentation framework, preferred extensions, naive extensions, recasting, irreducible self-defending sets.

1 Introduction

For the last few years, argumentation frameworks have become a major research trend within the field of artificial intelligence [4, 27, 28]. In his pioneering work [10], Dung defines an argumentation framework as a set of arguments connected together by a binary attack relation. To estimate the relevance (or the acceptability) of a group of arguments, he also introduces semantics and extensions. More precisely, a semantic is a collection of criteria and the extensions of a semantic are the sets of arguments meeting its requirements.

Extensions and semantics are at the core of numerous decision and search problems revolving around argumentation frameworks. Unfortunately, most of these problems are hard to solve for several argumentation semantics [15, 18, 23]. Nevertheless, these hardness results have motivated the search for tractable classes of argumentation frameworks among

which we quote the frameworks being without even cycles [12], symmetric [7] or bipartite [11].

In this context, our paper draws inspiration from the work of Dunne *et al.* [13] on realizability and recasting problems for argumentation frameworks. More precisely, we seek to express the set of preferred extensions of a given framework using a bijective correspondence with the naive extensions of another argumentation framework. For example, symmetric argumentation frameworks admit a bijection between their naive and preferred extensions [7]. The benefit of our approach is to obtain polynomial delay and polynomial space algorithms enumerating the preferred extensions of an argumentation framework. On this purpose, we obtain the following contributions:

1. Naive-realizable argumentation frameworks (Section 3). An argumentation framework is *naive-realizable* if its naive extensions are equal to its preferred extensions. For instance, symmetric argumentation frameworks are naive-realizable [7]. We give an exact characterization of naive-realizable argumentation frameworks. We deduce an algorithm to recognize this class of argumentation frameworks, running in polynomial time in frameworks with bounded in-degree. In contrast, we prove that this recognition problem is coNP-complete in general.

2. Naive-recasting and admissible-closed argumentation frameworks (Section 4). An argumentation framework is *naive-recasting* if its preferred extensions are the naive extensions of another argumentation framework. This class of argumentation framework has been introduced by Dunne *et al.* [13], and studied in a series of works [2, 14, 21, 26]. In particular, the authors in [13] show that deciding whether a given argumentation framework is naive-recasting is coNP-hard. In our contribution, we introduce *admissible-closed* argumentation frameworks. An argumentation framework belongs to this class if its admissible sets are closed under intersection. Using implicational systems [17, 5], we show that admissible-closed argumentation frameworks are naive-recasting. Eventually, we prove that recognizing admissible-closed argumentation frameworks is a coNP-complete problem.

3. Irreducible self-defending sets (Section 5). Using the lattice structure of self-defending sets [8, 10] we introduce irreducible self-defending sets. A self-defending set is irreducible if it cannot be obtained as the union of other self-defending sets. We construct a bijection between the preferred extensions of an argumentation framework and the naive extensions of another framework on its irreducible self-defending sets. We deduce that when an argumentation framework meets some lattice properties, it is possible to enumerate its preferred extensions with polynomial delay and polynomial space.

2 Background

All the objects considered in this paper are finite. Let \mathcal{A} be a set. If S is a subset of \mathcal{A} , we write \bar{S} The complement of S in \mathcal{A} , that is $\bar{S} = \mathcal{A} \setminus S$. The complementary set of a set family \mathcal{F} over \mathcal{A} is the set system defined $\bar{\mathcal{F}}$ by $\bar{\mathcal{F}} = \{\bar{S} \mid S \in \mathcal{F}\}$.

Argumentation frameworks We follow [10]. An *argumentation framework* is a pair $F = \langle \mathcal{A}, \mathcal{R} \rangle$ where \mathcal{A} is a set of arguments, and \mathcal{R} (attacks) is a binary relation on \mathcal{A} , i.e. $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$. A pair (x, y) in \mathcal{R} depicts an attack of the argument x against the argument y . We can represent an argumentation framework F by a directed graph $G = (\mathcal{A}, \mathcal{R})$, known as the *attack graph* of F . Given a subset S of arguments, we denote by S^+ the set of arguments attacked by S , i.e. $S^+ = \{x \in \mathcal{A} \mid \exists y \in S, (y, x) \in \mathcal{R}\}$. Likewise,

the set of arguments that attack at least one argument of S is denoted by S^- , that is $S^- = \{x \in \mathcal{A} \mid \exists y \in S, (x, y) \in \mathcal{R}\}$. Finally, we put $\Gamma(S) = S^+ \cup S^-$.

All extensions in argumentation frameworks are based on two important concepts: being *acceptable* or *conflict-free*. Let $S \subseteq \mathcal{A}$. An argument $x \in \mathcal{A}$ is *acceptable* with respect to S if for each $y \in \mathcal{A}$, $(y, x) \in \mathcal{R}$ implies that $(z, y) \in \mathcal{R}$ for some z in S . We say that S is *conflict-free* (an *independent set* in graph theory's terminology) if $S \cap S^+ = \emptyset$. We denote by $\text{CF}(F)$ the family of all conflict-free sets of F . We call $\mathcal{S}(F)$ the set of all self-attacking arguments, that is $\mathcal{S}(F) = \{x \in \mathcal{A} \mid (x, x) \in \mathcal{R}\}$. The inclusion-wise maximal conflict-free subsets of \mathcal{A} are called *naive extensions* in [6] (*maximal independent set* in graph theoretic terms). The family of all naive extensions of F is denoted by $\text{NAIV}(F)$. A set $S \subseteq \mathcal{A}$ is *self-defending* if $S^- \subseteq S^+$. The family of all self-defending sets of F is denoted by $\text{SD}(F)$. Using conflict-free and acceptable sets, Dung [10] defines several extensions. A conflict-free set $S \subseteq \mathcal{A}$ is *admissible* if $S^- \subseteq S^+$. The family of all admissible sets of F is denoted by $\text{ADM}(F)$. The inclusion-wise maximal admissible subsets of \mathcal{A} are called *preferred extensions*, and the family of all preferred extensions of F is denoted by $\text{PREF}(F)$. We refer the reader to Baroni *et al.* [1] for more discussion on semantics.

Closure systems and implicational systems A closure system over a set \mathcal{A} is a family of subsets of \mathcal{A} closed under intersection and containing \mathcal{A} . When ordered by inclusion, a closure system (or its complement) forms a *lattice*. An *implication* over \mathcal{A} is an expression of the form $X \rightarrow y$ where $X \subseteq \mathcal{A}$ and $y \in \mathcal{A}$. In $X \rightarrow y$, X is the *premise* and y the *conclusion* of the implication. An *implicational system* Σ over \mathcal{A} is a set of implications over \mathcal{A} . For more details on implicational systems, see [5, 29]. Let $\Sigma = \{X_1 \rightarrow x_1, \dots, X_n \rightarrow x_n\}$ be an implicational system on a set \mathcal{A} and let $S \subseteq \mathcal{A}$. The Σ -closure of S , denoted S^Σ , is the inclusion-wise minimal set containing S and satisfying:

$$X_j \subseteq S^\Sigma \implies x_j \in S^\Sigma \quad \text{for every } 1 \leq j \leq n$$

The family $\mathcal{F}_\Sigma = \{S^\Sigma \mid S \subseteq \mathcal{A}\}$ is a closure system. The sets in \mathcal{F}_Σ are the closed sets of Σ . In [17], the authors show that the self-defending sets of an argumentation frameworks coincide with the closed sets of an implicational system.

Theorem 2.1. [17] *Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework, and let $\Sigma = \{y^- \rightarrow z \mid (y, z) \in \mathcal{R} \text{ and } (z, y) \notin \mathcal{R}\}$ be its associated implicational system. Then $\bar{\mathcal{F}}_\Sigma = \text{SD}(F)$.*

Enumeration complexity We use the definitions of [20]. Let \mathbf{A} be an algorithm with input x and output a set of solutions $R(x)$. We denote by $|R(x)|$ the number of solutions in $R(x)$. We assume that each solution in $R(x)$ has size $\text{poly}(|x|)$. The algorithm \mathbf{A} is running in *output-polynomial* time if its execution time is bounded by $\text{poly}(|x| + |R(x)|)$. If the delay between two solutions output and after the last one is $\text{poly}(|x|)$, \mathbf{A} has *polynomial-delay*.

3 Naive-realizable argumentation frameworks

In this section, we show that checking whether the family of preferred extensions of a given argumentation framework is equal to the family of its naive extensions is a coNP -complete problem. In contrast, we give a polynomial time algorithm to analyze the bounded in-degree argumentation frameworks.

Definition 1. Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework. We say that F is naive-realizable if $\text{NAIV}(F) = \text{PREF}(F)$.

Lemma 1. Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework and let Σ be its associated implicational system, obtained with Theorem 2.1. If $\emptyset^\Sigma \not\subseteq \mathcal{S}(F)$, then F is not naive-realizable.

Proof. The result follows from the fact that an argument $x \in \emptyset^\Sigma \setminus \mathcal{S}(F)$ belongs to at least one conflict-free set in F and not to any admissible set in F . \square

An argument x in \emptyset^Σ cannot appear in a self-defending set. In particular, it is not contained in any admissible set. Thus, adding the self-attack (x, x) to \mathcal{R} does not change $\text{ADM}(F)$. Therefore, an argumentation framework where $\emptyset^\Sigma \not\subseteq \mathcal{S}$ can be transformed in polynomial time to an argumentation framework satisfying $\emptyset^\Sigma \subseteq \mathcal{S}$ without changing the admissible and preferred extensions. This is done by adding a self-attack to each argument x in \emptyset^Σ . The next theorem characterizes naive-realizable argumentation frameworks.

Theorem 3.1. Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework and let Σ be its associated implicational system, obtained with Theorem 2.1. Suppose that $\emptyset^\Sigma \subseteq \mathcal{S}(F)$. Then F is naive-realizable if and only if for each $x, y \in \mathcal{A}$ such that $(y, x) \in \mathcal{R}$ and $(x, y) \notin \mathcal{R}$, there does not exist a subset $S \subseteq \Gamma(y^-)$ satisfying $S \cup \{x\} \in \text{CF}(F)$ and $S \cup \{z\} \notin \text{CF}(F)$ for every $z \in y^-$.

Proof. We begin with the only if part. We use contrapositive. Let $x, y \in \mathcal{A}$ such that $(x, y) \in \mathcal{R}$ and $(y, x) \notin \mathcal{R}$ and suppose that there exists $S \subseteq \Gamma(y^-)$ such that $S \cup \{x\} \in \text{CF}(F)$ and for all $z \in y^-$, $S \cup \{z\} \notin \text{CF}(F)$. Then, there exists a naive extension S' such that $S \cup \{x\} \subseteq S'$ and $S' \cap y^- = \emptyset$. It follows that S' is not self-defending and hence that it is not preferred either.

Now we prove the if part. Suppose that for all $x, y \in \mathcal{A}$ such that $(x, y) \in \mathcal{R}$ and $(y, x) \notin \mathcal{R}$, there does not exist $S \subseteq \Gamma(y^-)$ such that $S \cup \{x\} \in \text{CF}(F)$ and $S \cup \{z\} \notin \text{CF}(F)$ for every $z \in y^-$. If S is a conflict-free set of F such that $x \in S$, there must exist $z \in y^-$ such that $S \cup \{z\}$ is a conflict-free set of F . However, a naive extension is an inclusion-wise maximal conflict-free set. Thus, if S' is a naive extension of F and $x \in S'$, $S' \cap y^- \neq \emptyset$ must hold. Therefore, every naive extension of F is also self-defending. We deduce that the family of naive extensions of F equals the family of its preferred extensions, concluding the proof. \square

Now we consider the problem of recognizing naive-realizable argumentation frameworks. Formally, The problem reads as follows:

Naive-realizable argumentation framework recognition (NAF)

Input: An argumentation framework $F = \langle \mathcal{A}, \mathcal{R} \rangle$.

Question: Is F naive-realizable?

On the basis of Theorem 3.1, we deduce the following strategy to solve NAF: for each $x, y \in \mathcal{A}$ such that $(y, x) \in \mathcal{R}$ and $(x, y) \notin \mathcal{R}$, check whether there exists a conflict-free set $S \subseteq \Gamma(y^-)$ such that $S \cup \{x\}$ is a conflict-free set of F and for all $z \in y^-$, $S \cup \{z\}$ is not a conflict-free set of F . Return "No" if such a S exists and "Yes" otherwise. Quite clearly, this algorithm has exponential-time complexity. Yet, it runs in polynomial time if the in-degree of F is bounded by a constant.

Proposition 1. Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework. If the in-degree of F is bounded by a constant, there exists a polynomial time algorithm (in the size of F) to check whether F is naive-realizable.

Proof. By assumption, the sizes of y^- and $\Gamma(y^-)$ are bounded by a constant. Thus, for each $x \in \mathcal{A}$, the condition of Theorem 3.1 can be verified by testing a constant number of subsets of \mathcal{A} . Since \mathcal{A} is given in the input, the whole procedure runs in polynomial time as required. \square

For the general case, we settle the hardness of NAF in the subsequent theorem.

Theorem 3.2. *The problem NAF is coNP-complete.*

Proof. The proof is a reduction from 3-SAT. Let $C = \bigwedge_{i=1}^m C_i$ be a CNF formula, where each clause C_i is a disjunction of three literals from $\{x_1, \dots, x_n, \neg x_1, \dots, \neg x_n\}$. Let F be the argumentation framework $\langle \mathcal{A}, \mathcal{R} \rangle$ where (see also Figure 1):

$$\begin{aligned} \mathcal{A} = & \{x, y\} \cup \{C_i \mid 1 \leq i \leq m\} \\ & \cup \{x_j, \neg x_j \mid 1 \leq j \leq n\}; \\ \mathcal{R} = & \{(y, x)\} \cup \{(C_i, y), (y, C_i) \mid 1 \leq i \leq m\} \\ & \cup \{(x_j, C_i), (C_i, x_j) \mid x_j \text{ occurs in } C_i\} \\ & \cup \{(\neg x_j, C_i), (C_i, \neg x_j) \mid \neg x_j \text{ occurs in } C_i\} \\ & \cup \{(x_j, \neg x_j), (\neg x_j, x_j) \mid 1 \leq j \leq n\}. \end{aligned}$$

The construction of F is done in polynomial time in the size of C . We show that F has a non-admissible naive extension if and only if C is satisfiable. Let S be a naive extension of F . We have two cases. First, $y \in S$. Then, S is a conflict-free extension of the subargumentation framework $F|_x = \langle \mathcal{A} \setminus \{x\}, \mathcal{R} \setminus \{(y, x)\} \rangle$. Since $F|_x$ is symmetric and $x^+ = \emptyset$, each subset of $\mathcal{A} \setminus \{x\}$ is self-defending in F (see [7]). In particular, every conflict-free set of $F|_x$ is admissible in F . Therefore, S must be admissible in F . The second case is, $y \notin S$. Then, $x \in S$ and S is not admissible in F if and only if $S \cap \{C_1, \dots, C_m\} = \emptyset$. By construction of F , it follows that S is not admissible if and only if for each C_i , $1 \leq i \leq m$, there exists some $1 \leq j \leq n$ such that $\ell_j \in C_i$ and $\ell_j \in S$ with $\ell_j \in \{x_j, \neg x_j\}$. As $(x_j, \neg x_j) \in \mathcal{R}$ for each $1 \leq j \leq n$ and S is conflict-free, $\{x_j, \neg x_j\} \subseteq S$ cannot hold. We deduce that S is not admissible if and only if C is satisfiable, concluding the proof. \square

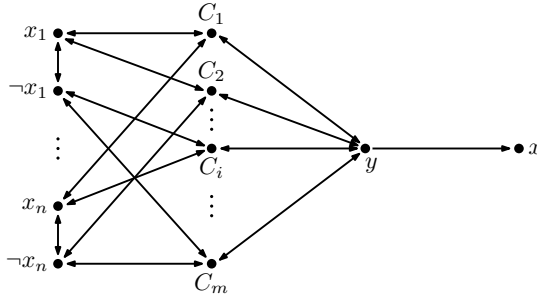


Figure 1: Attack graph of F .

4 Naive-recasting and admissible-closed argumentation frameworks

In this section, we study the class of argumentation frameworks whose preferred extensions are the naive extensions of another argumentation framework. This class of argumentation frameworks is called naive-recasting.

Definition 2. Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework. We say that F is naive-recasting if there exists an argumentation framework $F' = \langle \mathcal{A}', \mathcal{R}' \rangle$ such that $\text{PREF}(F) = \text{NAIV}(F')$.

Dunne *et al.* [13] show that it is coNP-hard to recognize naive-recasting argumentation frameworks. This problem is defined as follows:

Naive-recasting argumentation framework recognition (NRAF)

Input: An argumentation framework $F = \langle \mathcal{A}, \mathcal{R} \rangle$.

Question: Is F naive-recasting?

Theorem 4.1. [13] *The problem NRAF is coNP-hard.*

Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework. We prove that F is naive-recasting in the restricted case where $\text{ADM}(F)$ is closed under intersection. First, we formally define this class of argumentation frameworks.

Definition 3. Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework. We say that F is admissible-closed if its admissible sets are closed under intersection.

Lemma 2. Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an admissible-closed argumentation framework. Then, there exists a family of self-defending sets $\text{SD}'(F) \subseteq \text{SD}(F)$ closed under union and intersection such that $\text{SD}'(F) \cap \text{CF}(F) = \text{ADM}(F)$.

Proof. Let $\text{SD}'(F)$ be the set system obtained from $\text{ADM}(F)$ after closing under union, that is $\text{SD}'(F) = \{\bigcup \mathcal{I} \mid \mathcal{I} \subseteq \text{ADM}(F)\}$. Since self-defending sets are closed under union [10] and $\text{ADM}(F) \subseteq \text{SD}(F)$, $\text{SD}'(F) \subseteq \text{SD}(F)$ readily holds. Then, the fact that $\text{SD}'(F) \cap \text{CF}(F) = \text{ADM}(F)$ follows from $\text{ADM}(F) \subseteq \text{SD}'(F) \subseteq \text{SD}(F)$ and $\text{SD}(F) \cap \text{CF}(F) = \text{ADM}(F)$.

It remains to prove that $\text{SD}'(F)$ is closed under intersection. Let $S_1, S_2 \in \text{SD}'(F)$. We show that $S_1 \cap S_2 \in \text{SD}'(F)$. By construction of $\text{SD}'(F)$, $S_1 = \bigcup_{i=1}^k A_i$ and $S_2 = \bigcup_{j=1}^\ell B_j$ with $A_i, B_j \in \text{ADM}(F)$ for every $1 \leq i \leq k$ and every $1 \leq j \leq \ell$. Therefore, we have:

$$S_1 \cap S_2 = \left(\bigcup_{i=1}^k A_i \right) \cap \left(\bigcup_{j=1}^\ell B_j \right) = \bigcup_{i=1}^k \bigcup_{j=1}^\ell (A_i \cap B_j)$$

However, $\text{ADM}(F)$ is closed under intersection by assumption. Consequently, $A_i \cap B_j \in \text{ADM}(F)$ for every $1 \leq i \leq k$ and every $1 \leq j \leq \ell$. We deduce that $S_1 \cap S_2 \in \text{SD}'(F)$ as required. \square

Theorem 4.2. *Every admissible-closed argumentation framework is naive-recasting.*

Proof. Consider an admissible-closed argumentation framework $F = \langle \mathcal{A}, \mathcal{R} \rangle$. According to Lemma 2, there is a collection of self-defending sets $\text{SD}'(F) \subseteq \text{SD}(F)$ which completely includes $\text{ADM}(F)$. Since $\text{SD}'(F)$ is closed under union and intersection, it is possible to represent the closure system $\langle \text{SD}'(F), \subseteq \rangle$ by an implicational system Σ_d with singleton premises, as shown in [29]. Let S^* be the inclusion-wise maximal set of $\text{SD}'(F)$. Consider now the argumentation framework $F' = \langle \mathcal{A}, \mathcal{R}' \rangle$ constructed from F , Σ_d and S^* , by extending \mathcal{R} to \mathcal{R}' with the following rules:

Rule 1: if $x \in \mathcal{A} \setminus S^*$, then add (x, x) to \mathcal{R}' .

Rule 2: if $x \rightarrow z \in \Sigma_d$ and $y \in \Gamma(z)$, then add (x, y) and (y, x) to \mathcal{R}' .

To show that F is naive-recasting, it is sufficient to prove $\text{NAIV}(F') \subseteq \text{ADM}(F) \subseteq \text{CF}(F')$.

We begin with the first inclusion. Let $S \in \text{NAIV}(F')$. Since $\mathcal{R} \subseteq \mathcal{R}'$, $S \in \text{CF}(F)$ is clear. By **Rule 1**, $S \subseteq S^*$. We show that S is closed for Σ_d and hence that S is self-defending in F , in virtue of Lemma 2. Let $x \rightarrow z$ be an implication of Σ_d such that $x \in S$. Since S is conflict-free in F' , **Rule 2** implies that $S \cap \Gamma(z) = \emptyset$. Thus, $S \cup \{z\}$ is also conflict-free in F' . As S is an inclusion-wise maximal conflict-free set of F' , we deduce that $z \in S$. Therefore, S is a closed set of Σ_d so that $S \in \text{SD}(F)$ holds. Consequently, we have $S \in \text{ADM}(F)$ as required.

We move to the second inclusion. Let $S \in \text{ADM}(F)$. We show that $S \in \text{CF}(F')$. By assumption on F , $S \subseteq S^*$ and S is a closed set of Σ_d . Suppose for contradiction that S is not conflict-free in F' . Since S is conflict-free in F and $S \subseteq S^*$, it must be that S contains a pair (x, y) of \mathcal{R}' which has been added by **Rule 2**. Consequently, there exists an implication $x \rightarrow z$ in Σ_d such that $x \in S$ and $y \in \Gamma(z)$. However, S is a closed set of Σ_d by assumption on F and by construction of Σ_d . Therefore, $x \in S$ implies $z \in S$ so that $\{y, z\} \subseteq S$ also holds. As $y \in \Gamma(z)$ in F , this contradicts $S \in \text{CF}(F)$. We deduce that $S \in \text{CF}(F')$, which concludes the proof. \square

In the following result, we show that it is **coNP**-complete to recognize admissible-closed argumentation frameworks.

Admissible-closed argumentation framework recognition (ACAF)

Input: An argumentation framework $F = \langle \mathcal{A}, \mathcal{R} \rangle$.

Question: Is F admissible-closed?

Theorem 4.3. *The problem ACAF is coNP-complete.*

Proof. The proof is a reduction from 3-SAT. Let $C = \bigwedge_{i=1}^m C_i$ be a CNF formula, where each clause C_i is a disjunction of three literals from $\{x_1, \dots, x_n, \neg x_1, \dots, \neg x_n\}$. Let F be the argumentation framework $\langle \mathcal{A}, \mathcal{R} \rangle$ where (see also Figure 2):

$$\begin{aligned} \mathcal{A} &= \{x, y, z_1, z_2\} \cup \{C_i \mid 1 \leq i \leq m\} \\ &\quad \cup \{x_j, \neg x_j \mid 1 \leq j \leq n\} \\ \mathcal{R} &= \{(x_j, C_i), (C_i, x_j) \mid x_j \text{ occurs in } C_i\} \\ &\quad \cup \{(\neg x_j, C_i), (C_i, \neg x_j) \mid \neg x_j \text{ occurs in } C_i\} \\ &\quad \cup \{(z_1, y), (z_2, y), (z_1, z_2), (z_2, z_1), (y, x)\} \\ &\quad \cup \{(C_i, z_1) \mid 1 \leq i \leq m\} \\ &\quad \cup \{(x_j, \neg x_j), (\neg x_j, x_j) \mid 1 \leq j \leq n\} \end{aligned}$$

The construction of F can be done in polynomial time in the size of C . Using the same reasoning as in Theorem 3.2, we can show that F has an admissible set S with $\{x, z_1\} \subseteq S$ if and only if C is satisfiable. It remains to show that F is admissible-closed if and only if such a S does not exist.

We prove the only if part using contrapositive. Assume that there exists an admissible set S such that $\{z_1, x\} \subseteq S$. Since $(z_1, z_2) \in \mathcal{R}$ and S is admissible, $z_2 \notin S$. As $\{x, z_2\}$ is also admissible, we have that $S \cap \{x, z_2\}$ is not self-defending as $x \in S \cap \{x, z_2\}$ and $\Gamma(y) \cap S = \emptyset$. Hence, F is not admissible-closed.

We move to the if part. Suppose that S does not exist. By assumption, an admissible set which contains x also contains z_2 . Moreover, $y^- = \{z_1, z_2\}$ and $(z_1, z_2) \in \mathcal{R}$ so that no admissible set in F contains y . Thus, the family of admissible sets of F is not closed under

intersection if and only if there exists two admissible sets $S_1, S_2 \subseteq \mathcal{B} = \mathcal{A} \setminus \{x, y, z_1, z_2\}$ such that their intersection $S_1 \cap S_2$ is not an admissible set of F . However, there is no attack from $\{x, y, z_1, z_2\}$ to \mathcal{B} and the sub-argumentation framework $F' = \langle \mathcal{B}, \mathcal{R} \cap (\mathcal{B} \times \mathcal{B}) \rangle$ is symmetric (see, [7]). Therefore, each conflict-free set of F' is admissible in F . As the intersection of two conflict-free sets is conflict-free, we deduce that for every admissible sets $S_1, S_2 \subseteq \mathcal{B}$, $S_1 \cap S_2$ is also admissible. As a consequence, F is admissible-closed, concluding the proof. \square

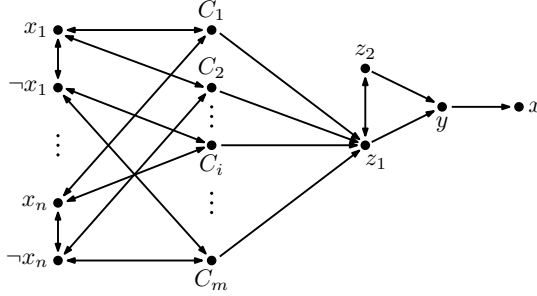


Figure 2: Attack graph of F .

To conclude this section, let F be an admissible-closed argumentation framework. According to Theorem 4.2, we can build an argumentation framework F' (on the same set of arguments) such that $\text{NAIV}(F') = \text{PREF}(F)$. Thus, we can compute $\text{PREF}(F)$ with polynomial delay from F' using the algorithm of Johnson *et al.* [20]. However, whether we can find F' in polynomial time in the size of F is open.

Open problem Let F be an admissible-closed argumentation framework and let F' be an argumentation framework such that $\text{NAIV}(F') = \text{PREF}(F)$. Is it possible to find F' in polynomial time from F ?

5 Irreducible self-defending sets and preferred extensions

In this section we introduce the irreducible self-defending sets of an argumentation framework. They are those self-defending sets that cannot be obtained as the union of other self-defending sets. In lattice theoretic terms [8], they are the join-irreducible elements of the lattice of self-defending sets [10]. Using these irreducible elements, we construct a new argumentation framework whose naive extensions are in a one-to-one correspondence with the preferred extensions of the input framework. We deduce an algorithm to list the preferred extensions of an argumentation framework using its irreducible self-defending sets. This algorithm has polynomial-delay and polynomial space when the irreducible sets are given. First, we formally define irreducible self-defending sets.

Definition 4. Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework. A self-defending set $S \in \text{SD}(F)$ with $S \neq \emptyset$ is irreducible if for every $S_1, S_2 \in \text{SD}(F)$, $S = S_1 \cup S_2$ implies $S = S_1$ or $S = S_2$. We denote by $\text{IRR}(F)$ the family of irreducible self-defending sets of F .

Remark 1. The initial admissible sets [30] are irreducible self-defending sets. A set of arguments is an initial admissible set if it is an inclusion-wise minimal non-empty admissible set.

Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework. The family $\text{SD}(F)$ can be generated from $\text{IRR}(F)$ by taking the union of all possible subsets of $\text{IRR}(F)$, that is $\text{SD}(F) = \{\bigcup \mathcal{I} \mid \mathcal{I} \subseteq \text{IRR}(F)\}$, see [8]. Since a self-defending set in $\text{IRR}(F)$ cannot be obtained by union, $\text{IRR}(F)$ is in fact the most compact representation of $\text{SD}(F)$ by self-defending sets. For a given $S \in \text{SD}(F)$, we put $\text{IRR}(S) = \{S' \in \text{IRR}(F) \mid S' \subseteq S\}$. We have $S = \bigcup \text{IRR}(S)$.

Remark 2. *This representation by irreducible elements is common in several fields of computer science. It is used for instance in: posets of irreducible elements of lattices [25], (reduced) contexts in Formal Concept Analysis [19], characteristic models of a Horn function [22], bases of knowledge spaces [9], Armstrong relations in database theory [24].*

Example 1. *Let F be the argumentation framework on the left of Figure 3. On the right, we represent the family $\text{SD}(F)$ ordered by inclusion. The irreducible self-defending sets are highlighted. We have $\text{IRR}(F) = \{1, 2, 14, 24\}$. On the other hand, 12 is not irreducible as $12 = 1 \cup 2$.*

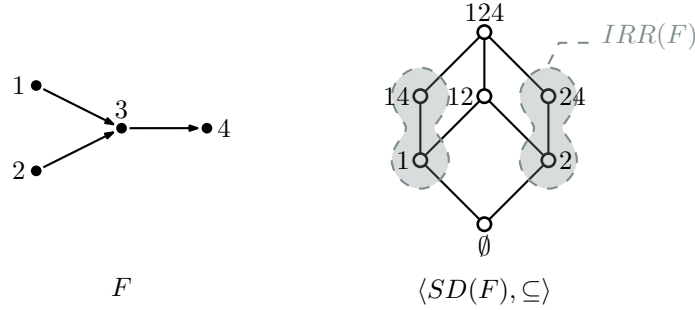


Figure 3: An argumentation framework F , its self-defending sets $\text{SD}(F)$ with $\text{IRR}(F)$ highlighted

Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework. We construct an argumentation framework $F_{\text{IRR}} = \langle \text{IRR}(F), \mathcal{R}_{\text{IRR}} \rangle$ from which $\text{PREF}(F)$ can be easily recovered with $\text{NAIV}(F_{\text{IRR}})$. We start with the following lemma.

Lemma 3. *Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework. A self-defending set S is not admissible if and only if there exists $S_1, S_2 \in \text{IRR}(S)$ such that $S_1 \cup S_2$ is not an admissible extension of F .*

Proof. The if part is clear. Let $S \in \text{SD}(F)$ with $S \notin \text{ADM}(F)$. By definition, S contains some pair (x, y) such that $(x, y) \in \mathcal{R}$. Since $S = \bigcup \text{IRR}(S)$, there exist $S_1, S_2 \in \text{IRR}(S)$ such that $x \in S_1$ and $y \in S_2$ (possibly $S_1 = S_2$). Thus, $S_1 \cup S_2$ is a self-defending set which is not admissible, concluding the proof. \square

In the next lemma, we use Lemma 3 to devise an argumentation framework F_{IRR} on $\text{IRR}(F)$ and show the relationship between $\text{NAIV}(F_{\text{IRR}})$ and $\text{PREF}(F)$.

Lemma 4. *Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework. Let $F_{\text{IRR}} = \langle \text{IRR}(F), \mathcal{R}_{\text{IRR}} \rangle$ be a new argumentation framework where $\mathcal{R}_{\text{IRR}} = \{(S_1, S_2) \mid S_1, S_2 \in \text{IRR}(F) \text{ and } S_1 \cup S_2 \notin \text{ADM}(F)\}$. Then, the following equality holds and it is moreover a bijection between $\text{PREF}(F)$ and $\text{NAIV}(F_{\text{IRR}})$:*

$$\text{PREF}(F) = \left\{ \bigcup \mathcal{I} \mid \mathcal{I} \in \text{NAIV}(F_{\text{IRR}}) \right\}$$

Proof. First we show the equality. Let F_{IRR} be the argumentation framework defined in the lemma. We begin with the \subseteq part of the equality. Let $S \in \text{PREF}(F)$ and consider $\text{IRR}(S)$. We show that $\text{IRR}(S)$ is a naive extension of F_{IRR} . Since S is admissible, $\text{IRR}(S)$ is a conflict-free extension of F_{IRR} by Lemma 3 and by construction of F_{IRR} . If $\text{IRR}(S) = \text{IRR}(F)$, then it is clearly the unique naive extension of F_{IRR} . Suppose now there exists $S' \in \text{IRR}(F)$ such that $S' \notin \text{IRR}(S)$ and consider the set $S'' = S' \cup \text{IRR}(S) = S' \cup S$. Since the union of two self-defending set is self-defending, S'' is a self-defending set of F . Moreover, we have $S \subset S''$ by definition of $\text{IRR}(S)$. Since $S \in \text{PREF}(F)$ and $S \subset S''$, we have $S'' \notin \text{ADM}(F)$. Thus, there exists a pair (x, y) in \mathcal{R} such that $\{x, y\} \subseteq S''$ while $\{x, y\} \not\subseteq S$. As S is admissible and $S'' = S \cup S'$ we deduce that either $x \in S'$ or $y \in S'$. If both x and y belong to S' , then $\text{IRR}(S) \cup \{S'\}$ is not a conflict-free extension of F_{IRR} . Assume now that $x \in S'$ but $y \notin S'$, without loss of generality. Observe that we must have $y \in S$ as otherwise $\{x, y\} \subseteq S''$ would not hold. Because $S = \bigcup \text{IRR}(S)$ and $y \in S$, we deduce that there exists $T \in \text{IRR}(S)$ such that $y \in T$. Hence, $\{x, y\} \subseteq S' \cup T$ which entails that $(T, S') \in \mathcal{R}_{\text{IRR}}$ and $\text{IRR}(S) \cup \{S'\}$ is not a conflict-free extension of F_{IRR} . Consequently, for every $S' \in \text{IRR}(F) \setminus \text{IRR}(S)$, we have that $\text{IRR}(S) \cup \{S'\}$ is not a conflict-free extension of F_{IRR} . Together with the fact that $\text{IRR}(S)$ is conflict-free, we conclude that $\text{IRR}(S)$ is a naive extension of F_{IRR} as expected.

We move to the \supseteq part. Let $\mathcal{I} \in \text{NAIV}(F_{\text{IRR}})$. Let $S = \bigcup \mathcal{I}$. We show that $S \in \text{PREF}(F)$. Since $\text{IRR}(F)$ generates $\text{SD}(F)$, $S \in \text{SD}(F)$ readily holds. Assume for contradiction that $S \notin \text{ADM}(F)$. Then, there exists $(x, y) \in \mathcal{R}$ such that $\{x, y\} \subseteq S$. Since $S = \bigcup \mathcal{I}$, there exists $S_1, S_2 \in \mathcal{I}$ such that $\{x, y\} \subseteq S_1 \cup S_2$. In particular, it must be that $(S_1, S_2) \in \mathcal{R}_{\text{IRR}}$ by construction of F_{IRR} so that \mathcal{I} is not a conflict-free extension of F_{IRR} , a contradiction. Thus, $S \in \text{ADM}(F)$ must hold. Now let $S' \in \text{IRR}(F)$ such that $S' \notin \mathcal{I}$. If such a S' does not exist, the result $S \in \text{PREF}(F)$ is clear. Otherwise, we have two cases. First, $\{S'\} \notin \text{CF}(F_{\text{IRR}})$. Then $S \cup S'$ is not admissible in F by construction of F_{IRR} . Second, $\{S'\} \in \text{CF}(F_{\text{IRR}})$. Since $\mathcal{I} \in \text{NAIV}(F_{\text{IRR}})$, we have that $S \cup \{S'\}$ contains some pair of \mathcal{R}_{IRR} and hence that $S \cup S' \notin \text{ADM}(F)$ again by construction of F_{IRR} . In both cases, we deduce that $S \cup S' \notin \text{ADM}(F)$. As $S \in \text{ADM}(F)$, we deduce that $S \in \text{PREF}(F)$ as expected. This concludes the \supseteq part of the proof, and the equality of the lemma holds.

Now we show this equality defines a bijection between $\text{PREF}(F)$ and $\text{NAIV}(F_{\text{IRR}})$. As $\text{IRR}(S)$ is uniquely defined for every $S \in \text{SD}(F)$, it is sufficient to prove that for every $\mathcal{I} \in \text{NAIV}(F_{\text{IRR}})$, $\mathcal{I} = \text{IRR}(S)$ for some $S \in \text{PREF}(F)$. Let $\mathcal{I} \in \text{NAIV}(F)$ and let $S = \bigcup \mathcal{I}$. By previous discussion $S \in \text{PREF}(F)$. Moreover, $\mathcal{I} \subseteq \text{IRR}(S)$ by definition of $\text{IRR}(S)$. To prove that $\mathcal{I} = \text{IRR}(S)$, we show that $S' \notin \mathcal{I}$ implies $S' \notin \text{IRR}(S)$, for every $S' \in \text{IRR}(F)$. First, observe that if $\mathcal{I} = \text{IRR}(F)$, the implication is clear. Now assume there exists some S' in $\text{IRR}(F) \setminus \mathcal{I}$. We show that $S \cup S' \neq S$. Because $\mathcal{I} \in \text{NAIV}(F)$ and $S = \bigcup \mathcal{I}$, it follows from the construction of F_{IRR} that $S \cup S' \notin \text{ADM}(F)$. As $S \in \text{ADM}(F)$, we deduce that $S \cup S' \neq S$ must hold. Thus, $S' \notin \text{IRR}(S)$ for every $S' \notin \mathcal{I}$. We obtain $\mathcal{I} = \text{IRR}(S)$, concluding the proof. \square

Example 2. Let F be the argumentation framework represented on the top of Figure 4. We give the family $\text{SD}(F)$ on the bottom of the same figure. We have $\text{PREF}(F) = \{1234, 4xyz\}$ and $\text{IRR}(F) = \{4, 124, 134, 23, xyz\}$. In Figure 5, we give the argumentation framework F_{IRR} from Lemma 4. It is symmetric by construction. We have $\text{NAIV}(F_{\text{IRR}}) = \{\{4, xyz\}, \{4, 124, 23, 134\}\}$. Then $4 \cup xyz = 4xyz$ and $4 \cup 124 \cup 23 \cup 134 = 1234$, corresponding to $\text{PREF}(F)$ as proven in Lemma 4.

We use Lemma 4 to find $\text{PREF}(F)$. If $\text{IRR}(F)$ is given, F_{IRR} can be computed in

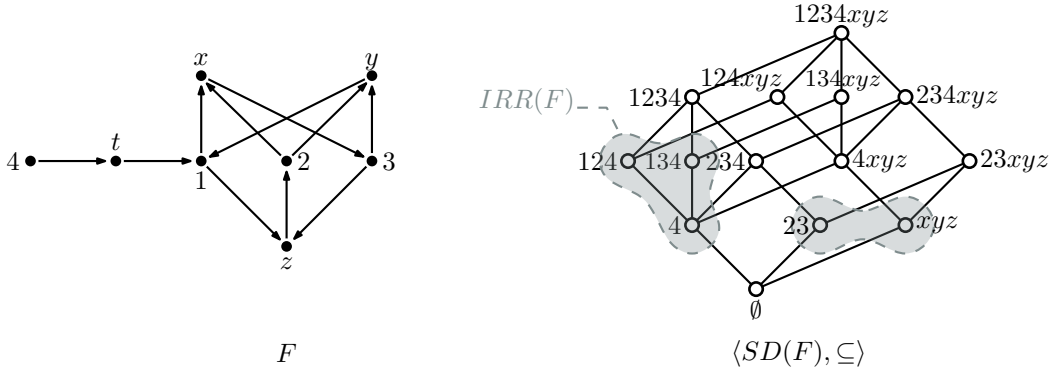


Figure 4: An argumentation framework and its family of self-defending sets (ordered by inclusion).

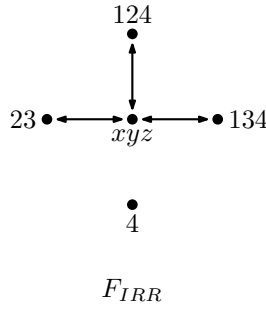


Figure 5: The argumentation framework F_{IRR} .

polynomial time in the size of F and $IRR(F)$. Applying the algorithm of Jonhson *et al.* [20], we derive:

Corollary 1. *Let $F = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework. Then, $PREF(F)$ can be computed with polynomial delay and space in the size of $IRR(F)$ and F .*

Proof. We describe the algorithm. First, we construct the argumentation framework $F_{IRR} = \langle IRR(F), \mathcal{R}_{IRR} \rangle$ in polynomial time in the size of $IRR(F)$ and F . Then, we use the algorithm in [20] to list $NAIV(F_{IRR})$. However, instead of outputting a subset \mathcal{I} of $IRR(F)$, we compute the set $\bigcup \mathcal{I}$. Due to Lemma 4, each preferred extension of F will be given exactly once. As computing \mathcal{I} is done in polynomial time in the size of $IRR(F)$ and F , the polynomial delay and space of the whole algorithm follows. \square

We conclude by mentioning cases where $IRR(F)$ can be computed in polynomial time in the size of F . In general, this problem is open and relates to hypergraph dualization [16, 22]. Yet, there are types of argumentation frameworks where the irreducible self-defending sets can be computed in polynomial time. When F is symmetric, the lattice $\langle SD(F), \subseteq \rangle$ is Boolean. In this case, the irreducible self-defending sets are singleton elements and they can be computed in polynomial time. Using implicational systems [17] and previous results rooted in lattice theory [3, 29], we can compute the irreducible self-defending sets in polynomial time for argumentation frameworks with prescribed lattice properties such as distributivity and meet-semidistributivity.

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