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# REGULAR AUTOMORPHISMS AND CALOGERO-MOSER FAMILIES

by

CÉDRIC BONNAFÉ

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**Abstract.** — We study the subvariety of fixed points of an automorphism of a Calogero-Moser space induced by a regular element of finite order of the normalizer of the associated complex reflection group  $W$ . We determine some of (and conjecturally all) the  $\mathbb{C}^\times$ -fixed points of its unique irreducible component of maximal dimension in terms of the character table of  $W$ . This is inspired by the mysterious relations between the geometry of Calogero-Moser spaces and unipotent representations of finite reductive groups, which is the theme of another paper [Bon3].

Let  $V$  be a finite dimensional vector space and let  $W$  be a finite subgroup of  $\mathbf{GL}_{\mathbb{C}}(V)$  generated by reflections. To some parameter  $k$ , Etingof and Ginzburg [EtGi] have associated a normal irreducible affine complex variety  $\mathcal{X}_k = \mathcal{X}_k(V, W)$  called a (generalized) *Calogero-Moser space*. If  $\tau$  is an element of finite order of the normalizer of  $W$  in  $\mathbf{GL}_{\mathbb{C}}(V)$  stabilizing the parameter  $k$ , it induces an automorphism of  $\mathcal{X}_k$ .

We denote by  $V_{\text{reg}}^\tau$  the open subset of  $V$  on which  $W$  acts freely, and we assume that  $V_{\text{reg}}^\tau \neq \emptyset$  (then  $\tau$  is called *regular*). In this case, there exists a unique irreducible component  $(\mathcal{X}_k^\tau)_{\text{max}}$  of  $\mathcal{X}_k^\tau$  of maximal dimension (as it will be explained in Section 2). Recall that  $\mathcal{X}_k$  is endowed with a  $\mathbb{C}^\times$ -action and that we have a surjective map  $\text{Irr}(W) \rightarrow \mathcal{X}_k^{\mathbb{C}^\times}$  defined by Gordon [Gor] (induced by the action of the center of a rational Cherednik algebra on *baby Verma modules*) whose fibers are called the *Calogero-Moser  $k$ -families* of  $W$ . If  $p \in \mathcal{X}_k^{\mathbb{C}^\times}$ , we denote by  $\mathfrak{F}_p$  its associated Calogero-Moser  $k$ -family. It is a natural question to wonder which  $\mathbb{C}^\times$ -fixed points of  $\mathcal{X}_k^\tau$  belong to  $(\mathcal{X}_k^\tau)_{\text{max}}$ . The aim of this note is to provide a partial answer in terms of the character table of  $W$ :

**Theorem A.** — Assume that  $V_{\text{reg}}^\tau \neq \emptyset$ . Let  $p \in \mathcal{X}_k^{\mathbb{C}^\times}$  be such that  $\tau(p) = p$ . If  $\sum_{\chi \in \mathfrak{F}_p} |\tilde{\chi}(\tau)|^2 \neq 0$ , then  $p \in (\mathcal{X}_k^\tau)_{\text{max}}$ .

In this statement, if  $\chi$  is a  $\tau$ -stable irreducible character of  $W$ , we denote by  $\tilde{\chi}$  an extension of  $\chi$  to the finite group  $W\langle\tau\rangle$  (note that  $|\tilde{\chi}(\tau)|^2$  does not depend on the choice of  $\tilde{\chi}$ ). Our proof of Theorem A makes an extensive use of the Gaudin operators introduced in [BoRo, §8.3.B]. This result is also inspired by the theory of unipotent representations of finite reductive groups and some conjectures of Broué-Michel [BrMi] on the cohomology of Deligne-Lusztig varieties associated with regular elements in the sense of Springer [Spr] and

by [BMM2, Rem. 4.21] (this will also be discussed in [Bon3]). If we believe in this analogy, we can conjecture that the converse of Theorem A holds:

**Conjecture B.** — Assume that  $V_{\text{reg}}^\tau \neq \emptyset$ . Let  $p \in \mathfrak{X}_k^{\mathbb{C}^\times}$  be such that  $\tau(p) = p$ . Then  $p \in (\mathfrak{X}_k^\tau)_{\text{max}}$  if and only if  $\sum_{\chi \in \mathfrak{S}_p^\tau} |\tilde{\chi}(\tau)|^2 \neq 0$ .

**General notation.** Throughout this paper, we will abbreviate  $\otimes_{\mathbb{C}}$  as  $\otimes$  and all varieties will be algebraic, complex, quasi-projective and reduced. If  $\mathfrak{X}$  is an affine variety, we denote by  $\mathbb{C}[\mathfrak{X}]$  its coordinate ring.

If  $X$  is a subset of a vector space  $V$  (or of its dual  $V^*$ ), and if  $\Gamma$  is a subgroup of  $\mathbf{GL}_{\mathbb{C}}(V)$ , we denote by  $\Gamma_X$  the pointwise stabilizer of  $X$ . If moreover  $\Gamma$  is finite, we will identify  $(V^\Gamma)^*$  and  $(V^*)^\Gamma$ .

## 1. Set-up

**Hypothesis and notation.** We fix in this paper a finite dimensional complex vector space  $V$  and a finite subgroup  $W$  of  $\mathbf{GL}_{\mathbb{C}}(V)$ . We set

$$\text{Ref}(W) = \{s \in W \mid \text{codim}_{\mathbb{C}} V^s = 1\}$$

and we assume throughout this paper that

$$W = \langle \text{Ref}(W) \rangle,$$

i.e. that  $W$  is a complex reflection group.

**1.A. About  $W$ .** — We set  $\varepsilon : W \rightarrow \mathbb{C}^\times$ ,  $w \mapsto \det(w)$ . We identify  $\mathbb{C}[V]$  (resp.  $\mathbb{C}[V^*]$ ) with the symmetric algebra  $S(V^*)$  (resp.  $S(V)$ ).

We denote by  $\mathcal{A}$  the set of reflecting hyperplanes of  $W$ , namely

$$\mathcal{A} = \{V^s \mid s \in \text{Ref}(W)\}.$$

If  $H \in \mathcal{A}$ , we denote by  $\alpha_H$  an element of  $V^*$  such that  $H = \text{Ker}(\alpha_H)$  and by  $\alpha_H^\vee$  an element of  $V$  such that  $V = H \oplus \mathbb{C}\alpha_H^\vee$  and the line  $\mathbb{C}\alpha_H^\vee$  is  $W_H$ -stable. We set  $e_H = |W_H|$ . Note that  $W_H$  is cyclic of order  $e_H$  and that  $\text{Irr}(W_H) = \{\text{Res}_{W_H}^W \varepsilon^j \mid 0 \leq j \leq e_H - 1\}$ . We denote by  $\varepsilon_{H,j}$  the (central) primitive idempotent of  $\mathbb{C}W_H$  associated with the character  $\text{Res}_{W_H}^W \varepsilon^{-j}$ , namely

$$\varepsilon_{H,j} = \frac{1}{e_H} \sum_{w \in W_H} \varepsilon(w)^j w \in \mathbb{C}W_H.$$

If  $\Omega$  is a  $W$ -orbit of reflecting hyperplanes, we write  $e_\Omega$  for the common value of all the  $e_H$ , where  $H \in \Omega$ . We denote by  $\aleph$  the set of pairs  $(\Omega, j)$  where  $\Omega \in \mathcal{A}$  and  $0 \leq j \leq e_\Omega - 1$ . The vector space of families of complex numbers indexed by  $\aleph$  will be denoted by  $\mathbb{C}^\aleph$ : elements of  $\mathbb{C}^\aleph$  will be called *parameters*. If  $k = (k_{\Omega,j})_{(\Omega,j) \in \aleph} \in \mathbb{C}^\aleph$ , we define  $k_{H,j}$  for all  $H \in \Omega$  and  $j \in \mathbb{Z}$  by  $k_{H,j} = k_{\Omega,j_0}$  where  $\Omega$  is the  $W$ -orbit of  $H$  and  $j_0$  is the unique element of  $\{0, 1, \dots, e_H - 1\}$  such that  $j \equiv j_0 \pmod{e_H}$ .

We denote by  $V_{\text{reg}}$  the set of elements  $v$  of  $V$  such that  $W_v = 1$ . It is an open subset of  $V$  and recall from Steinberg-Serre Theorem [Bro, Theo. 4.7] that

$$(1.1) \quad V_{\text{reg}} = V \setminus \bigcup_{H \in \mathcal{A}} H.$$

**1.B. Rational Cherednik algebra at  $t = 0$ .** — Let  $k \in \mathbb{C}^\times$ . We define the *rational Cherednik algebra*  $\mathbf{H}_k$  (at  $t = 0$ ) to be the quotient of the algebra  $\mathbb{T}(V \oplus V^*) \rtimes W$  (the semi-direct product of the tensor algebra  $\mathbb{T}(V \oplus V^*)$  with the group  $W$ ) by the relations

$$(1.2) \quad \begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = \sum_{H \in \mathcal{A}} \sum_{j=0}^{e_H-1} e_H(k_{H,j} - k_{H,j+1}) \frac{\langle y, \alpha_H \rangle \cdot \langle \alpha_H^\vee, x \rangle}{\langle \alpha_H^\vee, \alpha_H \rangle} \varepsilon_{H,j}, \end{cases}$$

for all  $x, x' \in V^*$ ,  $y, y' \in V$ . Here  $\langle \cdot, \cdot \rangle : V \times V^* \rightarrow \mathbb{C}$  is the standard pairing. The first commutation relations imply that we have morphisms of algebras  $\mathbb{C}[V] \rightarrow \mathbf{H}_k$  and  $\mathbb{C}[V^*] \rightarrow \mathbf{H}_k$ . Recall [EtGi, Theo. 1.3] that we have an isomorphism of  $\mathbb{C}$ -vector spaces

$$(1.3) \quad \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} \mathbf{H}_k$$

induced by multiplication (this is the so-called *PBW-decomposition*).

**Remark 1.4.** — Let  $(l_\Omega)_{\Omega \in \mathcal{A}/W}$  be a family of complex numbers and let  $k' \in \mathbb{C}^\times$  be defined by  $k'_{\Omega,j} = k_{\Omega,j} + l_\Omega$ . Then  $\mathbf{H}_k = \mathbf{H}_{k'}$ . This means that there is no restriction to generality if we consider for instance only parameters  $k$  such that  $k_{\Omega,0} = 0$  for all  $\Omega$ , or only parameters  $k$  such that  $k_{\Omega,0} + k_{\Omega,1} + \cdots + k_{\Omega,e_\Omega-1} = 0$  for all  $\Omega$  (as in [BoRo]). ■

**1.C. Calogero-Moser space.** — We denote by  $\mathbf{Z}_k$  the center of the algebra  $\mathbf{H}_k$ : it is well-known [EtGi, Theo 3.3 and Lem. 3.5] that  $\mathbf{Z}_k$  is an integral domain, which is integrally closed. Moreover, it contains  $\mathbb{C}[V]^W$  and  $\mathbb{C}[V^*]^W$  as subalgebras [Gor, Prop. 3.6] (so it contains  $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ ), and it is a free  $\mathbf{P}$ -module of rank  $|W|$ . We denote by  $\mathcal{X}_k$  the affine algebraic variety whose ring of regular functions  $\mathbb{C}[\mathcal{X}_k]$  is  $\mathbf{Z}_k$ : this is the *Calogero-Moser space* associated with the datum  $(V, W, k)$ . It is irreducible and normal.

We set  $\mathcal{P} = V/W \times V^*/W$ , so that  $\mathbb{C}[\mathcal{P}] = \mathbf{P}$  and the inclusion  $\mathbf{P} \hookrightarrow \mathbf{Z}_k$  induces a morphism of varieties

$$\Upsilon_k : \mathcal{X}_k \longrightarrow \mathcal{P}$$

which is finite and flat.

**1.D. Calogero-Moser families.** — Using the PBW-decomposition, we define a  $\mathbb{C}$ -linear map  $\Omega^{\mathbf{H}_k} : \mathbf{H}_k \rightarrow \mathbb{C}W$  by

$$\Omega^{\mathbf{H}_k}(fwg) = f(0)g(0)w$$

for all  $f \in \mathbb{C}[V]$ ,  $g \in \mathbb{C}[V^*]$  and  $w \in \mathbb{C}W$ . This map is  $W$ -equivariant for the action on both sides by conjugation, so it induces a well-defined  $\mathbb{C}$ -linear map

$$\Omega^k : \mathbf{Z}_k \longrightarrow Z(\mathbb{C}W).$$

Recall from [BoRo, Cor. 4.2.11] that  $\Omega^k$  is a morphism of algebras.

Calogero-Moser families were defined by Gordon using his theory of *baby Verma modules* [Gor, §4.2 and §5.4]. We explain here an equivalent definition given in [BoRo, §7.2]. If  $\chi \in \text{Irr}(W)$ , we denote by  $\omega_\chi : Z(\mathbb{C}W) \rightarrow \mathbb{C}$  its central character (i.e.,  $\omega_\chi(z) = \chi(z)/\chi(1)$  is the scalar by which  $z$  acts on an irreducible representation affording the character  $\chi$ ). We say that two characters  $\chi$  and  $\chi'$  belong to the same *Calogero-Moser  $k$ -family* if  $\omega_\chi \circ \Omega^k = \omega_{\chi'} \circ \Omega^k$ .

In other words, the map  $\omega_\chi \circ \Omega^k : \mathbf{Z}_k \rightarrow \mathbb{C}$  is a morphism of algebras, so it might be viewed as a point  $\varphi_k(\chi)$  of  $\mathcal{F}_k$ , which is easily checked to be  $\mathbb{C}^\times$ -fixed. This defines a surjective map

$$\varphi_k : \text{Irr}(W) \longrightarrow \mathcal{F}_k^{\mathbb{C}^\times}$$

whose fibers are the Calogero-Moser  $k$ -families. If  $p \in \mathcal{F}_k^{\mathbb{C}^\times}$ , we denote by  $\mathfrak{F}_p$  the corresponding Calogero-Moser  $k$ -family.

**1.E. Other parameters.** — Let  $\mathcal{C}$  denote the space of maps  $\text{Ref}(W) \rightarrow \mathbb{C}$  which are constant on conjugacy classes of reflections. The element

$$\sum_{(\Omega, j) \in \mathbb{N}} \sum_{H \in \Omega} (k_{H, j} - k_{H, j+1}) e_{H \varepsilon_{H, j}}$$

of  $Z(\mathbb{C}W)$  is supported only by reflections, so there exists a unique map  $c_k \in \mathcal{C}$  such that

$$\sum_{(\Omega, j) \in \mathbb{N}} \sum_{H \in \Omega} (k_{H, j} - k_{H, j+1}) e_{H \varepsilon_{H, j}} = \sum_{s \in \text{Ref}(W)} (\varepsilon(s) - 1) c_k(s) s.$$

Then the map  $\mathbb{C}^\mathbb{N} \rightarrow \mathcal{C}$ ,  $k \mapsto c_k$  is linear and surjective. With this notation, we have

$$(1.5) \quad [y, x] = \sum_{s \in \text{Ref}(W)} (\varepsilon(s) - 1) c_k(s) \frac{\langle y, \alpha_s \rangle \cdot \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s,$$

for all  $y \in V$  and  $x \in V^*$ . Here,  $\alpha_s = \alpha_{V^s}$  and  $\alpha_s^\vee = \alpha_{V^s}^\vee$ .

**1.F. Actions on the Calogero-Moser space.** — The Calogero-Moser space  $\mathcal{F}_k$  is endowed with a  $\mathbb{C}^\times$ -action and an action of the stabilizer of  $k$  in  $N_{\text{GL}_{\mathbb{C}}(V)}(W)$ , which are described below.

*1.F.1. Grading,  $\mathbb{C}^\times$ -action.* — The algebra  $T(V \oplus V^*) \rtimes W$  can be  $\mathbb{Z}$ -graded in such a way that the generators have the following degrees

$$\begin{cases} \deg(y) = -1 & \text{if } y \in V, \\ \deg(x) = 1 & \text{if } x \in V^*, \\ \deg(w) = 0 & \text{if } w \in W. \end{cases}$$

This descends to a  $\mathbb{Z}$ -grading on  $\mathbf{H}_k$ , because the defining relations (1.2) are homogeneous. Since the center of a graded algebra is always graded, the subalgebra  $\mathbf{Z}_k$  is also  $\mathbb{Z}$ -graded. So the Calogero-Moser space  $\mathcal{F}_k$  inherits a regular  $\mathbb{C}^\times$ -action. Note also that by definition  $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$  is clearly a graded subalgebra of  $\mathbf{Z}_k$ .

1.F.2. *Action of the normalizer.* — The group  $N_{\mathrm{GL}_{\mathbb{C}}(V)}(W)$  acts on the set  $\mathfrak{N}$  and so on the space of parameters  $\mathbb{C}^{\mathfrak{N}}$ . If  $\tau \in N_{\mathrm{GL}_{\mathbb{C}}(V)}(W)$ , then  $\tau$  induces an isomorphism of algebras  $\mathbf{H}_k \rightarrow \mathbf{H}_{\tau(k)}$ . So, if  $\tau(k) = k$ , then it induces an action on the algebra  $\mathbf{H}_k$  (and so on its center  $\mathbf{Z}_k$  and on the Calogero-Moser space  $\mathfrak{X}_k$ ).

We say that  $\tau$  is a *regular* element of  $N_{\mathrm{GL}_{\mathbb{C}}(V)}(W)$  if  $V_{\mathrm{reg}}^{\tau} \neq \emptyset$ .

**Notation.** From now on, and until the end of this paper, we fix a parameter  $k \in \mathbb{C}^{\mathfrak{N}}$  and a *regular* element  $\tau$  of *finite* order of  $N_{\mathrm{GL}_{\mathbb{C}}(V)}(W)$  such that  $\tau(k) = k$ .

We denote by  $\mathfrak{X}_k^{\tau}$  the variety of fixed points of  $\tau$  in  $\mathfrak{X}_k$ , endowed with its reduced structure. All the above constructions are  $\tau$ -equivariant: for instance, the map  $\varphi_k : \mathrm{Irr}(W) \rightarrow \mathfrak{X}_k^{\mathbb{C}^{\times}}$  is  $\tau$ -equivariant.

Let us recall the following consequence [Spr, Prop. 3.5 and Theo. 4.2] of the above hypothesis:

**Theorem 1.6 (Springer).** — *The group  $W^{\tau}$  acts as a reflection group on  $V^{\tau}$  and the natural map  $V^{\tau}/W^{\tau} \rightarrow (V/W)^{\tau}$  is an isomorphism of varieties.*

**Corollary 1.7.** — *The natural map  $(V_{\mathrm{reg}}^{\tau} \times V^{*\tau})/W^{\tau} \rightarrow ((V_{\mathrm{reg}} \times V^*)/W)^{\tau}$  is an isomorphism of varieties.*

*Proof.* — Since  $W$  acts freely on  $V_{\mathrm{reg}} \times V^*$ , the quotient  $(V_{\mathrm{reg}} \times V^*)/W$  is smooth. Consequently, the variety of fixed points  $((V_{\mathrm{reg}} \times V^*)/W)^{\tau}$  is also smooth. Similarly,  $(V_{\mathrm{reg}}^{\tau} \times V^{*\tau})/W^{\tau}$  is smooth. Since a bijective morphism between smooth varieties is an isomorphism, we only need to show that the above natural map is bijective.

First, if  $(v_1, v_1^*)$  and  $(v_2, v_2^*)$  are two elements of  $V_{\mathrm{reg}}^{\tau} \times V^{*\tau}$  belonging to the same  $W$ -orbit, there exists  $w \in W$  such that  $v_2 = w(v_1)$ . Since  $v_1$  and  $v_2$  are  $\tau$ -stable, we also have  $\tau(w)(v_1) = v_2$ , and so  $v_1 = w^{-1}\tau(w)(v_1)$ . Since  $v_1 \in V_{\mathrm{reg}}$ , this forces  $\tau(w) = w$  and the injectivity follows.

Now, if  $(v, v^*) \in V_{\mathrm{reg}} \times V^*$  is such that its  $W$ -orbit is  $\tau$ -stable, then the  $W$ -orbit of  $v$  is  $\tau$ -stable. So Theorem 1.6 shows that we may assume that  $\tau(v) = v$ . The hypothesis implies that there exists  $w \in W$  such that  $\tau(v) = w(v)$  and  $\tau(v^*) = w(v^*)$ . But  $\tau(v) = v \in V_{\mathrm{reg}}$ , so  $w = 1$ . In particular,  $\tau(v^*) = v^*$ , and the surjectivity follows.  $\square$

## 2. Irreducible component of maximal dimension

Let  $(\mathfrak{X}_k)_{\mathrm{reg}}$  denote the open subset  $\Upsilon_k^{-1}(V_{\mathrm{reg}}/W \times V^*/W)$ . By [EtGi, Prop. 4.11], we have a  $\mathbb{C}^{\times}$ -equivariant and  $\tau$ -equivariant isomorphism

$$(2.1) \quad (\mathfrak{X}_k)_{\mathrm{reg}} \simeq (V_{\mathrm{reg}} \times V^*)/W.$$

This shows that  $(\mathfrak{X}_k)_{\mathrm{reg}}$  is smooth and so  $(\mathfrak{X}_k)_{\mathrm{reg}}^{\tau}$  is also smooth. By Corollary 1.7, this implies that

$$(2.2) \quad (\mathfrak{X}_k)_{\mathrm{reg}}^{\tau} \simeq (V_{\mathrm{reg}}^{\tau} \times V^{*\tau})/W^{\tau}.$$

In particular it is irreducible. We denote by  $(\mathfrak{X}_k^{\tau})_{\mathrm{max}}$  its closure: it is an irreducible closed subvariety of  $\mathfrak{X}_k^{\tau}$ .

Moreover,  $(\mathcal{X}_k)_{\text{reg}}^\tau$  has dimension  $2 \dim V^\tau$  by Corollary 1.7. So  $\dim \mathcal{X}_k^\tau \geq 2 \dim V^\tau = \dim(\mathcal{X}_k^\tau)_{\text{max}}$ . But, on the other hand,  $\Upsilon_k(\mathcal{X}_k^\tau) \subset (V/W)^\tau \times (V^*/W)^\tau$ . Since  $\Upsilon_k$  is a finite morphism, we get from Theorem 1.6 that  $\dim \mathcal{X}_k^\tau \leq 2 \dim V^\tau$ . Hence

$$(2.3) \quad \dim \mathcal{X}_k^\tau = \dim(\mathcal{X}_k^\tau)_{\text{max}} = 2 \dim V^\tau.$$

This shows that  $(\mathcal{X}_k^\tau)_{\text{max}}$  is an irreducible component of maximal dimension of  $\mathcal{X}_k^\tau$ .

**Proposition 2.4.** — *The closed subvariety  $(\mathcal{X}_k^\tau)_{\text{max}}$  of  $\mathcal{X}_k^\tau$  is the unique irreducible component of maximal dimension.*

*Proof.* — Let  $\mathcal{X}$  be an irreducible component of  $\mathcal{X}_k^\tau$  of dimension  $2 \dim V^\tau$ . Since  $\Upsilon_k$  is finite, the image  $\Upsilon_k(\mathcal{X})$  is closed in  $V/W \times V^*/W$ , irreducible of dimension  $2 \dim(V^\tau)$  and contained in  $(V/W)^\tau \times (V^*/W)^\tau$ . By Theorem 1.6, we get that  $\Upsilon_k(\mathcal{X}) = (V/W)^\tau \times (V^*/W)^\tau$ .

Let  $\mathcal{U} = \Upsilon_k^{-1}(V_{\text{reg}}/W \times V^*/W) \cap \mathcal{X}$ . Then  $\mathcal{U}$  is a non-empty open subset of  $\mathcal{X}$ : since  $\mathcal{X}$  is irreducible, this forces  $\mathcal{U}$  to have dimension  $2 \dim(V^\tau)$ . But  $\mathcal{U}$  is contained in  $(\mathcal{X}_k)_{\text{reg}}^\tau$  which is irreducible of the same dimension, so the closure of  $\mathcal{U}$  contains  $(\mathcal{X}_k)_{\text{reg}}^\tau$ . This proves that  $\mathcal{X} = (\mathcal{X}_k^\tau)_{\text{max}}$ .  $\square$

**Corollary 2.5.** —  $\Upsilon_k((\mathcal{X}_k^\tau)_{\text{max}}) = (V/W)^\tau \times (V^*/W)^\tau$ .

It is natural to ask which  $\mathbb{C}^\times$ -fixed points of  $\mathcal{X}_k$  belong to  $(\mathcal{X}_k^\tau)_{\text{max}}$ . Inspired by the representation theory of finite reductive groups (see [BrMi] and [BMM2, Rem. 4.21]), we propose an answer to this question in terms of the character table of the finite group  $W\langle\tau\rangle$  (see [Bon3, Ex. 12.9] for some explanations). We first need some notation.

If  $\chi \in \text{Irr}(W)$ , we denote by  $E_\chi$  a  $\mathbb{C}W$ -module affording the character  $\chi$ . If moreover  $\chi$  is  $\tau$ -stable, we fix a structure of  $\mathbb{C}W\langle\tau\rangle$ -module on  $E_\chi$  extending the structure of  $\mathbb{C}W$ -module, and we denote by  $\tilde{\chi}$  its associated irreducible character of  $W\langle\tau\rangle$ . Note that the real number  $|\tilde{\chi}(\tau)|^2$  does not depend on the choice of  $\tilde{\chi}$ .

**Conjecture 2.6.** — *Recall that  $\tau$  is regular. Let  $p \in \mathcal{X}_k^{\mathbb{C}^\times}$  be such that  $\tau(p) = p$ . Then  $p$  belongs to  $(\mathcal{X}_k^\tau)_{\text{max}}$  if and only if  $\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 \neq 0$ .*

**Remark 2.7.** — Let  $\mathfrak{F}$  be a  $\tau$ -stable Calogero-Moser family. Then  $\mathfrak{F}$  contains a unique irreducible character  $\chi_{\mathfrak{F}}$  with minimal  $b$ -invariant [BoRo, Theo. 7.4.1], where the  $b$ -invariant of an irreducible character  $\chi$  is the minimal natural number  $j$  such that  $\chi$  occurs in the  $j$ -th symmetric power of the natural representation  $V$  of  $W$ . From this characterization, we see that  $\chi_{\mathfrak{F}}$  is  $\tau$ -stable. In particular, any  $\tau$ -stable Calogero-Moser family contains at least one  $\tau$ -stable character.  $\blacksquare$

In general, we are only able to prove the “if” part of Conjecture 2.6.

**Theorem 2.8.** — *Recall that  $\tau$  is regular. Let  $p \in \mathcal{X}_k^{\mathbb{C}^\times}$  be such that  $\tau(p) = p$ . If  $\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 \neq 0$ , then  $p$  belongs to  $(\mathcal{X}_k^\tau)_{\text{max}}$ .*

The next two sections are devoted to the proof of Theorem 2.8.

### 3. Verma modules

**3.A. Definition.** — Recall that  $\mathbb{C}[V] \rtimes W$  is a subalgebra of  $\mathbf{H}_k$  (it is the image of  $1 \otimes \mathbb{C}W \otimes \mathbb{C}[V]$  by the PBW-decomposition 1.3). If  $E$  is a  $\mathbb{C}W$ -module, we denote by  $E^\#$  the  $(\mathbb{C}[V^*] \rtimes W)$ -module extending  $E$  by letting any element  $f \in \mathbb{C}[V^*]$  acting by multiplication by  $f(0)$ . If  $\chi \in \text{Irr}(W)$ , we define an  $\mathbf{H}_k$ -module  $\Delta(\chi)$  as follows:

$$\Delta(\chi) = \mathbf{H}_k \otimes_{\mathbb{C}[V^*] \rtimes W} E_\chi^\#.$$

Then  $\Delta(\chi)$  is called a *Verma module* of  $\mathbf{H}_k$  (see [BoRo, §5.4.A]: in this reference,  $\Delta(\chi)$  is denoted by  $\Delta(E_\chi^\#)$ ). Let  $\mathbf{H}_k^{\text{reg}}$  denote the localization of  $\mathbf{H}_k$  at  $\mathbf{P}_{\text{reg}} = \mathbb{C}[V_{\text{reg}}/W] \otimes \mathbb{C}[V^*]$ . By [EtGi, Prop. 4.11], we have an isomorphism  $\mathbb{C}[V_{\text{reg}} \times V^*] \rtimes W \simeq \mathbf{H}_k^{\text{reg}}$ . We denote by  $\Delta^{\text{reg}}(\chi)$  the localization of  $\Delta(\chi)$  at  $\mathbf{H}_k^{\text{reg}}$ . So, by restriction to  $\mathbb{C}[V_{\text{reg}} \times V^*]$ , the Verma module  $\Delta(\chi)$  might be viewed as a  $W$ -equivariant coherent sheaf on  $V_{\text{reg}} \times V^*$ . We also view  $e\Delta(\chi)$  as a coherent sheaf on  $\mathcal{X}_k$ , so that  $e\Delta^{\text{reg}}(\chi)$  may be viewed as a coherent sheaf on  $(V_{\text{reg}} \times V^*)/W$ . If  $p \in \mathcal{X}_k$  (or if  $(v, v^*) \in V_{\text{reg}} \times V^*$ ), we denote by  $e\Delta(\chi)_p$  (respectively  $e\Delta(\chi)_{W \cdot (v, v^*)} = e\Delta^{\text{reg}}(\chi)_{W \cdot (v, v^*)}$ , respectively  $\Delta^{\text{reg}}(\chi)_{v, v^*}$ ) the restriction of  $e\Delta(\chi)$  (respectively of  $e\Delta(\chi)$  or  $e\Delta^{\text{reg}}(\chi)$ , respectively  $\Delta^{\text{reg}}(\chi)$ ) at the point  $p$  (respectively  $W \cdot (v, v^*) \in (V_{\text{reg}} \times V^*)/W \simeq (\mathcal{X}_k)_{\text{reg}}$ , respectively  $(v, v^*)$ ). It follows from the definition that the support of  $e\Delta(\chi)$  is contained in  $\Upsilon_k^{-1}(V/W \times 0)$ , and recall that, through the isomorphism  $\mathcal{X}_k^{\text{reg}} \simeq (V_{\text{reg}} \times V^*)/W$ ,  $\Upsilon_k^{-1}(V_{\text{reg}}/W \times 0)$  is not necessarily contained in  $(V_{\text{reg}} \times \{0\})/W$ .

**Lemma 3.1.** — Let  $\chi \in \text{Irr}(W)$  and let  $p \in \mathcal{X}_k^{\mathbb{C}^\times}$ . Then  $e\Delta(\chi)_p \neq 0$  if and only if  $\chi \in \mathfrak{F}_p$ .

*Proof.* — Let  $\mathfrak{p}_0$  denote the maximal ideal of the algebra  $\mathbf{P} = \mathbb{C}[\mathcal{P}]$  consisting of functions which vanish at 0. Then  $\Delta(\chi)/\mathfrak{p}_0\Delta(\chi)$  is a representation of the *restricted rational Cherednik algebra*  $\mathbf{H}_k/\mathfrak{p}_0\mathbf{H}_k$  which coincides with the *baby Verma module* defined by Gordon [Gor, §4.2]. As  $\mathcal{X}_k^{\mathbb{C}^\times} = \Upsilon_k^{-1}(0)$ , the result follows from the very definition of Calogero-Moser families in terms of baby Verma modules and the fact that it is equivalent to the definition given in §1.D.  $\square$

**3.B. Bialynicki-Birula decomposition.** — We denote by  $\mathcal{X}_k^{\text{att}}$  the *attracting set* of  $\mathcal{X}_k$  for the action of  $\mathbb{C}^\times$ , namely

$$\mathcal{X}_k^{\text{att}} = \{p \in \mathcal{X}_k \mid \lim_{\xi \rightarrow 0} \xi p \text{ exists}\}.$$

Recall from [BoRo, Chap. 14] the following facts:

**Proposition 3.2.** — With the above notation, we have:

- (a) The map  $\lim : \mathcal{X}_k^{\text{att}} \rightarrow \mathcal{X}_k^{\mathbb{C}^\times}$ ,  $p \mapsto \lim_{\xi \rightarrow 0} \xi p$  is a morphism of varieties.
- (b)  $\mathcal{X}_k^{\text{att}} = \Upsilon_k^{-1}(V/W \times \{0\})$ .
- (c) If  $\mathcal{F}$  is an irreducible component of  $\mathcal{X}_k^{\text{att}}$ , then  $\mathcal{F}$  is  $\mathbb{C}^\times$ -stable and  $\Upsilon_k(\mathcal{F}) = V/W \times \{0\}$  and  $\lim(\mathcal{F})$  is a single point.
- (d) If  $\chi \in \text{Irr}(W)$ , then the support of  $e\Delta(\chi)$  is a union of irreducible components of  $\mathcal{X}_k^{\text{att}}$ .
- (e) If  $\mathcal{F}$  is an irreducible component of  $\mathcal{X}_k^{\text{att}}$ , then there exists  $\chi \in \text{Irr}(W)$  such that the support of  $e\Delta(\chi)$  contains  $\mathcal{F}$ .



We first propose a characterization of points  $p \in \mathcal{X}_k^{\mathbb{C}^\times}$  which belong to  $(\mathcal{X}_k^\tau)_{\max}$  in terms of Verma modules.

**Lemma 3.3.** — *Let  $p \in \mathcal{X}_k^{\mathbb{C}^\times}$  and assume that  $\tau(p) = p$ . Then  $p \in (\mathcal{X}_k^\tau)_{\max}$  if and only if there exist  $\chi \in \mathfrak{F}_p^k$  and  $(v, v^*) \in V_{\text{reg}}^\tau \times V^{*\tau}$  such that  $e\Delta(\chi)_{W \cdot (v, v^*)} \neq 0$ .*

*Proof.* — Let  $(\mathcal{X}_k^\tau)_{\max}^{\text{att}}$  denote the attracting set of  $(\mathcal{X}_k^\tau)_{\max}$ . Then Corollary 2.5 implies that  $\Upsilon_k((\mathcal{X}_k^\tau)_{\max}) = (V/W)^\tau \times \{0\}$ . Since  $\Upsilon_k$  is a finite morphism, the same arguments used in [BoRo, Chap. 14] to prove the Proposition 3.2 above yields the following statements:

- (a) The map  $\lim : (\mathcal{X}_k^\tau)_{\max}^{\text{att}} \rightarrow (\mathcal{X}_k^\tau)_{\max}^{\mathbb{C}^\times}, p \mapsto \lim_{\xi \rightarrow 0} \xi p$  is a morphism of varieties.
- (b)  $(\mathcal{X}_k^\tau)_{\max}^{\text{att}} = (\mathcal{X}_k^\tau)_{\max} \cap \Upsilon_k^{-1}((V/W)^\tau \times \{0\})$ .
- (c) If  $\mathcal{F}$  is an irreducible component of  $(\mathcal{X}_k^\tau)_{\max}^{\text{att}}$ , then  $\mathcal{F}$  is  $\mathbb{C}^\times$ -stable and  $\Upsilon_k(\mathcal{F}) = (V/W)^\tau \times \{0\}$  and  $\lim(\mathcal{F})$  is a single point.

Assume that  $p \in (\mathcal{X}_k^\tau)_{\max}$ . Let  $\mathcal{F}$  be an irreducible component of  $(\mathcal{X}_k^\tau)_{\max}^{\text{att}} \cap \lim^{-1}(p)$ . Then  $\mathcal{F}$  is contained in an irreducible component  $\mathcal{F}'$  of  $(\mathcal{X}_k^\tau)_{\max}^{\text{att}}$ . Since  $\lim(\mathcal{F}')$  is a single point by (c), we have  $\lim(\mathcal{F}') = \{p\}$  and so  $\mathcal{F} = \mathcal{F}'$ . Still by (c), this says that  $\Upsilon_k(\mathcal{F}) = (V/W)^\tau \times \{0\}$ . So let  $q \in \mathcal{F}$  be such that  $\Upsilon_k(q) \in (V_{\text{reg}}/W)^\tau \times \{0\}$ .

Now, let  $\mathcal{F}$  be an irreducible component of  $\mathcal{X}_k^{\text{att}}$  containing  $\mathcal{F}$ . By Proposition 3.2(e), there exists  $\chi \in \text{Irr}(W)$  such that the support of  $e\Delta(\chi)$  contains  $\mathcal{F}$ . In particular,  $e\Delta(\chi)_p \neq 0$  and so  $\chi \in \mathfrak{F}_p$  by Lemma 3.1. But also  $e\Delta(\chi)_q \neq 0$ . Since  $q \in (\mathcal{X}_k^\tau)_{\max}$  and  $\Upsilon_k(q) \in V_{\text{reg}}/W$ , it follows that there exists  $(v, v^*) \in V_{\text{reg}}^\tau \times V^{*\tau}$  such that  $e\Delta(\chi)_{W \cdot (v, v^*)} \neq 0$ , as desired.

Conversely, assume that there exist both  $\chi \in \mathfrak{F}_p^k$  and  $(v, v^*) \in V_{\text{reg}}^\tau \times V^{*\tau}$  such that  $e\Delta(\chi)_{W \cdot (v, v^*)} \neq 0$ . Let  $\mathcal{F}$  be an irreducible component of  $\mathcal{X}_k^{\text{att}}$  contained in the support of  $e\Delta(\chi)$ . Then  $p \in \mathcal{F}$  and so  $p = \lim W \cdot (v, v^*)$ . Since  $W \cdot (v, v^*) \in (\mathcal{X}_k^\tau)_{\max}$  by the definition of  $(\mathcal{X}_k^\tau)_{\max}$ , this implies that  $p \in (\mathcal{X}_k^\tau)_{\max}$ , as desired.  $\square$

## 4. Gaudin algebra

**4.A. Definition.** — We recall here the definition of *Gaudin algebra* [BoRo, §8.3.B]. First, let  $\mathbb{C}[V_{\text{reg}}][W]$  denote the group algebra of  $W$  over the algebra  $\mathbb{C}[V_{\text{reg}}]$  (and not the semi-direct product  $\mathbb{C}[V_{\text{reg}}] \rtimes W$ ). For  $y \in V$ , let

$$\mathcal{D}_y^k = \sum_{s \in \text{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\alpha_s} s \in \mathbb{C}[V_{\text{reg}}][W].$$

Now, let  $\text{Gau}_k(W)$  be the sub- $\mathbb{C}[V_{\text{reg}}]$ -algebra of  $\mathbb{C}[V_{\text{reg}}][W]$  generated by the  $\mathcal{D}_y^k$ 's (where  $y$  runs over  $V$ ): it will be called the *Gaudin algebra (with parameter  $k$ )* associated with  $W$ .

Let  $\mathbb{C}(V)$  denote the function field of  $V$  (which is the fraction field of  $\mathbb{C}[V]$  or of  $\mathbb{C}[V_{\text{reg}}]$ ) and let  $\mathbb{C}(V)\text{Gau}_k(W)$  denote the subalgebra  $\mathbb{C}(V) \otimes_{\mathbb{C}[V_{\text{reg}}]} \text{Gau}_k(W)$  of the group algebra  $\mathbb{C}(V)[W]$ . Recall [BoRo, §8.3.B] that

$$(4.1) \quad \text{Gau}_k(W) \text{ is a commutative algebra,}$$

but that  $\text{Gau}_k(W)$  is generally non-split, as shown by the examples treated in [Bon1, §4] and [Lac].

**4.B. Generalized eigenspaces.** — If  $v \in V_{\text{reg}}$ , we denote by  $\mathcal{D}_y^{k,v}$  the specialization of  $\mathcal{D}_y^k$  at  $v$ , namely  $\mathcal{D}_y^{k,v}$  is the element of the group algebra  $\mathbb{C}W$  equal to

$$\mathcal{D}_y^k = \sum_{s \in \text{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\langle v, \alpha_s \rangle} s.$$

Now, if  $v^* \in V^*$  and if  $M$  is a  $\mathbb{C}W$ -module, we define  $M^{k,v,v^*}$  to be the common generalized eigenspace of the operators  $\mathcal{D}_y^{k,v}$  for the eigenvalue  $\langle y, v^* \rangle$ , for  $y$  running over  $V$ . Namely,

$$M^{k,v,v^*} = \{m \in M \mid \forall y \in V, (\mathcal{D}_y^{k,v} - \langle y, v^* \rangle \text{Id}_M)^{\dim(M)}(m) = 0\}.$$

Then

$$(4.2) \quad M = \bigoplus_{v^* \in V^*} M^{k,v,v^*},$$

since  $\text{Gau}_k(W)$  is commutative.

**Lemma 4.3.** — Let  $\chi \in \text{Irr}(W)$  and let  $(v, v^*) \in V_{\text{reg}} \times V^*$ . Then the following are equivalent:

- (1)  $e\Delta(\chi)_{W \cdot (v, v^*)} \neq 0$ .
- (2)  $\Delta^{\text{reg}}(\chi)_{v, v^*} \neq 0$ .
- (3)  $E_\chi^{k,v,v^*} \neq 0$ .

*Proof.* — The equivalence between (1) and (2) follows from the Morita equivalence between  $\mathbb{C}[V_{\text{reg}} \times V^*]^W$  and  $\mathbb{C}[V_{\text{reg}} \times V^*] \rtimes W$  proved in [BoRo, Lem. 3.1.8(b)]. Now, as a  $\mathbb{C}[V_{\text{reg}}]$ -module,  $\Delta^{\text{reg}}(\chi) \simeq \mathbb{C}[V_{\text{reg}}] \otimes E_\chi$ , and the equivalence between (2) and (3) follows from the computations in [BoRo, §8.3.B].  $\square$

**4.C. Proof of Theorem A (i.e. Theorem 2.8).** — Let  $\chi \in \text{Irr}(W)$  be  $\tau$ -stable and such that  $\tilde{\chi}(\tau) \neq 0$  and let  $v \in V_{\text{reg}}^\tau$ . By Lemmas 3.1 and 4.3, it is sufficient to show that there exists  $v^* \in V^{*\tau}$  such that  $E_\chi^{k,v,v^*} \neq 0$ .

For this, let  $\mathcal{E}$  denote the set of  $v^* \in V^*$  such that  $E_\chi^{k,v,v^*} \neq 0$ . Then it follows from (4.2) that

$$(*) \quad E_\chi = \bigoplus_{v^* \in \mathcal{E}} E_\chi^{k,v,v^*}.$$

Since  $\tau(v) = v$ , we have

$$\tau \mathcal{D}_y^{k,v} = \sum_{s \in \text{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \alpha_s \rangle}{\langle v, \alpha_s \rangle} \tau s \tau^{-1} = \sum_{s \in \text{Ref}(W)} \varepsilon(s) c_k(s) \frac{\langle y, \tau^{-1}(\alpha_s) \rangle}{\langle v, \tau^{-1}(\alpha_s) \rangle} s = \mathcal{D}_{\tau(y)}^{k,v}.$$

Consequently,

$$\tau E_\chi^{k,v,v^*} = E_\chi^{k,v,\tau(v^*)}.$$

But  $\tilde{\chi}(\tau) = \text{Tr}(\tau, E_\chi) \neq 0$ , so  $\tau$  must fix at least one of the generalized eigenspaces in the decomposition (\*). In other words, this implies that there exists  $v^* \in \mathcal{E}$  such that  $\tau(v^*) = v^*$ , as desired. The proof is complete.

## 5. Complements

**5.A. Conjectures.** — The variety  $\mathcal{X}_k$  is endowed with a Poisson structure [EtGi, §1] and so the variety of fixed points  $\mathcal{X}_k^\tau$  inherits a Poisson structure too, as well as all its irreducible components. Recall from Springer Theorem 1.6 that  $W^\tau$  is a reflection group for its action on  $V^\tau$ , so we can define a set of pairs  $\aleph_\tau$  for the pair  $(V^\tau, W^\tau)$  as well as  $\aleph$  has been defined for the pair  $(V, W)$  and, for each parameter  $l \in \mathbb{C}^{\aleph_\tau}$ , we can define a Calogero-Moser space  $\mathcal{X}_l(V^\tau, W^\tau)$ . The following conjecture is a particular case of [Bon2, Conj. B] (see [Bon2] for a discussion about the cases where this conjecture is known to hold):

**Conjecture 5.1.** — *Recall that  $\tau$  is regular. Then there exists a linear map  $\lambda : \mathbb{C}^\aleph \rightarrow \mathbb{C}^{\aleph_\tau}$  and, for each  $k \in \mathbb{C}^\aleph$ , a  $\mathbb{C}^\times$ -equivariant isomorphism of Poisson varieties*

$$\iota_k : (\mathcal{X}_k^\tau)_{\max} \xrightarrow{\sim} \mathcal{X}_{\lambda(k)}(V^\tau, W^\tau).$$

Assume that Conjecture 5.1 holds and keep its notation. Then  $\iota_k$  restricts to a map  $\iota_k : (\mathcal{X}_k)_{\max}^{\mathbb{C}^\times} \xrightarrow{\sim} \mathcal{X}_{\lambda(k)}(V^\tau, W^\tau)^{\mathbb{C}^\times}$ . If  $p \in \mathcal{X}_{\lambda(k)}(V^\tau, W^\tau)^{\mathbb{C}^\times}$ , we denote by  $\mathfrak{F}_{\lambda(k)}^{(\tau)}$  the corresponding Calogero-Moser  $\lambda(k)$ -family of  $W^\tau$ . The next conjecture, still inspired by the representation theory of finite reductive groups (see again [Bon3, Ex. 12.9] for some explanations), makes Conjecture B more precise:

**Conjecture 5.2.** — *Recall that  $\tau$  is regular and assume that Conjecture 5.1 holds. If  $p \in (\mathcal{X}_k^\tau)_{\max}^{\mathbb{C}^\times}$ , then*

$$\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 = \sum_{\psi \in \mathfrak{F}_{\lambda(k)}^{(\tau)}} \psi(1)^2.$$

Note that this last conjecture is compatible with the fact that

$$\sum_{\chi \in \text{Irr}(W)^\tau} |\tilde{\chi}(\tau)|^2 = |W^\tau| = \sum_{\psi \in \text{Irr}(W^\tau)} \psi(1)^2,$$

where the first equality follows from the second orthogonality relation for characters.

**5.B. Roots of unity.** — We consider in this subsection a particular (but very important) case of the general situation studied in this paper. We fix a natural number  $d \geq 1$  and a primitive  $d$ -th root of unity  $\zeta_d$ . The group of  $d$ -th roots of unity is denoted by  $\mu_d$ . An element  $w \in W$  is called  $\zeta_d$ -regular if the element  $\zeta_d^{-1}w$  of  $N_{\text{GL}_{\mathbb{C}}(V)}(W)$  is regular. In other words,  $w$  is  $\zeta_d$ -regular if and only if its  $\zeta_d$ -eigenspace meets  $V_{\text{reg}}$ . The existence of a  $\zeta_d$ -regular element is not guaranteed: we say that  $d$  is a *regular number* of  $W$  if such an element exists.

**Hypothesis.** *We assume in this subsection, and only in this subsection, that  $d$  is a regular number of  $W$ . We denote by  $w_d$  a  $\zeta_d$ -regular element and we also set  $\tau_d = \zeta_d^{-1}w_d$ , so that  $\tau_d$  is a regular element of  $N_{\text{GL}_{\mathbb{C}}(V)}(W)$ .*

Recall from [Spr] that  $w_d$  is uniquely defined up to conjugacy. Note that

$$(5.3) \quad V^{\tau_d} = \text{Ker}(w_d - \zeta_d \text{Id}_V), \quad W^{\tau_d} = C_W(w_d) \quad \text{and} \quad \mathcal{X}_k^{\tau_d} = \mathcal{X}_k^{\mu_d}.$$

Since  $\tau_d$  induces an inner automorphism of  $W$ , all the irreducible characters are  $\tau_d$ -stable. Moreover, if  $\chi \in \text{Irr}(W)$ , then  $\tilde{\chi}(\tau_d) = \xi\chi(w_d)$  for some root of unity  $\xi$ , so  $|\tilde{\chi}(\tau_d)|^2 = |\chi(w_d)|^2$ . This allows to reformulate both Theorem A and Conjecture B in this case:

**Conjecture 5.4.** — Recall that  $d$  is a regular number. Let  $p \in \mathfrak{F}_k^{\mathbb{C}^\times}$ . Then  $p$  belongs to  $(\mathfrak{F}_k^{\mu_d})_{\max}$  if and only if  $\sum_{\chi \in \mathfrak{F}_p} |\chi(w_d)|^2 \neq 0$ .

**Theorem 5.5.** — Recall that  $d$  is regular. Let  $p \in \mathfrak{F}_k^{\mathbb{C}^\times}$  be such that  $\sum_{\chi \in \mathfrak{F}_p} |\chi(w_d)|^2 \neq 0$ . Then  $p$  belongs to  $(\mathfrak{F}_k^{\mu_d})_{\max}$ .

**Example 5.6 (Symmetric group).** — We assume here, and only here, that  $W = \mathfrak{S}_n$  acting on  $V = \mathbb{C}^n$  by permutation of the coordinates, for some  $n \geq 2$ . The canonical basis of  $\mathbb{C}^n$  is denoted by  $(y_1, \dots, y_n)$ . Then there is a unique orbit of hyperplanes, that we denote by  $\Omega$ , and  $e_\Omega = 2$ . To avoid too easy cases, we also assume that  $k_{\Omega,0} \neq k_{\Omega,1}$  (so that  $\mathfrak{F}_k$  is smooth [EtGi, Cor. 1.14]) and that  $d \geq 2$ . Saying that  $d$  is a regular number is equivalent to say that  $d$  divides  $n$  or  $n - 1$ . Therefore, we will denote by  $j$  the unique element of  $\{0, 1\}$  such that  $d$  divides  $n - j$  and we set  $r = (n - j)/d$ . Then  $w_d$  is the product of  $r$  disjoint cycles of length  $d$ , so one can choose for instance

$$w_d = (1, 2, \dots, d)(d + 1, d + 2, \dots, 2d) \cdots ((r - 1)d + 1, (r - 1)d + 2, \dots, rd).$$

Then  $V^{\tau_d}$  is  $r$ -dimensional, with basis  $(v_1, \dots, v_r)$  where  $v_a = \sum_{b=1}^d \zeta_d^{-b} e_{(a-1)d+b}$  and the group  $C_W(w_d) \simeq G(d, 1, r)$  acting “naturally” as a reflection group on  $V^{\tau_d} = \bigoplus_{a=1}^r \mathbb{C}v_a$ .

We also need some combinatorics. We denote by  $\text{Part}(n)$  (resp.  $\text{Part}^d(r)$ ) the set of partitions of  $n$  (resp. of  $d$ -partitions of  $r$ ). If  $\lambda \in \text{Part}(n)$ , we denote by  $\chi_\lambda$  the irreducible character of  $\mathfrak{S}_n$  (with the convention of [GePf]: for instance  $\chi_n = 1$  and  $\chi_{1^n} = \varepsilon$ ), by  $\text{cor}_d(\lambda)$  the  $d$ -core of  $\lambda$ , by  $\text{quo}_d(\lambda)$  its  $d$ -quotient. We let  $\text{Part}(n, d)$  denote the set of partitions of  $n$  whose  $d$ -core is the unique partition of  $j \in \{0, 1\}$ . Then the map

$$\text{quo}_d : \text{Part}(n, d) \longrightarrow \text{Part}^d(r)$$

is bijective. Finally, if  $\mu \in \text{Part}^d(r)$ , we denote by  $\chi_\mu$  the associated irreducible character of  $C_W(w_d) = G(d, 1, r)$  (with the convention of [GeJa]). It follows from Murnaghan-Nakayama rule that

$$(5.7) \quad \chi_\lambda(w_d) \neq 0 \text{ if and only if } \lambda \in \text{Part}(n, d),$$

and that

$$(5.8) \quad \chi_\lambda(w_d) = \pm \chi_{\text{quo}_d(\lambda)}(1)$$

for all  $\lambda \in \text{Part}(n, d)$  (see for instance [BMM1, Page 47]).

Now, the smoothness of  $\mathfrak{F}_k$  implies that the map  $\varphi_k : \text{Irr}(\mathfrak{S}_n) \longrightarrow \mathfrak{F}_k^{\mathbb{C}^\times}$  is bijective (so that Calogero-Moser  $k$ -families of  $\mathfrak{S}_n$  are singleton) and it follows from [BoMa] that Conjecture 5.1 holds (except that we do not know if the isomorphism respects the Poisson structure), so that we have a  $\mathbb{C}^\times$ -equivariant isomorphism of varieties

$$\iota_k : (\mathfrak{F}_k^{\mu_d})_{\max} \xrightarrow{\sim} \mathfrak{F}_{\lambda(k)}(V^{\tau_d}, G(d, 1, r))$$

for some explicit  $\lambda(k) \in \mathbb{C}^{\mathbb{N}^{\tau_d}}$ . Moreover,  $\mathcal{X}_{\lambda(k)}(V^{\tau_d}, G(d, 1, r))$  is smooth so that the map  $\varphi_{\lambda(k)}^{\tau_d} : \text{Irr}(G(d, 1, r)) \rightarrow \mathcal{X}_{\lambda(k)}(V^{\tau_d}, G(d, 1, r))^{\mathbb{C}^\times}$  is bijective (that is, Calogero-Moser  $\lambda(k)$ -families of  $G(d, 1, r)$  are singleton). Now, by [BoMa], we have that

$$(5.9) \quad \varphi_k(\chi_\lambda) \in (\mathcal{X}_k^{\mu_d})_{\max} \text{ if and only if } \lambda \in \text{Part}(n, d),$$

and that

$$(5.10) \quad \iota_k(\varphi_k(\chi_\lambda)) = \varphi_{\lambda(k)}^{\tau_d}(\chi_{\text{quo}_d(\lambda)})$$

for all  $\lambda \in \text{Part}(n, d)$ . Then (5.7), (5.8), (5.9) and (5.10) show that Conjectures 5.4 and 5.2 hold for the symmetric group. ■

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