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► **To cite this version:**

Cédric Bonnafé. Regular automorphisms and Calogero-Moser families. Revista de la Unión Matemática Argentina, In press, 10.48550/arXiv.2112.13685 . hal-03565132v2

HAL Id: hal-03565132

<https://hal.science/hal-03565132v2>

Submitted on 14 Oct 2023

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REGULAR AUTOMORPHISMS AND CALOGERO-MOSER FAMILIES

by

CÉDRIC BONNAFÉ

Abstract. — We study the subvariety of fixed points of an automorphism of a Calogero-Moser space induced by a regular element of finite order of the normalizer of the associated complex reflection group W . We determine some of (and conjecturally all) the \mathbb{C}^\times -fixed points of its unique irreducible component of maximal dimension in terms of the character table of W . This is inspired by the mysterious relations between the geometry of Calogero-Moser spaces and unipotent representations of finite reductive groups, which is the theme of another paper [Bon3].

If \mathbf{G} is a split reductive group over a finite field with q elements \mathbb{F}_q with Weyl group W , Deligne and Lusztig [DeLu] have defined a particular class of irreducible characters of the finite group $G = \mathbf{G}(\mathbb{F}_q)$, called the *unipotent characters* of G . To W , one can also associate a Calogero-Moser space \mathcal{X} at equal parameters, which is a complex irreducible normal affine Poisson variety endowed with a \mathbb{C}^\times -action [EtGi]. The main theme of a previous paper of the author [Bon3] is the observation that many aspects of the combinatorics of unipotent characters of G have a conjectural analogue in the geometry of \mathcal{X} , thanks to the Poisson structure and the \mathbb{C}^\times -action. Here are two examples:

- Unipotent characters have been partitioned by Lusztig [Lus] into *families* and it has been conjectured by Gordon-Martino [GoMa] that these families are in bijection with \mathbb{C}^\times -fixed points of \mathcal{X} . Note that this conjecture has been proved in all cases except type E_6 , E_7 and E_8 (see [GoMa, Bel, BoTh]).
- For d a natural number, Broué-Malle-Michel [BMM1] defined a partition of unipotent characters into *d -Harish-Chandra series* (generalizing the classical partition into *Harish-Chandra series*, which correspond to $d = 1$ case). This partition is conjecturally related to the stratification of \mathcal{X}^{μ_d} by symplectic leaves (here, μ_d denotes the group of complex d -th roots of unity): the reader may find more details in [Bon3, §12.C]. See [Bon3, Part IV] for a list of cases where this conjecture is proved.

In the second point, whenever d is a *regular number* in the sense of Springer [Spr], it has been observed by Broué-Malle-Michel [BMM2, Rem. 4.21] that the families of unipotent characters which meet the principal d -Harish-Chandra series are characterized by a property involving character values of W (again, more details may be found in [Bon3, Ex. 12.9]). If one believes in the analogy between unipotent characters and geometry of \mathcal{X} , this suggests [Bon3, Conj. 7.5] a conjectural characterization of \mathbb{C}^\times -fixed points meeting the unique irreducible component of \mathcal{X}^{μ_d} of maximal dimension in terms of the character table of W . The proof of one direction of this conjecture is the theme of the present paper.

Note that the conjecture [Bon3, Conj. 7.5] involves only the Calogero-Moser space and can be studied without any reference to unipotent characters. Moreover, since Calogero-Moser spaces are defined for any finite complex reflection group (and not only for Weyl groups) and for a bigger family of parameters, this conjecture is somewhat more general than what has been explained above in this introduction, which can be seen as a motivation for the results obtained here. Therefore, from now on, we will work in this more general context of complex reflection groups. Let V be a finite dimensional complex vector space and let W be a finite subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$ generated by reflections (i.e. automorphisms of V whose fixed points space is an hyperplane). To some parameter k , Etingof and Ginzburg [EtGi] have associated a normal irreducible affine complex variety $\mathcal{X}_k = \mathcal{X}_k(V, W)$ called a (generalized) *Calogero-Moser space*. If τ is an element of finite order of the normalizer of W in $\mathbf{GL}_{\mathbb{C}}(V)$ stabilizing the parameter k , it induces an automorphism of \mathcal{X}_k .

We denote by V_{reg}^{τ} the open subset of V on which W acts freely, and we assume that $V_{\text{reg}}^{\tau} \neq \emptyset$ (then τ is called *regular*). In this case, there exists a unique irreducible component $(\mathcal{X}_k^{\tau})_{\text{max}}$ of \mathcal{X}_k^{τ} of maximal dimension (as it will be explained in Section 2). Recall that \mathcal{X}_k is endowed with a \mathbb{C}^{\times} -action and that we have a surjective map $\text{Irr}(W) \rightarrow \mathcal{X}_k^{\mathbb{C}^{\times}}$ defined by Gordon [Gor] (induced by the action of the center of a rational Cherednik algebra on *baby Verma modules*) whose fibers are called the *Calogero-Moser k -families* of W . Here, $\text{Irr}(W)$ is the set of irreducible characters of W . If $p \in \mathcal{X}_k^{\mathbb{C}^{\times}}$, we denote by \mathfrak{F}_p its associated Calogero-Moser k -family. It is a natural question to wonder which \mathbb{C}^{\times} -fixed points of \mathcal{X}_k^{τ} belong to $(\mathcal{X}_k^{\tau})_{\text{max}}$. The aim of this note is to provide a partial answer in terms of the character table of W :

Theorem A. — Assume that $V_{\text{reg}}^{\tau} \neq \emptyset$. Let $p \in \mathcal{X}_k^{\mathbb{C}^{\times}}$ be such that $\tau(p) = p$. If there exists χ in \mathfrak{F}_p^{τ} such that $\tilde{\chi}(\tau) \neq 0$, then $p \in (\mathcal{X}_k^{\tau})_{\text{max}}$.

In this statement, if χ is a τ -stable irreducible character of W , we denote by $\tilde{\chi}$ an extension of χ to the finite group $W\langle\tau\rangle$ (see [Isa, Coro. 11.22] for the existence of $\tilde{\chi}$): note that $|\tilde{\chi}(\tau)|^2$ does not depend on the choice of $\tilde{\chi}$ (see [Isa, Coro. 6.17]). Our proof of Theorem A makes an extensive use of the Gaudin operators introduced in [BoRo, §8.3.B]: whenever $\tilde{\chi}(\tau) \neq 0$, the decomposition of a representation affording χ with respect to generalized eigenspaces of the Gaudin operators allows to construct a τ -fixed point p' in $(\mathcal{X}_k^{\tau})_{\text{max}}$ such that $p = \lim_{\xi \rightarrow 0} \xi \cdot p'$.

We conjecture that the converse of Theorem A holds [Bon3, Conj. 7.5]:

Conjecture B. — Assume that $V_{\text{reg}}^{\tau} \neq \emptyset$. Let $p \in \mathcal{X}_k^{\mathbb{C}^{\times}}$ be such that $\tau(p) = p$ and $p \in (\mathcal{X}_k^{\tau})_{\text{max}}$. Then there exists χ in \mathfrak{F}_p^{τ} such that $\tilde{\chi}(\tau) \neq 0$.

General notation. Throughout this paper, we will abbreviate $\otimes_{\mathbb{C}}$ as \otimes and all varieties will be algebraic, complex, quasi-projective and reduced. If \mathcal{X} is an affine variety, we denote by $\mathbb{C}[\mathcal{X}]$ its coordinate ring.

If X is a subset of a vector space V (or of its dual V^*), and if Γ is a subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$, we denote by Γ_X the pointwise stabilizer of X . If moreover Γ is finite, we will identify $(V^{\Gamma})^*$ and $(V^*)^{\Gamma}$.

1. Calogero-Moser spaces and families

Hypothesis and notation. We fix in this paper a finite dimensional complex vector space V and a finite subgroup W of $\mathbf{GL}_{\mathbb{C}}(V)$. We set

$$\mathrm{Ref}(W) = \{s \in W \mid \mathrm{codim}_{\mathbb{C}} V^s = 1\}$$

and we assume throughout this paper that

$$W = \langle \mathrm{Ref}(W) \rangle,$$

i.e. that W is a complex reflection group.

1.A. About W . — We set $\varepsilon : W \rightarrow \mathbb{C}^{\times}$, $w \mapsto \det(w)$. We identify $\mathbb{C}[V]$ (resp. $\mathbb{C}[V^*]$) with the symmetric algebra $S(V)$ (resp. $S(V^*)$).

We denote by \mathcal{A} the set of *reflecting hyperplanes* of W , namely

$$\mathcal{A} = \{V^s \mid s \in \mathrm{Ref}(W)\}.$$

If $H \in \mathcal{A}$, we denote by α_H an element of V^* such that $H = \mathrm{Ker}(\alpha_H)$ and by α_H^{\vee} an element of V such that $V = H \oplus \mathbb{C}\alpha_H^{\vee}$ and the line $\mathbb{C}\alpha_H^{\vee}$ is W_H -stable. We set $e_H = |W_H|$. Note that W_H is cyclic of order e_H and that $\mathrm{Irr}(W_H) = \{\mathrm{Res}_{W_H}^W \varepsilon^j \mid 0 \leq j \leq e_H - 1\}$. We denote by $\varepsilon_{H,j}$ the (central) primitive idempotent of $\mathbb{C}W_H$ associated with the character $\mathrm{Res}_{W_H}^W \varepsilon^{-j}$, namely

$$\varepsilon_{H,j} = \frac{1}{e_H} \sum_{w \in W_H} \varepsilon(w)^j w \in \mathbb{C}W_H.$$

If Ω is a W -orbit of reflecting hyperplanes, we write e_{Ω} for the common value of all the e_H , where $H \in \Omega$. We denote by \mathfrak{N} the set of pairs (Ω, j) where $\Omega \in \mathcal{A}$ and $0 \leq j \leq e_{\Omega} - 1$. The vector space of families of complex numbers indexed by \mathfrak{N} will be denoted by $\mathbb{C}^{\mathfrak{N}}$: elements of $\mathbb{C}^{\mathfrak{N}}$ will be called *parameters*. If $k = (k_{\Omega,j})_{(\Omega,j) \in \mathfrak{N}} \in \mathbb{C}^{\mathfrak{N}}$, we define $k_{H,j}$ for all $H \in \Omega$ and $j \in \mathbb{Z}$ by $k_{H,j} = k_{\Omega,j_0}$ where Ω is the W -orbit of H and j_0 is the unique element of $\{0, 1, \dots, e_H - 1\}$ such that $j \equiv j_0 \pmod{e_H}$.

We denote by V_{reg} the set of elements v of V such that $W_v = 1$. It is an open subset of V and recall from Steinberg-Serre Theorem [Bro, Theo. 4.7] that

$$(1.1) \quad V_{\mathrm{reg}} = V \setminus \bigcup_{H \in \mathcal{A}} H.$$

In particular, V_{reg} is a principal open affine subset of V and $\mathbb{C}[V_{\mathrm{reg}}] = \mathbb{C}[V][1/\prod_{H \in \mathcal{A}} \alpha_H]$.

1.B. Rational Cherednik algebra at $t = 0$. — Let $k \in \mathbb{C}^{\mathfrak{N}}$. We define the *rational Cherednik algebra* \mathbf{H}_k (at $t = 0$) to be the quotient of the algebra $\mathbb{T}(V \oplus V^*) \rtimes W$ (the semi-direct product of the tensor algebra $\mathbb{T}(V \oplus V^*)$ with the group W) by the relations

$$(1.2) \quad \begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = \sum_{H \in \mathcal{A}} \sum_{j=0}^{e_H-1} e_H (k_{H,j} - k_{H,j+1}) \frac{\langle y, \alpha_H \rangle \cdot \langle \alpha_H^{\vee}, x \rangle}{\langle \alpha_H^{\vee}, \alpha_H \rangle} \varepsilon_{H,j}, \end{cases}$$

for all $x, x' \in V^*$, $y, y' \in V$. Here $\langle \cdot, \cdot \rangle : V \times V^* \rightarrow \mathbb{C}$ is the standard pairing. The first commutation relations imply that we have morphisms of algebras $\mathbb{C}[V] \rightarrow \mathbf{H}_k$ and $\mathbb{C}[V^*] \rightarrow \mathbf{H}_k$. Recall [EtGi, Theo. 1.3] that we have an isomorphism of \mathbb{C} -vector spaces

$$(1.3) \quad \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} \mathbf{H}_k$$

induced by multiplication (this is the so-called *PBW-decomposition*).

Remark 1.4. — Let $(l_\Omega)_{\Omega \in \mathfrak{sl}/W}$ be a family of complex numbers and let $k' \in \mathbb{C}^{\mathfrak{N}}$ be defined by $k'_{\Omega,j} = k_{\Omega,j} + l_\Omega$. Then $\mathbf{H}_k = \mathbf{H}_{k'}$. This means that there is no loss of generality if we consider for instance only parameters k such that $k_{\Omega,0} = 0$ for all Ω , or only parameters k such that $k_{\Omega,0} + k_{\Omega,1} + \cdots + k_{\Omega,e_\Omega-1} = 0$ for all Ω (as in [BoRo]). ■

1.C. Calogero-Moser space. — We denote by \mathbf{Z}_k the center of the algebra \mathbf{H}_k : it is well-known [EtGi, Theo 3.3 and Lem. 3.5] that \mathbf{Z}_k is an integral domain, which is integrally closed. Moreover, it contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ as subalgebras [Gor, Prop. 3.6]. So, by the PBW-decomposition, \mathbf{Z}_k contains $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$, and it is a free \mathbf{P} -module of rank $|W|$ (see [EtGi, Prop. 4.15]). We denote by \mathcal{X}_k the affine algebraic variety whose ring of regular functions $\mathbb{C}[\mathcal{X}_k]$ is \mathbf{Z}_k : this is the *Calogero-Moser space* associated with the datum (V, W, k) . It is irreducible and normal.

We set $\mathcal{P} = V/W \times V^*/W$, so that $\mathbb{C}[\mathcal{P}] = \mathbf{P}$ and the inclusion $\mathbf{P} \hookrightarrow \mathbf{Z}_k$ induces a morphism of varieties

$$\Upsilon_k : \mathcal{X}_k \longrightarrow \mathcal{P}$$

which is finite and flat.

1.D. Calogero-Moser families. — Using the PBW-decomposition, we define a \mathbb{C} -linear map $\Omega^{\mathbf{H}_k} : \mathbf{H}_k \longrightarrow \mathbb{C}W$ by

$$\Omega^{\mathbf{H}_k}(f w g) = f(0)g(0)w$$

for all $f \in \mathbb{C}[V]$, $g \in \mathbb{C}[V^*]$ and $w \in \mathbb{C}W$. This map is W -equivariant for the action on both sides by conjugation, so it induces a well-defined \mathbb{C} -linear map

$$\Omega^k : \mathbf{Z}_k \longrightarrow Z(\mathbb{C}W).$$

Recall from [BoRo, Cor. 4.2.11] that Ω^k is a morphism of algebras.

Calogero-Moser families were defined by Gordon using his theory of *baby Verma modules* [Gor, §4.2 and §5.4]. We explain here an equivalent definition given in [BoRo, §7.2]. If $\chi \in \text{Irr}(W)$, we denote by $\omega_\chi : Z(\mathbb{C}W) \rightarrow \mathbb{C}$ its central character (i.e., $\omega_\chi(z) = \chi(z)/\chi(1)$ is the scalar by which z acts on an irreducible representation affording the character χ). We say that two characters χ and χ' belong to the same *Calogero-Moser k -family* if $\omega_\chi \circ \Omega^k = \omega_{\chi'} \circ \Omega^k$.

In other words, the map $\omega_\chi \circ \Omega^k : \mathbf{Z}_k \rightarrow \mathbb{C}$ is a morphism of algebras, so it might be viewed as a point $\varphi_k(\chi)$ of \mathcal{X}_k , which is easily checked to be \mathbb{C}^\times -fixed. This defines a surjective map [Gor, §5.4]

$$\varphi_k : \text{Irr}(W) \longrightarrow \mathcal{X}_k^{\mathbb{C}^\times}$$

whose fibers are the Calogero-Moser k -families. If $p \in \mathcal{X}_k^{\mathbb{C}^\times}$, we denote by \mathfrak{F}_p the corresponding Calogero-Moser k -family.

1.E. Alternative parameters. — Let \mathcal{C} denote the space of maps $\text{Ref}(W) \rightarrow \mathbb{C}$ which are constant on conjugacy classes of reflections. The element

$$\sum_{(\Omega, j) \in \aleph} \sum_{H \in \Omega} (k_{H, j} - k_{H, j+1}) e_{H \varepsilon_{H, j}}$$

of $Z(\mathbb{C}W)$ is supported only by reflections, so there exists a unique map $c_k \in \mathcal{C}$ such that

$$\sum_{(\Omega, j) \in \aleph} \sum_{H \in \Omega} (k_{H, j} - k_{H, j+1}) e_{H \varepsilon_{H, j}} = \sum_{s \in \text{Ref}(W)} (\varepsilon(s) - 1) c_k(s) s.$$

Then the map $\mathbb{C}^\aleph \rightarrow \mathcal{C}$, $k \mapsto c_k$ is linear and surjective. With this notation, we have

$$(1.5) \quad [y, x] = \sum_{s \in \text{Ref}(W)} (\varepsilon(s) - 1) c_k(s) \frac{\langle y, \alpha_s \rangle \cdot \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s,$$

for all $y \in V$ and $x \in V^*$. Here, $\alpha_s = \alpha_{V^s}$ and $\alpha_s^\vee = \alpha_{V^s}^\vee$.

1.F. Actions on the Calogero-Moser space. — The Calogero-Moser space \mathfrak{X}_k is endowed with a \mathbb{C}^\times -action and an action of the stabilizer of k in $N_{\text{GL}_{\mathbb{C}}(V)}(W)$, which are described below.

1.F.1. Grading, \mathbb{C}^\times -action. — The algebra $T(V \oplus V^*) \rtimes W$ can be \mathbb{Z} -graded in such a way that the generators have the following degrees

$$\begin{cases} \deg(y) = -1 & \text{if } y \in V, \\ \deg(x) = 1 & \text{if } x \in V^*, \\ \deg(w) = 0 & \text{if } w \in W. \end{cases}$$

This descends to a \mathbb{Z} -grading on \mathbf{H}_k , because the defining relations (1.2) are homogeneous. Since the center of a graded algebra is always graded, the subalgebra \mathbf{Z}_k is also \mathbb{Z} -graded. So the Calogero-Moser space \mathfrak{X}_k inherits a regular \mathbb{C}^\times -action. Note also that by definition $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ is clearly a graded subalgebra of \mathbf{Z}_k .

1.F.2. Action of the normalizer. — The group $N_{\text{GL}_{\mathbb{C}}(V)}(W)$ acts on the set \aleph and so on the space of parameters \mathbb{C}^\aleph . If $\tau \in N_{\text{GL}_{\mathbb{C}}(V)}(W)$, then τ induces an isomorphism of algebras $\mathbf{H}_k \rightarrow \mathbf{H}_{\tau(k)}$. So, if $\tau(k) = k$, then it induces an action on the algebra \mathbf{H}_k (and so on its center \mathbf{Z}_k and on the Calogero-Moser space \mathfrak{X}_k).

Notation. From now on, and until the end of this paper, we fix a parameter $k \in \mathbb{C}^\aleph$ and a **regular** element τ of **finite** order of $N_{\text{GL}_{\mathbb{C}}(V)}(W)$ such that $\tau(k) = k$.

We denote by \mathfrak{X}_k^τ the variety of fixed points of τ in \mathfrak{X}_k , endowed with its reduced structure. All the above constructions are τ -equivariant: for instance, the map $\varphi_k : \text{Irr}(W) \rightarrow \mathfrak{X}_k^{\mathbb{C}^\times}$ is τ -equivariant.

Let us recall the following consequence [Spr, Prop. 3.5 and Theo. 4.2] of the above hypothesis:

Theorem 1.6 (Springer). — *The group W^τ acts as a reflection group on V^τ and the natural map $V^\tau/W^\tau \rightarrow (V/W)^\tau$ is an isomorphism of varieties.*

Corollary 1.7. — *The natural map $(V_{\text{reg}}^\tau \times V^{*\tau})/W^\tau \rightarrow ((V_{\text{reg}} \times V^*)/W)^\tau$ is an isomorphism of varieties.*

Proof. — Since W acts freely on the variety $V_{\text{reg}} \times V^*$, the quotient variety $(V_{\text{reg}} \times V^*)/W$ is smooth. Consequently, the variety of fixed points $((V_{\text{reg}} \times V^*)/W)^\tau$ is also smooth. Similarly, $(V_{\text{reg}}^\tau \times V^{*\tau})/W^\tau$ is smooth. Since a bijective morphism between smooth complex varieties is an isomorphism (by Zariski's Main Theorem), we only need to show that the above natural map is bijective.

First, if (v_1, v_1^*) and (v_2, v_2^*) are two elements of $V_{\text{reg}}^\tau \times V^{*\tau}$ belonging to the same W -orbit, there exists $w \in W$ such that $v_2 = w(v_1)$. Since v_1 and v_2 are τ -stable, we also have $\tau(w)(v_1) = v_2$, and so $v_1 = w^{-1}\tau(w)(v_1)$. Since $v_1 \in V_{\text{reg}}$, this forces $\tau(w) = w$ and the injectivity follows.

Now, if $(v, v^*) \in V_{\text{reg}} \times V^*$ is such that its W -orbit is τ -stable, then the W -orbit of v is τ -stable. So Theorem 1.6 shows that we may assume that $\tau(v) = v$. The hypothesis implies that there exists $w \in W$ such that $\tau(v) = w(v)$ and $\tau(v^*) = w(v^*)$. But $\tau(v) = v \in V_{\text{reg}}$, so $w = 1$. In particular, $\tau(v^*) = v^*$, and the surjectivity follows. \square

2. Irreducible component of maximal dimension

Let $(\mathcal{X}_k)_{\text{reg}}$ denote the open subset $\Upsilon_k^{-1}(V_{\text{reg}}/W \times V^*/W)$. By [EtGi, Prop. 4.11], we have a \mathbb{C}^\times -equivariant and τ -equivariant isomorphism

$$(2.1) \quad (\mathcal{X}_k)_{\text{reg}} \simeq (V_{\text{reg}} \times V^*)/W.$$

This shows that $(\mathcal{X}_k)_{\text{reg}}$ is smooth and so $(\mathcal{X}_k)_{\text{reg}}^\tau$ is also smooth. By Corollary 1.7, this implies that

$$(2.2) \quad (\mathcal{X}_k)_{\text{reg}}^\tau \simeq (V_{\text{reg}}^\tau \times V^{*\tau})/W^\tau.$$

In particular it is irreducible. We denote by $(\mathcal{X}_k^\tau)_{\text{max}}$ its closure: it is an irreducible closed subvariety of \mathcal{X}_k^τ .

Moreover, $(\mathcal{X}_k)_{\text{reg}}^\tau$ has dimension $2 \dim V^\tau$ by Corollary 1.7. So $\dim \mathcal{X}_k^\tau \geq 2 \dim V^\tau = \dim(\mathcal{X}_k^\tau)_{\text{max}}$. But, on the other hand, $\Upsilon_k(\mathcal{X}_k^\tau) \subset (V/W)^\tau \times (V^*/W)^\tau$. Since Υ_k is a finite morphism, we get from Theorem 1.6 that $\dim \mathcal{X}_k^\tau \leq 2 \dim V^\tau$. Hence

$$(2.3) \quad \dim \mathcal{X}_k^\tau = \dim(\mathcal{X}_k^\tau)_{\text{max}} = 2 \dim V^\tau.$$

This shows that $(\mathcal{X}_k^\tau)_{\text{max}}$ is an irreducible component of maximal dimension of \mathcal{X}_k^τ and that

$$(2.4) \quad \Upsilon_k((\mathcal{X}_k^\tau)_{\text{max}}) = (V/W)^\tau \times (V^*/W)^\tau.$$

Proposition 2.5. — *The closed subvariety $(\mathcal{X}_k^\tau)_{\text{max}}$ of \mathcal{X}_k^τ is the unique irreducible component of maximal dimension.*

Proof. — Let \mathcal{X} be an irreducible component of \mathcal{X}_k^τ of dimension $2 \dim V^\tau$. Since Υ_k is finite, the image $\Upsilon_k(\mathcal{X})$ is closed in $V/W \times V^*/W$, irreducible of dimension $2 \dim(V^\tau)$ and contained in $(V/W)^\tau \times (V^*/W)^\tau$. By Theorem 1.6, we get that $\Upsilon_k(\mathcal{X}) = (V/W)^\tau \times (V^*/W)^\tau$.

Let $\mathcal{U} = \Upsilon_k^{-1}(V_{\text{reg}}/W \times V^*/W) \cap \mathcal{X}$. Then \mathcal{U} is a non-empty open subset of \mathcal{X} : since \mathcal{X} is irreducible, this forces \mathcal{U} to have dimension $2 \dim(V^\tau)$. But \mathcal{U} is contained in $(\mathcal{X}_k)_{\text{reg}}^\tau$ which is irreducible of the same dimension, so the closure of \mathcal{U} contains $(\mathcal{X}_k)_{\text{reg}}^\tau$. This proves that $\mathcal{X} = (\mathcal{X}_k^\tau)_{\text{max}}$. \square

It is natural to ask which \mathbb{C}^\times -fixed points of \mathcal{F}_k belong to $(\mathcal{F}_k^\tau)_{\max}$. Inspired by the representation theory of finite reductive groups (see [BrMi] and [BMM2, Rem. 4.21]), we propose an answer to this question in terms of the character table of the finite group $W\langle\tau\rangle$ (see [Bon3, Ex. 12.9] for some explanations). We first need some notation.

If $\chi \in \text{Irr}(W)$, we denote by E_χ a $\mathbb{C}W$ -module affording the character χ . If moreover χ is τ -stable, we fix a structure of $\mathbb{C}W\langle\tau\rangle$ -module on E_χ extending the structure of $\mathbb{C}W$ -module, and we denote by $\tilde{\chi}$ its associated irreducible character of $W\langle\tau\rangle$. Note that the real number $|\tilde{\chi}(\tau)|^2$ does not depend on the choice of $\tilde{\chi}$.

Conjecture 2.6. — *Recall that τ is regular. Let $p \in \mathcal{F}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$. Then p belongs to $(\mathcal{F}_k^\tau)_{\max}$ if and only if $\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 \neq 0$.*

Remark 2.7. — Let \mathfrak{F} be a τ -stable Calogero-Moser family. Then \mathfrak{F} contains a unique irreducible character $\chi_{\mathfrak{F}}$ with minimal b -invariant [BoRo, Theo. 7.4.1], where the b -invariant of an irreducible character χ is the minimal natural number j such that χ occurs in the j -th symmetric power of the natural representation V of W . From this characterization, we see that $\chi_{\mathfrak{F}}$ is τ -stable. In particular, any τ -stable Calogero-Moser family contains at least one τ -stable character. ■

In general, we are only able to prove the “if” part of Conjecture 2.6.

Theorem 2.8. — *Recall that τ is regular. Let $p \in \mathcal{F}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$. If $\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 \neq 0$, then p belongs to $(\mathcal{F}_k^\tau)_{\max}$.*

The next two sections are devoted to the proof of Theorem 2.8.

3. Verma modules

3.A. Definition. — Recall that $\mathbb{C}[V] \rtimes W$ is a subalgebra of \mathbf{H}_k (it is the image of $1 \otimes \mathbb{C}W \otimes \mathbb{C}[V]$ by the PBW-decomposition 1.3). If E is a $\mathbb{C}W$ -module, we denote by $E^\#$ the $(\mathbb{C}[V^*] \rtimes W)$ -module extending E by letting any element $f \in \mathbb{C}[V^*]$ act by multiplication by $f(0)$. If $\chi \in \text{Irr}(W)$, we define an \mathbf{H}_k -module $\Delta(\chi)$ as follows:

$$\Delta(\chi) = \mathbf{H}_k \otimes_{\mathbb{C}[V^*] \rtimes W} E_\chi^\#.$$

Then $\Delta(\chi)$ is called a *Verma module* of \mathbf{H}_k (see [BoRo, §5.4.A]: in this reference, $\Delta(\chi)$ is denoted by $\Delta(E_\chi^\#)$). Let $\mathbf{H}_k^{\text{reg}}$ denote the localization of \mathbf{H}_k at $\mathbf{P}_{\text{reg}} = \mathbb{C}[V_{\text{reg}}/W] \otimes \mathbb{C}[V^*/W]$. By [EtGi, Prop. 4.11], we have an isomorphism $\mathbb{C}[V_{\text{reg}} \times V^*] \rtimes W \simeq \mathbf{H}_k^{\text{reg}}$. We denote by $\Delta^{\text{reg}}(\chi)$ the localization of $\Delta(\chi)$ at $\mathbf{H}_k^{\text{reg}}$. So, by restriction to $\mathbb{C}[V_{\text{reg}} \times V^*]$, the localized Verma module $\Delta^{\text{reg}}(\chi)$ might be viewed as a W -equivariant coherent sheaf on $V_{\text{reg}} \times V^*$. We also view $e\Delta(\chi)$ as a coherent sheaf on \mathcal{F}_k , so that $e\Delta^{\text{reg}}(\chi)$ may be viewed as a coherent sheaf on $(V_{\text{reg}} \times V^*)/W$. If $p \in \mathcal{F}_k$ (or if $(v, v^*) \in V_{\text{reg}} \times V^*$), we denote by $e\Delta(\chi)_p$ (respectively $e\Delta(\chi)_{W \cdot (v, v^*)} = e\Delta^{\text{reg}}(\chi)_{W \cdot (v, v^*)}$, respectively $\Delta^{\text{reg}}(\chi)_{v, v^*}$) the restriction of $e\Delta(\chi)$ (respectively of $e\Delta(\chi)$ or $e\Delta^{\text{reg}}(\chi)$, respectively $\Delta^{\text{reg}}(\chi)$) at the point p (respectively $W \cdot (v, v^*) \in (V_{\text{reg}} \times V^*)/W \simeq (\mathcal{F}_k)_{\text{reg}}$, respectively (v, v^*)). It follows from the definition that the support of $e\Delta(\chi)$ is contained in $\Upsilon_k^{-1}(V/W \times 0)$, and recall that, through the isomorphism $\mathcal{F}_k^{\text{reg}} \simeq (V_{\text{reg}} \times V^*)/W$, $\Upsilon_k^{-1}(V_{\text{reg}}/W \times 0)$ is not necessarily contained in $(V_{\text{reg}} \times \{0\})/W$.

Lemma 3.1. — *Let $\chi \in \text{Irr}(W)$ and let $p \in \mathfrak{F}_k^{\mathbb{C}^\times}$. Then $e\Delta(\chi)_p \neq 0$ if and only if $\chi \in \mathfrak{F}_p$.*

Proof. — Let \mathfrak{p}_0 denote the maximal ideal of the algebra $\mathbf{P} = \mathbb{C}[\mathcal{P}]$ consisting of functions which vanish at 0. Then $\Delta(\chi)/\mathfrak{p}_0\Delta(\chi)$ is a representation of the *restricted rational Cherednik algebra* $\mathbf{H}_k/\mathfrak{p}_0\mathbf{H}_k$ which coincides with the *baby Verma module* defined by Gordon [Gor, §4.2]. As $\mathfrak{F}_k^{\mathbb{C}^\times} = \Upsilon_k^{-1}(0)$, the result follows from the very definition of Calogero-Moser families in terms of baby Verma modules and the fact that it is equivalent to the definition given in §1.D. \square

3.B. Bialynicki-Birula decomposition. — We denote by $\mathfrak{F}_k^{\text{att}}$ the *attracting set* of \mathfrak{F}_k for the action of \mathbb{C}^\times , namely

$$\mathfrak{F}_k^{\text{att}} = \{p \in \mathfrak{F}_k \mid \lim_{\xi \rightarrow 0} \xi p \text{ exists}\}.$$

Recall from [BoRo, Chap. 9] the following facts:

Proposition 3.2. — *With the above notation, we have:*

- (a) *The map $\lim : \mathfrak{F}_k^{\text{att}} \rightarrow \mathfrak{F}_k^{\mathbb{C}^\times}$, $p \mapsto \lim_{\xi \rightarrow 0} \xi p$ is a morphism of varieties.*
- (b) $\mathfrak{F}_k^{\text{att}} = \Upsilon_k^{-1}(V/W \times \{0\})$.
- (c) *If \mathcal{F} is an irreducible component of $\mathfrak{F}_k^{\text{att}}$, then \mathcal{F} is \mathbb{C}^\times -stable and $\Upsilon_k(\mathcal{F}) = V/W \times \{0\}$ and $\lim(\mathcal{F})$ is a single point.*
- (d) *If $\chi \in \text{Irr}(W)$, then the support of $e\Delta(\chi)$ is a union of irreducible components of $\mathfrak{F}_k^{\text{att}}$.*
- (e) *If \mathcal{F} is an irreducible component of $\mathfrak{F}_k^{\text{att}}$, then there exists $\chi \in \text{Irr}(W)$ such that the support of $e\Delta(\chi)$ contains \mathcal{F} .*

Proof. — (a) is classical (see for instance [BoRo, §9.1]). For (b), see [BoRo, Lem. 9.3.2]. (c) is explained at the end of [BoRo, §9.3]. For (d) and (e), see [BoRo, (8.1.3) and Prop. 9.3.3]. \square

We characterize points of $\mathfrak{F}_k^{\mathbb{C}^\times}$ belonging to $(\mathfrak{F}_k^\tau)_{\text{max}}$ in terms of Verma modules:

Lemma 3.3. — *Let $p \in \mathfrak{F}_k^{\mathbb{C}^\times}$ and assume that $\tau(p) = p$. Then $p \in (\mathfrak{F}_k^\tau)_{\text{max}}$ if and only if there exist $\chi \in \mathfrak{F}_p^k$ and $(v, v^*) \in V_{\text{reg}}^\tau \times V^{*\tau}$ such that $e\Delta(\chi)_{W \cdot (v, v^*)} \neq 0$.*

Proof. — Let $(\mathfrak{F}_k^\tau)_{\text{max}}^{\text{att}}$ denote the attracting set of $(\mathfrak{F}_k^\tau)_{\text{max}}$. Then (2.4) and Proposition 3.2(b) imply that $\Upsilon_k((\mathfrak{F}_k^\tau)_{\text{max}}^{\text{att}}) = (V/W)^\tau \times \{0\}$. Since Υ_k is a finite morphism, the same arguments used in [BoRo, Chap. 9] to prove the Proposition 3.2 above yields the following statements:

- (a) *The map $\lim : (\mathfrak{F}_k^\tau)_{\text{max}}^{\text{att}} \rightarrow (\mathfrak{F}_k^\tau)_{\text{max}}^{\mathbb{C}^\times}$, $p \mapsto \lim_{\xi \rightarrow 0} \xi p$ is a morphism of varieties.*
- (b) $(\mathfrak{F}_k^\tau)_{\text{max}}^{\text{att}} = (\mathfrak{F}_k^\tau)_{\text{max}} \cap \Upsilon_k^{-1}((V/W)^\tau \times \{0\})$.
- (c) *If \mathcal{F} is an irreducible component of $(\mathfrak{F}_k^\tau)_{\text{max}}^{\text{att}}$, then \mathcal{F} is \mathbb{C}^\times -stable and $\Upsilon_k(\mathcal{F}) = (V/W)^\tau \times \{0\}$ and $\lim(\mathcal{F})$ is a single point.*

Assume that $p \in (\mathcal{X}_k^\tau)_{\max}$. Let \mathcal{F} be an irreducible component of $(\mathcal{X}_k^\tau)_{\max}^{\text{att}} \cap \lim^{-1}(p)$. Then \mathcal{F} is contained in an irreducible component \mathcal{F}' of $(\mathcal{X}_k^\tau)_{\max}^{\text{att}}$. Since $\lim(\mathcal{F}')$ is a single point by (c), we have $\lim(\mathcal{F}') = \{p\}$ and so $\mathcal{F} = \mathcal{F}'$. Still by (c), this says that $\Upsilon_k(\mathcal{F}) = (V/W)^\tau \times \{0\}$. So let $q \in \mathcal{F}$ be such that $\Upsilon_k(q) \in (V_{\text{reg}}/W)^\tau \times \{0\}$.

Now, let \mathcal{F} be an irreducible component of $\mathcal{X}_k^{\text{att}}$ containing \mathcal{F} . By Proposition 3.2(e), there exists $\chi \in \text{Irr}(W)$ such that the support of $e\Delta(\chi)$ contains \mathcal{F} . In particular, $e\Delta(\chi)_p \neq 0$ and so $\chi \in \mathfrak{F}_p$ by Lemma 3.1. But also $e\Delta(\chi)_q \neq 0$. Since $q \in (\mathcal{X}_k^\tau)_{\max}$ and $\Upsilon_k(q) \in V_{\text{reg}}/W$, it follows that there exists $(v, v^*) \in V_{\text{reg}}^\tau \times V^{*\tau}$ such that $e\Delta(\chi)_{W \cdot (v, v^*)} \neq 0$, as desired.

Conversely, assume that there exist both $\chi \in \mathfrak{F}_p^k$ and $(v, v^*) \in V_{\text{reg}}^\tau \times V^{*\tau}$ such that $e\Delta(\chi)_{W \cdot (v, v^*)} \neq 0$. Let \mathcal{F} be an irreducible component of $\mathcal{X}_k^{\text{att}}$ contained in the support of $e\Delta(\chi)$. Then $p \in \mathcal{F}$ and so $p = \lim W \cdot (v, v^*)$. Since $W \cdot (v, v^*) \in (\mathcal{X}_k^\tau)_{\max}$ by the definition of $(\mathcal{X}_k^\tau)_{\max}$, this implies that $p \in (\mathcal{X}_k^\tau)_{\max}$, as desired. \square

4. Gaudin algebra

4.A. Definition. — We recall here the definition of *Gaudin algebra* [BoRo, §8.3.B]. First, let $\mathbb{C}[V_{\text{reg}}][W]$ denote the group algebra of W over the algebra $\mathbb{C}[V_{\text{reg}}]$ (and not the semi-direct product $\mathbb{C}[V_{\text{reg}}] \rtimes W$). For $y \in V$, let

$$\mathcal{D}_y^k = \sum_{s \in \text{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\alpha_s} s \in \mathbb{C}[V_{\text{reg}}][W].$$

Now, let $\text{Gau}_k(W)$ be the sub- $\mathbb{C}[V_{\text{reg}}]$ -algebra of $\mathbb{C}[V_{\text{reg}}][W]$ generated by the \mathcal{D}_y^k 's (where y runs over V): it will be called the *Gaudin algebra (with parameter k)* associated with W .

Let $\mathbb{C}(V)$ denote the function field of V (which is the fraction field of $\mathbb{C}[V]$ or of $\mathbb{C}[V_{\text{reg}}]$) and let $\mathbb{C}(V)\text{Gau}_k(W)$ denote the subalgebra $\mathbb{C}(V) \otimes_{\mathbb{C}[V_{\text{reg}}]} \text{Gau}_k(W)$ of the group algebra $\mathbb{C}(V)[W]$. Recall [BoRo, §8.3.B] that

$$(4.1) \quad \text{Gau}_k(W) \text{ is a commutative algebra,}$$

but that $\mathbb{C}(V)\text{Gau}_k(W)$ is generally non-split, as shown by the examples treated in [Bon1, §4] and [Lac].

4.B. Generalized eigenspaces. — If $v \in V_{\text{reg}}$, we denote by $\mathcal{D}_y^{k,v}$ the specialization of \mathcal{D}_y^k at v , namely $\mathcal{D}_y^{k,v}$ is the element of the group algebra $\mathbb{C}W$ equal to

$$\mathcal{D}_y^{k,v} = \sum_{s \in \text{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\langle v, \alpha_s \rangle} s.$$

Now, if $v^* \in V^*$ and if M is a $\mathbb{C}W$ -module, we define M^{k,v,v^*} to be the common generalized eigenspace of the operators $\mathcal{D}_y^{k,v}$ for the eigenvalue $\langle y, v^* \rangle$, for y running over V . Namely,

$$M^{k,v,v^*} = \{m \in M \mid \forall y \in V, (\mathcal{D}_y^{k,v} - \langle y, v^* \rangle \text{Id}_M)^{\dim(M)}(m) = 0\}.$$

Then

$$(4.2) \quad M = \bigoplus_{v^* \in V^*} M^{k,v,v^*},$$

since $\text{Gau}_k(W)$ is commutative.

Lemma 4.3. — *Let $\chi \in \text{Irr}(W)$ and let $(v, v^*) \in V_{\text{reg}} \times V^*$. Then the following are equivalent:*

- (1) $e\Delta(\chi)_{W \cdot (v, v^*)} \neq 0$.
- (2) $\Delta^{\text{reg}}(\chi)_{v, v^*} \neq 0$.
- (3) $E_{\chi}^{k, v, v^*} \neq 0$.

Proof. — The equivalence between (1) and (2) follows from the Morita equivalence between $\mathbb{C}[V_{\text{reg}} \times V^*]^W$ and $\mathbb{C}[V_{\text{reg}} \times V^*] \rtimes W$ proved in [BoRo, Lem. 3.1.8(b)]. Now, as a $(\mathbb{C}[V_{\text{reg}}] \rtimes W)$ -module, $\Delta^{\text{reg}}(\chi) \simeq \mathbb{C}[V_{\text{reg}}] \otimes E_{\chi}$. and the action of $y \in V \subset \mathbb{C}[V^*]$ is given by the operator $-\mathcal{D}_y^k \in \mathbb{C}[V_{\text{reg}}][W]$ (see [BoRo, §8.3.B]). Now, $\Delta^{\text{reg}}(\chi)_{v, v^*} \simeq E_{\chi}$ as a \mathbb{C} -vector space: on this vector space, the action of an element $f \in \mathbb{C}[V_{\text{reg}}]$ is given by multiplication by $f(v)$ while the action of an element $y \in V$ is given by the operator $\langle y, v^* \rangle \text{Id}_{E_{\chi}} - \mathcal{D}_y^{k, v}$ (see [BoRo, Theo. 4.1.7]). This shows the equivalence between (2) and (3). \square

4.C. Proof of Theorem A (i.e. Theorem 2.8). — Let $\chi \in \text{Irr}(W)$ be τ -stable and such that $\tilde{\chi}(\tau) \neq 0$ and let $v \in V_{\text{reg}}^{\tau}$. By Lemmas 3.1 and 4.3, it is sufficient to show that there exists $v^* \in V^{*\tau}$ such that $E_{\chi}^{k, v, v^*} \neq 0$.

For this, let \mathcal{E} denote the set of $v^* \in V^*$ such that $E_{\chi}^{k, v, v^*} \neq 0$. Then it follows from (4.2) that

$$(*) \quad E_{\chi} = \bigoplus_{v^* \in \mathcal{E}} E_{\chi}^{k, v, v^*}.$$

Since $\tau(v) = v$, we have

$$\tau \mathcal{D}_y^{k, v} = \sum_{s \in \text{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\langle v, \alpha_s \rangle} \tau s \tau^{-1} = \sum_{s \in \text{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \tau^{-1}(\alpha_s) \rangle}{\langle v, \tau^{-1}(\alpha_s) \rangle} s = \mathcal{D}_{\tau(y)}^{k, v}.$$

Consequently,

$$\tau E_{\chi}^{k, v, v^*} = E_{\chi}^{k, v, \tau(v^*)}.$$

But $\tilde{\chi}(\tau) = \text{Tr}(\tau, E_{\chi}) \neq 0$, so τ must fix at least one of the generalized eigenspaces in the decomposition (*). In other words, this implies that there exists $v^* \in \mathcal{E}$ such that $\tau(v^*) = v^*$, as desired. The proof is complete.

5. Complements: further conjectures, examples

5.A. Conjectures. — The variety \mathcal{E}_k is endowed with a Poisson structure [EtGi, §1] and so the variety of fixed points \mathcal{E}_k^{τ} inherits a Poisson structure too, as well as all its irreducible components. Recall from Springer Theorem 1.6 that W^{τ} is a reflection group for its action on V^{τ} , so we can define a set of pairs \mathfrak{N}_{τ} for the pair (V^{τ}, W^{τ}) as well as \mathfrak{N} has been defined for the pair (V, W) and, for each parameter $l \in \mathbb{C}^{\mathfrak{N}_{\tau}}$, we can define a Calogero-Moser space $\mathcal{E}_l(V^{\tau}, W^{\tau})$. The following conjecture is a particular case of [Bon2, Conj. B] (see [Bon2] for a discussion about the cases where this conjecture is known to hold):

Conjecture 5.1. — *Recall that τ is regular. Then there exists a linear map $\lambda : \mathbb{C}^{\mathfrak{N}} \rightarrow \mathbb{C}^{\mathfrak{N}_{\tau}}$ and, for each $k \in \mathbb{C}^{\mathfrak{N}}$, a \mathbb{C}^{\times} -equivariant isomorphism of Poisson varieties*

$$\iota_k : (\mathcal{E}_k^{\tau})_{\max} \xrightarrow{\sim} \mathcal{E}_{\lambda(k)}(V^{\tau}, W^{\tau}).$$

If the existence of such a \mathbb{C}^\times -equivariant isomorphism $\iota_k : (\mathcal{F}_k^\tau)_{\max} \xrightarrow{\sim} \mathcal{F}_{\lambda(k)}(V^\tau, W^\tau)$ is known but if it is not known that it preserves the Poisson structure, then we will say that “Conjecture 5.1⁻ holds”.

Assume now that Conjecture 5.1⁻ holds and keep its notation. Then ι_k restricts to a map $\iota_k : (\mathcal{F}_k)_{\max}^{\mathbb{C}^\times} \xrightarrow{\sim} \mathcal{F}_{\lambda(k)}(V^\tau, W^\tau)^{\mathbb{C}^\times}$. If $p \in \mathcal{F}_{\lambda(k)}(V^\tau, W^\tau)^{\mathbb{C}^\times}$, we denote by $\mathfrak{F}_{\iota_k(p)}^{(\tau)}$ the corresponding Calogero-Moser $\lambda(k)$ -family of W^τ . The next conjecture, still inspired by the representation theory of finite reductive groups (see again [Bon3, Ex. 12.9] for some explanations), makes Conjecture B more precise:

Conjecture 5.2. — *Recall that τ is regular and assume that Conjecture 5.1⁻ holds.*

If $p \in (\mathcal{F}_k^\tau)_{\max}^{\mathbb{C}^\times}$, then

$$\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 = \sum_{\psi \in \mathfrak{F}_{\iota_k(p)}^{(\tau)}} \psi(1)^2.$$

Note that this last conjecture is compatible with the fact that

$$\sum_{\chi \in \text{Irr}(W)^\tau} |\tilde{\chi}(\tau)|^2 = |W^\tau| = \sum_{\psi \in \text{Irr}(W^\tau)} \psi(1)^2,$$

where the first equality follows from the second orthogonality relation for characters applied to $W\langle\tau\rangle$. Indeed, $\sum_{\theta \in \text{Irr}(W\langle\tau\rangle)} |\theta(\tau)|^2 = |C_{W\langle\tau\rangle}(\tau)|$ and $\theta(\tau) \neq 0$ implies that θ is an extension of a τ -invariant character χ of W by [Isa, Theo. 6.11]: the equality then follows from the fact that $|\theta(\tau)|$ depends only on χ and that there are $|W\langle\tau\rangle|/|W|$ extensions of χ by [Isa, Coro. 6.17].

5.B. Roots of unity. — We consider in this subsection a particular (but very important) case of the general situation studied in this paper. We fix a natural number $d \geq 1$ and a primitive d -th root of unity ζ_d . The group of d -th roots of unity is denoted by μ_d . An element $w \in W$ is called ζ_d -regular if the element $\zeta_d^{-1}w$ of $N_{\text{GL}_{\mathbb{C}}(V)}(W)$ is regular. In other words, w is ζ_d -regular if and only if its ζ_d -eigenspace meets V_{reg} . The existence of a ζ_d -regular element is not guaranteed: we say that d is a *regular number* of W if such an element exists.

Hypothesis. *We assume in this subsection that d is a regular number of W . We denote by w_d a ζ_d -regular element and we also set $\tau_d = \zeta_d^{-1}w_d$, so that τ_d is a regular element of $N_{\text{GL}_{\mathbb{C}}(V)}(W)$.*

Recall from [Spr, Theo. 4.2(iv)] that w_d is uniquely defined up to conjugacy. Note that

$$(5.3) \quad V^{\tau_d} = \text{Ker}(w_d - \zeta_d \text{Id}_V), \quad W^{\tau_d} = C_W(w_d) \quad \text{and} \quad \mathcal{F}_k^{\tau_d} = \mathcal{F}_k^{\mu_d}.$$

Since τ_d induces an inner automorphism of W , all the irreducible characters are τ_d -stable. Moreover, if $\chi \in \text{Irr}(W)$, then $\tilde{\chi}(\tau_d) = \xi\chi(w_d)$ for some root of unity ξ , so $|\tilde{\chi}(\tau_d)|^2 = |\chi(w_d)|^2$. This allows to reformulate both Theorem A and Conjecture B in this case:

Conjecture 5.4. — *Recall that d is a regular number. Let $p \in \mathcal{F}_k^{\mathbb{C}^\times}$. Then p belongs to $(\mathcal{F}_k^{\mu_d})_{\max}$ if and only if $\sum_{\chi \in \mathfrak{F}_p} |\chi(w_d)|^2 \neq 0$.*

Theorem 5.5. — Recall that d is regular. Let $p \in \mathcal{X}_k^{\mathbb{C}^\times}$ be such that $\sum_{\chi \in \mathfrak{S}_p} |\chi(w_d)|^2 \neq 0$. Then p belongs to $(\mathcal{X}_k^{\mu_d})_{\max}$.

Example 5.6 (Symmetric group). — We assume here that $W = \mathfrak{S}_n$ acting on $V = \mathbb{C}^n$ by permutation of the coordinates, for some $n \geq 2$. The canonical basis of \mathbb{C}^n is denoted by (y_1, \dots, y_n) . Then there is a unique orbit of hyperplanes, that we denote by Ω , and $e_\Omega = 2$. To avoid too easy cases, we also assume that $k_{\Omega,0} \neq k_{\Omega,1}$ (so that \mathcal{X}_k is smooth [EtGi, Cor. 1.14]) and that $d \geq 2$. Saying that d is a regular number is equivalent to say that d divides n or $n-1$. Therefore, we will denote by j the unique element of $\{0, 1\}$ such that d divides $n-j$ and we set $r = (n-j)/d$. Then w_d is the product of r disjoint cycles of length d , so one can choose for instance

$$w_d = (1, 2, \dots, d)(d+1, d+2, \dots, 2d) \cdots ((r-1)d+1, (r-1)d+2, \dots, rd).$$

Then V^{τ_d} is r -dimensional, with basis (v_1, \dots, v_r) where $v_a = \sum_{b=1}^d \zeta_d^{-b} e_{(a-1)d+b}$ and the group $C_W(w_d) \simeq G(d, 1, r)$ acts “naturally” as a reflection group on $V^{\tau_d} = \bigoplus_{a=1}^r \mathbb{C}v_a$.

We also need some combinatorics. We denote by $\text{Part}(n)$ (resp. $\text{Part}^d(r)$) the set of partitions of n (resp. of d -partitions of r). If $\lambda \in \text{Part}(n)$, we denote by χ_λ the irreducible character of \mathfrak{S}_n (with the convention of [GePf]: for instance $\chi_n = 1$ and $\chi_{1^n} = \varepsilon$), by $\text{cor}_d(\lambda)$ the d -core of λ , by $\text{quo}_d(\lambda)$ its d -quotient. We let $\text{Part}(n, d)$ denote the set of partitions of n whose d -core is the unique partition of $j \in \{0, 1\}$. Then the map

$$\text{quo}_d : \text{Part}(n, d) \longrightarrow \text{Part}^d(r)$$

is bijective. Finally, if $\mu \in \text{Part}^d(r)$, we denote by χ_μ the associated irreducible character of $C_W(w_d) = G(d, 1, r)$ (with the convention of [GeJa]). It follows from Murnaghan-Nakayama rule that

$$(5.7) \quad \chi_\lambda(w_d) \neq 0 \text{ if and only if } \lambda \in \text{Part}(n, d),$$

and that

$$(5.8) \quad \chi_\lambda(w_d) = \pm \chi_{\text{quo}_d(\lambda)}(1)$$

for all $\lambda \in \text{Part}(n, d)$ (see for instance [BMM1, Page 47]).

Now, the smoothness of \mathcal{X}_k implies that the map $\varphi_k : \text{Irr}(\mathfrak{S}_n) \longrightarrow \mathcal{X}_k^{\mathbb{C}^\times}$ is bijective (so that Calogero-Moser k -families of \mathfrak{S}_n are singleton) and it follows from the main theorem of [BoMa] that Conjecture 5.1 holds (except that we do not know if the isomorphism respects the Poisson structure), so that we have a \mathbb{C}^\times -equivariant isomorphism of varieties

$$\iota_k : (\mathcal{X}_k^{\mu_d})_{\max} \xrightarrow{\sim} \mathcal{X}_{\lambda(k)}(V^{\tau_d}, G(d, 1, r))$$

for some explicit $\lambda(k) \in \mathbb{C}^{\mathbb{N}^{\tau_d}}$. Moreover, $\mathcal{X}_{\lambda(k)}(V^{\tau_d}, G(d, 1, r))$ is smooth so that the map $\varphi_{\lambda(k)}^{\tau_d} : \text{Irr}(G(d, 1, r)) \longrightarrow \mathcal{X}_{\lambda(k)}(V^{\tau_d}, G(d, 1, r))^{\mathbb{C}^\times}$ is bijective (that is, Calogero-Moser $\lambda(k)$ -families of $G(d, 1, r)$ are singleton). Now, by [BoMa, Theo. 4.21], we have that

$$(5.9) \quad \varphi_k(\chi_\lambda) \in (\mathcal{X}_k^{\mu_d})_{\max} \text{ if and only if } \lambda \in \text{Part}(n, d),$$

and that

$$(5.10) \quad \iota_k(\varphi_k(\chi_\lambda)) = \varphi_{\lambda(k)}^{\tau_d}(\chi_{\text{quo}_d(\lambda)})$$

for all $\lambda \in \text{Part}(n, d)$. Then (5.7), (5.8), (5.9) and (5.10) show that Conjectures 5.4 and 5.2 hold for the symmetric group. ■

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October 14, 2023

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