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CALOGERO-MOSER SPACES VS UNIPOTENT REPRESENTATIONS

by

CÉDRIC BONNAFÉ

To George Lusztig, with admiration

Abstract. — Lusztig’s classification of unipotent representations of finite reductive groups depends only on the associated Weyl group W (and the automorphism that the Frobenius automorphism induces on W). All the structural questions (families, Harish-Chandra series, partition into blocks...) have an answer in a combinatorics that can be entirely built directly from W .

Over the years, we have noticed that the same combinatorics seems to be encoded in the Poisson geometry of a Calogero-Moser space associated with W (families should correspond to \mathbb{C}^\times -fixed points, Harish-Chandra series should correspond to symplectic leaves, blocks should correspond to symplectic leaves in the fixed point subvariety under the action of a root of unity).

The aim of this survey is to gather all these observations, state precise conjectures and provide general facts and examples supporting these conjectures.

For this introduction, let us focus on the case where \mathbf{G} is a *split* reductive group over a finite field with q elements \mathbb{F}_q and let $G = \mathbf{G}(\mathbb{F}_q)$ be the finite group consisting of \mathbb{F}_q -rational points. Let W denote the Weyl group of \mathbf{G} and let \mathcal{X} denote the Calogero-Moser space associated with W at equal parameters (recall that it is an affine Poisson variety endowed with a \mathbb{C}^\times -action [EtGi, §4]). Let $\text{Unip}(G)$ denote the set of irreducible unipotent characters of G (as defined by Lusztig). A consequence of a conjecture of Gordon-Martino (2007, [GoMa]) is that the fixed point set $\mathcal{X}^{\mathbb{C}^\times}$ should be in bijection with the set of *Lusztig families* of $\text{Unip}(G)$. This first link was the starting point of our interest in the geometry of Calogero-Moser spaces.

In 2008, Gordon, following works of Haiman, obtained in type A a parametrization of the irreducible components of the fixed point subvariety \mathcal{X}^{μ_d} by d -cores of partitions. This fits perfectly with the partition of irreducible unipotent representations of $\mathbf{GL}_n(\mathbb{F}_q)$ into d -Harish-Chandra series (defined by Broué-Malle-Michel [BMM1]). In 2011, Bellamy and Losev (independently) obtained a parametrization *à la Harish-Chandra* of symplectic leaves of \mathcal{X} . In 2013, Rouquier and the author [BoRo1] constructed partitions of W into left, right and two-sided Calogero-Moser cells and conjectured they coincide with Kazhdan-Lusztig cells.

From then, the author has worked (with many different authors) on representations of Cherednik algebras at $t = 0$ and the geometry of Calogero-Moser spaces (see [BoRo1, BoRo2, Bon3, BoMa, BoSh]), for, as main motivation, understanding these strange analogies between the geometry of Calogero-Moser spaces and the representation theory of finite reductive groups. Over the years, the author has enriched these coincidences with several examples but has never exposed them in a paper. This is the aim of this survey to present them, state precise conjectures, and provide a list of examples that support these conjectures. A main reason for waiting for such a long time is that we needed to establish some theoretical background on Calogero-Moser space to state precise conjectures: this is done in [Bon4], where we generalize some results of Bellamy [Bel3] and Losev [Los] on symplectic leaves. We also needed some general results (cohomology, fixed points, regular automorphisms) in accordance with these conjectures [BoSh, BoMa, Bon5].

Let us explain one of the strangest (and most convincing) coincidences. Let ℓ be a prime number not dividing q and assume for simplicity that ℓ does not divide $|W|$. We denote by d the order of q modulo ℓ . Then each ℓ -block B of $\text{Unip}(G)$ should correspond to a *symplectic leaf* \mathcal{S}_B of the fixed point subvariety \mathcal{X}^{μ_d} of \mathcal{X} in such a way that:

- On one hand, the *d-Harish-Chandra theory* of Broué-Malle-Michel [BMM1] associates to B a complex reflection group \mathcal{W}_B whose irreducible characters are in bijection with B . Moreover, Broué-Malle-Michel also associate to B a Deligne-Lusztig variety \mathcal{X}_B and a parameter k_B and conjecture that the endomorphism algebra of the ℓ -adic cohomology of \mathcal{X}_B is isomorphic to a Hecke algebra of \mathcal{W}_B with parameter k_B . This association is motivated by Broué's abelian defect conjecture, and its geometric version for finite reductive groups [Bro2, §6] (see also [BMM1, BrMa2]).
- On the other hand, an analogue of a *d-Harish-Chandra theory for symplectic leaves* of \mathcal{X}^{μ_d} developed by the author [Bon4] (extending earlier works of Bellamy [Bel3] and Losev [Los] which deal with the case where $d = 1$) associates to \mathcal{S}_B a finite linear group \mathcal{W}'_B and a parameter k'_B . We conjecture [Bon4, Conj. B] that the normalization $\overline{\mathcal{S}_B}^{\text{nor}}$ of the closure of the symplectic leaf \mathcal{S}_B is the Calogero-Moser space for the pair (\mathcal{W}'_B, k'_B) .
- The main intriguing observation is that, in the cases where computations can be done, \mathcal{W}_B is a subgroup of \mathcal{W}'_B (in fact, in most cases, $\mathcal{W}_B = \mathcal{W}'_B$) and the parameter k_B is the restriction of the parameter k'_B . Our main conjecture is that this holds in general.

So, important features of the ℓ -modular representation theory of G seem to be encoded in the (Poisson) geometry of the affine variety \mathcal{X}^{μ_d} (where d and ℓ are linked by the fact that q is a primitive d -th root of unity modulo ℓ). Moreover, this correspondence seems to carry more properties, as explained in Section 12. To support our conjectures, we have the following examples available:

- We are able to prove most of them if W is of type A (see Section 15).
- They hold if W is of type B_2 or G_2 and d is the Coxeter number (see Section 13).
- They hold if W is of classical type and $d = 1$ (classical Harish-Chandra theory); see Section 16.
- In the *regular* case (see §7.C for the definition), we have a general result on Calogero-Moser spaces (see Theorem 7.4) which fits with observations made on the unipotent representations side (see Example 12.9).
- Our conjectures are compatible with Ennola duality.

The text is organized as follows. An introductory part presents the set-up and the notation involved all along the text. We summarize in the first part some general questions on the geometry of Calogero-Moser spaces (cohomology, geometry of symplectic leaves...), already contained in [BoRo2, BoSh, Bon4, Bon5]. The second part is a crash-course on unipotent representations of finite reductive groups (we hope it is understandable for non-specialists). The third part contains an explanation of the notion of *genericity* and also a detailed exposition of the different coincidences (stated as conjectures) we expect between the Poisson geometry of \mathcal{X} and the representation theory of G : this is the heart of this survey. The fourth part contains several very explicit examples confirming the conjectures. The last (short) part is an invitation to the *Spetses* theory of Broué-Malle-Michel [BMM2, BMM3], which have connections with the theme of this paper.

Contents

Set-up	4
1. General notation	4
2. Finite linear group, reflections	5
3. Rational Cherednik algebra at $t = 0$	6
4. Reflection groups	8
5. Braid group, Hecke algebra	9
 Part I. Questions about Calogero-Moser spaces	 12
6. Cohomology	12
7. Symplectic leaves and fixed points	14
8. Special features of Coxeter groups	17
 Part II. Unipotent representations of finite reductive groups	 19
9. Harish-Chandra theories	20
10. Families	24
 Part III. Genericity vs Calogero-Moser spaces	 25
11. Genericity	25
12. Coincidences, conjectures	30
 Part IV. Examples	 33
13. Rank 2	33
14. Some combinatorics	34
15. The smooth example: type A	37
16. Classical groups and Harish-Chandra theory	41
 Part V. Spetses	 44
17. What is a spets?	44
18. A primitive example	45
 References	 50

Commentary. Recently, Riche-Williamson [RiWi] provided a geometric proof of the *linkage principle* [Ver, Hum, Jan, And]: in their construction, blocks of the category of rational representations of $\mathbf{G}(\overline{\mathbb{F}}_q)$ are in bijection with the irreducible components of \mathcal{G}^{μ_p} , where p is the prime number dividing q and \mathcal{G} is the (complex) affine Grassmannian of the (complex) Langlands dual group to \mathbf{G} . So our observation has the same flavor as Riche-Williamson result: the blocks of some category of representations are controlled by the geometry of fixed points under the action of a group of roots of unity on some variety. Of course, the main difference is that Riche-Williamson proved a true theorem, built on the geometric Satake equivalence [Lus5, Gin, BeDr, MiVi] between representations of $\mathbf{G}(\overline{\mathbb{F}}_q)$ and some category of perverse sheaves on \mathcal{G} . Our observations are conjectural, and are only concerned with numerical/combinatorial coincidences. We lack of a *geometric Calogero-Moser equivalence*^(*)...

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SET-UP

1. General notation

Throughout this paper, we will abbreviate $\otimes_{\mathbb{C}}$ as \otimes .

If \mathcal{X} is a quasi-projective scheme of finite type over an algebraically closed field, we denote by \mathcal{X}_{red} its reduced subscheme. By an algebraic variety, we mean a quasi-projective reduced scheme of finite type over an algebraically closed field. If \mathcal{X} is an algebraic variety, we denote by \mathcal{X}^{nor} its normalization. If \mathcal{X} is affine we denote by $\mathbb{C}[\mathcal{X}]$ its coordinate ring.

If \mathcal{X} is a complex algebraic variety, we denote by $H^j(\mathcal{X})$ its j -th singular cohomology group with coefficients in \mathbb{C} . If \mathcal{X} carries a regular action of a torus \mathbf{T} , we denote by $H_{\mathbf{T}}^j(\mathcal{X})$ its j -th \mathbf{T} -equivariant cohomology group (still with coefficients in \mathbb{C}). Note that $H^{2\bullet}(\mathcal{X}) = \bigoplus_{j \geq 0} H^{2j}(\mathcal{X})$ is a graded \mathbb{C} -algebra and $H_{\mathbf{T}}^{2\bullet}(\mathcal{X}) = \bigoplus_{j \geq 0} H_{\mathbf{T}}^{2j}(\mathcal{X})$ is a graded $H_{\mathbf{T}}^{2\bullet}(\mathbf{pt})$ -algebra, where $\mathbf{pt} = \text{Spec}(\mathbb{C})$. We identify $H_{\mathbb{C}^{\times}}^{2\bullet}(\mathbf{pt})$ with $\mathbb{C}[\hbar]$ in the usual way (note that $H_{\mathbf{T}}^{2j+1}(\mathbf{pt}) = 0$ for all j). If \mathcal{Y} is another complex variety endowed with a regular \mathbf{T} -action and if $\varphi : \mathcal{Y} \rightarrow \mathcal{X}$ is a \mathbf{T} -equivariant morphism of varieties, we denote by $\varphi^* : H_{\mathbf{T}}^{\bullet}(\mathcal{X}) \rightarrow H_{\mathbf{T}}^{\bullet}(\mathcal{Y})$ the induced morphism in equivariant cohomology.

^(*)Recent works of Dudas-Rouquier relate the category of coherent sheaves on the Hilbert scheme of points in the plane (which is diffeomorphic to the Calogero-Moser space associated with the symmetric group) and representations of finite general linear or unitary groups. This work is still unpublished, but the interested reader might look at numerous videos of some of their talks:

<https://www.birs.ca/events/2017/5-day-workshops/17w5003/videos/watch/201710181031-Rouquier.html>

<https://www.msri.org/workshops/820/schedules/23934>

<https://www.youtube.com/watch?v=CMBV5JC6EX0>

2. Finite linear group, reflections

Notation. We fix in this paper a finite dimensional \mathbb{C} -vector space V and a finite subgroup W of $\mathrm{GL}_{\mathbb{C}}(V)$.

2.A. Reflections, hyperplanes. — We set $\varepsilon : W \rightarrow \mathbb{C}^\times$, $w \mapsto \det(w)$ and

$$\mathrm{Ref}(W) = \{s \in W \mid \dim_{\mathbb{C}} V^s = n - 1\}.$$

Note that, for the moment, we do not assume that $W = \langle \mathrm{Ref}(W) \rangle$. We identify $\mathbb{C}[V]$ (resp. $\mathbb{C}[V^*]$) with the symmetric algebra $S(V^*)$ (resp. $S(V)$).

We denote by \mathcal{A} the set of *reflecting hyperplanes* of W , namely

$$\mathcal{A} = \{V^s \mid s \in \mathrm{Ref}(W)\}.$$

If $H \in \mathcal{A}$, we denote by W_H its inertia group, i.e. the group consisting of elements $w \in W$ such that $w(v) = v$ for all $v \in H$. We denote by α_H an element of V^* such that $H = \mathrm{Ker}(\alpha_H)$ and by α_H^\vee an element of V such that $V = H \oplus \mathbb{C}\alpha_H^\vee$ and the line $\mathbb{C}\alpha_H^\vee$ is W_H -stable. We set $e_H = |W_H|$. Note that W_H is cyclic of order e_H and that $\mathrm{Irr}(W_H) = \{\mathrm{Res}_{W_H}^W \varepsilon^j \mid 0 \leq j \leq e_H - 1\}$. We denote by $\varepsilon_{H,j}$ the (central) primitive idempotent of $\mathbb{C}W_H$ associated with the character $\mathrm{Res}_{W_H}^W \varepsilon^{-j}$, namely

$$\varepsilon_{H,j} = \frac{1}{e_H} \sum_{w \in W_H} \varepsilon(w)^j w \in \mathbb{C}W_H.$$

If Ω is a W -orbit of reflecting hyperplanes, we write e_Ω for the common value of all the e_H , where $H \in \Omega$. We denote by \aleph the set of pairs (Ω, j) where $\Omega \in \mathcal{A}/W$ and $0 \leq j \leq e_\Omega - 1$. The vector space of families of complex numbers indexed by \aleph will be denoted by \mathbb{C}^\aleph , elements of \mathbb{C}^\aleph will be called *parameters*. If $k = (k_{\Omega,j})_{(\Omega,j) \in \aleph} \in \mathbb{C}^\aleph$, we define $k_{H,j}$ for all $H \in \Omega$ and $j \in \mathbb{Z}$ by $k_{H,j} = k_{\Omega,j_0}$ where Ω is the W -orbit of H and j_0 is the unique element of $\{0, 1, \dots, e_H - 1\}$ such that $j \equiv j_0 \pmod{e_H}$.

2.B. Filtration. — Let $\mathrm{cod} : W \rightarrow \mathbb{Z}_{\geq 0}$ be defined by

$$\mathrm{cod}(w) = \mathrm{codim}_{\mathbb{C}}(V^w)$$

(note that $\mathrm{Ref}(W) = \mathrm{cod}^{-1}(1)$) and we define a filtration $\mathcal{F}_\bullet(\mathbb{C}W)$ of the group algebra of W as follows: let

$$\mathcal{F}_j(\mathbb{C}W) = \bigoplus_{\mathrm{cod}(w) \leq j} \mathbb{C}w.$$

Then

$$\mathbb{C}\mathrm{Id}_V = \mathcal{F}_0(\mathbb{C}W) \subset \mathcal{F}_1(\mathbb{C}W) \subset \dots \subset \mathcal{F}_n(\mathbb{C}W) = \mathbb{C}W = \mathcal{F}_{n+1}(\mathbb{C}W) = \dots$$

is a filtration of $\mathbb{C}W$. For any subalgebra A of $\mathbb{C}W$, we set $\mathcal{F}_j(A) = A \cap \mathcal{F}_j(\mathbb{C}W)$, so that

$$\mathbb{C}\mathrm{Id}_V = \mathcal{F}_0(A) \subset \mathcal{F}_1(A) \subset \dots \subset \mathcal{F}_n(A) = A = \mathcal{F}_{n+1}(A) = \dots$$

is also a filtration of A . Write

$$\mathrm{Rees}_{\mathcal{F}}^\bullet(A) = \bigoplus_{j \geq 0} \hbar^j \mathcal{F}_j(A) \subset \mathbb{C}[\hbar] \otimes A \quad (\text{the Rees algebra}),$$

$$\mathrm{gr}_{\mathcal{F}}^\bullet(A) = \bigoplus_{j \geq 0} \mathcal{F}_j(A) / \mathcal{F}_{j-1}(A).$$

Recall that $\mathrm{gr}_{\mathcal{F}}^\bullet(A) \simeq \mathrm{Rees}_{\mathcal{F}}^\bullet(A) / \hbar \mathrm{Rees}_{\mathcal{F}}^\bullet(A)$.

3. Rational Cherednik algebra at $t = 0$

Notation. Throughout this paper, we fix a parameter $k \in \mathbb{C}^{\mathfrak{N}}$.

3.A. Definition. — We define the *rational Cherednik algebra* \mathbf{H}_k to be the quotient of the algebra $T(V \oplus V^*) \rtimes W$ (the semi-direct product of the tensor algebra $T(V \oplus V^*)$ with the group W) by the relations

$$(3.1) \quad \begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = \sum_{H \in \mathfrak{A}} \sum_{j=0}^{e_H-1} e_H (k_{H,j} - k_{H,j+1}) \frac{\langle y, \alpha_H \rangle \cdot \langle \alpha_H^\vee, x \rangle}{\langle \alpha_H^\vee, \alpha_H \rangle} \varepsilon_{H,j}, \end{cases}$$

for all $x, x' \in V^*$, $y, y' \in V$. Here $\langle \cdot, \cdot \rangle : V \times V^* \rightarrow \mathbb{C}$ is the standard pairing. The first commutation relations imply that we have morphisms of algebras $\mathbb{C}[V] \rightarrow \mathbf{H}_k$ and $\mathbb{C}[V^*] \rightarrow \mathbf{H}_k$. Recall [EtGi, Theo. 1.3] that we have an isomorphism of \mathbb{C} -vector spaces

$$(3.2) \quad \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} \mathbf{H}_k$$

induced by multiplication (this is the so-called *PBW-decomposition*).

Remark 3.3. — Let $(l_\Omega)_{\Omega \in \mathfrak{A}/W}$ be a family of complex numbers and let $k' \in \mathbb{C}^{\mathfrak{N}}$ be defined by $k'_{\Omega,j} = k_{\Omega,j} + l_\Omega$. Then $\mathbf{H}_k = \mathbf{H}_{k'}$. This means that there is no restriction to generality if we consider for instance only parameters k such that $k_{\Omega,0} = 0$ for all Ω , or only parameters k such that $k_{\Omega,0} + k_{\Omega,1} + \cdots + k_{\Omega,e_\Omega-1} = 0$ for all Ω (as in [BoRo2]). ■

3.B. Calogero-Moser space. — We denote by \mathbf{Z}_k the center of the algebra \mathbf{H}_k : it is well-known [EtGi, Theo 3.3 and Lem. 3.5] that \mathbf{Z}_k is an integral domain, which is integrally closed. Moreover, it contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ as subalgebras [Gor1, Prop. 3.6] (so it contains $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$), and it is a free \mathbf{P} -module of rank $|W|$. We denote by \mathfrak{X}_k the affine algebraic variety whose ring of regular functions $\mathbb{C}[\mathfrak{X}_k]$ is \mathbf{Z}_k : this is the *Calogero-Moser space* associated with the datum (V, W, k) . It is irreducible and integrally closed.

We set $\mathcal{P} = V/W \times V^*/W$, so that $\mathbb{C}[\mathcal{P}] = \mathbf{P}$ and the inclusion $\mathbf{P} \hookrightarrow \mathbf{Z}_k$ induces a finite and flat morphism of varieties

$$\Upsilon_k : \mathfrak{X}_k \longrightarrow \mathcal{P}.$$

Using the PBW-decomposition, we define a \mathbb{C} -linear map $\Omega^{\mathbf{H}_k} : \mathbf{H}_k \longrightarrow \mathbb{C}W$ by

$$\Omega^{\mathbf{H}_k}(fwg) = f(0)g(0)w$$

for all $f \in \mathbb{C}[V]$, $g \in \mathbb{C}[V^*]$ and $w \in \mathbb{C}W$. This map is W -equivariant with respect to the action on both sides by conjugation, so it induces a well-defined \mathbb{C} -linear map

$$\Omega^k : \mathbf{Z}_k \longrightarrow Z(\mathbb{C}W).$$

Recall from [BoRo2, Cor. 4.2.11] that Ω^k is a morphism of algebras, and that

$$(3.4) \quad \mathfrak{X}_k \text{ is smooth if and only if } \Omega^k \text{ is surjective.}$$

The “only if” part is essentially due to Gordon [Gor1, Cor. 5.8] (but the reader must see take [BoRo2, Prop. 9.6.6 and (16.1.2)] into account for translating Gordon’s result in terms of Ω^k) while the “if” part follows from the work of Bellamy, Schedler and Thiel [BeScTh, Cor. 1.4].

3.C. Other parameters. — Let \mathcal{C} denote the space of maps $\text{Ref}(W) \rightarrow \mathbb{C}$ which are constant on conjugacy classes of reflections. The element

$$\sum_{(\Omega, j) \in \mathbb{N}} \sum_{H \in \Omega} (k_{H, j} - k_{H, j+1}) e_{H \varepsilon_{H, j}}$$

of $Z(\mathbb{C}W)$ is supported only by reflections, so there exists a unique map $c_k \in \mathcal{C}$ such that

$$\sum_{(\Omega, j) \in \mathbb{N}} \sum_{H \in \Omega} (k_{H, j} - k_{H, j+1}) e_{H \varepsilon_{H, j}} = \sum_{s \in \text{Ref}(W)} (\varepsilon(s) - 1) c_k(s) s.$$

Then the map $\mathbb{C}^{\mathbb{N}} \rightarrow \mathcal{C}$, $k \mapsto c_k$, is linear and surjective. With this notation, we have

$$(3.5) \quad [y, x] = \sum_{s \in \text{Ref}(W)} (\varepsilon(s) - 1) c_k(s) \frac{\langle y, \alpha_s \rangle \cdot \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s,$$

for all $y \in V$ and $x \in V^*$. Here, $\alpha_s = \alpha_{V^s}$ and $\alpha_s^\vee = \alpha_{V^s}^\vee$.

3.D. Extra-structures on the Calogero-Moser space. — The Calogero-Moser space \mathcal{X}_k is endowed with a \mathbb{C}^\times -action, a Poisson bracket and an Euler element which are described below.

3.D.1. Grading, \mathbb{C}^\times -action. — The algebra $T(V \oplus V^*) \rtimes W$ can be \mathbb{Z} -graded in such a way that the generators have the following degrees

$$\begin{cases} \deg(y) = -1 & \text{if } y \in V, \\ \deg(x) = 1 & \text{if } x \in V^*, \\ \deg(w) = 0 & \text{if } w \in W. \end{cases}$$

This descends to a \mathbb{Z} -grading on \mathbf{H}_k because the defining relations (3.1) are homogeneous. Since the center of a graded algebra is always graded, the subalgebra \mathbf{Z}_k is also \mathbb{Z} -graded. So the Calogero-Moser space \mathcal{X}_k inherits a regular \mathbb{C}^\times -action. Note also that by definition $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ is clearly a graded subalgebra of \mathbf{Z}_k .

3.D.2. Poisson structure. — Let $t \in \mathbb{C}$. One can define a deformation $\mathbf{H}_{t, k}$ of \mathbf{H}_k as follows: $\mathbf{H}_{t, k}$ is the quotient of the algebra $T(V \oplus V^*) \rtimes W$ by the relations

$$(3.6) \quad \begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = t \langle y, x \rangle + \sum_{H \in \mathcal{S}} \sum_{j=0}^{e_H-1} e_H (k_{H, j} - k_{H, j+1}) \frac{\langle y, \alpha_H \rangle \cdot \langle \alpha_H^\vee, x \rangle}{\langle \alpha_H^\vee, \alpha_H \rangle} \varepsilon_{H, j}, \end{cases}$$

for all $x, x' \in V^*$, $y, y' \in V$. It is well-known [EtGi, Theo 1.3] that the PBW decomposition (as in (3.2)) still holds so that the family $(\mathbf{H}_{t, k})_{t \in \mathbb{C}}$ is a flat deformation of $\mathbf{H}_k = \mathbf{H}_{0, k}$. This allows to define a Poisson bracket $\{, \}$ on \mathbf{Z}_k as follows: if $z_1, z_2 \in \mathbf{Z}_k$, we denote by z_1^t, z_2^t the corresponding element of $\mathbf{H}_{t, k}$ through the PBW decomposition and we define

$$\{z_1, z_2\} = \lim_{t \rightarrow 0} \frac{[z_1^t, z_2^t]}{t}.$$

Finally, note that

$$(3.7) \quad \text{The Poisson bracket is } \mathbb{C}^\times\text{-equivariant.}$$

3.D.3. *Euler element.* — Let (y_1, \dots, y_m) be a basis of V and let (x_1, \dots, x_m) denote its dual basis. As in [BoRo2, §3.3], we set

$$\mathbf{eu} = \sum_{j=1}^m x_j y_j + \sum_{s \in \text{Ref}(W)} \varepsilon(s) c_k(s) s = \sum_{j=1}^m x_j y_j + \sum_{H \in \mathcal{A}} \sum_{j=0}^{e_H-1} e_H k_{H,j} \varepsilon_{H,j}.$$

Recall that \mathbf{eu} does not depend on the choice of the basis of V . Also

$$(3.8) \quad \mathbf{eu} \in \mathbf{Z}_k, \quad \text{Frac}(\mathbf{Z}_k) = \text{Frac}(\mathbf{P})[\mathbf{eu}]$$

and

$$(3.9) \quad \{\mathbf{eu}, z\} = dz$$

if $z \in \mathbf{Z}_k$ is homogeneous of degree d (see for instance [BoRo2, Prop. 3.3.3]).

Notation. If $?$ is one of the above objects defined in this section ($\mathbf{H}_k, \mathcal{F}_k, \mathfrak{N}, \mathcal{A}, \mathcal{H}_k, \dots$), we will sometimes denote it by $?(W)$ or $?(V, W)$ if we need to emphasize the context.

4. Reflection groups

Recall that, for the moment, we did not assume that $W = \langle \text{Ref}(W) \rangle$ (this will be assumed only after this section). Let $W_{\text{ref}} = \langle \text{Ref}(W) \rangle$ be the maximal subgroup of W generated by reflections. Then the set \mathcal{A} depends only on W_{ref} and the finite group W/W_{ref} acts on $\mathfrak{N}(W_{\text{ref}})$ and $\mathfrak{N} = \mathfrak{N}(W_{\text{ref}})/(W/W_{\text{ref}})$. In other words, giving an element $k \in \mathfrak{N}$ is equivalent to giving an element $k \in \mathfrak{N}(W_{\text{ref}})$ which is W/W_{ref} -invariant. In this case, the relations (\mathcal{H}_k) only depend on W_{ref} . If we denote by $\mathbf{H}_k(W_{\text{ref}})$ the Cherednik algebra defined with W_{ref} instead of W , then $\mathbf{H}_k(W_{\text{ref}})$ is naturally a subalgebra of \mathbf{H}_k and, as a $\mathbb{C}W$ -module, $\mathbf{H}_k = \mathbb{C}W \otimes_{\mathbb{C}W_{\text{ref}}} \mathbf{H}_k(W_{\text{ref}})$. Note also that the finite group W/W_{ref} acts on $\mathbf{Z}_k(W_{\text{ref}})$ and on $\mathcal{F}_k(W_{\text{ref}})$, and that

$$(4.1) \quad \mathbf{Z}_k = \mathbf{Z}_k(W_{\text{ref}})^{W/W_{\text{ref}}} \quad \text{and} \quad \mathcal{F}_k = \mathcal{F}_k(W_{\text{ref}})/(W/W_{\text{ref}}).$$

We deduce from this the following facts:

Proposition 4.2. — Let $q : \mathcal{F}_k(W_{\text{ref}}) \longrightarrow \mathcal{F}_k$ denote the quotient map. Then:

- (a) We have $\mathcal{F}_k^{\mathbb{C}^\times} = q(\mathcal{F}_k(W_{\text{ref}})^{\mathbb{C}^\times})$ and $q^{-1}(\mathcal{F}_k^{\mathbb{C}^\times}) = \mathcal{F}_k^{\mathbb{C}^\times}$.
- (b) The morphism q induces isomorphisms

$$q_* : \mathbf{H}^\bullet(\mathcal{F}_k) \xrightarrow{\sim} \mathbf{H}^\bullet(\mathcal{F}_k(W_{\text{ref}}))^{W/W_{\text{ref}}}$$

and

$$q_* : \mathbf{H}_{\mathbb{C}^\times}^\bullet(\mathcal{F}_k) \xrightarrow{\sim} \mathbf{H}_{\mathbb{C}^\times}^\bullet(\mathcal{F}_k(W_{\text{ref}}))^{W/W_{\text{ref}}}.$$

Proof. — (a) follows since q is a finite morphism and since an action of \mathbb{C}^\times on a finite set is necessarily trivial. (b) is a classical property of cohomology [Bre, Theo. III.2.4]. \square

Continuing this reduction, we denote by $W(k)$ the subgroup of W_{ref} generated by the reflections $s \in \text{Ref}(W)$ such that $c_k(s) \neq 0$. It is a normal subgroup of W and W_{ref} . Also, the formula (3.5) shows that, as a $\mathbb{C}W$ -module, $\mathbf{H}_k = \mathbb{C}W \otimes_{\mathbb{C}W_{\text{ref}}} \mathbf{H}_{k^b}(W_{\text{ref}})$. Here, $k^b \in \mathbb{C}^{\mathcal{N}(W(k))}$ is such that $c_{k^b}^{W(k)} \in \mathcal{C}(W(k))$ is the restriction of c_k to $\text{Ref}(W(k))$. Therefore, as above, we have

$$(4.3) \quad \mathbf{Z}_k = \mathbf{Z}_{k^b}(W(k))^{W/W(k)} \quad \text{and} \quad \mathcal{X}_k = \mathcal{X}_{k^b}(W(k))/(W/W(k)).$$

We deduce from this the following facts:

Proposition 4.4. — *Let $q^b : \mathcal{X}_{k^b}(W(k)) \rightarrow \mathcal{X}_k$ denote the quotient map. Then:*

- (a) *We have $\mathcal{X}_k^{\mathbb{C}^\times} = q^b(\mathcal{X}_{k^b}(W(k))^{\mathbb{C}^\times})$ and $(q^b)^{-1}(\mathcal{X}_k^{\mathbb{C}^\times}) = \mathcal{X}_{k^b}^{\mathbb{C}^\times}$.*
- (b) *The morphism q^b induces isomorphisms*

$$q^b : \mathbf{H}^\bullet(\mathcal{X}_k) \xrightarrow{\sim} \mathbf{H}^\bullet(\mathcal{X}_{k^b}(W(k)))^{W/W(k)}$$

and

$$q_*^b : \mathbf{H}_{\mathbb{C}^\times}^\bullet(\mathcal{X}_k) \xrightarrow{\sim} \mathbf{H}_{\mathbb{C}^\times}^\bullet(\mathcal{X}_{k^b}(W(k)))^{W/W(k)}.$$

Even though the case where $k = 0$ serves as a base of our conjectures/questions, the really interesting case is when $W(k) = W$: equations (4.3) and Proposition 4.4 help us to recover properties of $\mathcal{X}_k(W)$ from those of $\mathcal{X}_{k^b}(W(k))$. For instance, Etingof-Ginzburg proved that, if \mathcal{X}_k is smooth, then $W = W(k)$ (see [EtGi, Prop. 3.10]).

5. Braid group, Hecke algebra

Hypothesis and notation. *From now on, and until the end of this paper, we assume that*

$$W = \langle \text{Ref}(W) \rangle$$

and we fix $k \in \mathbb{C}^{\mathcal{N}}$. We set

$$V_{\text{reg}} = V \setminus \bigcup_{H \in \mathcal{A}} H$$

and we recall that V_{reg} is the set of elements of V whose stabilizer in W is trivial (this is Steinberg's Theorem: see for instance [Bro1, Theo. 4.7]).

We fix $v_0 \in V_{\text{reg}}$ and we denote by \bar{v}_0 its image in V_{reg}/W . We set

$$\mathbb{B} = \pi_1(V_{\text{reg}}/W, \bar{v}_0) \quad \text{and} \quad \mathbb{P} = \pi_1(V_{\text{reg}}, v_0).$$

*Then the group \mathbb{B} (resp. \mathbb{P}) is called the **braid group** (resp. the **pure braid group**) of W .*

5.A. Generators of \mathbb{B} and \mathbb{P} . — If $H \in \mathcal{A}$, we denote by s_H the generator of W_H of determinant $\zeta_{e_H} = \exp(2i\pi/e_H)$ and by s_H a *braid reflection* around H (as defined in [Bro1, Def. 4.13]: they are called *generator of the monodromy* around H in [BrMaRo]). Through the exact sequence

$$(5.1) \quad 1 \longrightarrow \mathbb{P} \longrightarrow \mathbb{B} \longrightarrow W \longrightarrow 1$$

induced by the unramified covering $V_{\text{reg}} \rightarrow V_{\text{reg}}/W$, the image of \mathbf{s}_H is s_H and so $\mathbf{s}_H^{e_H} \in \mathbb{P}$. Moreover,

$$(5.2) \quad \mathbb{B} = \langle (\mathbf{s}_H)_{H \in \mathcal{A}} \rangle \quad \text{and} \quad \mathbb{P} = \langle (\mathbf{s}_H^{e_H})_{H \in \mathcal{A}} \rangle.$$

5.B. Hecke algebra. — Let F denote the number field generated by the traces of the elements of W (it is generally called the *character field* of W). It is known [Ben, Bes] that the algebra FW is split. We denote by \mathbb{O} the ring of algebraic integers in F and let $R = \mathbb{O}[\mathbf{q}^{\mathbb{C}}]$ be the group algebra of $(\mathbb{C}, +)$ over \mathbb{O} , denoted with an exponential notation: namely, we have $\mathbf{q}^a \mathbf{q}^{a'} = \mathbf{q}^{a+a'}$ for all $a, a' \in \mathbb{C}$. We set $\mathbf{q} = \mathbf{q}^1$. The *Hecke algebra* with parameter k , denoted by $\mathcal{H}_k(W)$, is the quotient of the group algebra $R\mathbb{B}$ of \mathbb{B} over R by the ideal generated by the elements

$$\prod_{j=0}^{e_H-1} (\mathbf{s}_H - \zeta_{e_H}^j \mathbf{q}^{k_H, j}),$$

where H runs over \mathcal{A} .

We denote by T_H the image of \mathbf{s}_H in $\mathcal{H}_k(W)$. We have

$$(5.3) \quad \prod_{j=0}^{e_H-1} (T_H - \zeta_{e_H}^j \mathbf{q}^{k_H, j}) = 0.$$

If q is a non-zero complex number, let $\mathcal{H}_k(W, q)$ denote a *specialization of $\mathcal{H}_k(W)$ at q* . Namely, we choose a complex logarithm v of q and we denote by $\text{ev}_v : R \rightarrow \mathbb{C}$ the morphism of \mathbb{C} -algebras such that $\mathbf{q}^a \mapsto q^a = \exp(av)$ for all $a \in \mathbb{C}$. Then $\mathcal{H}_k(W, q)$ is the \mathbb{C} -algebra obtained by specialisation through ev_v . This is clearly an abuse of notation, as the specialization might depend on the choice of the logarithm v of q (for instance whenever the parameter k has some non-integer values). But it turns out that, in this survey, this notation will occur only whenever the specialization does not depend on this choice.

Recall that $\mathcal{H}_k(W)$ is a free R -module of rank $|W|$ (see [Ari], [ArKo], [BrMaRo], [Cha1], [Cha2], [Cha3], [Mar1], [Mar2], [Mar3], [MaPf] and [Tsu]) such that its specialization $\mathcal{H}_k(W, 1)$ is just the group algebra $\mathbb{C}W$ of W over \mathbb{C} .

5.C. Hecke families. — Whenever $k_{\Omega, j} \in \mathbb{Z}$ for all $(\Omega, j) \in \mathfrak{N}$, Broué and Kim [BrKi] defined a partition of $\text{Irr}(W)$ into families, which they call *Rouquier k -families*. In [BoRo2, §6.5], Rouquier and the author extended (easily) the definition of these families to general parameters k , and decided to call them *Hecke k -families*. We will stick to this last terminology in this paper. Let us explain this definition.

Let K denote the fraction field of R . The K -algebra $K\mathcal{H}_k(W)$ is split semisimple [Mal3, Theo. 5.2] so, by Tits deformation Theorem [GePf, Theo. 7.4.6], it is isomorphic to the group algebra KW . Therefore, its irreducible characters are in bijection with $\text{Irr}(W)$. If $\chi \in \text{Irr}(W)$, we denote by χ_k the corresponding irreducible character of $K\mathcal{H}_k(W)$. Now, let R_{cyc} denote the localization of R defined by

$$R_{\text{cyc}} = R[(1 - \mathbf{q}^a)^{-1}]_{a \in \mathbb{C} \setminus \{0\}}.$$

We say the χ and χ' are *in the same Hecke k -family* if there is a primitive central idempotent b of $R_{\text{cyc}}\mathcal{H}_k(W)$ such that $\chi_k(b) = \chi'_k(b) \neq 0$.

We denote by $\text{Fam}_k^{\text{Hec}}(W)$ the set of Hecke k -families.

5.D. Calogero-Moser families. — Calogero-Moser families were defined by Gordon using *baby Verma modules* [Gor1, §4.2 and §5.4]. We explain here an equivalent definition given in [BoRo2, §7.2]. If $\chi \in \text{Irr}(W)$, we denote by $\omega_\chi : Z(\mathbb{C}W) \rightarrow \mathbb{C}$ its central character (i.e., $\omega_\chi(z) = \chi(z)/\chi(1)$ is the scalar by which z acts on an irreducible representation affording the character χ). We denote by e_χ (or e_χ^W if necessary) the primitive central idempotent such that $\omega_\chi(e_\chi) = 1$. We say that two irreducible characters χ and χ' of W belong to the same *Calogero-Moser k -family* if $\omega_\chi \circ \Omega^k = \omega_{\chi'} \circ \Omega^k$. If \mathfrak{F} is a subset of $\text{Irr}(W)$, we set

$$e_{\mathfrak{F}} = \sum_{\chi \in \mathfrak{F}} e_\chi \in Z(\mathbb{C}W).$$

Finally, we denote by $\text{Fam}_k^{\text{CM}}(W)$ the set of Calogero-Moser k -families. Then [BoRo2, (16.1.2)]

$$(5.4) \quad \text{Im}(\Omega^k) = \bigoplus_{\mathfrak{F} \in \text{Fam}_k^{\text{CM}}(W)} \mathbb{C}e_{\mathfrak{F}}$$

and $\text{Im}(\Omega^k)$ can be identified with the algebra of functions on $\mathfrak{F}_k^{\mathbb{C}^\times}$.

In other words, this defines a surjective map

$$\mathfrak{z}_k : \text{Irr}(W) \longrightarrow \mathfrak{F}_k^{\mathbb{C}^\times}$$

whose fibers are the Calogero-Moser k -families. If $p \in \mathfrak{F}_k^{\mathbb{C}^\times}$, we denote by \mathfrak{F}_p (or \mathfrak{F}_p^k if we need to emphasize the parameter) the corresponding Calogero-Moser k -family. The next conjecture can be found in [Mart1]:

Conjecture 5.5 (Martino). — Let k^\sharp be the parameter $(k_{\Omega, -j})_{(\Omega, j) \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}$, where the index j is viewed modulo e_Ω . Then each Calogero-Moser k -family is a union of Hecke k^\sharp -families.

Theorem 5.6. — Conjecture 5.5 is known to hold in the following cases^(*):

- (1) If W is of type $G(de, e, r)$, with $d, e, r \geq 1$ and e odd whenever $r = 2$.
- (2) If W is of type $G_4, G_{12}, G_{13}, G_{20}, G_{22}, G_{23} = W(H_3)$ or $G_{28} = W(F_4)$.
- (3) If W is of type $G_5, G_6, G_7, G_8, G_9, G_{10}, G_{14}, G_{15}, G_{16}$ or G_{24} for generic values of k .

Proof. — For (1), see [Mart1, Bel2, Mart2]. For (2) and (3), see [Thi] (except for $G_{28} = W(F_4)$: for this one, see [BoTh]). □

^(*)We refer to Shephard-Todd notation for irreducible complex reflection groups [ShTo].

PART I
QUESTIONS ABOUT CALOGERO-MOSER SPACES

6. Cohomology

6.A. Localization. — We denote by $i_k : \mathcal{X}_k^{\mathbb{C}^\times} \hookrightarrow \mathcal{X}_k$ the closed immersion (here, $\mathcal{X}_k^{\mathbb{C}^\times}$ denotes the reduced zero-dimensional variety of \mathbb{C}^\times -fixed points). As explained in §5.D, we have a natural isomorphism of algebras

$$(6.1) \quad H_{\mathbb{C}^\times}(\mathcal{X}_k^{\mathbb{C}^\times}) \simeq \mathbb{C}[\hbar] \otimes \mathrm{Im}(\Omega^k).$$

So we view the map i_k^* as a morphism of algebras

$$i_k^* : H_{\mathbb{C}^\times}(\mathcal{X}_k) \longrightarrow \mathbb{C}[\hbar] \otimes \mathrm{Im}(\Omega^k).$$

We can now state the following conjecture (see [BoRo2, §16.1] and [BoSh, Conj. 3.3]).

Conjecture 6.2. — *With the above notation, we have:*

- (a) *If $i \geq 0$, then $H^{2i+1}(\mathcal{X}_k) = 0$.*
- (b) *$\mathrm{Im}(i_k^*) = \mathrm{Rees}_{\mathcal{F}}(\mathrm{Im}(\Omega^k))$.*

Recall from standard arguments [BoSh, Prop. 2.4] that this conjecture would imply a description of both the cohomology and the equivariant cohomology of \mathcal{X}_k :

Proposition 6.3. — *Assume that Conjecture 6.2 holds. Then:*

- (a) *If $i \geq 0$, then $H_{\mathbb{C}^\times}^{2i+1}(\mathcal{X}_k) = 0$.*
- (b) *$H_{\mathbb{C}^\times}^{2\bullet}(\mathcal{X}_k) \simeq \mathrm{Rees}_{\mathcal{F}}(\mathrm{Im}(\Omega^k))$ as $\mathbb{C}[\hbar]$ -algebras.*
- (c) *$H^{2\bullet}(\mathcal{X}_k) \simeq \mathrm{gr}_{\mathcal{F}}(\mathrm{Im}(\Omega^k))$.*

Theorem 6.4. — *Conjecture 6.2 is known to hold in the following cases:*

- (a) *If $k = 0$.*
- (b) *If $\dim V = 1$.*
- (c) *If \mathcal{X}_k is smooth.*

Proof. — (a) follows from the fact that $\mathcal{X}_0 = (V \times V^*)/W$ is contractible and $\mathrm{Im} \Omega^0 = \mathbb{C}$. For (b), see [BoRo2, Theo. 18.5.8] and [BoSh, Prop. 1.6]. For (c), see [BoSh, Theo. A]. \square

Example 6.5. — It might be tempting to conjecture that the Calogero-Moser space \mathcal{X}_k is rationally smooth and p -smooth if p is a prime number not dividing $|W|$. Indeed, \mathcal{X}_k is a deformation of $\mathcal{X}_0 = (V \times V^*)/W$ which tends to be smoother and smoother as k becomes more and more generic. However, both statements are false in general:

- (1) If $\dim V = 1$ and $m = |W| \geq 2$ (so that $\aleph = ((0, j))_{0 \leq j \leq m-1}$ and we write $k_j = k_{0,j}$ for simplicity), then it follows from [BoRo2, Theo. 18.2.4] that

$$\mathcal{X}_k = \{(x, y, z) \in \mathbb{C}^3 \mid \prod_{j=0}^{m-1} (z - mk_j) = xy\}.$$

Now, if p is a prime number not dividing m and smaller than m (this always exists if $m \geq 3$), and if we choose k such that $k_0 = k_1 = \dots = k_{p-1} = 0$ and $k_p = k_{p+1} = \dots = k_{m-1} = 1$, then

$$\mathcal{X}_k = \{(x, y, z) \in \mathbb{C}^3 \mid z^p(z - m)^{m-p} = xy\}$$

contains a simple singularity of type A_{p-1} and so \mathcal{X}_k is rationally smooth but not p -smooth while $\mathcal{X}_0 = \mathbb{C}^2/\mu_m$ is p -smooth because p does not divide m .

- (2) If W is of type B_2 and $(k_{\Omega,0}, k_{\Omega,1}) = (k_{\Omega',0}, k_{\Omega',1})$ and $k_{\Omega,0} \neq k_{\Omega,1}$ (where Ω and Ω' are the two orbits of reflecting hyperplanes), then \mathcal{X}_k admits a unique singular point and the singularity is equivalent to the singularity at 0 of the orbit closure of the minimal nilpotent orbit of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ (see [BBFJLS, Theo. 1.3(b)]): it is well-known that this orbit closure is not rationally smooth. ■

6.B. Morphisms between Calogero-Moser spaces. — Let (V', W') be another pair consisting of a finite dimensional complex vector space and a finite subgroup $W' \subset \mathbf{GL}_{\mathbb{C}}(V')$. We fix a parameter $k' \in \mathbb{C}^{\aleph(V', W')}$ and, in this subsection, we will denote by a prime $'$ the object $'$ defined using (V', W') instead of (V, W) , i.e. the object $'(V', W')$. For instance, $\mathcal{X}'_{k'} = \mathcal{X}_{k'}(V', W')$ and $\aleph' = \aleph(V', W')$.

Hypothesis. We assume in this subsection that we are given a \mathbb{C}^{\times} -equivariant morphism of varieties $\varphi : \mathcal{X}'_{k'} \rightarrow \mathcal{X}_k$.

We denote by $\varphi_{\text{fix}} : \mathcal{X}'_{k'}{}^{\mathbb{C}^{\times}} \rightarrow \mathcal{X}_k{}^{\mathbb{C}^{\times}}$ the induced map. Then φ_{fix} induces a morphism of algebras

$$\varphi_{\text{fix}}^{\#} : \text{Im } \Omega^k \longrightarrow \text{Im } \Omega'^{k'}$$

through the formula

$$\varphi_{\text{fix}}^{\#}(e_{\mathfrak{S}_p}) = \sum_{p' \in \varphi_{\text{fix}}^{-1}(p)} e'_{\mathfrak{S}'_{p'}}.$$

The following proposition should be compared with [BoMa, Cor. 1.5]:

Proposition 6.6. — Assume that Conjecture 6.2 holds for both \mathcal{X}_k and $\mathcal{X}'_{k'}$. Then

$$\varphi_{\text{fix}}^{\#}(\mathcal{F}_j \text{Im } \Omega^k) \subset \mathcal{F}'_j \text{Im } \Omega'^{k'}$$

for all j .

Proof. — The maps φ and φ_{fix} induce maps between equivariant cohomology groups which we denote by

$$\varphi^* : H_{\mathbb{C}^\times}^\bullet(\mathcal{X}_k) \longrightarrow H_{\mathbb{C}^\times}^\bullet(\mathcal{X}'_{k'})$$

and

$$\varphi_{\text{fix}}^* : H_{\mathbb{C}^\times}^\bullet(\mathcal{X}_k^{\mathbb{C}^\times}) \longrightarrow H_{\mathbb{C}^\times}^\bullet(\mathcal{X}'_{k'}{}^{\mathbb{C}^\times}).$$

The functoriality properties of cohomology imply that the diagram

$$\begin{array}{ccc} H_{\mathbb{C}^\times}^\bullet(\mathcal{X}_k) & \xrightarrow{\varphi^*} & H_{\mathbb{C}^\times}^\bullet(\mathcal{X}'_{k'}) \\ \downarrow i_k^* & & \downarrow i_{k'}^* \\ H_{\mathbb{C}^\times}^\bullet(\mathcal{X}_k^{\mathbb{C}^\times}) & \xrightarrow{\varphi_{\text{fix}}^*} & H_{\mathbb{C}^\times}^\bullet(\mathcal{X}'_{k'}{}^{\mathbb{C}^\times}) \end{array}$$

is commutative. Recall from (6.1) that we identify $H_{\mathbb{C}^\times}^\bullet(\mathcal{X}_k^{\mathbb{C}^\times})$ (resp. $H_{\mathbb{C}^\times}^\bullet(\mathcal{X}'_{k'}{}^{\mathbb{C}^\times})$) with $\mathbb{C}[\hbar] \otimes \text{Im } \Omega^k$ (resp. $\mathbb{C}[\hbar] \otimes \text{Im } \Omega'^{k'}$). Through this identification, the map φ_{fix}^* becomes $\text{Id}_{\mathbb{C}[\hbar]} \otimes \varphi_{\text{fix}}^\#$ by construction. Therefore, the above commutative diagram yields a commutative diagram

$$\begin{array}{ccc} H_{\mathbb{C}^\times}^\bullet(\mathcal{X}_k) & \xrightarrow{\varphi^*} & H_{\mathbb{C}^\times}^\bullet(\mathcal{X}'_{k'}) \\ \downarrow i_k^* & & \downarrow i_{k'}^* \\ \mathbb{C}[\hbar] \otimes \text{Im } \Omega^k & \xrightarrow{\text{Id}_{\mathbb{C}[\hbar]} \otimes \varphi_{\text{fix}}^\#} & \mathbb{C}[\hbar] \otimes \text{Im } \Omega'^{k'}. \end{array}$$

So $\text{Id}_{\mathbb{C}[\hbar]} \otimes \varphi_{\text{fix}}^\#(\text{Im } i_k^*) \subset \text{Im } i_{k'}^*$. As we assume that Conjecture 6.2 holds for both \mathcal{X}_k and $\mathcal{X}'_{k'}$, this is exactly the statement of the proposition. \square

Remark 6.7. — In the next section, we propose some conjecture which would give many examples of morphisms between Calogero-Moser spaces. In all the cases where these conjectures are proved, the above Proposition 6.6 gives a highly non-trivial link between the character tables of W and W' (see for instance [BoMa, Cor. 4.22] for the case where $W = G(l, 1, n)$). \blacksquare

7. Symplectic leaves and fixed points

If $\tau \in N_{\text{GL}_{\mathbb{C}}(V)}(W)$ has finite order and satisfies $\tau(k) = k$, then τ acts on the Calogero-Moser space \mathcal{X}_k . We are interested in this section in the variety of fixed points \mathcal{X}_k^τ of τ (endowed with its reduced structure) and its symplectic leaves. Since W acts trivially on \mathcal{X}_k , the action of τ on \mathcal{X}_k depends only on its coset $W\tau$.

We say that τ is *W-full* if $\dim(V^\tau) = \max_{w \in W} \dim(V^{w\tau})$ (see [Bon4, §1.F.4]). Recall that τ is *W-full* if and only if the natural map $V^\tau \rightarrow (V/W)^\tau$ is onto [Bon4, (3.2)], (the argument is due to Springer [Spr]). Since W acts trivially on \mathcal{X}_k , we may replace τ by

any $w\tau$ and assume that τ is W -full. Therefore, we will work in this section under the following hypothesis:

Hypothesis and notation. *We fix in this section, and only in this section, a W -full element τ of finite order in $N_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ and we assume that $\tau(k) = k$.*

As in [Bon4], let W_τ denote the quotient Σ/Π , where Σ (resp. Π) is the setwise (resp. pointwise) stabilizer of V^τ . A parabolic subgroup P of W is called τ -split if P is the stabilizer in W of a vector belonging to V^τ . Note that a τ -split parabolic subgroup is τ -stable. If P is a τ -split parabolic subgroup of W , we set

$$\bar{N}_{W_\tau}(P_\tau) = N_{W_\tau}(P_\tau)/P_\tau.$$

Then $\bar{N}_{W_\tau}(P_\tau)$ acts faithfully on the vector space $(V^P)^\tau$. So one can define Calogero-Moser spaces associated with the pair $((V^P)^\tau, \bar{N}_{W_\tau}(P_\tau))$, even though $\bar{N}_{W_\tau}(P_\tau)$ is not necessarily a reflection group for its action on $(V^P)^\tau$.

7.A. Symplectic leaves. — Brown-Gordon [BrGo, §3.5] defined a stratification of any complex affine Poisson variety into *symplectic leaves*. They also proved that the Calogero-Moser space \mathcal{X}_k has only finitely many symplectic leaves [BrGo, Theo. 7.8]. As explained in [Bon4, §4.A], this implies that the variety \mathcal{X}_k^τ admits a stratification into symplectic leaves and that there are only finitely many of them. We denote by $\mathit{Sym}(\mathcal{X}_k^\tau)$ the set of its symplectic leaves. Such a symplectic leaf is called τ -cuspidal if it has dimension 0 (we also talk about τ -cuspidal points^(*)). Note that this notion can be defined even if τ is not W -full. Let $\mathit{Cus}_k^\tau(V, W)$ denote the set of pairs (P, p) where P is a τ -split parabolic subgroup of W and p is a τ -cuspidal point of $\mathcal{X}_{k_P}(V/V^P, P)$ (here k_P is the restriction of k to $\mathfrak{N}(V/V^P, P)$).

Then W_τ acts on $\mathit{Cus}_k^\tau(V, W)$ and the next result is proved in [Bon4, Theo. A] (whenever $\tau = \text{Id}_V$, it is independently due to Bellamy [Bel3] and Losev [Los]).

Theorem 7.1. — *Recall that τ is W -full. Then there is a natural bijection*

$$\mathit{Sym}(\mathcal{X}_k^\tau) \xrightarrow{\sim} \mathit{Cus}_k^\tau(V, W)/W_\tau.$$

Moreover, the dimension of the symplectic leaf associated with the W_τ -orbit of (P, p) through this bijection is equal to $2 \dim(V^P)^\tau$.

We refer to [Bon4, Lem. 8.4] for the explicit description of the bijection: if $(P, p) \in \mathit{Cus}_k^\tau(V, W)$, we denote by $\mathcal{S}_{P,p}$ its associated symplectic leaf. Recall also [Bon4, Rem. 4.2] that all symplectic leaves are \mathbb{C}^\times -stable.

^(*)We can also say that a Calogero-Moser k -family is τ -cuspidal if it is associated to a τ -cuspidal point. Of course, a τ -cuspidal family is τ -stable.

7.B. Normalization. — Let $(P, p) \in \text{Cus}_k^\tau(V, W)$. Then $\overline{\mathcal{S}_{P,p}}$ carries a Poisson structure and so does its normalization $\overline{\mathcal{S}_{P,p}}^{\text{nor}}$ (see [Kal]). We proposed in [Bon4, Conj. B] the following conjecture:

Conjecture 7.2. — *There exists a parameter $k_{P,p} \in \mathbb{C}^{\mathbb{N}((V^P)^\tau, \overline{N}_{W_\tau}(P_\tau))}$ such that the varieties $\overline{\mathcal{S}_{P,p}}^{\text{nor}}$ and $\mathcal{X}_{k_{P,p}}((V^P)^\tau, \overline{N}_{W_\tau}(P_\tau))$ are isomorphic as Poisson varieties endowed with a \mathbb{C}^\times -action.*

Theorem 7.3. — *Conjecture 7.2 is known to hold in the following cases:*

- (a) *If $k = 0$.*
- (b) *If \mathcal{X}_k is smooth.*
- (c) *If W is a Weyl group of type B (i.e. C) and $\tau = \text{Id}_V$.*
- (d) *If W is of type D and τ is a diagram automorphism.*
- (e) *If W is dihedral and τ is the non-trivial diagram automorphism.*
- (f) *If W is of type G_4 .*

Proof. — See [Bon4, Prop. 6.7 and §9] for more details: this relies on works of Bellamy-Maksimau-Schedler [BeMaSc], Maksimau and the author [BoMa] and Thiel and the author [BoTh]. □

7.C. Regular case. — We say that the element τ of $N_{\text{GL}_{\mathbb{C}}(V)}(W)$ is *regular* if $V_{\text{reg}}^\tau \neq \emptyset$. In this case, \mathcal{X}_k^τ admits a unique irreducible component of maximal dimension [Bon5, Prop. 2.4]: we denote by $(\mathcal{X}_k^\tau)_{\text{max}}$. It has dimension $2 \dim(V^\tau)$. The following result has been proved in [Bon5, Theo. 2.8] (here, if χ is a τ -stable character of W , we choose an extension $\tilde{\chi}$ of χ to $W\langle\tau\rangle$):

Theorem 7.4. — *Assume that τ is a regular element. Let $p \in \mathcal{X}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$ and $\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 \neq 0$. Then p belongs to $(\mathcal{X}_k^\tau)_{\text{max}}$.*

Moreover, it is conjectured in [Bon5, Conj. 2.6] that the converse holds:

Conjecture 7.5. — *Assume that τ is a regular element. Let $p \in \mathcal{X}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$. Then p belongs to $(\mathcal{X}_k^\tau)_{\text{max}}$ if and only if $\sum_{\chi \in \mathfrak{F}_p^\tau} |\tilde{\chi}(\tau)|^2 \neq 0$.*

Note that Conjecture 7.5 holds for $W = \mathfrak{S}_n$ by [Bon5, Exam. 5.7].

8. Special features of Coxeter groups

Hypothesis and notation. *We assume in this section, and only in this section, that there exists a W -stable \mathbb{R} -vector subspace $V_{\mathbb{R}}$ of V such that $V = \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}}$ as a W -module, that k takes only **real values** and that $c_k(s) \geq 0$ for all $s \in \text{Ref}(W)$.*

First, note that the reflections of W have order 2 (so that $e_H = 2$ for all $H \in \mathcal{A}$) and that we have a bijection between $\text{Ref}(W)$ and \mathcal{A} . This implies that

$$c_k(s) = k_{H,1} - k_{H,0},$$

where H is the reflecting hyperplane of s . Note that $k = k^{\sharp}$.

8.A. Lusztig families. — If $\chi \in \text{Irr}(W)$, we denote by $\text{sch}_{\chi}^{(k)} \in \mathbb{O}[\mathfrak{q}^{\mathbb{R}}]$ the Schur element associated with the irreducible character χ_k of the Hecke algebra $\mathcal{H}_k(W)$ (see [GePf, §7.2]): since \mathbb{R} is an ordered group, we can set $a_{\chi}^{(k)} = \text{val sch}_{\chi}^{(k)}$ and $A_{\chi}^{(k)} = \text{deg sch}_{\chi}^{(k)}$: this defines two maps $a^{(k)}, A^{(k)} : \text{Irr}(W) \rightarrow \mathbb{R}$. Using the map $a^{(k)}$ and the notion of J -induction, Lusztig [Lus8, §22] defined the notion of k -constructible character (or c_k -constructible character) of W . Let $\mathcal{Frr}_k(W)$ denote the graph defined as follows:

- The set of vertices of $\mathcal{Frr}_k(W)$ is $\text{Irr}(W)$.
- There is an edge between two vertices if they both occur in a same k -constructible character.

A Lusztig k -family is a subset of $\text{Irr}(W)$ consisting of the vertices of a connected component of $\mathcal{Frr}_k(W)$. We denote by $\text{Fam}_k^{\text{Lus}}(W)$ the set of Lusztig k -families of W . It turns out that

$$(8.1) \quad \text{Fam}_k^{\text{Hec}}(W) = \text{Fam}_k^{\text{Lus}}(W)$$

(see [Chl] and [Lus8]). Martino's Conjecture 5.5 has a more precise version in the Coxeter case:

Conjecture 8.2 (Gordon-Martino). — *If W is a Coxeter group, then*

$$\text{Fam}_k^{\text{CM}}(W) = \text{Fam}_k^{\text{Lus}}(W).$$

It follows from the definition [Lus8] that

$$(8.3) \quad \text{the two maps } a^{(k)}, A^{(k)} : \text{Irr}(W) \rightarrow \mathbb{R} \text{ are constant on Lusztig } k\text{-families.}$$

So the next proposition is a strong argument in favor of Conjecture 8.2:

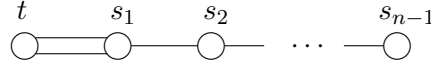
Proposition 8.4. — *The map $a^{(k)} + A^{(k)} : \text{Irr}(W) \rightarrow \mathbb{R}$ is constant on Calogero-Moser k -families.*

Proof. — If χ and χ' belong to the same Calogero-Moser family, then $\omega_{\chi}(\Omega^k(\mathbf{eu})) = \omega_{\chi'}(\Omega^k(\mathbf{eu}))$. But the scalar $\omega_{\chi}(\Omega^k(\mathbf{eu}))$ is, up to a suitable renormalization by a fixed affine transformation of \mathbb{R} , equal to $a_{\chi}^{(k)} + A_{\chi}^{(k)}$ (see [BoRo2, Lem. 7.2.1] and [BrMi, 4.21]). So the result follows. \square

Remark 8.5. — Using the Kazhdan-Lusztig basis of the Hecke algebra $\mathcal{H}_k(W)$ and the associated partition of W into two-sided cells [Lus8, §8] (see also [Bon2, Def. 6.1.4]), Lusztig defined a partition of $\text{Irr}(W)$ into *Kazhdan-Lusztig k -families* as follows: two irreducible characters χ and χ' of W belong to the same Kazhdan-Lusztig k -family if χ_k and χ'_k both occur in the left module associated to a same two-sided cell. Let $\text{Fam}_k^{\text{KL}}(W)$ denote the set of Kazhdan-Lusztig k -families of W .

Lusztig conjectured [Lus8, §23] that $\text{Fam}_k^{\text{Lus}}(W) = \text{Fam}_k^{\text{KL}}(W)$. This conjecture is known to hold in the following cases:

- If c_k is constant [Lus8, Prop. 23.3] (note that this covers the case where W has only one conjugacy class of reflections, i.e. if W is of type ADE or $I_2(2m+1)$).
- If W is dihedral [Lus8].
- If W is of type F_4 [Gec].
- If W is of type B_n and $c_k(t) > (n-1)c_k(s_1)$, where the Coxeter graph is given by



(see [BoIa] and [Bon1]).

Note that this conjecture involves only the Hecke algebra and is not related to the geometry of the Calogero-Moser space and the theme of this paper. ■

8.B. Cuspidal families. — Lusztig also introduced the important notion of τ -cuspidal Lusztig k -family [Lus6, §8.1] (note that the definition in [Lus6, §8.1] is for the *equal parameter* case, but the definition can easily be extended to general parameters, as explained in [BeTh, §2.5]). As there is also a notion of τ -cuspidal Calogero-Moser family (see §7.A), it is natural to expect that the equality predicted by Gordon-Martino conjecture preserves this feature:

Conjecture 8.6. — *If W is a finite Coxeter group, and if $\tau \in N_{\text{GL}_{\mathbb{R}}(V_{\mathbb{R}})}(W)$ is W -full and satisfies $\tau(k) = k$, then the τ -cuspidal Lusztig k -families coincide with the τ -cuspidal Calogero-Moser k -families.*

If $\tau = \text{Id}_V$, this conjecture has been proposed by Bellamy and Thiel [BeTh, Conj. B]. The τ -cuspidal Lusztig k -families have been classified (see [Lus6, §8.1] for the equal parameter case and [BeTh, §6, §7] for the unequal parameter case) and it turns out that there is at most one τ -cuspidal Lusztig k -family. So Conjecture 8.6 would imply that there is at most one τ -cuspidal point in \mathcal{X}_k whenever W is a Coxeter group [BeTh, Conj. D].

Theorem 8.7. — *Conjectures 8.2 and 8.6 are known to hold for W of type $A, B = C, D, I_2(m), H_3$ or F_4 (with the restriction that $\tau = \text{Id}_V$ if W is of type F_4).*

Proof. — For Conjecture 8.2, see [Gor1, Theo. 5.6] for type A , see [GoMa] for types $B = C$ and D , see [Bel2] for type $I_2(m)$ and [BoTh] for types H_3 and F_4 .

For Conjecture 8.6, see [BeTh, Theo. A] for types $A, B = C, D$ and $I_2(m)$, and [BoTh] for types H_3 and F_4 . □

PART II

UNIPOTENT REPRESENTATIONS OF FINITE REDUCTIVE GROUPS

Throughout this part, we will only consider algebraic varieties and algebraic groups defined over an algebraic closure of a finite field. If \mathbf{G} is an algebraic group, we denote by $Z(\mathbf{G})$ its center. If \mathbf{S} is a torus, we denote by $Y(\mathbf{S})$ its lattice of one-parameter subgroups.

Let *Groups* denote the class of triples (q, \mathbf{G}, F) where:

- q is a power of some prime number p .
- \mathbf{G} is a connected reductive group defined over an algebraic closure \mathbb{F} of the finite field \mathbb{F}_p with p elements.
- $F : \mathbf{G} \rightarrow \mathbf{G}$ is a Frobenius endomorphism of \mathbf{G} endowing \mathbf{G} with a rational structure over the finite subfield \mathbb{F}_q of \mathbb{F} with q elements.

This part provides a quick survey on unipotent representations of the finite reductive group \mathbf{G}^F (where $(q, \mathbf{G}, F) \in \text{Groups}$) and their associated structures (cuspidal representations, Harish-Chandra theory, d -Harish-Chandra theory...)^(*)

Hypothesis and notation. We fix in this part, and only in this part, a triple $\mathcal{G} = (q, \mathbf{G}, F) \in \text{Groups}$. We denote by p the unique prime number dividing q and by \mathbb{F} the algebraic closure of \mathbb{F}_p over which \mathbf{G} is defined. We fix a prime number ℓ different from p and we denote by $\overline{\mathbb{Q}_\ell}$ an algebraic closure of \mathbb{Q}_ℓ . Note that \mathbf{G} is not necessarily split over \mathbb{F}_q .

If \mathbf{X} is an algebraic variety over \mathbb{F} , we denote by $H_c^j(\mathbf{X})$ its j -th ℓ -adic cohomology group with compact support with coefficients in $\overline{\mathbb{Q}_\ell}$: it is a finite dimensional $\overline{\mathbb{Q}_\ell}$ -vector space. We set $H_c^\bullet(\mathbf{X}) = \bigoplus_{j \geq 0} H_c^j(\mathbf{X})$.

We fix an F -stable Borel subgroup \mathbf{B} of \mathbf{G} and an F -stable maximal torus \mathbf{T} of \mathbf{B} . Let $\mathfrak{B} = \mathbf{G}/\mathbf{B}$ denote the flag variety. Let $\mathcal{W} = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ denote the Weyl group of \mathbf{G} . It is acted on by F and we denote by τ the automorphism of \mathcal{W} induced by F . If \mathfrak{O} is a \mathbf{G} -orbit in $\mathfrak{B} \times \mathfrak{B}$, we denote by

$$\mathbf{X}_{\mathfrak{O}} = \{g\mathbf{B} \in \mathfrak{B} \mid (g\mathbf{B}, F(g)\mathbf{B}) \in \mathfrak{O}\}.$$

Then $\mathbf{X}_{\mathfrak{O}}$ is called a *Deligne-Lusztig variety*: it is acted on the left by the finite group \mathbf{G}^F . Hence, the vector spaces $H_c^j(\mathbf{X}_{\mathfrak{O}})$ inherits a structure of $\overline{\mathbb{Q}_\ell}\mathbf{G}^F$ -module. An irreducible representation of \mathbf{G}^F (over $\overline{\mathbb{Q}_\ell}$) is called *unipotent* if it appears in such a $\overline{\mathbb{Q}_\ell}\mathbf{G}^F$ -module, for some \mathfrak{O} and some j . The set of isomorphism classes of irreducible unipotent representations of \mathbf{G}^F will be denoted by $\mathcal{Unip}(\mathcal{G})$. We define a *unipotent* representation of \mathbf{G}^F to be a direct sum of irreducible unipotent representations.

We define a *Levi subgroup* of \mathcal{G} to be a triple $\mathcal{L} = (q, \mathbf{L}, F)$ where \mathbf{L} is an F -stable Levi complement of a parabolic subgroup of \mathbf{G} . In this case, we set $W_{\mathcal{G}}(\mathcal{L}) = N_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}^F$: this group acts on $\mathcal{Unip}(\mathcal{L})$ and, if $\lambda \in \mathcal{Unip}(\mathcal{L})$, we denote by $W_{\mathcal{G}}(\mathcal{L}, \lambda)$ its stabilizer.

The set of unipotent representations admits several interesting partitions, which are related to the different problems one may consider: Harish-Chandra series for an algebraic parametrization, d -Harish-Chandra series for blocks in transverse characteristic, families for computing character values through the theory of character sheaves... All these partitions interconnect in a very subtle way.

^(*)Our formalism here does not include the Suzuki and Ree groups

9. Harish-Chandra theories

9.A. Classical Harish-Chandra theory. — If \mathbf{L} is an F -stable Levi complement of an F -stable parabolic subgroup \mathbf{P} of \mathbf{G} , we denote by

$$\begin{aligned} \mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}} : \overline{\mathbb{Q}}_{\ell} \mathbf{L}^F\text{-mod} &\longrightarrow \overline{\mathbb{Q}}_{\ell} \mathbf{G}^F\text{-mod} \\ M &\longmapsto \text{Ind}_{\mathbf{P}^F}^{\mathbf{G}^F} \tilde{M} \end{aligned}$$

the *Harish-Chandra induction* functor [DiMi1, Chap. 4]. Here, \tilde{M} denotes the inflation of M through the surjective morphism $\mathbf{P}^F \twoheadrightarrow \mathbf{L}^F$. If we denote by $\mathbf{U}_{\mathbf{P}}$ the unipotent radical of \mathbf{P} , then $\mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}}$ admits a left and right adjoint functor ${}^* \mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}}$ given by

$$\begin{aligned} {}^* \mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}} : \overline{\mathbb{Q}}_{\ell} \mathbf{G}^F\text{-mod} &\longrightarrow \overline{\mathbb{Q}}_{\ell} \mathbf{L}^F\text{-mod} \\ N &\longmapsto N^{\mathbf{U}_{\mathbf{P}}^F}. \end{aligned}$$

The functor ${}^* \mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}}$ is called the *Harish-Chandra restriction*. Note that both functors send a unipotent representation to a unipotent representation. An irreducible unipotent representation N is called *cuspidal* if ${}^* \mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}} N = 0$ for any F -stable Levi complement of a proper F -stable parabolic subgroup \mathbf{P} of \mathbf{G} . We denote by $\mathcal{U}nir_{\text{cus}}(\mathcal{G})$ the set of (isomorphism classes of) cuspidal unipotent irreducible representations of \mathbf{G}^F .

It turns out that both Harish-Chandra functors do not depend on the choice of the F -stable parabolic subgroup \mathbf{P} admitting \mathbf{L} as a Levi complement. We denote by $\mathcal{C}us(\mathcal{G})$ the set of pairs (\mathcal{L}, λ) , where $\mathcal{L} = (q, \mathbf{L}, F)$ and \mathbf{L} is an F -stable Levi complement of an F -stable parabolic subgroup of \mathbf{G} , and $\lambda \in \mathcal{U}nir_{\text{cus}}(\mathcal{L})$. We denote by $\mathcal{U}nir(\mathcal{G}, \mathcal{L}, \lambda)$ the set of unipotent irreducible representations occurring in $\mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}} \lambda$, where \mathbf{P} is any F -stable parabolic subgroup admitting \mathbf{L} as a Levi complement: this set is called the *Harish-Chandra series* associated with the pair (\mathcal{L}, λ) . This set depends on the pair (\mathcal{L}, λ) up to \mathbf{G}^F -conjugacy. We denote by $\mathcal{C}us(\mathcal{G})/\sim$ the set of \mathbf{G}^F -conjugacy classes of elements of $\mathcal{C}us(\mathcal{G})$. The *Harish-Chandra theory* can then be summarized as follows [Lus1, §5]:

Theorem 9.1 (Lusztig). — *We have*

$$\mathcal{U}nir(\mathcal{G}) = \dot{\bigcup}_{(\mathcal{L}, \lambda) \in \mathcal{C}us(\mathcal{G})/\sim} \mathcal{U}nir(\mathcal{G}, \mathcal{L}, \lambda),$$

where $\dot{\bigcup}$ means a disjoint union.

Moreover, if $(\mathcal{L}, \lambda) \in \mathcal{C}us(\mathcal{G})$, with $\mathcal{L} = (q, \mathbf{L}, F)$ and if \mathbf{P} is any F -stable parabolic subgroup admitting \mathbf{L} as a Levi complement, then:

- (a) The group $W_{\mathcal{G}}(\mathcal{L})$ is a Weyl group for its action on the lattice

$$\{y \in Y(\mathbf{Z}(\mathbf{L})) \mid F(y) = qy\}.$$

Moreover, $W_{\mathcal{G}}(\mathcal{L}, \lambda) = W_{\mathcal{G}}(\mathcal{L})$.

- (b) The endomorphism algebra $\text{End}_{\mathbf{G}^F} \mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}} \lambda$ is isomorphic to a Hecke algebra of the form $\mathcal{H}_{k_{\mathcal{L}}}(W_{\mathcal{G}}(\mathcal{L}), q)$, where $k_{\mathcal{L}} \in \mathfrak{N}(W_{\mathcal{G}}(\mathcal{L}))$ does not depend on λ (and is integer valued).
- (c) The statement (a) induces a bijection $\text{HC}^{\mathcal{G}, \mathcal{L}, \lambda} : \text{Irr } W_{\mathcal{G}}(\mathcal{L}) \xrightarrow{\sim} \mathcal{U}nir(\mathcal{G}, \mathcal{L}, \lambda)$. In other words,

$$\mathcal{U}nir(\mathcal{G}) = \dot{\bigcup}_{(\mathcal{L}, \lambda) \in \mathcal{C}us(\mathcal{G})/\sim} \text{HC}^{\mathcal{G}, \mathcal{L}, \lambda}(\text{Irr } W_{\mathcal{G}}(\mathcal{L})).$$

Example 9.2 (Principal series). — As \mathbf{T} is an F -stable maximal torus of the F -stable Borel subgroup \mathbf{B} , we may set $\mathcal{T} = (q, \mathbf{T}, F)$ and so there is a Harish-Chandra series associated with the trivial character (denoted by 1) of \mathbf{T}^F (indeed, as there is no proper parabolic subgroup of \mathbf{T} , 1 is a cuspidal unipotent representation). Then $W_{\mathcal{G}}(\mathcal{T}) = \mathcal{W}^\tau$ and the injective map $\mathrm{HC}^{\mathcal{G}, \mathcal{T}, 1} : \mathrm{Irr}(\mathcal{W}^\tau) \rightarrow \mathcal{U}n\mathrm{i}p(\mathcal{G})$ will be simply denoted by $\mathrm{HC}^{\mathcal{G}}$. Its image is called the *principal series* of unipotent representations of \mathcal{G} .

If $\tau = \mathrm{Id}_{\mathcal{W}}$ (i.e., if $\mathbf{G}/\mathbf{Z}(\mathbf{G})$ is split), then the parameter $k_{\mathcal{T}}$ of Theorem 9.1(b) is given by $(k_{\mathcal{T}})_{H,0} = 1$ and $(k_{\mathcal{T}})_{H,1} = 0$ for any reflecting hyperplane H of \mathcal{W} . This parameter will be denoted by k_{sp} in the next part. ■

9.B. Deligne-Lusztig induction. — For going further, we need to recall the construction of Deligne-Lusztig induction. If \mathbf{P} is a (not necessarily F -stable) parabolic subgroup of \mathbf{G} admitting an F -stable Levi complement \mathbf{L} , then we define the variety $\mathbf{Y}_{\mathbf{P}}$ (also called a *Deligne-Lusztig variety*) by

$$\mathbf{Y}_{\mathbf{P}} = \{g\mathbf{U}_{\mathbf{P}} \in \mathbf{G}/\mathbf{U}_{\mathbf{P}} \mid g^{-1}F(g) \in \mathbf{U}_{\mathbf{P}} \cdot F(\mathbf{U}_{\mathbf{P}})\},$$

where $\mathbf{U}_{\mathbf{P}}$ denotes the unipotent radical of \mathbf{P} . Then $\mathbf{Y}_{\mathbf{P}}$ inherits a left action of \mathbf{G}^F and a right action of \mathbf{L}^F which commute, endowing the ℓ -adic cohomology groups with compact support $\mathrm{H}_c^j(\mathbf{Y}_{\mathbf{P}})$ with a structure of a $(\overline{\mathbb{Q}}_{\ell}\mathbf{G}^F, \overline{\mathbb{Q}}_{\ell}\mathbf{L}^F)$ -bimodule. This allows us to define a map $R_{\mathbf{LCP}}^{\mathbf{G}} : \mathbb{Z} \mathrm{Irr}(\mathbf{L}^F) \rightarrow \mathbb{Z} \mathrm{Irr}(\mathbf{G}^F)$ between Grothendieck groups by the formula

$$R_{\mathbf{LCP}}^{\mathbf{G}}([M]) = \sum_{k \geq 0} (-1)^j [\mathrm{H}_c^j(\mathbf{Y}_{\mathbf{P}}) \otimes_{\overline{\mathbb{Q}}_{\ell}\mathbf{L}^F} M].$$

This map is called the *Deligne-Lusztig induction* and is the shadow of a functor $\mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}} : \mathrm{D}^b(\overline{\mathbb{Q}}_{\ell}\mathbf{L}^F\text{-mod}) \rightarrow \mathrm{D}^b(\overline{\mathbb{Q}}_{\ell}\mathbf{G}^F\text{-mod})$ between bounded derived categories, which is defined by

$$\mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}}(M) = \mathrm{H}_c^{\bullet}(\mathbf{Y}_{\mathbf{P}}) \otimes_{\overline{\mathbb{Q}}_{\ell}\mathbf{L}^F} M.$$

Note that we work in a semisimple world, so that any complex of $\overline{\mathbb{Q}}_{\ell}\Gamma$ -modules (where Γ is a finite group) is quasi-isomorphic to its cohomology: here, $\mathrm{H}_c^{\bullet}(\mathbf{Y}_{\mathbf{P}})$ is viewed as a complex of $(\overline{\mathbb{Q}}_{\ell}\mathbf{G}^F, \overline{\mathbb{Q}}_{\ell}\mathbf{L}^F)$ -bimodules whose j -th term is $\mathrm{H}_c^j(\mathbf{Y}_{\mathbf{P}})$ and whose differential is zero.

If the parabolic subgroup \mathbf{P} is F -stable, then the Deligne-Lusztig functor $\mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}}$ is just the functor induced by the Harish-Chandra functor at the level of derived categories (this justifies the use of the same notation). Both $R_{\mathbf{LCP}}^{\mathbf{G}}$ and $\mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}}$ admits adjoints (in two different significations of *adjoint*) which we denote by ${}^*R_{\mathbf{LCP}}^{\mathbf{G}}$ and ${}^*\mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}}$ and which are defined by

$${}^*\mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}}(N) = \mathrm{H}_c^{\bullet}(\mathbf{Y}_{\mathbf{P}})^* \otimes_{\overline{\mathbb{Q}}_{\ell}\mathbf{G}^F} N$$

and

$${}^*R_{\mathbf{LCP}}^{\mathbf{G}}([N]) = \sum_{k \geq 0} (-1)^j [\mathrm{H}_c^j(\mathbf{Y}_{\mathbf{P}})^* \otimes_{\overline{\mathbb{Q}}_{\ell}\mathbf{G}^F} N].$$

9.C. d -Harish-Chandra theory. — We fix a natural number $d \geq 1$ and we denote by ζ_d a primitive d -th root of unity. The d -Harish-Chandra theory of Broué-Malle-Michel [BMM1] is an analogue of Harish-Chandra theory where the F -stable Levi subgroups of F -stable parabolic subgroups are replaced by a larger class of F -stable Levi subgroups, using Deligne-Lusztig induction instead of Harish-Chandra induction. Whenever $d = 1$, we retrieve the usual Harish-Chandra theory. Let us summarize it.

Let \mathbf{S} be a torus over \mathbb{F} , endowed with a Frobenius endomorphism $F : \mathbf{S} \rightarrow \mathbf{S}$ associated with an \mathbb{F}_q -structure on \mathbf{S} . Let Φ_d denote the d -th cyclotomic polynomial. Then \mathbf{S} is called a Φ_d -torus if the following two conditions are satisfied:

- \mathbf{S} is split over \mathbb{F}_{q^d} .
- If \mathbf{S}' is an F -stable subtorus of \mathbf{S} different from 1, and if e divides d and is different from d , then \mathbf{S}' is not split over \mathbb{F}_{q^e} .

A Levi subgroup $\mathcal{L} = (q, \mathbf{L}, F)$ of \mathcal{G} is called d -split if \mathbf{L} is the centralizer of an F -stable Φ_d -torus of \mathbf{G} .

Remark 9.3. — Let \mathbf{L} be an F -stable Levi subgroup of \mathbf{G} . Then (q, \mathbf{L}, F) is 1-split if and only if \mathbf{L} is the Levi complement of an F -stable parabolic subgroup of \mathbf{G} . ■

An irreducible unipotent representation N of \mathbf{G}^F is called d -cuspidal if $*R_{\mathbf{LCP}}^{\mathbf{G}}[N] = 0$ for any pair (\mathbf{L}, \mathbf{P}) where \mathbf{P} is a parabolic subgroup of \mathbf{G} and \mathbf{L} is an F -stable Levi complement of \mathbf{P} which is d -split as a Levi subgroup of \mathbf{G} . We denote by $\mathcal{U}nip_{\text{cus}}^d(\mathcal{G})$ the set of (isomorphism classes of) d -cuspidal irreducible unipotent representations of \mathbf{G}^F .

We denote by $\mathcal{C}us^d(\mathcal{G})$ the set of pairs (\mathcal{L}, λ) , where $\mathcal{L} = (q, \mathbf{L}, F)$ is a d -split Levi subgroup of \mathbf{G} and λ is a d -cuspidal irreducible unipotent representation of \mathbf{L}^F . We denote by $\mathcal{U}nip(\mathcal{G}, \mathcal{L}, \lambda)$ the set of unipotent irreducible representations occurring in $R_{\mathbf{LCP}}^{\mathbf{G}}[\lambda]$, where \mathbf{P} is a parabolic subgroup admitting \mathbf{L} as a Levi complement: this set is called the d -Harish-Chandra series associated with the pair (\mathcal{L}, λ) . This set depends on the pair (\mathcal{L}, λ) up to \mathbf{G}^F -conjugacy. We denote by $\mathcal{C}us^d(\mathcal{G})/\sim$ the set of \mathbf{G}^F -conjugacy classes of elements of $\mathcal{C}us^d(\mathcal{G})$. The d -Harish-Chandra theory can then be summarized as follows [BMM1, Theo. 3.2]:

Theorem 9.4 (Broué-Malle-Michel). — We have

$$\mathcal{U}nip(\mathcal{G}) = \dot{\bigcup}_{(\mathcal{L}, \lambda) \in \mathcal{C}us^d(\mathcal{G})/\sim} \mathcal{U}nip(\mathcal{G}, \mathcal{L}, \lambda),$$

where $\dot{\bigcup}$ means a disjoint union.

Moreover, if $(\mathcal{L}, \lambda) \in \mathcal{C}us^d(\mathcal{G})$ with $\mathcal{L} = (q, \mathbf{L}, F)$ and if \mathbf{P} is a parabolic subgroup admitting \mathbf{L} as a Levi complement, then:

- (a) The group $W_{\mathcal{G}}(\mathcal{L}, \lambda)$ is a complex reflection group for its action on

$$\{y \in \mathbb{C} \otimes_{\mathbb{Z}} Y(Z(\mathbf{L})) \mid F(y) = \zeta_d q y\}.$$

- (b) There exists a bijection $\text{HC}_d^{\mathcal{G}, \mathcal{L}, \lambda} : \text{Irr } W_{\mathcal{G}}(\mathcal{L}, \lambda) \xrightarrow{\sim} \mathcal{U}nip(\mathcal{G}, \mathcal{L}, \lambda)$ and a sign function $\text{Irr } W_{\mathcal{G}}(\mathcal{L}, \lambda) \rightarrow \{1, -1\}$ intertwining ordinary induction and Lusztig induction [BMM1]^(*). In other words,

$$\mathcal{U}nip(\mathcal{G}) = \dot{\bigcup}_{(\mathcal{L}, \lambda) \in \mathcal{C}us^d(\mathcal{G})/\sim} \text{HC}_d^{\mathcal{G}, \mathcal{L}, \lambda}(\text{Irr } W_{\mathcal{G}}(\mathcal{L}, \lambda)).$$

^(*)Sorry for being somewhat vague in this survey.

Note that this theorem sounds like a miracle and is proved through a case-by-case analysis: it would be better explained by the following conjecture (which is true for $d = 1$ by Theorem 9.1).

Conjecture 9.5 (Broué-Malle-Michel). — *Let $(\mathcal{L}, \lambda) \in \mathcal{Cus}^d(\mathcal{G})$ with $\mathcal{L} = (q, \mathbf{L}, F)$. Then there exists a parabolic subgroup \mathbf{P} of \mathbf{G} , admitting \mathbf{L} as a Levi complement, and such that:*

- (a) *The $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$ -modules $H_c^j(\mathbf{Y}_{\mathbf{P}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{L}^F} \lambda$ and $H_c^{j'}(\mathbf{Y}_{\mathbf{P}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{L}^F} \lambda$ have no common irreducible constituent if $j \neq j'$.*
- (b) *The endomorphism algebra of the complex of $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$ -modules $\mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}}(\lambda)$ is isomorphic to a Hecke algebra $\mathcal{H}_{k_{\mathcal{L}, \lambda}}(W_{\mathcal{G}}(\mathcal{L}, \lambda), \zeta_d^{-1}q)$ for some parameter $k_{\mathcal{L}, \lambda} \in \mathfrak{N}(W_{\mathcal{G}}(\mathcal{L}, \lambda))$.*

Of course, Conjecture 9.5 can easily be reduced to the case where \mathbf{G} is quasi-simple. In this case, and for $d > 1$, the full Conjecture 9.5 is known only whenever d is the Coxeter number (see Lusztig's fundamental paper [Lus1], which served as an inspiration for the conjecture) or whenever \mathbf{G} is of type A_d (see [DiMi2]). Part (a) of Conjecture 9.5 is known if \mathbf{G} is of type A and \mathcal{L} is a torus (see [BDR], which relies in an essential way on the work of Dudas [Dud]). For this part (a), some other cases have been solved by Digne-Michel-Rouquier [DiMiRo] and Digne-Michel [DiMi2].

For part (b), note also that, in many important cases, a map from the group algebra of the braid group of $W_{\mathcal{G}}(\mathcal{L}, \lambda)$ to the endomorphism algebra of the complex of $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$ -modules $\mathcal{R}_{\mathbf{LCP}}^{\mathbf{G}}(\lambda)$ has been constructed [BrMi, BrMa2, DiMi2], but it is generally not known whether it is onto and if it factorizes (exactly) through the Hecke algebra $\mathcal{H}_{k_{\mathcal{L}, \lambda}}(W_{\mathcal{G}}(\mathcal{L}, \lambda), \zeta_d^{-1}q)$. There are, however, partial results in this direction [BrMi, BrMa2, DiMi2].

Also, extra-properties that should be satisfied by the Hecke algebra involved in Conjecture 9.5 imply that, if $\chi \in \text{Irr}(W_{\mathcal{G}}(\mathcal{L}, \lambda))$, then

$$(9.6) \quad \deg H^{\mathcal{G}, \mathcal{L}, \lambda}(\chi) = \pm \frac{R_{\mathbf{LCP}}^{\mathbf{G}}(\lambda)(1)}{\text{sch}_{\chi}^{k_{\mathcal{L}, \lambda}}}.$$

This imposes huge constraints on the parameter $k_{\mathcal{L}, \lambda}$ and allows to determine it explicitly in almost all cases [BMM1] (including the classical groups).

Remark 9.7. — Let ℓ be a prime number different from p and assume for simplicity that $\ell \geq 5$ and ℓ is very good for \mathbf{G} . Let d denote the smallest integer such that ℓ divides $q^d - 1$. Then two irreducible unipotent representations of \mathbf{G}^F belong to the same ℓ -block if and only if they belong to the same d -Harish-Chandra series (see [BMM1, Theo. 5.24] for the case where ℓ does not divide the order of the Weyl group and [CaEn, Theo. 22.4] for the general case). ■

10. Families

10.A. Almost characters. — If χ is a τ -stable irreducible character of \mathcal{W} , we fix once and for all an extension $\tilde{\chi}$ of χ to $\mathcal{W} \rtimes \langle \tau \rangle$. If $w \in \mathcal{W}$, we set $\mathbf{T}_w = g_w \mathbf{T} g_w^{-1}$, where $g_w \in \mathbf{G}$ is chosen so that $g_w^{-1} F(g_w) \in \mathbf{N}_{\mathbf{G}}(\mathbf{T})$ is a representative of w . Then \mathbf{T}_w is an F -stable maximal torus so, using Deligne-Lusztig induction, we can define

$$(10.1) \quad R_{\chi}^{\mathcal{G}} = \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \tilde{\chi}(w\tau) R_{\mathbf{T}_w}^{\mathbf{G}}(1_{\mathbf{T}_w^F}) \in \mathbb{C} \mathcal{U}n\mathcal{I}p(\mathcal{G}).$$

Then $R_{\chi}^{\mathcal{G}}$ is called an *almost character* of \mathbf{G}^F . Note that $R_{\chi}^{\mathcal{G}}$ depends on the choice of $\tilde{\chi}$, but only up to multiplication by a root of unity.

10.B. Families. — Lusztig [Lus6, Chap. 4] has defined a partition of $\mathcal{U}n\mathcal{I}p(\mathcal{G})$ into *families*: let us recall his construction. Define a graph $\mathfrak{G}r_{\mathcal{G}}$ on $\text{Irr}(\mathcal{W})^{\tau}$ as follows:

- (G1) The set of vertices of $\mathfrak{G}r_{\mathcal{G}}$ is $\text{Irr}(\mathcal{W})^{\tau}$.
- (G2) Two τ -stable irreducible characters χ and χ' are linked by an edge in the graph $\mathfrak{G}r_{\mathcal{G}}$ if $R_{\chi}^{\mathcal{G}}$ and $R_{\chi'}^{\mathcal{G}}$ have a common irreducible constituent.

If \mathcal{C} is a connected component of $\mathfrak{G}r_{\mathcal{G}}$, we denote by $\mathfrak{F}_{\mathcal{C}}^{\mathcal{G}}$ the set of irreducible unipotent representations $\gamma \in \mathcal{U}n\mathcal{I}p(\mathcal{G})$ such that $\langle R_{\chi}^{\mathcal{G}}, \gamma \rangle_{\mathbf{G}^F} \neq 0$ for some $\chi \in \mathcal{C}$. The subset $\mathfrak{F}_{\mathcal{C}}^{\mathcal{G}}$ of $\mathcal{U}n\mathcal{I}p(\mathcal{G})$ is called a *unipotent Lusztig family* of \mathcal{G} . We denote by $\mathcal{F}am_{\text{un}}(\mathcal{G})$ the set of such families. By construction, the unipotent Lusztig families form a partition of $\mathcal{U}n\mathcal{I}p(\mathcal{G})$.

One of the main results in Lusztig's work on unipotent representations is the list of following compatibilities between this partition and Harish-Chandra series [Lus6] (some of them are proved by a case-by-case analysis):

Theorem 10.2 (Lusztig). — *With the above notation, we have:*

- (a) *If $(\mathcal{L}, \lambda) \in \mathcal{C}us(\mathcal{G})$ and $\mathfrak{F} \in \mathcal{F}am_{\text{un}}(\mathcal{G})$, then $(\text{HC}^{\mathcal{G}, \mathcal{L}, \lambda})^{-1}(\mathfrak{F})$ is empty or belongs to $\text{Fam}_{k_{\mathcal{L}}}^{\text{Lus}}(W_{\mathcal{G}}(\mathcal{L}))$ (recall that $W_{\mathcal{G}}(\mathcal{L}) = W_{\mathcal{G}}(\mathcal{L}, \lambda)$ and that $k_{\mathcal{L}} = k_{\mathcal{L}, \lambda}$ does not depend on λ).*
- (b) *If $\mathcal{U}n\mathcal{I}p_{\text{cus}}(\mathcal{G})$ is non-empty, then it is contained in a single family, which will be denoted by $\mathfrak{F}_{\text{cus}}^{\mathcal{G}}$.*
- (c) *If $\tau = \text{Id}_V$, then the principal series (see Example 9.2) satisfies the following property: for any $\mathfrak{F} \in \mathcal{F}am_{\text{un}}(\mathcal{G})$ and any $\chi \in \text{Irr}(\mathcal{W})$, then $R_{\chi}^{\mathcal{G}} \in \mathbb{C}\mathfrak{F}$ if and only if $\text{HC}^{\mathcal{G}}(\chi) \in \mathfrak{F}$. In particular, every family meets the principal series.*

Note that the analogue of statement (a) for d -Harish-Chandra theory (instead of classical Harish-Chandra theory) is probably true (by replacing Lusztig families Calogero-Moser families) but it is still not known up to now. The analogue of statement (b) for d -Harish-Chandra theory is false in general (for instance, if d is large enough, then all irreducible unipotent representations are d -cuspidal). The analogue of statement (c) for $\tau \neq \text{Id}_V$ is false in general (for instance, in twisted type A_{n-1} with $n \geq 3$, every family is a singleton but there are irreducible unipotent representations not belonging to the principal series).

PART III
GENERICITY VS CALOGERO-MOSER SPACES

Hypothesis. We assume in this third part that there exists a rational structure $V_{\mathbb{Q}}$ on V which is stable under the action of W (i.e., W is a Weyl group). We also assume that $V^W = 0$. We denote by $k_{\text{sp}} \in \mathbb{C}^{\times}$ the *spetsial* parameter, that is, the parameter such that $(k_{\text{sp}})_{H,0} = 1$ and $(k_{\text{sp}})_{H,1} = 0$ for all $H \in \mathcal{A}$.
We also fix an element $\tau \in N_{\mathbf{GL}_{\mathbb{Q}}(V_{\mathbb{Q}})}(W)$ of finite order.

We denote by $\mathcal{G}roups(W\tau)$ the class of triples $(q, \mathbf{G}, F) \in \mathcal{G}roups$ such that, if \mathbf{T} is any F -stable maximal torus of \mathbf{G} , then the pair $(\mathbb{Q} \otimes Y(\mathbf{T}/Z(\mathbf{G})), N_{\mathbf{G}}(\mathbf{T})/\mathbf{T})$ is isomorphic to $(V_{\mathbb{Q}}, W)$ and, moreover, τ stabilizes the lattice $Y(\mathbf{T}/Z(\mathbf{G}))$ of $V_{\mathbb{Q}}$ and there exists $w \in W$ such that $F(y) = qw\tau(y)$ for all $y \in Y(\mathbf{T}/Z(\mathbf{G}))$.

This part may be viewed as the aim of this survey article, where we propose several conjectures which compare the geometry (fixed points, symplectic leaves) of the Calogero-Moser space $\mathcal{X}_{k_{\text{sp}}}$ with the different partitions (families, d -Harish-Chandra series) of unipotent characters of triples belonging to $\mathcal{G}roups(W\tau)$. A first general remark (due to Lusztig) is that most of these partitions do not depend that much on the triple $\mathcal{G} \in \mathcal{G}roups(W\tau)$: they mainly depend only on the coset $W\tau$. This phenomenon, called *genericity*, was developed and formalized by Broué-Malle-Michel [BMM1] and will be explained in Section 11.

The conjectures will be stated precisely in Section 12. If $\mathcal{G} \in \mathcal{G}roups(W\tau)$, they propose conjectural links between:

- the partition of $\mathcal{U}nip(\mathcal{G})$ into families and the \mathbb{C}^{\times} -fixed points in $\mathcal{X}_{k_{\text{sp}}}^{\tau}$;
- the partition into d -Harish-Chandra series and symplectic leaves of $\mathcal{X}_{k_{\text{sp}}}^{\zeta_d \tau}$, where ζ_d is a primitive d -th root of unity.

In the second point, the most spectacular conjecture relates the parameter involved in the description of the normalization of the closure of a symplectic leaf as a Calogero-Moser space (i.e. the parameter $k_{P,p}$ of Conjecture 7.2) and the parameter of the Hecke algebra which conjecturally describes the endomorphism algebra of the cohomology of some Deligne-Lusztig variety.

11. Genericity

11.A. Rough definition. — The notions of *generic groups*, *generic unipotent representations*... have been defined rigorously in [BMM1]. In this survey, we will not recall this precise definition, which would require to introduce again much more notation. We will use throughout this part a rather vague definition: when some structure associated with any $\mathcal{G} = (q, \mathbf{G}, F) \in \mathcal{G}roups(W\tau)$ depends only on $W\tau$ and not on the triple \mathcal{G} , we will say that this structure behaves *generically*.

A first example is the order of \mathbf{G}'^F , where \mathbf{G}' is the derived subgroup of \mathbf{G} . Indeed, if $m = \dim_{\mathbb{C}} V$, there exists a choice of algebraically independent homogeneous generators f_1, \dots, f_m of $\mathbb{C}[V]^W$ which are eigenvectors for the action of τ (and we denote by d_j the degree of f_j and by ξ_j the eigenvalue corresponding to f_j). We can then define the following polynomial $\text{Ord}_{W\tau}(\mathbf{q}) \in \mathbb{Q}[\mathbf{q}]$:

$$\text{Ord}_{W\tau}(\mathbf{q}) = q^{|\mathfrak{sl}|} \prod_{j=1}^m (\mathbf{q}^{d_j} - \xi_j).$$

Then this polynomial does not depend on the precise choice of the f_j 's, and

$$(11.1) \quad |\mathbf{G}'^F| = \text{Ord}_{W\tau}(q)$$

for all $(q, \mathbf{G}, F) \in \mathcal{G}\text{roups}(W\tau)$.

Remark 11.2 (d-splitness). — Let $d \geq 1$ and let ζ_d denote a primitive d -th root of unity. We set

$$\delta(d) = \max_{w \in W} \dim V^{\zeta_d w \tau}$$

and we denote by w_d an element of W such that $\dim V^{\zeta_d w_d \tau} = \delta(d)$. Recall **[Spr]** that $\delta(d)$ is the number of $j \in \{1, 2, \dots, m\}$ such that $\zeta_d^{d_j} = \xi_j$.

Then $w_d \tau$ is well-defined up to W -conjugacy **[Spr]** and any subspace of the form $V^{\zeta_d w \tau}$ for some $w \in W$ is contained in a subspace of the form $x(V^{\zeta_d w_d \tau})$ for some $x \in W$. We set $\tau_d = \zeta_d w_d \tau \in N_{\text{GL}_{\mathbb{C}}(V)}(W)$. Note that this choice of w_d implies that the element τ_d is W -full **[Spr]**. Note also that $\mathcal{F}_{k_{\text{sp}}}^{\tau_d} = \mathcal{F}_{k_{\text{sp}}}^{\zeta_d \tau}$.

Then, for any $\mathcal{G} \in \mathcal{G}\text{roups}(W\tau)$, the conjugacy classes of d -split Levi subgroups of \mathcal{G} are in bijection with the W_{τ_d} -orbits of τ_d -split parabolic subgroups of W : the correspondence assigns to the conjugacy class of the d -split Levi subgroup its Weyl group, suitably embedded in W (see **[BrMa1]** for details). ■

11.B. Genericity of unipotent representations. — For our purpose, the most important result about genericity is the following theorem, which says that the unipotent representations of $\mathcal{G} \in \mathcal{G}\text{roups}(W\tau)$ and their degree behave generically.

Theorem 11.3 (Lusztig). — *There exists a finite set $\text{Unip}(W\tau)$ endowed with a map $\text{deg}_{W\tau} : \text{Unip}(W\tau) \rightarrow \mathbb{Q}[\mathbf{q}]$, both depending only on the coset $W\tau$ and such that, for each triple $\mathcal{G} \in \mathcal{G}\text{roups}(W)$, there exists a well-defined bijection*

$$\rho^{\mathcal{G}} : \text{Unip}(W\tau) \xrightarrow{\sim} \text{Unip}(\mathcal{G})$$

satisfying

$$\text{deg } \rho_{\gamma}^{\mathcal{G}} = (\text{deg}_{W\tau} \gamma)(q)$$

for all $\gamma \in \text{Unip}(W)$.

This Theorem 11.3 follows from the classification of unipotent representations obtained by Lusztig **[Lus6]** and a case-by-case analysis. More recent works of Lusztig **[Lus9]** provide general explanations for the existence of the finite set $\text{Unip}(W\tau)$ and the bijection $\rho^{\mathcal{G}}$ but do not explain the polynomial behaviour of the degree of the unipotent representations.

It turns out that the different structures on $\mathcal{Unip}(\mathcal{G})$ (families, Harish-Chandra series, partitions into ℓ -blocks...) can also be read only from the finite set $\text{Unip}(W\tau)$, as it will be explained below. In other words, they behave generically. Our aim here is to provide numerical evidences that these extra-structures can be read from the geometry of the Calogero-Moser space $\mathcal{X}_{k_{\text{sp}}}$ (\mathbb{C}^\times -fixed points, symplectic leaves, fixed points under the action of μ_d ...).

11.C. Almost characters. — We denote by $\mathbb{C}\text{Unip}(W\tau)$ the set of formal \mathbb{C} -linear combinations of elements of $\text{Unip}(W\tau)$ and we still denote by $\rho^{\mathcal{G}} : \mathbb{C}\text{Unip}(W\tau) \rightarrow \mathbb{C}\mathcal{Unip}(\mathcal{G})$ the \mathbb{C} -linear extension of the bijection $\rho^{\mathcal{G}}$.

With this notation, the almost characters behave generically. In other words, there exists a (necessarily unique) family $(R_\chi)_{\chi \in \text{Irr}(W)\tau}$ of elements of $\mathbb{C}\text{Unip}(W\tau)$ such that

$$(11.4) \quad \rho^{\mathcal{G}}(R_\chi) = R_\chi^{\mathcal{G}}$$

for any $\mathcal{G} \in \text{Groups}(W\tau)$ (see [Lus6, Chap. 4]). In particular, the graph $\mathfrak{G}_\mathcal{G}$ constructed in §10.B is generic: it will also be denoted by $\mathfrak{G}_{W\tau}$.

11.D. Families. — Families of unipotent characters behave generically. Indeed, it follows from (11.4) that there exists a partition of $\text{Unip}(W\tau)$ into *unipotent Lusztig families* (we denote by $\text{Fam}_{\text{un}}(W\tau)$ the set of such families) such that,

$$(11.5) \quad \rho^{\mathcal{G}}(\text{Fam}_{\text{un}}(W\tau)) = \mathcal{Fam}_{\text{un}}(\mathcal{G})$$

for any $\mathcal{G} \in \text{Groups}(W\tau)$. If \mathcal{C} is a connected component of the graph $\mathfrak{G}_{W\tau}$, we denote by $\mathfrak{F}_{\mathcal{C}}^{\text{un}} \subset \text{Unip}(W\tau)$ the associated generic family. Lusztig proved the following important result [Lus6]:

Theorem 11.6 (Lusztig). — *The map*

$$\begin{array}{ccc} \text{Fam}_{k_{\text{sp}}}^{\text{Lus}}(W)\tau & \longrightarrow & \text{Fam}_{\text{un}}(W\tau) \\ \mathcal{C} & \longmapsto & \mathfrak{F}_{\mathcal{C}}^{\text{un}} \end{array}$$

is well-defined and bijective.

This Theorem contains in particular the fact that, if \mathcal{C} is a τ -stable Lusztig k_{sp} -family of characters of W , then $\mathcal{C}^\tau \neq \emptyset$. But this follows directly from the fact that every Lusztig k_{sp} -family contains a unique character with minimal b -invariant (the *b-invariant* of an irreducible character χ is the minimal value of j such that χ occurs in the j -th symmetric power $S^j(V)$), called the *special* character of the family [Lus4, §12]: if the family is τ -stable, then its special character is necessarily τ -stable.

11.E. Lusztig's a-function. — If $\gamma \in \text{Unip}(W\tau)$, we denote by a_γ (resp. A_γ) the valuation (resp. the degree) of the polynomial $\text{deg}_{W\tau} \mathbf{q}$. It follows from Lusztig's work [Lus6] that

$$(11.7) \quad \text{the functions } a, A : \text{Unip}(W\tau) \rightarrow \mathbb{Z}_{\geq 0} \text{ are constant on families.}$$

11.F. d -Harish-Chandra series. — It turns out that d -Harish-Chandra theory behaves also generically [BMM1, Theo. 3.2]. More precisely, if $\mathcal{G} = (q, \mathbf{G}, F) \in \mathcal{G}roups(W\tau)$, then:

- Let $\text{Unip}_{\text{cus}}^d(W\tau)$ denote the set of $\gamma \in \text{Unip}(W\tau)$ such that $\mathbf{deg}_{W\tau} \gamma$ is divisible by $(q - \zeta_d)^{\dim V^{\tau_d}}$. Then $(\rho^{\mathcal{G}})(\text{Unip}_{\text{cus}}^d(W\tau)) = \mathcal{Unip}_{\text{cus}}^d(\mathcal{G})$ (see [BMM1, Prop. 2.9]).
- The \mathbf{G}^F -conjugacy classes of d -split Levi subgroups of \mathcal{G} are in bijection with the W_{τ_d} -conjugacy classes of d -split parabolic subgroups of W (see Remark 11.2): if the d -split Levi subgroup \mathcal{L} of \mathcal{G} corresponds to the d -split parabolic subgroup P under this bijection, then $\mathcal{L} \in \mathcal{G}roups(P\tau_d)$ and $W_{\mathcal{G}}(\mathcal{L}) \simeq \overline{N}_{W_{\tau_d}}(P\tau_d)$ (see [BrMa1, Prop. 3.12]).

Through this isomorphism and the bijection $\rho^{\mathcal{L}}$, the group $\overline{N}_{W_{\tau_d}}(P\tau_d)$ acts on $\text{Unip}(P\tau_d)$ and stabilizes the subset $\text{Unip}_{\text{cus}}^d(\mathcal{L})$. Moreover, the action of $\overline{N}_{W_{\tau_d}}(P\tau_d)$ on $\text{Unip}(P\tau_d)$ is generic. If $\lambda \in \text{Unip}(P\tau_d)$, we denote by $\overline{N}_{W_{\tau_d}}(P\tau_d, \lambda)$ its stabilizer in $\overline{N}_{W_{\tau_d}}(P\tau_d)$.

- If we denote by $\text{Cus}^d(W\tau)$ the set of pairs (P, λ) where P is a d -split parabolic subgroup of W and $\lambda \in \text{Unip}_{\text{cus}}^d(P\tau_d)$, then the previous point defines a natural bijection between $\text{Cus}^d(W\tau)/\sim$ and $\mathcal{Cus}^d(\mathcal{G})/\sim$.
- If $(P, \lambda) \in \text{Cus}^d(W\tau)$ corresponds to $(\mathcal{L}, \rho_{\lambda}^{\mathcal{L}}) \in \mathcal{Cus}^d(\mathcal{G})$, then the maps $\text{HC}_d^{\mathcal{G}, \mathcal{L}, \rho_{\lambda}^{\mathcal{L}}}$ and $\rho^{\mathcal{G}}$ define an injection $\text{HC}_d^{W, P, \lambda} : \text{Irr}(\overline{N}_{W_{\tau_d}}(P\tau_d, \lambda)) \hookrightarrow \text{Unip}(W\tau)$ which behaves generically. Its image is denoted by $\text{Unip}_d(W\tau, P, \lambda)$. Then

$$(11.8) \quad \text{Unip}(W\tau) = \bigcup_{(P, \lambda) \in \mathcal{Cus}^d(W)} \text{HC}_d^{W, P, \lambda}(\text{Irr}(\overline{N}_{W_{\tau_d}}(P\tau_d, \lambda))).$$

Moreover, the parameter $k_{\mathcal{L}, \rho_{\lambda}^{\mathcal{L}}}$ in Conjecture 9.5 is generic, i.e. depends only on (P, λ) . It will be denoted by $k_{P, \lambda}$.

Here, all the statements stated without reference can be found in [BMM1, Theo. 3.2].

Remark 11.9. — Let $(P, \lambda) \in \text{Cus}^d(W\tau)$. It follows from the classification of such pairs (see [BMM1]) that:

- $\overline{N}_{W_{\tau_d}}(P\tau_d, \lambda)$ is always a reflection group for its action on $(V^P)^{\tau_d}$.
- Examples where $\overline{N}_{W_{\tau_d}}(P\tau_d, \lambda) \neq \overline{N}_{W_{\tau_d}}(P\tau_d)$ are very rare. For instance, this never happens if $d = 1$ (see Theorem 9.1(a)) or if W is of type A (see [BMM1, §3.A]). ■

Example 11.10 (Principal series). — Let us describe the generic version of Example 9.2. First, $\text{Unip}(1)$ consists of a single element that we may (and will) denote by 1. Then the cuspidal pair $(1, 1) \in \text{Cus}(W\tau)$ corresponds to the pair $(\mathcal{T}, 1) \in \mathcal{Cus}(\mathcal{G})$ associated with an F -stable maximal torus of an F -stable Borel subgroup and the map $\text{HC}^{\mathcal{G}} = \text{HC}^{\mathcal{G}, \mathcal{T}, 1}$ will be simply denoted by $\text{HC}^W : \text{Irr}(W\tau) \hookrightarrow \text{Unip}(W\tau)$, instead of $\text{HC}^{W, 1, 1}$.

If $\tau = \text{Id}_V$, then the parameter $k_{1, 1}$ is equal to k_{sp} (see Example 9.2). ■

11.G. d -Harish-Chandra theory and filtration. — Assume in this subsection, and only in this subsection, that $\tau = \text{Id}_V$ (on the reductive group side, this means that we work in the split case). Let $Z(\mathbb{C}W)^{\text{Lus}}$ denote the subalgebra of $Z(\mathbb{C}W)$ whose basis is given by $(e_{\mathcal{C}}^W)_{\mathcal{C} \in \text{Fam}_{k_{\text{sp}}}^{\text{Lus}}(W)}$. Let $(P, \lambda) \in \text{Cus}^d(W)$. We define a morphism of algebras

$$(\text{HC}_d^{W,P,\lambda})^{\#} : Z(\mathbb{C}W)^{\text{Lus}} \longrightarrow Z(\mathbb{C}\bar{N}_{W\tau_d}(P_{\tau_d}, \lambda))$$

by

$$(\text{HC}_d^{W,P,\lambda})^{\#}(e_{\mathcal{C}}^W) = \sum_{\substack{\chi \in \text{Irr } \bar{N}_{W\tau_d}(P_{\tau_d}, \lambda) \\ \text{such that } \text{HC}_d^{W,P,\lambda}(\chi) \in \mathfrak{F}_{\mathcal{C}}^{\text{un}}}} e_{\chi}^{\bar{N}_{W\tau_d}(P_{\tau_d}, \lambda)}.$$

Conjecture 11.11. — Assume that $\tau = \text{Id}_V$ and fix $(P, \lambda) \in \text{Cus}^d(W)$. Then

$$(\text{HC}_d^{W,P,\lambda})^{\#}(\mathfrak{F}_j Z(\mathbb{C}W)^{\text{Lus}}) \subset \mathfrak{F}_j Z(\mathbb{C}\bar{N}_{W\tau_d}(P_{\tau_d}, \lambda))$$

for all j .

Remark 11.12. — This conjecture seems to come from nowhere and provides a strange link between the character tables of W and $\bar{N}_{W\tau_d}(P_{\tau_d}, \lambda)$. However, if we believe in the links between representation theory of finite reductive groups and geometry of Calogero-Moser spaces (as developed in the next section), then this Conjecture 11.11 is just a consequence of this philosophy (see the upcoming Proposition 12.6) and of Conjecture 6.2. This is an example where the geometry of Calogero-Moser spaces suggests unexpected properties of the representation theory of finite reductive groups.

Note that Conjecture 11.11 holds in the following cases:

- If W is of type A (see [BoMa, Cor. 4.22] and the explanations given in Section 15).
- A pretty convincing result is that it holds if $\dim V \leq 8$ (so this includes the type E_8): this has been checked through computer calculations based on all the functions implemented by Jean Michel [Mic]^(*). ■

Example 11.13. — Let $z = \sum_{s \in \text{Ref}(W)} s \in Z(\mathbb{C}W)$. Then $z = \sum_{\chi \in \text{Irr}(W)} (|\mathcal{A}| - (a_{\chi}^{(k_{\text{sp}})} + A_{\chi}^{(k_{\text{sp}})})e_{\chi}^W$ (see [BMM2, Cor.6.9] and [BoRo2, Lem. 7.2.1]). So $z \in \mathfrak{F}_1 Z(\mathbb{C}W)^{\text{Lus}}$ by (8.3). Now, if $\psi \in \text{Irr } \bar{N}_{W\tau_d}(P_{\tau_d}, \lambda)$ is such that $\text{HC}_d^{W,P,\lambda}(\psi)$ belongs to the same family as $\text{HC}^W(\chi)$, then it follows from (9.6) that $|\mathcal{A}| - (a_{\chi}^{(k_{\text{sp}})} - A_{\chi}^{(k_{\text{sp}})}) = M - (a_{\psi}^{(k_{P,\lambda})} + A_{\psi}^{(k_{P,\lambda})})$ for some M which does not depend on ψ or χ (and only on (W, P, λ)). Therefore,

$$(\text{HC}_d^{W,P,\lambda})^{\#}(z) = \sum_{\chi \in \text{Irr } \bar{N}_{W\tau_d}(P_{\tau_d}, \lambda)} (M - (a_{\psi}^{(k_{P,\lambda})} + A_{\psi}^{(k_{P,\lambda})}))e_{\psi}^{\bar{N}_{W\tau_d}(P_{\tau_d}, \lambda)}.$$

In other words, it follows from [BMM2, Cor.6.9] and [BoRo2, Lem. 7.2.1] that there exists $M' \in \mathbb{C}$ such that

$$(\text{HC}_d^{W,P,\lambda})^{\#}(z) = M' + \sum_{s \in \text{Ref}(\bar{N}_{W\tau_d}(P_{\tau_d}, \lambda))} c_{k_{P,\lambda}}(s)s \in \mathfrak{F}_1 Z(\mathbb{C}\bar{N}_{W\tau_d}(P_{\tau_d}, \lambda)),$$

as desired. ■

^(*)We wish to thank again warmly Jean Michel for writing the programs for performing these calculations.

12. Coincidences, conjectures

12.A. Families. — By Theorem 10.2(b), the set $\text{Fam}_{\text{un}}(W\tau)$ is in bijection with the set $\text{Fam}_{k_{\text{sp}}}^{\text{Lus}}(W)^\tau$. We conjecture the first link between the geometry of $\mathcal{X}_{k_{\text{sp}}}$ and the representation theory of finite reductive groups:

Conjecture 12.1. — *There exists a unique bijection*

$$\Phi : (\mathcal{X}_{k_{\text{sp}}}^{\mathbb{C}^\times})^\tau \xrightarrow{\sim} \text{Fam}_{\text{un}}(W\tau)$$

such that, for any $p \in (\mathcal{X}_{k_{\text{sp}}}^{\mathbb{C}^\times})^\tau$ and for any $\chi \in (\mathfrak{S}_p^{k_{\text{sp}}})^\tau$, the almost character R_χ belongs to $\mathbb{C}\Phi(p)$.

Whenever $\tau = \text{Id}_V$, this Conjecture 12.1 is equivalent to Gordon-Martino Conjecture 8.2 (see Theorem 11.6). For the rest of this section, we assume that Conjecture 12.1 holds, and we keep the notation $\Phi : (\mathcal{X}_{k_{\text{sp}}}^{\mathbb{C}^\times})^\tau \xrightarrow{\sim} \text{Fam}_{\text{un}}(W\tau)$.

12.B. Fixed points and d -cuspidality. — We expect that d -cuspidality of unipotent representations and τ_d -cuspidality of points in $\mathcal{X}_{k_{\text{sp}}}^{\tau_d} = \mathcal{X}_{k_{\text{sp}}}^{\zeta_d\tau}$ are linked as follows:

Conjecture 12.2. — *Assume here that Conjecture 12.1 holds. Let $p \in (\mathcal{X}_{k_{\text{sp}}}^{\mathbb{C}^\times})^\tau$ be such that there exists $\lambda \in \Phi(p) \subset \text{Unip}(W\tau)$ which is d -cuspidal. Then p is τ_d -cuspidal.*

Note that the converse to Conjecture 12.2 does not hold in general, even for $d = 1$ (for instance for W of type D , as it will be explained in §16.C).

12.C. d -Harish-Chandra theory and symplectic leaves. — Assume in this subsection that Conjectures 12.1 and 12.2 hold for W and all its parabolic subgroups. Fix $(P, \lambda) \in \text{Cus}^d(W\tau)$ and let p be the point of $\mathcal{X}_{k_{\text{sp}}}(V/V^P, P)$ corresponding to the Lusztig family of λ through Conjecture 12.1. Then p is τ_d -cuspidal by Conjecture 12.2. Therefore, one can associate to the pair (P, p) a symplectic leaf $\mathcal{S}_{P,p}$ of $\mathcal{X}_{k_{\text{sp}}}^{\tau_d}$.

Conjecture 12.3. — *Recall that we assume that Conjectures 12.1 and 12.2 hold for W and all its parabolic subgroups. Then, with the above notation:*

- (a) *Let $p' \in (\mathcal{X}_{k_{\text{sp}}}^{\mathbb{C}^\times})^\tau$. Then $p' \in \overline{\mathcal{S}}_{P,p}$ if and only if the d -Harish-Chandra series $\text{HC}^{W,P,\lambda}(\text{Irr}(\overline{\mathbb{N}}_{W\tau_d}(P_{\tau_d}, \lambda)))$ meets the family $\Phi(p')$.*
- (b) *There exists a parameter $k_{P,p} \in \mathfrak{N}((V^P)^{\tau_d}, \overline{\mathbb{N}}_{W\tau_d}(P_{\tau_d}))$ such that:*
 - (b1) *$\overline{\mathcal{S}}_{P,p}^{\text{nor}} \simeq \mathcal{X}_{k_{P,p}}((V^P)^{\tau_d}, \overline{\mathbb{N}}_{W\tau_d}(P_{\tau_d}))$ as Poisson varieties endowed with a \mathbb{C}^\times -action.*
 - (b2) *The parameter $k_{P,\lambda} \in \mathfrak{N}((V^P)^{\tau_d}, \overline{\mathbb{N}}_{W\tau_d}(P_{\tau_d}, \lambda))$ involved in Conjecture 9.5(b) (and which is generic by the previous section) is the restriction of $k_{P,p}$ to $\overline{\mathbb{N}}_{W\tau_d}(P_{\tau_d}, \lambda)$.*

Commentary 12.4. — In the previous conjecture, the existence of a parameter $k_{P,p}$ satisfying (b1) is just a restatement of Conjecture 7.2: the main point of the above conjecture is that its restriction should coincide with the parameter of Conjecture 9.5(b), which has to do with a completely different context (ℓ -adic cohomology of Deligne-Lusztig varieties). In some sense, this is a justification of this long paper. ■

The correspondence outlined in Conjecture 12.3 should also be compatible in a more precise way with Harish-Chandra theory. For this survey, keep the notation of the above conjecture and assume moreover that $\overline{N}_{W\tau_d}(P_{\tau_d}, \lambda) = \overline{N}_{W\tau_d}(P_{\tau_d})$ (recall from Remark 11.9 that this is the most probable situation). Assume also that Conjecture 12.3 holds. Then, by (b1), we get a \mathbb{C}^\times -equivariant morphism of varieties

$$\psi : \mathfrak{X}_{k_{P,p}}((V^P)^{\tau_d}, \overline{N}_{W\tau_d}(P_{\tau_d})) \longrightarrow \mathfrak{X}_{k_{\text{sp}}}$$

whose image is the closure $\overline{\mathcal{S}}_{P,p}$ of the symplectic leaf $\mathcal{S}_{P,p}$ of $\mathfrak{X}_{k_{\text{sp}}}^{\tau_d}$. By restriction to the \mathbb{C}^\times -fixed points, we get a map

$$\psi_{\text{fix}} : \mathfrak{X}_{k_{P,p}}((V^P)^{\tau_d}, \overline{N}_{W\tau_d}(P_{\tau_d}))^{\mathbb{C}^\times} \longrightarrow \mathfrak{X}_{k_{\text{sp}}}^{\mathbb{C}^\times}$$

whose image is contained in $(\mathfrak{X}_{k_{\text{sp}}}^{\mathbb{C}^\times})^\tau$. On the other hand, Conjecture 12.1 provides a surjective map $\Phi^* : \text{Unip}(W\tau) \longrightarrow (\mathfrak{X}_{k_{\text{sp}}}^{\mathbb{C}^\times})^\tau$ (whose fibers are the unipotent Lusztig families) and the definition of Calogero-Moser families provides a surjective map

$$\mathfrak{z}_{k_{P,p}} : \text{Irr}(\overline{N}_{W\tau_d}(P_{\tau_d})) \longrightarrow \mathfrak{X}_{k_{P,p}}((V^P)^{\tau_d}, \overline{N}_{W\tau_d}(P_{\tau_d}))^{\mathbb{C}^\times}.$$

Finally, recall that d -Harish-Chandra theory (see Theorem 9.4 and its generic version) provides an injective map

$$\text{HC}^{W,P,\lambda} : \text{Irr}(\overline{N}_{W\tau_d}(P_{\tau_d})) \hookrightarrow \text{Unip}(W\tau).$$

We expect all these maps to be compatible in the following sense:

Conjecture 12.5. — Assume that Conjectures 12.1, 12.2 and 12.3 hold and that $\overline{N}_{W\tau_d}(P_{\tau_d}, \lambda) = \overline{N}_{W\tau_d}(P_{\tau_d})$. Then the diagram

$$\begin{array}{ccc} \text{Irr}(\overline{N}_{W\tau_d}(P_{\tau_d})) & \xrightarrow{\mathfrak{z}_{k_{P,p}}} & \mathfrak{X}_{k_{P,p}}((V^P)^{\tau_d}, \overline{N}_{W\tau_d}(P_{\tau_d}))^{\mathbb{C}^\times} \\ \text{HC}^{W,P,\lambda} \downarrow & & \downarrow \psi_{\text{fix}} \\ \text{Unip}(W\tau) & \xrightarrow{\Phi^*} & (\mathfrak{X}_{k_{\text{sp}}}^{\mathbb{C}^\times})^\tau \end{array}$$

is commutative.

The conjectures stated in this section, together with Conjecture 6.2 on the cohomology of Calogero-Moser spaces, imply the Conjecture 11.11. Let us give some details. First, as in §6.B, the morphism ψ induces a morphism of algebras

$$\psi_{\text{fix}}^\# : \text{Im } \Omega_W^{k_{\text{sp}}} \longrightarrow \text{Im } \Omega_{\overline{N}_{W\tau_d}(P_{\tau_d})}^{k_{P,p}}.$$

Then, if we assume that $\tau = \text{Id}_V$ and that Conjectures 8.2 and 12.1 hold, the map $\psi_{\text{fix}}^\#$ is just the map $(\text{HC}_d^{W,P,\lambda})^\#$ of Conjecture 11.11. So Proposition 6.6 has the following consequence:

Proposition 12.6. — *With the above notation, assume that $\bar{N}_{W_{\tau_d}}(P_{\tau_d}, \lambda) = \bar{N}_{W_{\tau_d}}(P_{\tau_d})$ and that Conjectures 6.2, 8.2, 12.1, 12.2, 12.3 and 12.5 hold. Then Conjecture 11.11 holds for the d -cuspidal triple (W, P, λ) .*

Remark 12.7. — The consequence of Proposition 12.6 does not involve anymore the geometry of Calogero-Moser spaces but only the representation theory of finite reductive groups. Therefore, the validity of Conjecture 11.11 in many cases (see Remark 11.12 and Example 11.13) is a good indication that the general philosophy of this paper has some reasonable foundation. ■

Example 12.8 (Principal series). — Assume in this example, and only in this example, that $\tau = \text{Id}_V$, that Conjecture 12.1 (i.e. Gordon-Martino's Conjecture 8.2) holds and that $(P, \lambda) = (1, 1) \in \text{Cus}(W)$. Then Conjecture 12.3(a) is a restatement of the fact that every family meets the principal series while Conjecture 12.3(b) and Conjecture 12.5 are vacuous. ■

Example 12.9 (Regular element). — Assume in this example, and only in this example, that $\tau = \text{Id}_V$ and d is chosen such that τ_d is regular (see §7.C for the definition: from this definition, the trivial subgroup of W is τ -split). Then, $(1, 1) \in \text{Cus}^d(W)$ and $W_{\tau_d} = C_W(w_d)$.

On the unipotent representation side, if \mathfrak{C} is a Lusztig k_{sp} -family, then it has been checked by J. Michel (unpublished) that the unipotent Lusztig family $\mathfrak{F}_{\mathfrak{C}}^{\text{un}}$ (see Theorem 11.6) meets the d -Harish-Chandra series $\text{Unip}_d(W, 1, 1)$ if and only if $\sum_{\chi \in \mathfrak{C}} |\chi(w_d)|^2 \neq 0$. Moreover, he also checked that

$$\sum_{\chi \in \mathfrak{C}} |\chi(w_d)|^2 = \sum_{\substack{\psi \in \text{Irr } C_W(w_d) \\ \text{such that } \text{HC}_d^{W,1,1}(\psi) \in \mathfrak{F}_{\mathfrak{C}}^{\text{un}}}} \psi(1)^2.$$

On the Calogero-Moser space side, the closure of the symplectic leaf associated with $(1, 1)$ is just the irreducible component of maximal dimension $(\mathcal{X}_{k_{\text{sp}}}^\tau)_{\text{max}}$ defined in §7.C. So the above facts about unipotent representations justify, through the philosophy of this section, Conjecture 7.5 and [Bon5, Conj 5.2]. ■

PART IV EXAMPLES

Hypothesis. *As in the third part, we assume that there exists a rational structure $V_{\mathbb{Q}}$ on V which is stable under the action of W (i.e., W is a Weyl group) and that $V^W = 0$. We also fix an element $\tau \in N_{\mathrm{GL}_{\mathbb{Q}}(V_{\mathbb{Q}})}(W)$ of finite order, an integer $d \geq 1$ and a primitive d -th root of unity ζ_d .*

We aim to illustrate the Conjectures stated in Section 12 by several examples:

- (a) We prove that Conjectures 12.1, 12.2, 12.3 and 12.5 hold in rank 2 for d equal to the Coxeter number.
- (b) We also prove that, assuming Broué-Malle-Michel Conjecture 9.5 (and particularly the conjectural value of $k_{P,\lambda}$), they hold in type A.
- (c) For classical types, we only prove Conjectures 12.1, as well as Conjecture 12.2 whenever $d = 1$ (classical Harish-Chandra theory).

As explained in Commentary 12.4, the most intriguing question is Conjecture 12.3(b), which predicts the equality of parameters coming from two extremely different contexts (cohomology of some Deligne-Lusztig variety vs symplectic leaves of Calogero-Moser spaces). Even for classical Harish-Chandra theory (i.e. whenever the Deligne-Lusztig variety is zero-dimensional), this is somewhat unexpected and certainly reflects some deep connections. In the examples treated in this part, we will mainly focus on this question.

13. Rank 2

The case of type A being treated in the upcoming Section 15, we will just consider here the types B_2 and G_2 . We will not fill the details for proving all the Conjectures: indeed, the groups are small enough so that the remaining details can be filled by the reader. So, as explained in the introduction to this part, we only give the details for Conjecture 12.3(b).

Theorem 13.1. — *Assume that W is of type B_2 or G_2 and that d is the Coxeter number. Then Conjectures 12.1, 12.2, 12.3 and 12.5 hold.*

Proof. — Let s and t be the two simple reflections of W . Let $c = st$ be a standard Coxeter element of W and let \mathfrak{G}_c denote its corresponding \mathbf{G} -orbit in $\mathfrak{B} \times \mathfrak{B}$. Then $\zeta_d c$ is W -full (so we may take $\tau_d = \zeta_d c$) and $W_{\tau_d} = C_W(\zeta_d c) = \langle c \rangle$ is the cyclic group of order d . As there is only one reflecting hyperplane for W_{τ_d} , the parameters for W_{τ_d} will be denoted by $k = (k_0, k_1, \dots, k_{d-1})$. We denote by k^{cox} the parameter given by:

$$k^{\mathrm{cox}} = \begin{cases} (0, 1, 2, 1) & \text{if } d = 4, \\ (0, 1, 2, 1, 1, 1) & \text{if } d = 6. \end{cases}$$

Also, there is (up to \mathbf{G}^F -conjugacy), only one proper d -split Levi subgroup, namely the Coxeter torus \mathbf{T}_c . Computing the Deligne-Lusztig induction of the trivial character of

\mathbf{T}_c^F amounts to determining the cohomology of the Deligne-Lusztig variety $\mathbf{X}_{\mathfrak{G}_c}$. This has been done by Lusztig [Lus1], and it follows from his work that Conjecture 9.5 holds in this case.

Let us give more details. First, he proved Conjecture 9.5(a) about the disjointness of the cohomology groups [Lus1, Theo. 6.1] and that the endomorphism algebra of the \mathbf{G}^F -module $H_c^\bullet(\mathfrak{X}_{\mathfrak{G}_c})$ is generated by the Frobenius endomorphism F and he computed the eigenvalues of F in all cases [Lus1, Table 7.3]. This leads to the following presentation for this endomorphism algebra:

$$\begin{cases} \text{Generator: } F \text{ (the Frobenius endomorphism),} \\ \text{Relation: } \prod_{j=0}^{d-1} (F - \zeta_d^j (\zeta_d^{-1} q)^{k_j^{\text{cox}}}) = 0. \end{cases}$$

In other words,

$$(13.2) \quad \text{End}_{\mathbf{G}^F} H_c^\bullet(\mathfrak{X}_{\mathfrak{G}_c}) \simeq \mathcal{H}_{k^{\text{cox}}}(W_{\tau_d}, \zeta_d^{-1} q).$$

On the other hand, the computation of the fixed point subvariety $\mathfrak{X}_{k_{\text{sp}}}^{\mu_d}$ has been done in [Bon3, Theo. 7.1] and the result is given by:

$$\mathfrak{X}_{k_{\text{sp}}}^{\mu_d} \simeq \{(x, y, z) \in \mathbb{C}^3 \mid (z^2 - d^2)z^{d-2} = xy\}.$$

Setting $z' = z + d$, we get

$$\mathfrak{X}_{k_{\text{sp}}}^{\mu_d} \simeq \{(x, y, z') \in \mathbb{C}^3 \mid z'(z' - 2d)(z' - d)^{d-2} = xy\}.$$

In other words,

$$(13.3) \quad \mathfrak{X}_{k_{\text{sp}}}^{\mu_d} \simeq \mathfrak{X}_{k^{\text{cox}}}(V^{\tau_d}, W_{\tau_d})$$

(see Example 6.5(a)).

We see that the same parameter occurs in (13.2) and (13.3): this shows that Conjecture 12.3(b) holds in this case, as desired. \square

14. Some combinatorics

We refer to [JaKe, §2.7] for facts about abaci, d -cores, d -quotients of partitions that will be used here.

14.A. Notation. — A *partition* is a sequence $\lambda = (\lambda_k)_{k \geq 1}$ of non-negative integers such that $\lambda_k \geq \lambda_{k+1}$ for all k and $\lambda_k = 0$ for $k \gg 0$. Let Part denote the set of all partitions. If $\lambda \in \text{Part}$, we set $|\lambda| = \sum_{k \geq 1} \lambda_k$ and $a_\lambda = \sum_{k \geq 1} (k-1)\lambda_k$, and we denote by $Y(\lambda)$ the *Young diagram* of λ , that is, the set of pairs of natural numbers (i, j) such that $j \geq 1$ and $1 \leq i \leq \lambda_j$. If $y \in Y(\lambda)$, we denote by $\text{hk}_\lambda(y)$ the *hook length* of λ based at y , i.e. the number of $(i', j') \in Y(\lambda)$ such that $i' \geq i$, $j' \geq j$ and $(i-i')(j-j') = 0$. Let

$$\text{deg } \lambda = \mathbf{q}^{a_\lambda} \frac{\prod_{k=1}^{|\lambda|} (\mathbf{q}^k - 1)}{\prod_{y \in Y(\lambda)} (\mathbf{q}^{\text{hk}_\lambda(y)} - 1)}.$$

It turns out that $\text{deg } \lambda \in \mathbb{Z}[\mathbf{q}]$.

Let $d \geq 1$. A partition λ is called a d -core if $\text{hk}_\lambda(y) \neq d$ for all $y \in Y(\lambda)$. The subset of Part consisting of d -cores is denoted by Cor_d . An element $\lambda = (\lambda^{(1)}, \dots, \lambda^{(d)})$ of the set Part^d of d -uples of partitions is called a d -partition: we set $|\lambda| = |\lambda^{(1)}| + \dots + |\lambda^{(d)}|$. If $\lambda \in \text{Part}$, we denote by $\text{cor}_d(\lambda) \in \text{Cor}_d$ its d -core and by $\text{quo}_d(\lambda) \in \text{Part}^d$ its d -quotient. The map

$$(14.1) \quad \begin{aligned} \text{cor}_d \times \text{quo}_d : \text{Part} &\longrightarrow \text{Cor}_d \times \text{Part}^d \\ \lambda &\longmapsto (\text{cor}_d(\lambda), \text{quo}_d(\lambda)) \end{aligned}$$

is bijective. Its inverse will be denoted by

$$\text{par}_d : \text{Cor}_d \times \text{Part}^d \xrightarrow{\sim} \text{Part}.$$

It follows from the definition of both maps that

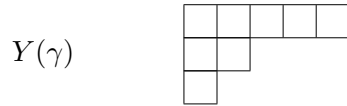
$$(14.2) \quad |\lambda| = |\text{cor}_d(\lambda)| + d |\text{quo}_d(\lambda)|.$$

If $r \geq 0$, we denote by $\text{Part}(r)$ (resp. $\text{Part}^d(r)$, resp. $\text{Cor}_d(r)$) the set of $\lambda \in \text{Part}$ (resp. $\lambda \in \text{Part}^d$, resp. $\lambda \in \text{Cor}_d$) such that $|\lambda| = r$. We also set $\text{Cor}_d(\equiv r)$ for the set of $\lambda \in \text{Cor}_d$ such that $|\lambda| \leq r$ and $|\lambda| \equiv r \pmod{d}$. In other words, $\text{Cor}_d(\equiv r) = \text{cor}_d(\text{Part}(r))$.

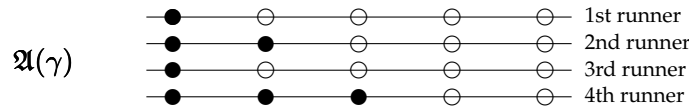
If $\lambda \in \text{Part}^d(r)$, we denote by χ_λ the associated irreducible character of the complex reflection group $G(d, 1, r)$, following the convention in [GeJa]. If $k \in \mathbb{C}^{\mathbb{N}(G(d, 1, r))}$ and $\lambda \in \text{Part}^d(r)$, we denote by z_λ^k the element of $\mathfrak{z}_k(\chi_\lambda) \in \mathfrak{X}_k(d, 1, r)^{\mathbb{C}^\times}$ defined in §5.D. Note that we do not need to emphasize d or r , as they are determined by λ .

14.B. Abaci. — A d -abacus is an abacus with d runners. If $\gamma \in \text{Cor}_d$, we denote by $\mathfrak{A}(\gamma)$ its d -abacus, with the convention that the first runner contains the first empty box. Let $b(\gamma) = (b_0(\gamma), b_1(\gamma), \dots, b_{d-1}(\gamma))$ denote the sequence defined as follows: $b_j(\gamma)$ is the number of beads on the $(j + 1)$ -th runner of $\mathfrak{A}(\gamma)$ minus the number of beads on the first runner. Let $\text{Res}_d(\gamma) = (\rho_0(\gamma), \dots, \rho_{d-1}(\gamma))$ denote the d -residue of γ . It is defined as follows: $\rho_k(\gamma)$ is the number of pairs $(i, j) \in Y(\gamma)$ such that $i - j \equiv k \pmod{d}$.

Example 14.3. — Let $\gamma = (5, 2, 1) \in \text{Part}(8)$. Its Young diagram is



It is easily seen that γ is a 4-core, and its 4-abacus $\mathfrak{A}(\gamma)$ is given by



Then $b(\gamma) = (0, 1, 0, 2)$ while $\text{Res}_d(\gamma) = (3, 2, 2, 1)$. ■

Let us define two sequences $k^\gamma = (k_j^\gamma)_{0 \leq j \leq d-1}$ and $l^\gamma = (l_j^\gamma)_{0 \leq j \leq d-1}$ associated with a d -core γ :

- $k_j^\gamma = db_j(\gamma) + j$.
- $l_j^\gamma = b_0(\gamma) + b_1(\gamma) + \dots + b_{d-1}(\gamma) + \begin{cases} d(\rho_{1-j}(\gamma) - \rho_{-j}(\gamma)) + j - 1 & \text{if } 1 \leq j \leq d - 1, \\ d(\rho_1(\gamma) - \rho_0(\gamma)) + d - 1 & \text{if } j = 0. \end{cases}$

Here, the index in $\rho_{1-j}(\gamma)$ or $\rho_{-j}(\gamma)$ must be understood modulo d . The next result will be useful in the next section:

Proposition 14.4. — *Let γ be a d -core and let m denote its length (i.e., the number of non-zero parts). Then*

$$k_j^\gamma = l_{j+1-m}^\gamma$$

for all $j \in \mathbb{Z}/d\mathbb{Z}$.

Proof. — If $j \in \mathbb{Z}/d\mathbb{Z}$, we denote by \bar{j} its unique representative in $\{0, 1, \dots, d-1\}$. Note that $m = b_0(\gamma) + b_1(\gamma) + \dots + b_{d-1}(\gamma)$. We argue by induction on the length m of γ .

If $\gamma = \emptyset$, then $k^\gamma = (0, 1, \dots, d-1)$ and $l^\gamma = (d-1, 0, 1, \dots, d-2)$. This shows the result whenever $m = 0$.

Assume now that $m \geq 1$ and write $\gamma = (\gamma_1, \gamma_2, \gamma_3, \dots)$ with $\gamma_1 > 0$. Let $\gamma' = (\gamma_2, \gamma_3, \dots)$. Then γ' is a d -core so, by the induction hypothesis, Proposition 14.4 holds for γ' . In other words, $k_j^{\gamma'} = l_{j+2-m}^{\gamma'}$ for all $j \in \mathbb{Z}/d\mathbb{Z}$.

Let us first compare the sequences k^γ and $k^{\gamma'}$. For this, let y denote the unique element of $\{0, 1, \dots, d-1\}$ such that $b_y(\gamma) = b_y(\gamma') + 1$. Then, if $j \neq y$, we get $b_j(\gamma) = b_j(\gamma')$. Therefore,

$$(\#) \quad k_j^\gamma = \begin{cases} k_j^{\gamma'} & \text{if } j \neq y, \\ k_j^{\gamma'} + d & \text{if } j = y. \end{cases}$$

Let us now compare the sequences l^γ and $l^{\gamma'}$. For this, let x denote the unique element of $\{0, 1, \dots, d-1\}$ such that $\gamma_1 \equiv x \pmod{d}$. Then, for $j \in \mathbb{Z}/d\mathbb{Z}$, we get

$$\rho_j(\gamma) = \rho_{j-1}(\gamma') + (\gamma_1 - x)/d + \delta_x(j)$$

where δ_x is identically 0 if $x = 0$ and

$$\delta_x(j) = \begin{cases} 1 & \text{if } \bar{j} = 0 \text{ or } 1 \leq d - \bar{j} \leq x - 1, \\ 0 & \text{if } x \leq d - \bar{j} \leq d - 1, \end{cases}$$

if $x \geq 1$. Also, note that $\gamma_1 = d(b_y(\gamma) - 1) + y + 1 - m$, so $y + 1 - m \equiv x \pmod{d}$. Let us also write l_j^γ , for $j \in \mathbb{Z}/d\mathbb{Z}$, as follows:

$$l_j^\gamma = m + d(\rho_{1-j}(\gamma) - \rho_{-j}(\gamma)) + \bar{j} - 1 + d\delta_{j,0}$$

where $\delta_{j,0}$ is the Kronecker symbol. Putting things together, one gets:

$$\begin{aligned} l_j^\gamma &= m + d(\rho_{-j}(\gamma') - \rho_{-1-j}(\gamma')) + \delta_x(1-j) - \delta_x(-j) + \bar{j} - 1 + d\delta_{j,0} \\ &= 1 + l_{j+1}^{\gamma'} - \overline{\bar{j} + 1} + 1 - d\delta_{j+1,0} + d(\delta_x(1-j) - \delta_x(-j)) + \bar{j} - 1 + d\delta_{j,0}. \end{aligned}$$

But $\overline{\bar{j} + 1} = \bar{j} + 1 - d\delta_{j+1,0}$, so

$$l_j^\gamma = l_{j+1}^{\gamma'} + d(\delta_x(1-j) - \delta_x(-j)) + d\delta_{j,0}.$$

Two cases may occur:

- If $x = 0$, then δ_x is identically 0 and so $d(\delta_x(1-j) - \delta_x(-j)) + d\delta_{j,0} = d\delta_{j,0} = d\delta_{j,-x}$.
- If $x > 0$, then there are only two values of $j \in \mathbb{Z}/d\mathbb{Z}$ for which $\delta_x(1-j) - \delta_x(-j)$ is non-zero, namely $j = 0$ and $j = x$. If $j = 0$, this returns -1 while, if $j = x$, this returns 1. Therefore, we have again $d(\delta_x(1-j) - \delta_x(-j)) + d\delta_{j,0} = d\delta_{j,x}$.

Finally, $l_j^\gamma = l_{j+1}^{\gamma'} + d\delta_{j,-x}$ and $x \equiv y + 1 - m \pmod{d}$ so, by the induction hypothesis and (#),

$$l_j^\gamma = k_{j+m-1}^{\gamma'} + d\delta_{j,m-y} = k_{j+m-1}^{\gamma'} - d\delta_{j+m-1,y} + d\delta_{j,x} = k_{j+m-1}^{\gamma'}$$

as expected. \square

15. The smooth example: type A

The Calogero-Moser space $\mathcal{X}_{k_{\text{sp}}}$ is smooth if and only if W is a Weyl group of type A. This simplifies drastically its geometry (\mathbb{C}^\times -fixed points, symplectic leaves,...) and all the conjectures proposed in Part I are true in this case [BoSh, BoMa].

On the other hand, the almost characters of some $\mathcal{G} \in \text{Groups}$ are all irreducible characters if and only if \mathcal{G} is of type A. This also simplifies drastically its representation theory (unipotent Lusztig families, d -Harish-Chandra theory, blocks,...).

In the spirit of this paper, these two facts should be the shadow of a common phenomenon. We do not propose an explanation for it, but we give details about how the combinatorics on both sides fit perfectly in type A, so that all the conjectures stated in Part III hold in this case (provided that Conjecture 9.5 holds).

Hypothesis. *From now on, and until the end of this section, we fix a natural number $n \geq 2$ and we assume that*

$$V = \{(\xi_1, \dots, \xi_n) \in \mathbb{C}^n \mid \xi_1 + \dots + \xi_n = 0\}$$

and that $W = \mathfrak{S}_n$ acting on V by permutation of the coordinates. The Calogero-Moser space $\mathcal{X}_{k_{\text{sp}}}(V, \mathfrak{S}_n)$ will be simply denoted by $\mathcal{X}(n)$.

We set $r = \lfloor n/d \rfloor$ and we denote by w_d a product of r disjoint cycles of length d . Then $\zeta_d w_d$ is \mathfrak{S}_n -split.

The $\zeta_d w_d$ -split parabolic subgroups of \mathfrak{S}_n are those of the form \mathfrak{S}_m , where $m \leq n$ and $m \equiv n \pmod{d}$. In this case, $(\mathfrak{S}_n)_{\zeta_d w_d} \simeq G(d, 1, r)$ and

$$(15.1) \quad \overline{N}_{(\mathfrak{S}_n)_{\zeta_d w_d}}((\mathfrak{S}_m)_{\zeta_d w_d}) \simeq G(d, 1, (n-m)/d).$$

If $\gamma \in \text{cor}_d(\equiv n)$, we set $r_\gamma = r_\gamma(n) = (n - |\gamma|)/d$ and we denote by $k_\gamma \in \mathbb{C}^{\mathfrak{N}(G(d,1,r_\gamma))}$ the parameter defined by:

$$\begin{cases} ((k_\gamma)_{\text{Ker}(x_1-x_2),0}, (k_\gamma)_{\text{Ker}(x_1-x_2),1}) = (d, 0), & \text{if } r_\gamma \geq 2, \\ ((k_\gamma)_{\text{Ker}(x_1),0}, (k_\gamma)_{\text{Ker}(x_1),1}, \dots, (k_\gamma)_{\text{Ker}(x_1),d-1}) = k^\gamma, & \text{if } r_\gamma \geq 1 \text{ and } d \geq 2, \end{cases}$$

where k^γ is the sequence defined in §14.B. If $r_\gamma = 0$, then k_γ is the zero parameter of the trivial group.

If $\lambda \in \text{Part}(n)$, we denote by z_λ the image of χ_λ through the map $\mathfrak{z}_{k_{\text{sp}}} : \text{Irr}(\mathfrak{S}_n) \rightarrow \mathcal{X}(n)^{\mathbb{C}^\times}$. Similarly, if $\mu \in \text{Part}^d(m)$, and if $k \in \mathbb{C}^{\mathfrak{N}(G(d,1,m))}$, we denote by z_μ^k the image of χ_μ through the map $\mathfrak{z}_k : \text{Irr } G(d, 1, m) \rightarrow \mathcal{X}_k(G(d, 1, m))^{\mathbb{C}^\times}$.

15.A. Geometry of $\mathcal{X}(n)$. — Recall that $\mathcal{X}(n)$ is smooth [EtGi, Cor. 16.2]. This has several consequences:

- First, the map $\mathfrak{z}_{k_{\text{sp}}} : \text{Irr}(\mathfrak{S}_n) \longrightarrow \mathcal{X}(n)^{\mathbb{C}^\times}$ defined in §5.D is bijective [Gor1, Cor. 5.8]. This means that

$$(15.2) \quad \text{the map } \text{Part}(n) \longrightarrow \mathcal{X}(n)^{\mathbb{C}^\times}, \lambda \longmapsto z_\lambda \text{ is bijective.}$$

- The variety $\mathcal{X}(n)$ has only one symplectic leaf (it is a symplectic variety). Therefore, if $d \geq 1$, then $\mathcal{X}(n)^{\mu_d}$ is also smooth and symplectic, so its symplectic leaves coincide with its irreducible components: in particular, they are closed normal subvarieties of $\mathcal{X}(n)$, so coincide with the normalization of their closure (which is involved in Conjecture 7.2).

The next theorem, which describes these symplectic leaves, has been obtained by Maksimau and the author [BoMa, Theo. 4.21]:

Theorem 15.3. — *With the above notation, we have:*

- Let $\lambda \in \text{Part}(n)$. Then z_λ is ζ_d -cuspidal if and only if λ is a d -core.
- The map

$$\begin{array}{ccc} \text{Cor}_d(\equiv n) & \longrightarrow & \text{Cus}_{k_{\text{sp}}}^{\zeta_d \text{wd}}(\mathcal{X}(n)) \\ \gamma & \longmapsto & (\mathfrak{S}_{|\gamma|}, z_\gamma) \end{array}$$

is bijective. We denote by $\mathcal{S}_\gamma(n)$ the symplectic leaf $\mathcal{S}_{\mathfrak{S}_{|\gamma|}, z_\gamma}$ of $\mathcal{X}(n)^{\mu_d}$.

- If $\gamma \in \text{Cor}_d(\equiv n)$, then there exists a \mathbb{C}^\times -equivariant isomorphism of varieties

$$i_\gamma : \mathcal{X}_{k_\gamma}(G(d, 1, r_\gamma)) \xrightarrow{\sim} \mathcal{S}_\gamma(n)$$

such that

$$i_\gamma(z_\mu) = z_{\text{par}_d(\gamma, \mu)}$$

for all $\mu \in \text{Part}^d(r_\gamma)$ (in particular, $\dim \mathcal{S}_\gamma(n) = 2r_\gamma$).

Proof. — All the results have been proved in [BoMa, Theo. 4.21], except that we need to make some comments about the parameter. So let $\gamma \in \text{Cor}_d(\equiv n)$ and let $l_\gamma \in \mathbb{C}^{\mathbb{N}(G(d, 1, r_\gamma))}$ be the parameter defined by:

$$\begin{cases} ((l_\gamma)_{\text{Ker}(x_1-x_2), 0}, (l_\gamma)_{\text{Ker}(x_1-x_2), 1}) = (d, 0), & \text{if } r_\gamma \geq 2, \\ ((l_\gamma)_{\text{Ker}(x_1), 0}, (l_\gamma)_{\text{Ker}(x_1), 1}, \dots, (l_\gamma)_{\text{Ker}(x_1), d-1}) = l^\gamma, & \text{if } r_\gamma \geq 1 \text{ and } d \geq 2, \end{cases}$$

where l^γ is the sequence defined in §14.B. Then the result of [BoMa, Theo. 4.21] says that $\mathcal{S}_{\mathfrak{S}_{|\gamma|}, z_\gamma} \simeq \mathcal{X}_{l_\gamma}(G(d, 1, r_\gamma))$. However, Proposition 14.4 says that l^γ is obtained from k^γ by a cyclic permutation, and so $\mathcal{X}_{k_\gamma}(G(d, 1, r_\gamma)) \simeq \mathcal{X}_{l_\gamma}(G(d, 1, r_\gamma))$ by [BoRo2, (3.5.4)]. \square

15.B. Unipotent representations: the split case. — We fix in this subsection a triple $\mathcal{G} = (q, \mathbf{G}, F) \in \mathcal{G}roups(\mathfrak{S}_n)$. In other words, \mathbf{G}^F is a split group of type A_{n-1} (recall that the definition of $\mathcal{G}roups(\mathfrak{S}_n)$ implies no restriction on the rational structure of the center of \mathbf{G}). Then it is well-known that

$$(15.4) \quad \text{Unip}(\mathcal{G}) = \{R_\chi^\mathcal{G} \mid \chi \in \text{Irr}(\mathfrak{S}_n)\} \quad \text{and} \quad \deg R_{\chi\lambda}^\mathcal{G} = (\mathbf{deg} \lambda)(q).$$

In other words, we may define the set $\text{Unip}(\mathfrak{S}_n)$, the bijection $\rho^\mathcal{G}$ and the map $\mathbf{deg}_{\mathfrak{S}_n}$ as follows:

$$\begin{cases} \text{Unip}(\mathfrak{S}_n) = \text{Part}(n), \\ \rho_\lambda^\mathcal{G} = R_{\chi\lambda}^\mathcal{G} \quad \text{for any } \lambda \in \text{Part}(n), \\ \mathbf{deg}_{\mathfrak{S}_n} = \mathbf{deg}. \end{cases}$$

The partition into families is pretty easy in this case:

$$(15.5) \quad \text{All the unipotent Lusztig families are singletons.}$$

A generic translation is given by:

$$(15.6) \quad \text{the map } \text{Part}(n) \longrightarrow \text{Fam}_{\text{un}}(\mathfrak{S}_n), \lambda \longmapsto \{\lambda\} \text{ is bijective.}$$

The following result has been proved in [BMM1, Pages 45-47]:

Theorem 15.7 (Broué-Malle-Michel). — *With the above notation, we have:*

- (a) *Let $\lambda \in \text{Part}(n)$. Then the unipotent character $R_{\chi\lambda}$ is d -cuspidal if and only if λ is a d -core.*
- (b) *The map*

$$\begin{array}{ccc} \text{Cor}_d(\equiv n) & \longrightarrow & \text{Cus}_d(\mathfrak{S}_n) \\ \gamma & \longmapsto & (\mathfrak{S}_{|\gamma|}, R_{\chi_\gamma}) \end{array}$$

is bijective. If $\gamma \in \text{Cor}_d(\equiv n)$, then

$$\overline{\mathbf{N}}_{(\mathfrak{S}_n)_{\zeta_d w_d}}((\mathfrak{S}_{|\gamma|})_{\zeta_d w_d}) \simeq G(d, 1, r_\gamma).$$

- (c) *If $\gamma \in \text{Cor}_d(\equiv n)$, let HC_d^γ denote the bijection $\text{HC}_d^{\mathfrak{S}_n, \mathfrak{S}_m, R_{\chi_\gamma}} : \text{Irr}(G(d, 1, r_\gamma)) \longrightarrow \text{Unip}(\mathfrak{S}_n, \mathfrak{S}_{|\gamma|}, R_{\chi_\gamma})$ defined by the d -Harish-Chandra theory. Then*

$$\text{HC}_d^\gamma(\chi\lambda) = R_{\chi_{\text{par}_d(\gamma, \lambda)}}$$

for all $\lambda \in \text{Part}^d(r_\gamma)$.

Now, fix a d -core $\gamma \in \text{Cor}_d(\equiv n)$ and let $\mathcal{L}_\gamma = (g, \mathbf{L}_\gamma, F) \in \mathcal{G}roups(\mathfrak{S}_{|\gamma|})$ be such that \mathbf{L}_γ is a d -split Levi subgroup of \mathbf{G} (if $\mathbf{G}^F = \mathbf{GL}_n(\mathbb{F}_q)$, then $\mathbf{L}_\gamma^F \simeq \mathbf{GL}_{|\gamma|}(\mathbb{F}_q) \times (\mathbb{F}_{q^d}^\times)^{r_\gamma}$). Conjecture 9.5 predicts the existence of a parabolic subgroup \mathbf{P}_γ such that the Deligne-Lusztig variety $\mathbf{Y}_{\mathbf{U}_{\mathbf{P}_\gamma}}$ satisfies the following two properties:

- (a) *The $\overline{\mathbb{Q}}\mathbf{G}^F$ -modules $\text{H}_c^j(\mathbf{Y}_{\mathbf{P}_\gamma}) \otimes_{\overline{\mathbb{Q}}\mathbf{L}_\gamma^F} R_{\chi_\gamma}^{\mathcal{L}_\gamma}$ and $\text{H}_c^{j'}(\mathbf{Y}_{\mathbf{P}}} \otimes_{\overline{\mathbb{Q}}\mathbf{L}^F} R_{\chi_\gamma}^{\mathcal{L}_\gamma}$ have no common irreducible constituent if $j \neq j'$.*
- (b) *$\text{End}_{\mathbf{G}^F} \mathcal{R}_{\mathbf{L}_\gamma \subset \mathbf{P}_\gamma}^{\mathbf{G}}(R_{\chi_\gamma}^{\mathcal{L}_\gamma}) \simeq \mathcal{H}_{k_\gamma^\#}(G(d, 1, r_\gamma), \zeta_d^{-1}q)$ for some parameter $k_\gamma^\#$.*

This conjecture is far from being proved (see the next remark) but Broué-Malle [BrMa2, Prop. 2.10] proposed a conjectural value for $k_\gamma^\#$:

Conjecture 15.8 (Broué-Malle). — *If $\gamma \in \text{Cor}_d(\equiv n)$, then Conjecture 9.5 holds for the pair $(\mathcal{L}_\gamma, R_{\chi_\gamma}^{\mathcal{L}_\gamma})$ with parameter $k_\gamma^\# = k_\gamma$.*

We find remarkable that the parameter k_γ predicted, in this particular case, by Broué-Malle in 1993 in the context of Deligne-Lusztig varieties coincides (up to a non-relevant cyclic permutation) with the parameter l_γ found in 2018 by Maksimau and the author when studying Calogero-Moser spaces.

Remark 15.9. — The disjunction of the cohomology (see the above statement (a)) has been proved in [BDR, Theo. 4.3] whenever \mathbf{L}_γ is a torus (i.e. $|\gamma| = 0$ or 1), based on earlier works of Dudas [Dud, Cor. 3.2]. The statement (b) is only known if $d = n$ (Lusztig [Lus1, §7.3]) or $d = n - 1$ (Digne-Michel [DiMi2, Theo. 10.1]). ■

The comparison between Theorems 15.3 and 15.7 yields:

Theorem 15.10. — *If $W\tau = \mathfrak{S}_n$, then:*

- (a) *Conjectures 12.1, 12.2, 12.3(a) and 12.5 hold.*
- (b) *If moreover Conjecture 15.8 holds, then Conjecture 12.2(b) holds.*

15.C. The non-split case. — In type A , the non-split case corresponds to the case where F acts on the Weyl group \mathfrak{S}_n by the diagram automorphism (i.e. conjugation by the longest element). In our generic description, this corresponds to the coset $-\mathfrak{S}_n$ and objects in $\mathcal{G}roup(-\mathfrak{S}_n)$ (for instance, we may take for $(q, \mathbf{G}, F) \in \mathcal{G}roup(-\mathfrak{S}_n)$ the triple where $\mathbf{G} = \mathbf{GL}_n(\overline{\mathbb{F}}_q)$ and \mathbf{G}^F is the general unitary group). Ennola duality establishes a bijection between $\text{Unip}(\mathfrak{S}_n)$ and $\text{Unip}(-\mathfrak{S}_n)$ and this bijection transforms d -Harish-Chandra series into d' -Harish-Chandra series, where

$$d' = \begin{cases} 2d & \text{if } d \equiv 1 \pmod{2}, \\ d/2 & \text{if } d \equiv 2 \pmod{4}, \\ d & \text{if } d \equiv 0 \pmod{4}. \end{cases}$$

Therefore, the non-split case follows directly from the split one by applying Ennola duality (see [BrMa2, Rem. 2.11]). More precisely:

Theorem 15.11. — *If $W\tau = -\mathfrak{S}_n$, then:*

- (a) *Conjectures 12.1, 12.2, 12.3(a) and 12.5 hold.*
- (b) *If moreover Conjecture 15.8 holds, then Conjecture 12.2(b) holds.*

16. Classical groups and Harish-Chandra theory

We aim to check the following result:

Theorem 16.1. — *Assume that W is a Weyl group of classical type and that $d = 1$. Then Conjectures 12.1, 12.2 and 12.3 hold.*

The rest of this section is devoted to the proof of this Theorem. Note that the most difficult part of the work has been previously done by Gordon-Martino [GoMa], Bellamy-Thiel [BeTh] and Bellamy-Maksimau-Schedler [BeMaSc] on the Calogero-Moser space side (as well as an application of Bellamy-Maksimau-Schedler result by the author [Bon4, Cor. 9.8]), and by Lusztig [Lus2, Lus4, Lus5] on the unipotent representations side. The main interest of this section is to relate all these results together following the philosophy of this survey.

Notation. *If $n \geq 0$, we set $W_n = G(2, 1, n)$ and $W'_n = G(2, 2, n)$, so that W_n is a Weyl group of type B_n (i.e. C_n) and W'_n is a Weyl group of type D_n . We denote by $\tau = \text{diag}(-1, 1, \dots, 1) \in W_n$: it induces the non-trivial involutive diagram automorphism of W'_n . The canonical basis of $V = \mathbb{C}^n$ is denoted by (y_1, \dots, y_n) and its dual basis is denoted by (x_1, \dots, x_n) .*

Note that $W_n = \langle \tau \rangle \rtimes W'_n$.

16.A. Families. — First, Conjectures 12.1 and 12.2 for $d = 1$ follow immediately from Theorem 8.7 and [BeTh, Theo. A].

16.B. Type B or C. — We denote by $\mathbf{BC}(n)$ the set of $r \in \mathbb{Z}_{\geq 0}$ such that $r^2 + r \leq n$. If $r, m \geq 0$, we denote by $k[r]$ the element of $\mathbb{C}^{\mathbf{N}(W_m)}$ defined by

$$\begin{cases} k[r]_{\Omega,0} = r, & k[r]_{\Omega,1} = 0, \\ k[r]_{\Omega',1} = 1, & k[r]_{\Omega',0} = 0, \end{cases}$$

where Ω (resp. Ω') is the orbit of the reflecting hyperplane $\text{Ker}(x_1)$ (resp. $\text{Ker}(x_1 - x_2)$). Note that $k_{\text{sp}} = k[1]$. The notation $k[r]$ is somewhat ambiguous, as it does not refer to the natural number m , but it will be used only whenever m is clear from the context. The generic version of Harish-Chandra theory can be summarized as follows [Lus2], [Lus3, Tab. II] (note that $\bar{\mathbf{N}}_{W_n}(W_m) \simeq W_{n-m}$ for all $m \leq n$):

Theorem 16.2 (Lusztig). — *Let $n \geq 2$. Then:*

- (a) $\text{Unip}_{\text{cus}}(W_n)$ is non-empty if and only if $n = r^2 + r$ for some $r \in \mathbb{Z}_{\geq 0}$. In this case, it contains only one element, which will be denoted by cus_n .

(b) *The map*

$$\begin{array}{ccc} \mathbf{BC}(n) & \longrightarrow & \text{Cus}(W_n)/W_n \\ r & \longmapsto & (W_{r^2+r}, \text{cus}_{r^2+r}) \end{array}$$

is bijective.

(c) *Let $r \in \mathbf{BC}(n)$. If $\mathcal{G} = (q, \mathbf{G}, F) \in \text{Group}\mathfrak{d}(W_n)$ and if $\mathcal{L} = (q, \mathbf{L}, F) \in \text{Group}\mathfrak{d}(W_{r^2+r})$ is such that \mathbf{L} is a 1-split Levi subgroup of \mathbf{G} , then*

$$\text{End}_{\mathbf{G}^F} \mathcal{R}_{\mathbf{L}}^{\mathbf{G}} \rho_{\text{cus}_{r^2+r}}^{\mathcal{G}} \simeq \mathcal{H}_{k[2r+1]}(W_{n-(r^2+r)}).$$

On the Calogero-Moser space side, Bellamy-Thiel [BeTh, Theo. 6.24] and Bellamy-Maksimau-Schedler [BeMaSc] proved the following result (note that the proof of (c) by Bellamy-Maksimau-Schedler relies on the description of $\mathcal{X}_k(W_n)$ as quiver varieties):

Theorem 16.3 (Bellamy-Thiel, Bellamy-Maksimau-Schedler). — *Let $n \geq 2$. Then:*

- (a) $\mathcal{X}_{k_{\text{sp}}}(W_n)$ contains a cuspidal point if and only if $n = r^2 + r$ for some $r \in \mathbb{Z}_{\geq 0}$. In this case, it contains only one cuspidal point, which will be denoted by z_n^{cus} . It is equal to $z_{(r^2+r, \emptyset)}^{k_{\text{sp}}}$.
- (b) *The map*

$$\begin{array}{ccc} \mathbf{BC}(n) & \longrightarrow & \text{Cus}_{\text{Id}_V}(\mathcal{X}_{k_{\text{sp}}}(W_n))/W_n \\ r & \longmapsto & (W_{r^2+r}, z_{r^2+r}^{\text{cus}}) \end{array}$$

is bijective. We denote by $\mathcal{S}_r(n)$ the symplectic leaf of $\mathcal{X}_{k_{\text{sp}}}(W_n)$ indexed by $(W_{r^2+r}, z_{r^2+r}^{\text{cus}})$ through Theorem 7.1.

(c) *Let $r \in \mathbf{BC}(n)$. Then*

$$\overline{\mathcal{S}_r(n)}^{\text{nor}} \simeq \mathcal{X}_{k[2r+1]}(W_{n-(r^2+r)}).$$

The comparison between the above two theorems proves Conjecture 12.3 in type B or C , up to the verification that the cuspidal unipotent representation belongs the same unipotent Lusztig family as $\text{HC}^W(\chi_{(r^2+r, \emptyset)})$. We need for this the combinatorics of symbols [Lus2, §3] and its link with unipotent representations [Lus2, Theo. 8.2]. Whenever $n = r(r+1)$, then the cuspidal unipotent representation cus_n is parametrized by the symbol $\begin{pmatrix} 1 & 2 & \cdots & 2r & 2r+1 \\ & & & & \emptyset \end{pmatrix}$ (with defect $2r+1$) while $\text{HC}^W(\chi_{(r^2+r, \emptyset)})$ is parametrized by the symbol $\begin{pmatrix} r+1 & r+2 & \cdots & 2r & 2r+1 \\ 1 & 2 & \cdots & r & \end{pmatrix}$ (with defect 1). Since both symbols have the same entries, $\text{cus}_{r(r+1)}$ and $\text{HC}^W(\chi_{(r^2+r, \emptyset)})$ belong to the same unipotent Lusztig family [Lus4, Theo. 5.8], as desired.

16.C. Type D. — We denote by $\mathbf{D}(n)$ the set of $r \in \mathbb{Z}_{\geq 0}$ such that $r^2 \leq n$ and $r \neq 1$. For $j \in \{0, 1\}$, we set

$$\mathbf{D}_j(n) = \{0\} \cup \{r \in \mathbf{D}(n) \mid r \geq 2 \text{ and } r \equiv j \pmod{2}\}.$$

In type D , there are two kinds of possible rational structures (and a third one, if $n = 4$, inducing an order 3 automorphism of W'_4 : it will not be considered here), a split one and

a non-split one. They correspond respectively to the elements $\text{Id}_V = \tau^0$ and τ of the normalizer of W'_n . Note that

$$\overline{N}_{(W'_n)_{\tau^j}}((W'_m)_{\tau^j}) \simeq \begin{cases} W'_n & \text{if } (m, j) = (0, 0), \\ W_{n-1} & \text{if } (m, j) = (0, 1), \\ W_{n-m} & \text{if } m \geq 2. \end{cases}$$

We summarize the generic version of Harish-Chandra theory in both cases [Lus2], [Lus3, Tab. II]:

Theorem 16.4 (Lusztig). — *We have:*

- (a) $\text{Unip}_{\text{cus}}(W'_n)$ (resp. $\text{Unip}_{\text{cus}}(W'_n \tau_n)$) is non-empty if and only if $n = r^2$ for some $r \in \mathbb{Z}_{\geq 0}$, r even (resp. r odd or $r = 0$). In this case, it contains only one element, which will be denoted by cus'_n .
- (b) If $j \in \{0, 1\}$, the map

$$\begin{array}{ccc} \mathbf{D}_j(n) & \longrightarrow & \text{Cus}(W'_n \tau_n^j) / W'_n \\ r & \longmapsto & (W'_{r^2}, \text{cus}'_{r^2}) \end{array}$$

is bijective.

- (c) Let $j \in \{0, 1\}$ and let $r \in \mathbf{D}_j(n)$. If $\mathcal{G} = (q, \mathbf{G}, F) \in \text{Groups}(W'_n \tau_n^j)$ and if $\mathcal{L} = (q, \mathbf{L}, F) \in \text{Groups}(W'_{r^2})$ is such that \mathbf{L} is a 1-split Levi subgroup of \mathbf{G} , then

$$\text{End}_{\mathbf{G}^F} \mathcal{R}_{\mathbf{L}}^{\mathbf{G}} \rho_{\text{cus}'_{r^2}}^{\mathcal{G}} \simeq \begin{cases} \mathcal{H}_{k_{\text{sp}}}(W'_n) & \text{if } (r, j) = (0, 0), \\ \mathcal{H}_{k[2]}(W_{n-1}) & \text{if } (r, j) = (0, 1), \\ \mathcal{H}_{k[2r]}(W_{n-r^2}) & \text{otherwise.} \end{cases}$$

In [Bon4, Cor. 9.8], the author determined the partition into symplectic leaves (as well as their structure) of both $\mathcal{X}_{k_{\text{sp}}}(W'_n)$ and $\mathcal{X}_{k_{\text{sp}}}(W'_n)^{\tau_n}$, but it must be said that the essential part of the work was done by Bellamy-Thiel [BeTh, Prop. 4.17] and Bellamy-Maksimau-Schedler [BeMaSc]:

Theorem 16.5 (Bellamy-Thiel, Bellamy-Maksimau-Schedler). — *Let $n \geq 4$. Then:*

- (a) $\mathcal{X}_{k_{\text{sp}}}(W'_n)$ (resp. $\mathcal{X}_{k_{\text{sp}}}(W'_n)^{\tau_n}$) admits a cuspidal point if and only if there exists $r \geq 2$ such that $n = r^2$. In this case, it contains only one element, which will be denoted by y_n^{cus} . By extension, we set y_0^{cus} for the unique cuspidal point of $\mathcal{X}_{k_{\text{sp}}}(0, 1)$.
- (b) If $j \in \{0, 1\}$, the map

$$\begin{array}{ccc} \mathbf{D}_j(n) & \longrightarrow & \text{Cus}^{\tau_n^j}(\mathcal{X}_{k_{\text{sp}}}(W'_n) / W'_n) \\ r & \longmapsto & (W'_{r^2}, y_{r^2}^{\text{cus}}) \end{array}$$

is bijective. We denote by $\mathcal{S}'_r(n)$ the cuspidal leaf of $\mathcal{X}_{k_{\text{sp}}}(W'_n)^{\tau_n^j}$ indexed by $(W'_{r^2}, y_{r^2}^{\text{cus}})$.

- (c) Let $j \in \{0, 1\}$ and let $r \in \mathbf{D}_j(n)$. Then

$$\overline{\mathcal{S}'_r(n)}^{\text{nor}} \simeq \begin{cases} \mathcal{X}_{k_{\text{sp}}}(W'_n) & \text{if } (r, j) = (0, 0), \\ \mathcal{X}_{k[2]}(W_{n-1}) & \text{if } (r, j) = (0, 1), \\ \mathcal{X}_{k[2r]}(W_{n-r^2}) & \text{otherwise.} \end{cases}$$

The comparison between the above two theorems proves Conjecture 12.3 in type D , in both the untwisted and the twisted case, up to the verification that the cuspidal families correspond on both side: this is done thanks to the combinatoric of symbols, in the same way as for type B or C .

PART V SPETSES

17. What is a spets?

As explained in Section 11, and as noticed in many papers on the subject, essential features of the representation theory of a finite reductive group are controlled by its Weyl group and can be recovered from structures built from it. The *Spetses*^(*) program initiated by Broué-Malle-Michel [Mal1, Mal2, BMM2, BMM3], which takes its origin in their work on genericity [BMM1, BrMa2], proposes to attach to some finite complex reflection groups (called *spetsial*, see below) some numerical data (“unipotent representations”, “degrees”) which admits partitions into “families”, “ d -Harish-Chandra series” satisfying the same kind of properties as in the case of Weyl groups. This was soon corroborated by computations done by Lusztig [Lus7] and Malle (unpublished) for finite Coxeter groups that are not Weyl groups. This suggests that there should be a mysterious object (the *spets*) admitting some kind of representation theory similar to the representation theory of finite reductive groups.

We try to summarize it (very) quickly in this section, and see what are the possible links with the material of this paper. We come back to the general situation where W is a complex reflection group. The *spetsial* parameter of W , denoted by k_{sp} , is defined by $(k_{\text{sp}})_{\Omega,0} = 1$ and $(k_{\text{sp}})_{\Omega,j} = 0$ for all $\Omega \in \mathcal{A}/W$ and $1 \leq j \leq e_{\Omega} - 1$. Broué-Malle-Michel asked whether one can associate with any reflection group W several combinatorial data:

- (S1) A set $\text{Unip}(W)$, whose elements are called *irreducible unipotent representations* of the *spets* attached to W , even though there is no group and no representation attached to them.
- (S2) A map $\text{deg} : \text{Unip}(W) \rightarrow \mathbb{C}[\mathbf{q}]$. For ζ a root of unity, an irreducible unipotent representation $\rho \in \text{Unip}(W)$ is called ζ -*cuspidal* if $\text{deg} \rho$ is divisible by $(\mathbf{q} - \zeta)^{\dim(V/W)^{\zeta}}$.
- (S3) A ζ -Harish-Chandra theory: in other words, a partition of $\text{Unip}(W)$ into ζ -Harish-Chandra series built on the same model as in Theorem 9.4. In particular, to each Harish-Chandra series is associated a Hecke algebra of the stabilizer of the corresponding ζ -cuspidal pair (P, λ) , with a well-defined parameter $k_{P,\lambda}$.
- (S4) Almost characters: these are formal complex linear combinations of irreducible unipotent representations that can be used to define a partition of $\text{Unip}(W)$ into unipotent Lusztig families in the same way as in §10.B.

All these data should satisfy compatibility conditions (axioms) which mimic what is known or conjectured for finite reductive groups. The group W is said to be *spetsial* if some divisibility property of its Schur elements $\text{sch}_{\chi}^{k_{\text{sp}}}$ holds [Mal2, §3]. It turns out that

^(*)Spetses is a greek island where a conference on finite groups was organized in 1993: this program started there, during a coffee break...

many complex reflection groups are not spetsial, but some of them are. The list of irreducible spetsial groups is as follows:

- The groups $G(e, 1, n)$ and $G(e, e, n)$ for any $e \geq 1$;
- The primitive groups G_j , for

$$j \in \{4, 6, 8, 14, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37\}.$$

Being spetsial is easily seen to be a necessary condition for admitting combinatorial data as in (S1), (S2), (S3) and (S4) satisfying the list of axioms, but it is somewhat astonishing that it is also a sufficient condition [Mal1, BMM2, BMM3]. The natural remaining question is to figure out if there is a category (the spets?) lying above all these combinatorial data. Note the first attempts in this direction using fusion systems and ℓ -compact groups by Kessar-Malle-Semeraro [Sem, KMS1, KMS2].

Of course, if W is a Weyl group (i.e. a finite rational reflection group), then W is spetsial and one recovers the generic theory of unipotent representations of finite groups of the form \mathbf{G}^F where $(q, \mathbf{G}, F) \in \mathcal{G}r\text{oup}s(W)^{(*)}$. In the late 80's, Lusztig associated to each finite Coxeter group which is not a Weyl group a combinatorial datum as in (S1) and (S2) satisfying a few axioms (this was finally published in 1993; see [Lus7]). For H_2, H_3 and H_4 , this was rediscovered by Malle in 1992 (unpublished). In this case, the almost characters as in (S4) were obtained for dihedral groups by Lusztig and for H_4 by Malle [LuMa]. About the same period, Malle [Mal1] proved that the imprimitive complex reflection groups $G(e, 1, n)$ and $G(e, e, n)$ can be endowed with data satisfying (S1), (S2), (S3), (S4). The case of the primitive complex reflection groups has been done by Broué-Malle-Michel [BMM2, BMM3].

We expect that, for spetsial groups, all the above Broué-Malle-Michel constructions are compatible with the geometry of the Calogero-Moser space associated with W at spetsial parameter, and that all the conjectures stated in Section 12 remain valid in this context. In other words, is the spets attached to W hidden in the (Poisson) geometry of $\mathcal{L}_{k_{\text{sp}}}(W)$?

In the upcoming section, we illustrate again these coincidences in the smallest non-cyclic primitive complex reflection group, namely the group G_4 .

18. A primitive example

Hypothesis. *We assume in this section, and only in this section, that W is of type G_4 . In other words, we set*

$$s = \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix} \quad \text{and} \quad t = \frac{1}{3} \begin{pmatrix} 2\zeta_3 + 1 & 2(\zeta_3 - 1) \\ \zeta_3 - 1 & \zeta_3 + 2 \end{pmatrix},$$

and we assume that $W = \langle s, t \rangle = G_4$. Here, ζ_3 is a primitive third root of unity.

If $(\delta, \beta) \in \{(1, 0), (1, 4), (1, 8), (2, 1), (2, 3), (2, 5), (3, 2)\}$, there is a unique irreducible character of G_4 of degree δ and b -invariant β : it will be denoted by $\phi_{\delta, \beta}$. We have

$$\text{Irr}(G_4) = \{\phi_{\delta, \beta} \mid (\delta, \beta) \in \{(1, 0), (1, 4), (1, 8), (2, 1), (2, 3), (2, 5), (3, 2)\}\}.$$

(*)Note that, in this case, the ζ -Harish-Chandra series depend only on the order d of ζ and coincide with d -Harish-Chandra series

Note that $\phi_{1,0} = 1$ is the trivial character, that $\phi_{1,4} = \varepsilon$, $\phi_{1,8} = \varepsilon^2$, that $\phi_{2,1}$ and $\phi_{2,3}$ are the characters afforded by the representations V and V^* respectively, that $\phi_{2,5}$ is the character afforded by $V \otimes \varepsilon \simeq V^* \otimes \varepsilon^2$ and that $\phi_{3,2}$ corresponds to the second symmetric power $S^2(V) \simeq S^2(V^*)$. We denote by C_3 the parabolic subgroup $\langle s \rangle$ of W : it is a cyclic group of order 3.

18.A. Unipotent representations. — All the facts stated without proof in this paragraph are taken from [BMM3, §A.4] (see also [Mic]). The set $\text{Unip}_{\text{cus}}(C_3)$ contains a single element (which will be denoted by cus_{C_3}) and the set $\text{Unip}_{\text{cus}}(G_4)$ contains also a single element (which will be denoted by cus_{G_4}). The (classical) 1-Harish-Chandra theory of the spets G_4 may be summarized as follows:

- There are three Harish-Chandra series, namely $\text{Unip}(G_4, 1, 1)$, $\text{Unip}(G_4, C_3, \text{cus}_{C_3})$ and $\{\text{cus}_{G_4}\}$.
- $\text{Unip}(G_4, 1, 1)$ is the principal series, and we set $\rho_{\delta,\beta} = \text{HC}^{G_4}(\phi_{\delta,\beta})$.
- We have $\overline{N}_{G_4}(C_3, \text{cus}_{C_3}) = \overline{N}_{G_4}(C_3) \simeq \mu_2$ and we set $\rho_{C_3,+} = \text{HC}^{G_4, C_3, \text{cus}_{C_3}}(1)$ and $\rho_{C_3,-} = \text{HC}^{G_4, C_3, \text{cus}_{C_3}}(\sigma)$, where σ is the inclusion $\mu_2 \hookrightarrow \mathbb{C}^\times$.

Therefore,

$$(18.1) \quad \text{Unip}(G_4) = \{\rho_{1,0}, \rho_{1,4}, \rho_{1,8}, \rho_{2,1}, \rho_{2,3}, \rho_{2,5}, \rho_{3,2}, \rho_{C_3,+}, \rho_{C_3,-}, \text{cus}_{G_4}\}.$$

The unipotent Lusztig families are the following four subsets $\mathfrak{F}_{\clubsuit}^{\text{un}}$, $\mathfrak{F}_{\diamond}^{\text{un}}$, $\mathfrak{F}_{\heartsuit}^{\text{un}}$, $\mathfrak{F}_{\spadesuit}^{\text{un}}$ of $\text{Unip}(G_4)$:

$$(18.2) \quad \begin{cases} \mathfrak{F}_{\clubsuit}^{\text{un}} = \{\rho_{1,0}\}, \\ \mathfrak{F}_{\diamond}^{\text{un}} = \{\rho_{3,2}\}, \\ \mathfrak{F}_{\heartsuit}^{\text{un}} = \{\rho_{2,1}, \rho_{2,3}, \rho_{C_3,+}\}, \\ \mathfrak{F}_{\spadesuit}^{\text{un}} = \{\rho_{1,4}, \rho_{1,8}, \rho_{2,5}, \rho_{C_3,-}, \text{cus}_{G_4}\}. \end{cases}$$

The next table summarizes the numerical data attached to the spets G_4 . Let us explain the information given in this table:

- The first column contains the list of irreducible unipotent representations ρ of the spets G_4 .
- The second column contains the degree of ρ , where Φ_e denotes the e -th cyclotomic polynomial, $\Phi_3' = \mathbf{q} - \zeta_3$, $\Phi_3'' = \mathbf{q} - \zeta_3^{-1}$, $\Phi_6' = \mathbf{q} - \zeta_6$, $\Phi_6'' = \mathbf{q} - \zeta_6^{-1}$ and $\sqrt{-3} = 2\zeta_3 + 1$.
- The third column gives the family of ρ .
- The fourth (resp. fifth) column gives the cuspidal pair (P, λ) parametrizing the ζ_4 -Harish-Chandra series (resp. ζ_6 -Harish-Chandra series) to which ρ belongs. Note that an empty box means that ρ is ζ_4 -cuspidal (resp. ζ_6 -cuspidal) and that $(1, 1)_d$ denotes the ζ_d -cuspidal pair associated with the trivial parabolic subgroup, which is τ_d -split^(*).

^(*)The interesting ζ -Harish-Chandra series are those attached to a root of unity ζ of order equal to 1, 2, 3, 4 or 6; the $\zeta = -1$ (resp. the $\zeta = \zeta_3$) case can be deduced from the $\zeta = 1$ (resp. $\zeta = \zeta_6$) case thanks to Ennola duality [BMM3, Axiom 5.13], which essentially amounts to replacing \mathbf{q} by $-\mathbf{q}$ in this case.

ρ	$\deg(\rho)$	Family	ζ_4 -series	ζ_6 -series
$\rho_{1,0}$	1	♣	$(1, 1)_4$	$(1, 1)_6$
$\rho_{1,4}$	$-\sqrt{-3}/6\mathbf{q}^4\Phi_3''\Phi_4\Phi_6''$	♠		$(1, 1)_6$
$\rho_{1,8}$	$\sqrt{-3}/6\mathbf{q}^4\Phi_3'\Phi_4\Phi_6'$	♠		
$\rho_{2,1}$	$(3 + \sqrt{-3})/6\mathbf{q}\Phi_3'\Phi_4\Phi_6''$	♥		$(1, 1)_6$
$\rho_{2,3}$	$(3 - \sqrt{-3})/6\mathbf{q}\Phi_3''\Phi_4\Phi_6'$	♥		
$\rho_{2,5}$	$1/2\mathbf{q}^4\Phi_2^2\Phi_6$	♠	$(1, 1)_4$	
$\rho_{3,2}$	$\mathbf{q}^2\Phi_3\Phi_6$	◇	$(1, 1)_4$	
$\rho_{C_{3,+}}$	$-\sqrt{-3}/3\mathbf{q}\Phi_1\Phi_2\Phi_4$	♥		$(1, 1)_6$
$\rho_{C_{3,-}}$	$-\sqrt{-3}/3\mathbf{q}^4\Phi_1\Phi_2\Phi_4$	♠		$(1, 1)_6$
cus_{G_4}	$-1/2\mathbf{q}^4\Phi_1^2\Phi_3$	♠	$(1, 1)_4$	$(1, 1)_6$

We conclude this subsection by giving the parameters $k_{P,\lambda}$ for all $(P, \lambda) \in \text{Cus}^d(G_4)$ and $d \in \{1, 4, 6\}$. Whenever the relative Weyl group $\bar{N}_{G_4}(P, \lambda)$ is cyclic and isomorphic to μ_d , then the parameter will be given as a list $(k_0, k_1, \dots, k_{d-1})$ of complex numbers:

- $k_{1,1} = k_{\text{sp}}$.
- $k_{C_3, \text{cus}_{C_3}} = (3, 0)$.
- $k_{(1,1)_4} = (3, 0, 1, 0)$.
- $k_{(1,1)_6} = (2, 0, 0, 1, 0, 1)$.

18.B. Calogero-Moser space. — As there is only one orbit of reflecting hyperplanes (call it Ω), we will simply denote parameters $k \in \mathbb{C}^{\mathbb{N}(G_4)}$ by a triple $(k_0, k_1, k_2) \in \mathbb{C}^3$ where $k_j = k_{\Omega, j}$. For instance, $k_{\text{sp}} = (1, 0, 0)$. Descriptions of the Calogero-Moser space $\mathcal{X}_k(G_4)$ have been given in [BoMa] and [BoTh]. Note that the descriptions given in both cases are for parameters $k = (k_0, k_1, k_2) \in \mathbb{C}^3$ satisfying $k_0 + k_1 + k_2 = 0$: this is not restrictive, thanks to Remark 3.3. So, we set $k_{\text{sp}}^\circ = (2/3, -1/3, -1/3)$, and then $\mathcal{X}_{k_{\text{sp}}}(G_4) = \mathcal{X}_{k_{\text{sp}}^\circ}(G_4)$. Specializing the presentation [BoTh] at k_{sp}° , we get that $\mathcal{X}_{k_{\text{sp}}^\circ}$ is the closed subvariety of

\mathbb{C}^8 consisting of points $(x_1, x_2, y_1, y_2, a, b, c, e) \in \mathbb{C}^8$ such that

$$\left\{ \begin{array}{l} ab + 12ce + 2x_1y_1 - 15e^4 + 234e^2 + 192e = 0, \\ 3ay_1e + 4bc - 9be^3 + 126be + 2x_1y_2 = 0, \\ 3a^2e - 2bx_2 + 8cx_1 - 9x_1e^3 - 108x_1e = 0, \\ 4ac - 9ae^3 + 126ae + 3bx_1e + 2x_2y_1 = 0, \\ 2ay_2 - 3b^2e - 8cy_1 + 9y_1e^3 - 108y_1e = 0, \\ -a^3 - 3ax_1e^2 + 48ax_1 + 2ay_1^2 - b^3 + 2bx_1^2 \\ \quad - 3by_1e^2 + 48by_1 - 8cx_2 - 8cy_2 + 10x_2e^3 \\ \quad - 156x_2e + 128x_2 + 10y_2e^3 - 156y_2e - 128y_2 = 0, \\ 16c^2 + 720ce + 9x_1y_1e^2 + 2x_2y_2 - 27e^6 + 864e^3 + 6804e^2 = 0, \\ -2ay_1^2 + b^3 + 3by_1e^2 - 48by_1 + 8cy_2 - 10y_2e^3 + 156y_2e + 128y_2 = 0, \\ 5a^2y_1 + 444ab + 5b^2x_1 + 280ce^3 + 4848ce - 1280c + 60x_1y_1e^2 \\ \quad + 648x_1y_1 + 10x_2y_2 - 360e^6 + 7200e^3 + 88776e^2 + 44928e = 0. \end{array} \right.$$

The action of \mathbb{C}^\times is given by

$$(18.3) \quad \xi \cdot (x_1, x_2, y_1, y_2, a, b, c, e) = (\xi^4 x_1, \xi^6 x_2, \xi^{-4} y_1, \xi^{-6} y_2, \xi^2 a, \xi^{-2} b, c, e).$$

An immediate computation shows that $\mathcal{X}_{k_{\text{sp}}}^{\mathbb{C}^\times}$ contains 4 points, given by

$$\begin{aligned} z_{\clubsuit} &= (0, 0, 0, 0, 0, 0, 468, 8), & z_{\diamond} &= (0, 0, 0, 0, 0, 0, 0, 0), \\ z_{\heartsuit} &:= (0, 0, 0, 0, 0, 0, -45, 2) & \text{and} & \quad z_{\spadesuit} = (0, 0, 0, 0, 0, 0, -18, -4). \end{aligned}$$

We denote by $\mathfrak{F}_{\star}^{\text{CM}}$ the Calogero-Moser k_{sp}° -family associated with z_{\star} . Then

$$(18.4) \quad \left\{ \begin{array}{l} \mathfrak{F}_{\clubsuit}^{\text{CM}} = \{\phi_{1,0}\}, \\ \mathfrak{F}_{\diamond}^{\text{CM}} = \{\phi_{3,2}\}, \\ \mathfrak{F}_{\heartsuit}^{\text{CM}} = \{\phi_{2,1}, \phi_{2,3}\} \\ \mathfrak{F}_{\spadesuit}^{\text{CM}} = \{\phi_{1,4}, \phi_{1,8}, \phi_{2,5}\}. \end{array} \right.$$

The comparison of (18.2) and (18.4) proves the *spetsial* analogue of Conjecture 12.1.

18.B.1. Symplectic leaves of $\mathcal{X}_{k_{\text{sp}}^\circ}$. — Let \mathcal{S} denote the singular locus of $\mathcal{X}_{k_{\text{sp}}^\circ}$. It has been computed in [BoTh] and it is proved there that it is irreducible of dimension 2 and that

$$(18.5) \quad z_{\clubsuit}, z_{\diamond} \notin \mathcal{S} \quad \text{and} \quad z_{\heartsuit}, z_{\spadesuit} \in \mathcal{S}.$$

Moreover, z_{\spadesuit} is the only singular point of \mathcal{S} . Therefore, there are three symplectic leaves:

- The smooth locus: through the parametrization of Theorem 7.1, it corresponds to the pair $(1, p)$, where p is the unique point of the Calogero-Moser space $\mathcal{X}_{k_{\text{sp}}^\circ}(0, 1)$.
- $\mathcal{S}^\circ = \mathcal{S} \setminus \{z_{\spadesuit}\}$: through the parametrization of Theorem 7.1, it corresponds to the pair (C_3, q) , where q is the unique cuspidal point of the Calogero-Moser space $\mathcal{X}_{k_{\text{sp}}^\circ}(V/V^{C_3}, C_3)$.
- $\{z_{\spadesuit}\}$: it is cuspidal.

This parametrization fits perfectly with the partition of $\text{Unip}(G_4)$ into Harish-Chandra series, so this proves the *spetsial* analogue of Conjecture 12.2 and Conjecture 12.3(a) for $d = 1$.

Concerning Conjecture 12.3(b), the only interesting case is the second one. Recall that $\overline{N}_{G_4}(C_3) = \mu_2$. It is proved in [BoTh] that

$$(18.6) \quad \mathcal{S}^{\text{nor}} \simeq \{(x, y, e) \in \mathbb{C}^3 \mid (e-2)(e+4) = xy\} \simeq \mathcal{F}_{k_{C_3, \text{cus}_{C_3}}}(\mu_2)$$

as Poisson varieties. Note that the computation in [BoTh] is done for the parameter $-3k_{\text{sp}}^\circ$, so the equation given here is just obtained after a rescaling. This proves that Conjecture 12.3(b) holds for $d = 1$.

18.B.2. *Symplectic leaves of $\mathcal{F}_{k_{\text{sp}}^\circ}^{\mu_4}$.* — The action of \mathbb{C}^\times being given by (18.3), the variety $\mathcal{F}_{k_{\text{sp}}^\circ}^{\mu_4}$ is defined, inside $\mathcal{F}_{k_{\text{sp}}^\circ}$, by the equations $a = b = x_2 = y_2 = 0$. This yields

$$(18.7) \quad \mathcal{F}_{k_{\text{sp}}^\circ}^{\mu_4} = \{z_\heartsuit\} \dot{\cup} \mathcal{S}_4,$$

where

$$(18.8) \quad \mathcal{S}_4 \simeq \{(x_1, y_1, e) \in \mathbb{C}^3 \mid 4/3x_1y_1 = e(e-8)(e+4)^2\}$$

So \mathcal{S}_4 admits only one singular point (namely, z_\spadesuit) and so $\mathcal{F}_{k_{\text{sp}}^\circ}^{\mu_4}$ admits three symplectic leaves:

- There are two τ_4 -cuspidal points, namely z_\heartsuit and z_\spadesuit .
- There is one 2-dimensional symplectic leaf, which is the smooth locus of \mathcal{S}_4 (i.e. $\mathcal{S}_4 \setminus \{z_\spadesuit\}$). Through the parametrization of Theorem 7.1, it corresponds to the pair $(1, p)$, where p is the unique point of the Calogero-Moser space $\mathcal{F}_{k_{\text{sp}}^\circ}(0, 1)$.

This parametrization fits perfectly with the partition of $\text{Unip}(G_4)$ into 4-Harish-Chandra series, so this proves the spetsial analogues of Conjectures 12.2 and 12.3(a) for $d = 4$.

Concerning Conjecture 12.3(b), the only interesting case is the second one. Recall that $W_{\tau_4} \simeq \mu_4$. The above description proves that \mathcal{S}_4 (which is the closure of the symplectic leaf $\mathcal{S}_4 \setminus \{z_\spadesuit\}$) is normal and that

$$(18.9) \quad \mathcal{S}_4 \simeq \mathcal{F}_{k_{(1,1)_4}}(\mu_4)$$

as Poisson varieties (for the Poisson bracket, see [BoTh]). So Conjecture 12.3(b) holds for $d = 4$.

18.B.3. *Symplectic leaves of $\mathcal{F}_{k_{\text{sp}}^\circ}^{\mu_6}$.* — The action of \mathbb{C}^\times being given by (18.3), the variety $\mathcal{F}_{k_{\text{sp}}^\circ}^{\mu_6}$ is defined, inside $\mathcal{F}_{k_{\text{sp}}^\circ}$, by the equations $a = b = x_1 = y_1 = 0$. This yields

$$(18.10) \quad \mathcal{F}_{k_{\text{sp}}^\circ}^{\mu_6} = \{z_\diamondsuit\} \dot{\cup} \mathcal{S}_6,$$

where

$$(18.11) \quad \mathcal{S}_6 \simeq \{(x_2, y_2, e) \in \mathbb{C}^3 \mid x_2y_2 = (e-8)(e-2)^2(e+4)^3\}$$

So \mathcal{S}_6 admits two singular points (namely, z_\heartsuit and z_\spadesuit) and so $\mathcal{F}_{k_{\text{sp}}^\circ}^{\mu_6}$ admits four symplectic leaves:

- There are three τ_6 -cuspidal points, namely z_\diamondsuit , z_\heartsuit and z_\spadesuit .
- There is one 2-dimensional symplectic leaf, which is the smooth locus of \mathcal{S}_6 (i.e. $\mathcal{S}_6 \setminus \{z_\heartsuit, z_\spadesuit\}$). Through the parametrization of Theorem 7.1, it corresponds to the pair $(1, p)$, where p is the unique point of the Calogero-Moser space $\mathcal{F}_{k_{\text{sp}}^\circ}(0, 1)$.

This parametrization fits perfectly with the partition of $\text{Unip}(G_4)$ into 6-Harish-Chandra series, so this proves the spetsial analogues of Conjectures 12.2 and 12.3(a) for $d = 6$.

Concerning Conjecture 12.3(b), the only interesting case is the second one. Recall that $W_{\tau_6} \simeq \mu_6$. The above description proves that \mathcal{S}_6 (which is the closure of the symplectic leaf $\mathcal{S}_6 \setminus \{z_\heartsuit, z_\spadesuit\}$) is normal and that

$$(18.12) \quad \mathcal{S}_6 \simeq \mathcal{F}_{k(1,1)_6}(\mu_6)$$

as Poisson varieties (for the Poisson bracket, see [BoTh]). So Conjecture 12.3(b) holds for $d = 6$.

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