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# CALOGERO-MOSER CELLS OF DIHEDRAL GROUPS AT EQUAL PARAMETERS

by

CÉDRIC BONNAFÉ & JÉRÔME GERMONI

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*Abstract.* — We prove that Calogero-Moser cells coincide with Kazhdan-Lusztig cells for dihedral groups in the equal parameter case.

Calogero-Moser cells have been defined by Rouquier and the first author for any finite complex reflection group and any parameter, based on ramification theory for Calogero-Moser spaces [BoRo1, BoRo2]. It is conjectured that, for Coxeter groups, Calogero-Moser cells coincide with Kazhdan-Lusztig cells [BoRo1, Conj. 3.1 and 3.2], [BoRo2, Conj. LR and L], which were defined by Kazhdan-Lusztig [KaLu] in the equal parameter case and by Lusztig [Lus] in the general case. The aim of this paper is to prove this conjecture for dihedral groups in the equal parameter case.

For Calogero-Moser *left* cells, an alternative (and partially conjectural) definition is proposed in [BoRo2, Theo. 13.3.2], based on Gaudin operators. This definition is recalled in Section 4. This is the point of view we adopt in this paper: in the relatively small case of dihedral groups, an explicit diagonalization of these operators is possible, and the computation of Calogero-Moser left cells becomes easy.

## 1. Setup

Let  $V$  be a finite dimensional Euclidean real vector space, whose positive definite symmetric bilinear form is denoted by  $(\cdot, \cdot)$  and let  $W$  be a finite subgroup of the orthogonal group  $\mathbf{O}(V)$  generated by reflections. For  $v \in V$ , we denote by  $v^*$  the element of the dual space  $V^*$  defined by  $v^*(y) = (y, v)$  for all  $y \in V$ . The map  $V \rightarrow V^*$ ,  $v \mapsto v^*$  is a  $W$ -equivariant isomorphism of vector spaces.

The set of reflections of  $W$  is denoted by  $\text{Ref}(W)$ . For  $\alpha \in V \setminus \{0\}$ , we denote by  $s_\alpha$  the orthogonal reflection such that  $s_\alpha(\alpha) = -\alpha$ . We set

$$\Phi = \{\alpha \in V \mid (\alpha, \alpha) = 1 \text{ and } s_\alpha \in W\}.$$

Then  $\Phi = -\Phi$ , and we fix a subset  $\Delta$  of  $\Phi$  of cardinality  $\dim \mathbb{R}\Phi$  such that every element of  $\Phi$  belongs to  $\sum_{\alpha \in \Delta} \mathbb{R}_{\geq 0} \alpha$  or to  $\sum_{\alpha \in \Delta} \mathbb{R}_{\leq 0} \alpha$ . We set

$$S = \{s_\alpha \mid \alpha \in \Delta\},$$

so that  $(W, S)$  is a finite Coxeter system. We set

$$\Phi^+ = \Phi \cap \sum_{\alpha \in \Delta} \mathbb{R}_{\geq 0} \alpha \quad \text{and} \quad \Phi^- = -\Phi^+,$$

so that  $\Phi = \Phi^+ \dot{\cup} \Phi^-$ , where  $\dot{\cup}$  means disjoint union. We set

$$v_0 = \sum_{\alpha \in \Phi^+} \alpha.$$

We denote by  $w_0$  the longest element of  $W$  (with respect to the length function  $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$  defined by the choice of  $S$ ). Then

$$(1.1) \quad w_0(v_0) = -v_0.$$

We set

$$V_{\text{reg}} = V \setminus \bigcup_{\alpha \in \Phi} V^{s_\alpha}$$

and

$$\mathfrak{C} = \{v \in V \mid \forall \alpha \in \Delta, (\alpha, v) > 0\}.$$

Then  $\mathfrak{C}$  is the *fundamental chamber* of  $W$  associated with  $S$  and  $v_0 \in \mathfrak{C}$ . Recall that its closure is a fundamental domain for the action of  $W$  on  $V$ .

We denote by  $\text{Reg}_W$  the character afforded by the regular representation and  $\text{Irr}(W)$  denotes the set of irreducible characters of  $W$ . We denote by  $\mathbf{1}_W$  the trivial character of  $W$  and we set  $\varepsilon : W \rightarrow \mu_2, w \mapsto \det(w)$ . We denote by  $\mathcal{C}_{\mathbb{R}}$  the vector space of maps  $c : \text{Ref}(W) \rightarrow \mathbb{R}$  such that  $c_s = c_t$  if  $s$  and  $t$  are conjugate in  $W$  (the elements of  $\mathcal{C}_{\mathbb{R}}$  are called *parameters*). Finally, if  $X$  is a subset of  $W$ , we set  $X^{-1} = \{w^{-1} \mid w \in X\}$ .

## 2. Recollection about Kazhdan-Lusztig cells

Let  $c \in \mathcal{C}_{\mathbb{R}}$ . To the datum  $(W, S, c)$  are associated three partitions of  $W$  into *Kazhdan-Lusztig left, right and two-sided  $c$ -cells* (see for instance [Bon2, Chap. 6]). To each Kazhdan-Lusztig left  $c$ -cell  $C$  is associated a *Kazhdan-Lusztig  $c$ -cellular character* that is denoted by  $\chi_C^{c, \text{KL}}$ . Then

$$(2.1) \quad \text{Reg}_W = \sum_C \chi_C^{c, \text{KL}},$$

where  $C$  runs over the set of Kazhdan-Lusztig left  $c$ -cells.

On the other hand, to each Kazhdan-Lusztig two-sided  $c$ -cell  $\Gamma$  of  $W$  is associated a subset  $\text{Irr}_{\Gamma}^{c, \text{KL}}(W)$  called the *Kazhdan-Lusztig  $c$ -family* associated with  $\Gamma$ . They form a partition of  $\text{Irr}(W)$ :

$$(2.2) \quad \text{Irr}(W) = \dot{\bigcup}_{\Gamma} \text{Irr}_{\Gamma}^{c, \text{KL}}(W),$$

where  $\Gamma$  runs over the set of Kazhdan-Lusztig two-sided  $c$ -cells. Here are some other properties of Kazhdan-Lusztig cells (see for instance [Bon2, §6.1, §6.2 and Chap. 10]).

**Proposition 2.3.** — *Let  $C$  (resp.  $\Gamma$ ) be a Kazhdan-Lusztig left (resp. two-sided)  $c$ -cell. Then:*

- (a)  $C^{-1}$  is a Kazhdan-Lusztig right  $c$ -cell and  $|C| = \chi_C^{c,\text{KL}}(1)$ .  
 (b)  $\Gamma$  is a union of Kazhdan-Lusztig left (or right)  $c$ -cells. Moreover,

$$|\Gamma| = \sum_{\chi \in \text{Irr}_{\Gamma}^{c,\text{KL}}(W)} \chi(1)^2.$$

- (c) If  $C \subset \Gamma$ , then every irreducible component of  $\chi_C^{c,\text{KL}}$  belongs to  $\text{Irr}_{\Gamma}^{c,\text{KL}}(W)$ .  
 (d)  $w_0\Gamma w_0 = \Gamma$ .  
 (e)  $Cw_0$  and  $w_0C$  (resp.  $w_0\Gamma = \Gamma w_0$ ) are Kazhdan-Lusztig left (resp. two-sided)  $c$ -cells. Moreover,

$$\chi_{Cw_0}^{c,\text{KL}} = \chi_{w_0C}^{c,\text{KL}} = \chi_C^{c,\text{KL}} \cdot \varepsilon \quad \text{and} \quad \text{Irr}_{\Gamma w_0}^{c,\text{KL}}(W) = \text{Irr}_{\Gamma}^{c,\text{KL}}(W) \cdot \varepsilon.$$

- (f) If  $c_s \neq 0$  for all  $s \in \text{Ref}(W)$ , then  $\{1\}$  and  $\{w_0\}$  are Kazhdan-Lusztig two-sided  $c$ -cells. If moreover  $c_s > 0$  for all  $s \in \text{Ref}(W)$ , then

$$\begin{cases} \chi_{\{1\}}^{c,\text{KL}} = \varepsilon, \\ \chi_{\{w_0\}}^{c,\text{KL}} = \mathbf{1}_W, \end{cases} \quad \text{and} \quad \begin{cases} \text{Irr}_{\{1\}}^{c,\text{KL}}(W) = \{\varepsilon\}, \\ \text{Irr}_{\{w_0\}}^{c,\text{KL}}(W) = \{\mathbf{1}_W\}. \end{cases}$$

- (g) If  $\tau : W \rightarrow \mu_2$  is a linear character, then  $C$  (resp.  $\Gamma$ ) is a Kazhdan-Lusztig left (resp. two-sided)  $\tau \cdot c$ -cell. Moreover

$$\chi_C^{\tau \cdot c,\text{KL}} = \chi_C^{c,\text{KL}} \cdot \tau \quad \text{and} \quad \text{Irr}_{\Gamma}^{\tau \cdot c,\text{KL}}(W) = \text{Irr}_{\Gamma}^{c,\text{KL}}(W) \cdot \tau.$$

In the above statement (g),  $\tau \cdot c$  denotes the element of  $\mathcal{C}_{\mathbb{R}}$  defined by  $(\tau \cdot c)_s = \tau(s)c_s$ .

### 3. Gaudin operators

For  $y \in V$ ,  $v \in V_{\text{reg}}$  and  $v' \in V$ , we define an endomorphism  $D_y^{c,v,v'}$  of the underlying vector space of the group algebra  $\mathbb{R}W$  by the following formula [BoRo2, §13.2]:

$$\forall w \in W, \quad D_y^{c,v,v'}(w) = (y, w^{-1}(v'))w - \sum_{\alpha \in \Phi^+} c_{s_{\alpha}} \frac{(y, \alpha)}{(v, \alpha)} w s_{\alpha}.$$

The endomorphism  $D_y^{c,v,v'}$  is called a *Gaudin operator* (and is somewhat similar to Dunkl operators (see for instance [BoRo2, 3.1.B]). Then the map  $D^{c,v,v'} : V \rightarrow \text{End}_{\mathbb{R}}(\mathbb{R}W)$  is linear and it follows from [BoRo2, §13.2]<sup>(1)</sup> that

$$(3.1) \quad [D_y^{c,v,v'}, D_{y'}^{c,v,v'}] = 0$$

for all  $y, y' \in V$ ,  $v \in V_{\text{reg}}$  and  $v' \in V$ . Now, for  $\lambda \in V^*$ , we set

$$E_{\lambda}^{c,v,v'} = \{e \in \mathbb{R}W \mid D_y^{c,v,v'}(e) = \lambda(y)e\}$$

and we define

$$\mathfrak{Sp}^{c,v,v'} = \{\lambda \in V^* \mid E_{\lambda}^{c,v,v'} \neq 0\}.$$

<sup>(1)</sup>Note that we have not used exactly the convention of [BoRo2, §13.2]: our operators are obtained from those in loc. cit. by conjugating by the  $\mathbb{R}$ -linear map extending the inversion  $w \mapsto w^{-1}$  in  $W$  and by identifying  $V$  and  $V^*$  thanks to the non-degenerate form  $(\cdot, \cdot)$ .

As all reflections of  $W$  have order 2, the matrix of  $D_y^{c,v,v'}$  in the canonical basis of  $\mathbb{R}W$  is real and symmetric, so it is diagonalizable. Therefore, for all  $(v, v') \in V_{\text{reg}} \times V$ , the family of commuting matrices  $\mathcal{D}^{c,v,v'} = (D_y^{c,v,v'})_{y \in V}$  is simultaneously diagonalizable. In other words,

$$(3.2) \quad \mathbb{R}W = \bigoplus_{\lambda \in \mathcal{S}\rho^{c,v,v'}} E_\lambda^{c,v,v'}$$

for any  $(v, v') \in V_{\text{reg}} \times V$ . The set  $\mathcal{S}\rho^{c,v,v'}$  is called the *spectrum* of the family  $\mathcal{D}^{c,v,v'}$ . We say that the family  $\mathcal{D}^{c,v,v'}$  has *simple spectrum* if  $|\mathcal{S}\rho^{c,v,v'}| = |W|$  (in other words, if  $\dim E_\lambda^{c,v,v'} = 0$  or 1 for all  $\lambda \in V^*$ ).

**Conjecture 3.3.** — *If  $c \in \mathcal{C}_{\mathbb{R}}$  and  $(v, v') \in V_{\text{reg}} \times V_{\text{reg}}$ , then the family  $\mathcal{D}^{c,v,v'}$  has simple spectrum.*

By the work of Mukhin-Tarasov-Varchenko [MuTaVa1, Coro. 7.4], [MuTaVa2], this conjecture is known to hold in type A. Here is a weaker form of this conjecture.

**Conjecture 3.4.** — *If  $c \in \mathcal{C}_{\mathbb{R}}$  and  $\xi, \xi' \in \mathbb{R}_{>0}$ , then the family  $\mathcal{D}^{c,\xi v_0, \xi' v_0}$  has simple spectrum.*

We will prove in this paper that this weaker form holds if  $W$  is dihedral and  $c$  is constant (which is the so-called “equal parameter case”).

**Example 3.5.** — The matrix of the endomorphism  $D_y^{0,v,v'}$  in the canonical basis of  $\mathbb{R}W$  is diagonal, and so its spectrum can be easily computed. We get

$$\mathcal{S}\rho^{0,v,v'} = \{w(v'^*) \mid w \in W\}.$$

In particular,  $\mathcal{D}^{0,v,v'}$  has simple spectrum if and only if  $v' \in V_{\text{reg}}$ . ■

We conclude this subsection by some relations between Gaudin operators. For  $w \in W$ , we denote by  $l_w$  (resp.  $r_w$ ) the automorphism of the  $\mathbb{R}$ -vector space  $\mathbb{R}W$  defined by left (resp. right) multiplication by  $w$  (resp.  $w^{-1}$ ). If  $\tau : W \rightarrow \mu_2$  is a linear character, we denote by  $\tau_\bullet$  the automorphism of the  $\mathbb{R}$ -algebra  $\mathbb{R}W$  defined by  $\tau_\bullet(w) = \tau(w)w$  for all  $w \in W$ . The following formulas are straightforward:

$$(3.6) \quad \begin{cases} l_w D_y^{c,v,v'} l_w^{-1} = D_y^{c,v,w(v')}, \\ r_w D_y^{c,v,v'} r_w^{-1} = D_{w(y)}^{c,w(v),v'}, \\ \tau_\bullet D_y^{c,v,v'} \tau_\bullet^{-1} = D_y^{\tau \cdot c, v, v'}. \end{cases}$$

## 4. Calogero-Moser cells

**4.A. Calogero-Moser cellular characters.** — The operator  $D_y^{c,v_0,0}$  commutes with left multiplication by  $\mathbb{R}W$ . So each subspace  $E_\lambda^{c,v_0,0}$  inherits a structure of  $\mathbb{R}W$ -module: we denote by  $\chi_\lambda^c$  the character afforded by this  $\mathbb{R}W$ -module. We define the *Calogero-Moser  $c$ -cellular characters* to be the characters of the form  $\chi_\lambda^c$  for some  $\lambda \in \mathcal{S}\rho^{c,v_0,0}$ . Note that we may have  $\chi_\lambda^c = \chi_\mu^c$  even if  $\lambda \neq \mu$ . Then (3.2) implies that

$$(4.1) \quad \text{Reg}_W = \sum_{\lambda \in \mathcal{S}\rho^{c,v_0,0}} \chi_\lambda^c.$$

In particular, every irreducible character of  $W$  occurs in some Calogero-Moser  $c$ -cellular character.

Replacing  $(c, v_0)$  by  $(\xi c, \xi' v_0)$  (with  $\xi, \xi' \in \mathbb{R}^\times$ ) amounts to multiplying the Gaudin operators by  $\xi/\xi'$ : this does not change the list of Calogero-Moser cellular characters. This shows that Calogero-Moser  $c$ -cellular characters coincide with Calogero-Moser  $\xi c$ -cellular characters.

**Remark 4.2.** — The family  $\mathcal{D}^{c,v_0,0}$  does not have a simple spectrum in general. Indeed, if  $W$  is not abelian, then an irreducible character of degree  $> 1$  occurs in some cellular character  $\chi_\lambda^c$ , which shows that  $\dim E_\lambda^{c,v_0,0} \geq 2$ . ■

**4.B. Left cells.** — In order to define Calogero-Moser left cells, we need to work under the following hypothesis:

**Hypothesis.** *In this subsection, and only in this subsection, we assume that Conjecture 3.4 holds.*

Let  $v_1, v_2 \in \mathbb{R}_{>0}v_0$ . We fix two continuous functions  $\gamma, \xi : [0, 1] \rightarrow \mathbb{R}_{\geq 0}$  such that  $\gamma(t) \geq 0$  and  $\xi(t) > 0$  for all  $t \in [0, 1)$  and

$$\begin{cases} \gamma(0) = 0, & \xi(0) = 1, \\ \gamma(1) = 1, & \xi(1) = 0. \end{cases}$$

Therefore, for  $t \in [0, 1)$ , the family  $\mathcal{D}^{\gamma(t)c, v_1, \xi(t)v_2}$  has simple spectrum (indeed, if  $\gamma(t) = 0$ , then this follows from Example 3.5 and, if  $\gamma(t) > 0$ , then  $D_y^{\gamma(t)c, v_1, \xi(t)v_2} = D_y^{c, \gamma(t)^{-1}v_1, \xi(t)v_2}$  and so this follows from the fact that we assume that Conjecture 3.4 holds). So this spectrum varies continuously according to the parameter  $t$ . But, for  $t = 0$ , we have  $\mathcal{S}\rho^{0, v_1, v_2} = \{w(v_2^*) \mid w \in W\}$  by Example 3.5. This means that, for each  $w \in W$ , there exists a unique continuous map  $\lambda_w : [0, 1] \rightarrow V^*$  such that

$$\begin{cases} \lambda_w(0) = w(v_2^*) \\ \lambda_w(t) \in \mathcal{S}\rho^{\gamma(t)c, v_1, \xi(t)v_2} \text{ for all } t \in [0, 1]. \end{cases}$$

and the family  $(\lambda_w)_{w \in W}$  satisfies that

$$(4.3) \quad \forall t \in [0, 1), \lambda_w(t) \neq \lambda_{w'}(t)$$

whenever  $w \neq w'$ . However, it may happen that  $\lambda_w(1) = \lambda_{w'}(1)$  even if  $w \neq w'$ . This leads to the following definition:

**Definition 4.4.** — Two elements  $w$  and  $w'$  are said to belong to the same **Calogero-Moser left  $c$ -cell** if  $\lambda_w(1) = \lambda_{w'}(1)$ .

If  $C$  is a Calogero-Moser left  $c$ -cell, we set  $\chi_C^{c,\text{CM}} = \chi_{\lambda_w(1)}^c$  (where  $w$  is some, or any, element of  $C$ ): it is called the **Calogero-Moser  $c$ -cellular character** associated with  $C$ .

**Remark 4.5.** — A simple choice would be to take  $v_1 = v_2 = v_0$ ,  $\gamma(t) = tc$  and  $\xi(t) = 1 - t$ . But we want to work with this slightly more general setting for more flexibility. Indeed, one could wonder whether the notion of Calogero-Moser left  $c$ -cell depends on the choices of  $v_1, v_2, \gamma, \xi$ . In fact, it does not, because the topological space  $\mathcal{C}_{\mathbb{R}} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$  is simply connected.

For instance, this shows that, if  $r \in \mathbb{R}_{>0}$ , then Calogero-Moser left  $rc$ -cells coincide with Calogero-Moser left  $c$ -cells, and their associated cellular characters agree.

If we assume moreover that Conjecture 3.3 holds, then we could have added some more flexibility, by taking  $v_1, v_2$  in  $\mathcal{C} \times \mathcal{C}$  and replaced the path  $t \mapsto \xi(t)v_2$  by any path  $\nu_2 : [0, 1] \rightarrow \overline{\mathcal{C}}$  such that  $\nu_2(t) \in \mathcal{C}$  for  $t \in [0, 1)$ ,  $\nu_2(0) = v_2$  and  $\nu_2(1) = 0$  and the path  $t \mapsto \gamma(t)c$  by any path  $[0, 1] \rightarrow \mathcal{C}_{\mathbb{R}}$  starting at 0 and ending at  $c$ . ■

The Formula (4.1) can be rewritten as follows:

$$(4.6) \quad \text{Reg}_W = \sum_C \chi_C^{c,\text{CM}},$$

where  $C$  runs over the set of Calogero-Moser left  $c$ -cells.

The following conjecture has been proposed in [BoRo1, Conj. 3.2] and [BoRo2, Conj. L]:

**Conjecture 4.7.** — Calogero-Moser left  $c$ -cells coincide with Kazhdan-Lusztig left  $c$ -cells. Moreover, if  $C$  is one of these, then  $\chi_C^{c,\text{CM}} = \chi_C^{c,\text{KL}}$ .

A very weak evidence for this Conjecture is the comparison between (2.1) and (4.6). Note also that it holds for  $c = 0$ , as easily shown in [BoRo2, Coro. 17.2.3]. A somewhat strong evidence for this Conjecture is that it holds in type  $A$ , by the work of Brochier-Gordon-White [BrGoWh]. The aim of this paper is to deal with the far easier (but still non-trivial) case of dihedral groups whenever  $c$  is constant. The following list of properties of Calogero-Moser left cells shows that Conjecture 4.7 is compatible with Proposition 2.3.

**Proposition 4.8.** — Let  $C$  be a Calogero-Moser left  $c$ -cell. Then:

(a)  $|C| = \chi_C^{c,\text{CM}}(1)$ .

(b)  $Cw_0$  and  $w_0C$  are Calogero-Moser left  $c$ -cells. Moreover,

$$\chi_{Cw_0}^{c,\text{CM}} = \chi_{w_0C}^{c,\text{CM}} = \chi_C^{c,\text{CM}} \cdot \varepsilon.$$

(c) If  $c_s \neq 0$  for all  $s \in \text{Ref}(W)$ , then  $\{1\}$  and  $\{w_0\}$  are Calogero-Moser left  $c$ -cells. If moreover  $c_s > 0$  for all  $s \in \text{Ref}(W)$ , then

$$\chi_{\{1\}}^{c,\text{CM}} = \varepsilon \quad \text{and} \quad \chi_{\{w_0\}}^{c,\text{CM}} = \mathbf{1}_W.$$

(d) If  $\tau : W \rightarrow \mu_2$  is a linear character, then  $C$  (resp.  $\Gamma$ ) is a Calogero-Moser left (resp. two-sided)  $\tau \cdot c$ -cell. Moreover

$$\chi_C^{\tau \cdot c, \text{CM}} = \chi_C^{c, \text{CM}} \cdot \tau.$$

*Proof.* — As explained in Remark 4.5, we may assume that  $v_1 = v_2 = v_0$  and that  $\gamma(t) = tc$  and  $\xi(t) = 1 - t$  for all  $t \in [0, 1]$ .

(a) is clear.

(b) Let  $\tau_0 = \varepsilon_\bullet \circ l_{w_0} : \mathbb{R}W \rightarrow \mathbb{R}W$ . Since  $\varepsilon \cdot c = -c$  and  $w_0(v_0) = -v_0$ , we get from (3.6) that

$$\tau_0 D_y^{tc, v_0, (1-t)v_0} \tau_0^{-1} = D_y^{-tc, v_0, (t-1)v_0} = D_{-y}^{tc, v_0, (1-t)v_0}.$$

This means that  $\lambda \in \mathcal{S}p^{tc, v_0, (1-t)v_0}$  if and only if  $-\lambda \in \mathcal{S}p^{tc, v_0, (1-t)v_0}$ . Since  $\lambda_{ww_0}(0) = ww_0(v_0^*) = -w(v_0^*) = -\lambda_w(0)$ , this shows that  $\lambda_{ww_0}(t) = -\lambda_w(t)$  for all  $t \in [0, 1]$ . In particular,  $Cw_0$  is a Calogero-Moser left  $c$ -cell.

Finally, if  $\lambda \in \mathcal{S}p^{c, v_0, 0}$ , then  $E_{-\lambda}^{c, v_0, 0} = \tau_0(E_\lambda^{c, v_0, 0}) = \varepsilon_\bullet(E_\lambda^{c, v_0, 0})$ . This proves that  $\chi_{-\lambda}^c = \chi_\lambda^c \cdot \varepsilon$ , and completes the proof of (a).

(d) follows from the third equality in (3.6) and the same argument as in (b).

(c) By using (d) and rectifying the signs if necessary thanks to a linear character, we may, and we will, assume that  $c_s > 0$  for all  $s \in \text{Ref}(W)$ . We have

$$D_{-v_0}^{tc, v_0, (1-t)v_0}(w) = (t-1)(v_0, w^{-1}(v_0))w + \sum_{\alpha \in \Phi^+} tc_{s_\alpha} ws_\alpha.$$

Let  $A$  denote the diagonal endomorphism  $D_{-v_0}^{0, v_0, v_0}$  and let  $B$  denote the Gaudin operator  $D_{-v_0}^{c, v_0, 0}$ , so that  $D_{-v_0}^{tc, v_0, (1-t)v_0} = (1-t)A + tB$ .

The matrix  $B$  is a real matrix with non-negative coefficients, which is primitive (because  $W$  is generated by  $\text{Ref}(W)$ ). Let  $\nu = (v_0, v_0) + 1$ . Then  $A + \nu \text{Id}_{\mathbb{R}W}$  is a diagonal matrix with positive coefficients. Therefore, if  $t > 0$ , the matrix  $D_{-v_0}^{tc, v_0, (1-t)v_0} + \nu \text{Id}_{\mathbb{R}W}$  is a real matrix with non-negative coefficients which is primitive. By Perron-Frobenius Theorem, its spectral radius  $\rho_t$  is an eigenvalue of  $D_{-v_0}^{tc, v_0, (1-t)v_0} + \nu \text{Id}_{\mathbb{R}W}$ , with multiplicity 1. Therefore,  $\rho_t$  varies continuously as  $t$  varies.

For  $t = 0$ ,  $A + \nu \text{Id}_{\mathbb{R}W}$  is diagonal and its biggest diagonal coefficient is  $(v_0, v_0) + \nu$ , which occurs with multiplicity 1 (and its eigenvector is  $w_0$ ). So the map  $\rho' : [0, 1] \rightarrow \mathbb{R}_{>0}$ ,  $t \mapsto \rho_t - \nu$  is continuous. Adding  $\nu \text{Id}_{\mathbb{R}W}$  was an artefact to obtain a matrix with non-negative coefficients and apply Perron-Frobenius Theorem. Coming back to  $D_{-v_0}^{tc, v_0, (1-t)v_0}$ , we have proven that  $\rho'_t$  is its biggest eigenvalue, that it has multiplicity 1, that it varies continuously for  $t \in [0, 1]$ , that  $\rho'_0 = (v_0, v_0)$  with eigenvector  $w_0$ .

Let us now prove that  $\rho'_1 = a$ , where  $a = \sum_{\alpha \in \Phi^+} c_{s_\alpha}$ . For this, note first that  $\sum_{w \in W} w$  is an eigenvector of  $B$  with eigenvalue  $a$ : this proves that  $\rho'_1 \geq a$ . Now, if  $\omega = \sum_{w \in W} p_w w$  is an eigenvector of  $B$  for the eigenvalue  $\rho'_1$ , then Perron-Frobenius Theorem says that  $p_w > 0$  for all  $w \in W$ . Let  $w_1 \in W$  be such that  $p_{w_1}$  is maximal. Then the coefficient of  $w_1$  in  $B(\omega)$  is equal to  $\sum_{\alpha \in \Phi^+} c_{s_\alpha} p_{w_1 s_\alpha} \leq a p_{w_1}$ . But this coefficient is  $\rho'_1 p_{w_1}$ . This proves that  $\rho'_1 \leq a$ . Consequently,  $\rho'_1 = a$  and a corresponding eigenvector is  $\sum_{w \in W} w$ .

This shows that  $\lambda_{w_0}(t)(-v_0) = \rho'_t$  and so  $\lambda_{w_0}(1)$  has multiplicity 1 in  $\mathcal{S}p^{0, v_0, v_0}$  and the corresponding eigenspace  $E_{\lambda_{w_0}(1)}^{0, v_0, v_0}$  is the line generated by  $\sum_{w \in W} w$ . This concludes the proof of the fact that  $\{w_0\}$  is alone in its Calogero-Moser left  $c$ -cell, and that the associated cellular character is  $1_W$ . The statement for  $\{1\}$  instead of  $\{w_0\}$  is now obtained by using (b).  $\square$



**4.C. Two-sided cells.** — Until now, there is no alternative definition of *Calogero-Moser two-sided  $c$ -cells* in terms of Gaudin operators or something related: the only available definition is based on the ramification theory of the Calogero-Moser space [BoRo2, Part III]. This depends on the choice of some prime ideal in some Galois closure of some ring extension. This choice can be adapted to the choice of the two continuous functions  $\gamma$  and  $\xi$  and we will follow this choice.

Moreover, to each Calogero-Moser two-sided  $c$ -cell  $\Gamma$  is associated a subset  $\text{Irr}_\Gamma^{c,\text{CM}}(W)$  of  $\text{Irr}(W)$ , which is called a *Calogero-Moser  $c$ -family*. They form a partition of  $\text{Irr}(W)$ :

$$(4.9) \quad \text{Irr}(W) = \dot{\bigcup}_{\Gamma} \text{Irr}_\Gamma^{c,\text{CM}}(W),$$

where  $\Gamma$  runs over the set of Calogero-Moser two-sided  $c$ -cells. The following properties are proved in [BoRo2, Theo. 10.2.7, Prop. 11.3.3, Prop. 11.4.2]:

**Proposition 4.10.** — *Let  $\Gamma$  be a Calogero-Moser two-sided  $c$ -cell and  $C$  be a Calogero-Moser left  $c$ -cell. Then:*

(a)  $\Gamma$  is a union of Calogero-Moser left  $c$ -cell. Moreover,

$$|\Gamma| = \sum_{\chi \in \text{Irr}_\Gamma^{c,\text{CM}}(W)} \chi(1)^2.$$

(b) If  $C \subset \Gamma$ , then every irreducible component of  $\chi_C^{c,\text{CM}}$  belongs to  $\text{Irr}_\Gamma^{c,\text{CM}}(W)$ .

It is conjectured [BoRo2, Conj. LR] that the analogue of Conjecture 4.7 also holds for two-sided cells:

**Conjecture 4.11.** — *Calogero-Moser two-sided  $c$ -cells coincide with Kazhdan-Lusztig two-sided  $c$ -cells. Moreover, if  $\Gamma$  is one of these, then  $\text{Irr}_\Gamma^{c,\text{CM}}(W) = \text{Irr}_\Gamma^{c,\text{KL}}(W)$ .*

This Conjecture holds in type  $A$  (see [BrGoWh]). In this paper, we prove it for dihedral groups whenever  $c$  is constant.

## 5. Dihedral groups, equal parameters

**Hypothesis and notation.** *From now on, and until the end of this paper, we assume that  $V = \mathbb{R}^2$ , endowed with its canonical Euclidean structure, and we denote by  $(e_1, e_2)$  its canonical basis (which is an orthonormal basis). We fix a natural number  $d \geq 3$  and, if  $k \in \mathbb{Z}$ , we set*

$$\alpha_k = \cos\left(\frac{k\pi}{d}\right)e_1 + \sin\left(\frac{k\pi}{d}\right)e_2$$

*and  $s_k = s_{\alpha_k}$ . We assume that  $\Delta = \{\alpha_0, \alpha_{d-1}\}$ , so that  $S = \{s_0, s_{d-1}\}$ . For simplification, we set  $s = s_0$  and  $s' = s_{d-1}$ .*

Then  $W = \langle s, s' \rangle$  is the dihedral group of order  $2d$ , and  $d$  is the order of  $ss'$ . Moreover,  $\alpha_{k+d} = -\alpha_k$  and

$$\Phi = \{\alpha_k \mid k \in \mathbb{Z}\} = \{\alpha_k \mid 0 \leq k \leq 2d-1\}$$

and  $\text{Ref}(W) = \{s_k \mid k \in \mathbb{Z}\} = \{s_k \mid 0 \leq k \leq d-1\}$ .

Moreover,

$$\Phi^+ = \{\alpha_k \mid 0 \leq k \leq d-1\}.$$

We aim to prove Conjectures 3.4, 4.7 and 4.11 whenever  $c$  is constant. The conjectures 3.4 and 4.7 hold for  $c = 0$  by Example 3.5 and the remark following Conjecture 4.7. For Conjecture 4.11, see [BoRo2, Coro. 15.3.3]. Thus we may assume that  $c$  is constant and non-zero and, by Remark 4.5, that  $c = 1$ , the constant function with value 1.

**5.A. Elements, characters.** — Recall that  $s = s_0$  and  $s' = s_{d-1}$ . For  $k \geq 0$ , we set

$$\sigma_k = \underbrace{ss's \cdots}_{k \text{ terms}} \quad \text{and} \quad \sigma'_k = \underbrace{s'ss' \cdots}_{k \text{ terms}}.$$

Then  $\sigma_0 = \sigma'_0 = 1$ ,  $\sigma_d = \sigma'_d = w_0$  and

$$(5.1) \quad W = \{1, \sigma_1, \sigma'_1, \sigma_2, \sigma'_2, \dots, \sigma_{d-1}, \sigma'_{d-1}, w_0\}.$$

Let  $W^+ = \langle ss' \rangle$ . It is a normal cyclic subgroup of order  $d$  of  $W$ . We fix a primitive  $d$ -th root of unity  $\omega$  and, for  $k \in \mathbb{Z}$ , we denote by  $\theta_k : W^+ \rightarrow \mathbb{C}^\times$  the linear character such that  $\theta_k(ss') = \omega^k$ . We set

$$\chi_k = \text{Ind}_{W^+}^W \theta_k.$$

If  $d$  is even, we denote by  $\varepsilon_s$  (resp.  $\varepsilon_{s'}$ ) the linear character of  $W$  such that  $\varepsilon_s(s) = -1 = -\varepsilon_s(s')$  (resp.  $\varepsilon_{s'}(s) = 1 = -\varepsilon_{s'}(s')$ ). Then  $\chi_k = \chi_{-k} = \chi_{k+d}$  and

$$(5.2) \quad \text{Irr}(W) = \begin{cases} \{\mathbf{1}_W, \varepsilon, \chi_1, \dots, \chi_{(d-1)/2}\} & \text{if } d \text{ is odd,} \\ \{\mathbf{1}_W, \varepsilon, \varepsilon_s, \varepsilon_{s'}, \chi_1, \dots, \chi_{(d-2)/2}\} & \text{if } d \text{ is even.} \end{cases}$$

We conclude this subsection with a fact that will be useful for our purpose: if  $k \geq 0$ , then

$$(5.3) \quad (\sigma_k - \sigma'_k) \sum_{s \in \text{Ref}(W)} s = 0.$$

*Proof.* — Note that  $\text{Ref}(W) = W \setminus W^+$ . As it is clear that  $(\sigma_k - \sigma'_k) \sum_{w \in W} w = 0$ , it is sufficient to prove that  $(\sigma_k - \sigma'_k) \sum_{w \in W^+} w = 0$ . But

$$(\sigma_k - \sigma'_k) \sum_{w \in W^+} w = \sum_{w \in \sigma_k W^+} w - \sum_{w \in \sigma'_k W^+} w,$$

so the result follows from the fact that  $\sigma_k W^+ = \sigma'_k W^+$ .  $\square$

**5.B. Kazhdan-Lusztig cells.** — Let

$$C_s = \{\sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_{d-1}^{-1}\} \quad \text{and} \quad C_{s'} = \{\sigma_1'^{-1}, \sigma_2'^{-1}, \dots, \sigma_{d-1}'^{-1}\}.$$

The Kazhdan-Lusztig left 1-cells as well as the Kazhdan-Lusztig 1-cellular characters are easily computed (see for instance [Bon2, Chap. 21]):

**Proposition 5.4.** — *The Kazhdan-Lusztig left 1-cells are*

$$\{1\}, \quad \{w_0\}, \quad C_s \quad \text{and} \quad C_{s'}.$$

Moreover:

(a) *If  $d$  is odd, then*

$$\chi_{C_s}^{1, \text{KL}} = \chi_{C_{s'}}^{1, \text{KL}} = \chi_1 + \dots + \chi_{(d-1)/2}.$$

(b) *If  $d$  is even, then*

$$\chi_{C_s}^{1, \text{KL}} = \varepsilon_{s'} + \chi_1 + \dots + \chi_{(d-2)/2} \quad \text{and} \quad \chi_{C_{s'}}^{1, \text{KL}} = \varepsilon_s + \chi_1 + \dots + \chi_{(d-2)/2}.$$

**5.C. Calogero-Moser cells.** — The main result of our paper is the following:

**Theorem 5.5.** — *Conjectures 3.4, 4.7 and 4.11 hold whenever  $W$  is dihedral and  $c$  is constant.*

The rest of this section is devoted to the proof of this Theorem.

**5.D. Preliminaries.** — We use the flexibility of the definition of left cells explained in Remark 4.5. We take

$$v_1 = v_2 = \sin\left(\frac{\pi}{2d}\right)e_1 + \cos\left(\frac{\pi}{2d}\right)e_2, \quad \gamma(t) = t \quad \text{and} \quad \xi(t) = 1 - t$$

(one can check that  $v_0$  is a positive multiple of  $v_1$ ). We also set

$$v_1^\perp = \cos\left(\frac{\pi}{2d}\right)e_1 - \sin\left(\frac{\pi}{2d}\right)e_2,$$

so that  $(v_1, v_1^\perp)$  is an orthonormal basis of  $V$ . We have

$$(5.6) \quad \begin{cases} (v_1, \alpha_k) = \sin\left(\frac{(1+2k)\pi}{2d}\right), \\ (v_1^\perp, \alpha_k) = \cos\left(\frac{(1+2k)\pi}{2d}\right) \end{cases}$$

for all  $k \geq 0$ . Also

$$(5.7) \quad \begin{cases} \sigma_k(v_1) = \sin\left(\frac{(1-2k)\pi}{2d}\right)e_1 + \cos\left(\frac{(1-2k)\pi}{2d}\right)e_2, \\ \sigma'_k(v_1) = \sin\left(\frac{(1+2k)\pi}{2d}\right)e_1 + \cos\left(\frac{(1+2k)\pi}{2d}\right)e_2. \end{cases}$$

*Proof of (5.7).* — This is easily checked by induction on  $k$  using the fact that  $\sigma_k = s\sigma'_{k-1}$  and  $\sigma'_k = s'\sigma_{k-1}$  for  $k \geq 1$ .  $\square$

An immediate consequence is that

$$(5.8) \quad \begin{cases} (\sigma_k(v_1), v_1) = (\sigma'_k(v_1), v_1) = \cos\left(\frac{k\pi}{d}\right), \\ (\sigma_k(v_1), v_1^\perp) = -\sin\left(\frac{k\pi}{d}\right), \\ (\sigma'_k(v_1), v_1^\perp) = \sin\left(\frac{k\pi}{d}\right). \end{cases}$$

Now, we set  $\rho = ss'$  (so that  $\rho$  is the rotation with angle  $2\pi/d$ ). Recall that  $\rho$  generates  $W^+$ . Also

$$(5.9) \quad s_k s_l = \rho^{l-k}$$

for all  $k, l \in \mathbb{Z}$ .

**5.E. Proof of Conjecture 3.4.** — Now, we set

$$A = D_{-v_1}^{0, v_1, v_1}, \quad B = D_{-v_1}^{1, v_1, 0}, \quad A^\perp = D_{v_1^\perp}^{0, v_1, v_1} \quad \text{and} \quad B^\perp = D_{v_1^\perp}^{1, v_1, 0}.$$

In particular,

$$D_{-v_1}^{b-1, v_1, av_1} = aA + bB \quad \text{and} \quad D_{v_1^\perp}^{b-1, v_1, av_1} = aA^\perp + bB^\perp.$$

Now, if  $1 \leq k \leq d-1$ , then

$$(5.10) \quad \begin{cases} \sigma_k^{-1} - \sigma'_k{}^{-1} \text{ is an eigenvector of } aA + bB \text{ for the eigenvalue } -a \cos(k\pi/d), \\ \sigma_k^{-1} - \sigma'_k{}^{-1} \text{ is not an eigenvector of } aA^\perp + bB^\perp \text{ if } a > 0. \end{cases}$$

*Proof of (5.10).* — Note that  $\sigma_k^{-1} - \sigma'_k{}^{-1} = \pm(\sigma_k - \sigma'_k)$ . By (5.8), we have  $A(\sigma_k^{-1} - \sigma'_k{}^{-1}) = -\cos(k\pi/d)(\sigma_k^{-1} - \sigma'_k{}^{-1})$ . Moreover,  $B(w) = w \sum_{s \in \text{Ref}(W)} s$ , so  $B(\sigma_k^{-1} - \sigma'_k{}^{-1}) = 0$  by (5.3). The first assertion follows.

Still by (5.8), the coefficient of  $\sigma_k^{-1}$  (or  $\sigma'_k{}^{-1}$ ) in  $(aA^\perp + bB^\perp)(\sigma_k^{-1} - \sigma'_k{}^{-1})$  is equal to  $-a \sin(k\pi/d)$ , so this proves the second statement because  $\sin(k\pi/d) \neq 0$ .  $\square$

After these preliminaries, we are ready to prove the theorem. So let us first prove that Conjecture 3.4 holds. Let us assume that  $a > 0$ . Let  $\rho_{a,b}$  denote the largest eigenvalue of  $aA + bB$  (as in the proof of Proposition 4.8(c)). Then it has multiplicity 1. By the proof of Proposition 4.8(b),  $-\rho_{a,b}$  is the smallest eigenvalue of  $aA + bB$ , and it has multiplicity 1. We denote by  $E_{\max}(a, b)$  (resp.  $E_{\min}(a, b)$ ) the  $\rho_{a,b}$ -eigenspace (resp. the  $-\rho_{a,b}$ -eigenspace) of  $aA + bB$ .

On the other hand, if  $1 \leq k \leq d-1$ , it follows from (5.10) that the vector space generated by  $\sigma_k^{-1} - \sigma'_k{}^{-1}$  and its image by  $aA^\perp + bB^\perp$  is contained in the  $-a \cos(k\pi/d)$ -eigenspace of  $aA + bB$ . So this eigenspace (let us denote it by  $E_k(a, b)$ ) has dimension  $\geq 2$ . Since  $a \neq 0$ ,  $a \cos(k\pi/d) \neq a \cos(l\pi/d)$  if  $1 \leq k < l \leq d-1$ . Therefore, the vector space

$$E_{\max}(a, b) \oplus E_{\min}(a, b) \oplus E_1(a, b) \oplus E_2(a, b) \oplus \cdots \oplus E_{d-1}(a, b)$$

has dimension  $\geq 2 + 2(d-1) = 2d = \dim \mathbb{R}W$ . So this proves that

$$(5.11) \quad \mathbb{R}W = E_{\max}(a, b) \oplus E_{\min}(a, b) \oplus E_1(a, b) \oplus E_2(a, b) \oplus \cdots \oplus E_{d-1}(a, b)$$

and that

$$(5.12) \quad \dim E_k(a, b) = 2$$

for  $1 \leq k \leq d-1$ . This describes the diagonalization of  $aA + bB$ . But now, the second statement of (5.10) shows that  $aA^\perp + bB^\perp$  has two distinct eigenvalues on each  $E_k(a, b)$ . So the family  $\mathcal{D}^{a,1,bv_1}$  has simple spectrum as soon as  $a > 0$ . This is exactly Conjecture 3.4.

**5.F. Cellular characters.** — Let us define two elements of the group algebra  $\mathbb{R}W$  by

$$\mathfrak{b} = \sum_{k=0}^{d-1} s_k \quad \text{and} \quad \mathfrak{b}^\perp = \sum_{k=0}^{d-1} \cot\left(\frac{(2k+1)\pi}{2d}\right) s_k.$$

Then  $B$  (resp.  $B^\perp$ ) is the right multiplication by  $\mathfrak{b}$  (resp.  $\mathfrak{b}^\perp$ ). For  $\chi \in \text{Irr}(W)$ , we denote by  $e_\chi \in Z(\mathbb{R}W)$  the corresponding central idempotent

$$e_\chi = \frac{\chi(1)}{|W|} \sum_{w \in W} \chi(w^{-1})w.$$

Then

$$\mathbb{R}W = \bigoplus_{\chi \in \text{Irr}(W)} \mathbb{R}W e_\chi$$

and  $\mathfrak{b} = d(e_{\mathbf{1}_W} - e_\varepsilon)$ . We denote by  $E_d$  (resp.  $E_{-d}$ ) the  $d$ -eigenspace (resp.  $(-d)$ -eigenspace) of  $B^\perp$ . Then:

**Lemma 5.13.** — *With the above notation, we have:*

(a)  $\mathbb{R}W = \mathbb{R}e_{\mathbf{1}_W} \oplus \mathbb{R}e_\varepsilon \oplus E_d \oplus E_{-d}$  and

$$E_d \oplus E_{-d} = \bigoplus_{\chi \in \text{Irr}(W) \setminus \{\mathbf{1}_W, \varepsilon\}} \mathbb{R}W e_\chi.$$

(b)  $B$  acts on  $\mathbb{R}e_{\mathbf{1}_W}$  (resp.  $\mathbb{R}e_\varepsilon$ , resp.  $E_d$ , resp.  $E_{-d}$ ) by multiplication by  $d$  (resp.  $-d$ , resp.  $0$ , resp.  $0$ ).

(c)  $B^\perp$  acts on  $\mathbb{R}e_{\mathbf{1}_W}$  and  $\mathbb{R}e_\varepsilon$  by multiplication by  $0$ .

*Proof.* — First, it follows from (5.9) and (I.1) that

$$(\mathfrak{b}^\perp)^2 = d^2 - d \sum_{w \in W^+} w.$$

Moreover,

$$\mathfrak{b}^2 = d \sum_{w \in W^+} w.$$

But

$$e_\chi \left( \sum_{w \in W^+} w \right) = \begin{cases} de_\chi & \text{if } \chi \in \{\mathbf{1}_W, \varepsilon\}, \\ 0 & \text{if } \chi \notin \{\mathbf{1}_W, \varepsilon\}. \end{cases}$$

This proves the lemma because  $B$  and  $B^\perp$  are diagonalizable.  $\square$

This shows that

$$\mathcal{S}p^{1,v_1,0} = \{dv_1^*, -dv_1^*, d(v_1^\perp)^*, -d(v_1^\perp)^*\}$$

and that the decomposition in (a) is the decomposition (3.2) for  $(c, v, v') = (1, v_1, 0)$ . Let us now determine the corresponding cellular characters:

**Lemma 5.14.** — *If  $d$  is odd, then  $E_d$  and  $E_{-d}$  both afford the character  $\chi_1 + \chi_2 + \cdots + \chi_{(d-1)/2}$ . If  $d$  is even, then the  $\mathbb{R}W$ -module  $E_d$  (resp.  $E_{-d}$ ) affords the character  $\varepsilon_{s'} + \chi_1 + \chi_2 + \cdots + \chi_{(d-2)/2}$  (resp.  $\varepsilon_s + \chi_1 + \chi_2 + \cdots + \chi_{(d-1)/2}$ ).*

*Proof.* — Let  $\eta$  denote the automorphism of  $W$  exchanging  $s$  and  $s'$ . Then  $\eta\mathfrak{b}^\perp = -\mathfrak{b}^\perp$ . So  $\eta$  exchanges  $E_d$  and  $E_{-d}$ . Moreover,  $\eta\chi_j = \chi_j$  for any  $j$ .

But, if  $d$  is odd, then  $E_d \oplus E_{-d}$  affords the character  $2(\chi_1 + \chi_2 + \cdots + \chi_{(d-1)/2})$ . So this forces that both  $E_d$  and  $E_{-d}$  afford the character  $\chi_1 + \chi_2 + \cdots + \chi_{(d-1)/2}$ .

Now, if  $d$  is even, then  $E_d \oplus E_{-d}$  affords the character  $\varepsilon_s + \varepsilon_{s'} + 2(\chi_1 + \chi_2 + \cdots + \chi_{(d-1)/2})$ . So this forces that  $E_d$  affords the character  $\varepsilon_s + \chi_1 + \chi_2 + \cdots + \chi_{(d-2)/2}$  or  $\varepsilon_{s'} + \chi_1 + \chi_2 + \cdots + \chi_{(d-2)/2}$  (and  $E_{-d}$  affords the other). To determine which one it is, we just need to compute  $e_{\varepsilon_s}\mathfrak{b}^\perp$ . But  $e_{\varepsilon_s}\mathfrak{b}^\perp = \lambda e_{\varepsilon_s}$  where

$$\lambda = \sum_{k=0}^{d-1} \cot\left(\frac{(2k+1)\pi}{2d}\right) \varepsilon_s(s_k).$$

Since  $\varepsilon_s(s_k) = (-1)^{k+1}$ , we get from (I.3) that  $\lambda = -d$ . This proves the result.  $\square$

Lemmas 5.13 and 5.14, together with Proposition 5.4, prove that the list of Calogero-Moser 1-cellular characters equals the list of Kazhdan-Lusztig 1-cellular characters.

**5.G. Left cells.** — Assume that  $a > 0$ . If  $1 \leq k \leq d-1$ , then  $E_k(a, b)$  has dimension 2 (see §5.E). Let us determine the action of  $aA^\perp + bB^\perp$  on  $E_k(a, b)$ . Let  $r_k(a, b) = (aA^\perp + bB^\perp)(\sigma_k^{-1} - \sigma_k'^{-1}) \in E_k(a, b)$ . Then

$$E_k(a, b) = \mathbb{R}(\sigma_k^{-1} - \sigma_k'^{-1}) \oplus \mathbb{R}r_k(a, b).$$

So this means that we need to determine  $(aA^\perp + bB^\perp)(r_k(a, b))$ , i.e. we need to determine the two real numbers  $z, z'$  such that

$$(aA^\perp + bB^\perp)(r_k(a, b)) = z(\sigma_k^{-1} - \sigma_k'^{-1}) + z'r_k(a, b).$$

For this, we will work modulo  $F_k = \bigoplus_{w \in W \setminus \{\sigma_k, \sigma_k'\}} \mathbb{R}w$ . Then

$$r_k(a, b) \equiv -a \sin\left(\frac{k\pi}{d}\right) (\sigma_k^{-1} + \sigma_k'^{-1}) \pmod{F_k}.$$

On the other hand,

$$(aA^\perp + bB^\perp)^2 = a^2(A^\perp)^2 + b^2(B^\perp)^2 + ab(A^\perp B^\perp + B^\perp A^\perp).$$

So

$$(aA^\perp + bB^\perp)(r_k(a, b)) \equiv a^2 \sin^2\left(\frac{k\pi}{d}\right) (\sigma_k^{-1} - \sigma_k'^{-1}) + b^2(B^\perp)^2 (\sigma_k^{-1} - \sigma_k'^{-1}) \pmod{F_k}.$$

But the formula for  $(\mathfrak{b}^\perp)^2$  given in the proof of Lemma 5.13 shows that

$$(B^\perp)^2(\sigma_k^{-1} - \sigma_k'^{-1}) = d^2(\sigma_k^{-1} - \sigma_k'^{-1}).$$

All this together shows that

$$z = a^2 \sin^2\left(\frac{k\pi}{d}\right) + b^2 d^2 \quad \text{and} \quad z' = 0.$$

So we have proved the following fact:

**Lemma 5.15.** — *Assume that  $a > 0$ . Then the restriction of  $aA^\perp + bB^\perp$  to  $E_k(a, b)$  has two eigenvalues, namely*

$$\pm \sqrt{a^2 \sin^2\left(\frac{k\pi}{d}\right) + b^2 d^2}.$$

Coming back to the family  $\mathfrak{D}^{t,c,v_1,(1-t)v_1}$  (which corresponds to the case  $a = 1 - t$  and  $b = t$ ), this shows that

$$(5.16) \quad \begin{cases} \lambda_{\sigma_k^{-1}}(t) = -(1-t) \cos\left(\frac{k\pi}{d}\right) v_1^* - \sqrt{(1-t)^2 \sin^2\left(\frac{k\pi}{d}\right) + t^2 d^2} (v_1^\perp)^*, \\ \lambda_{\sigma_k'^{-1}}(t) = -(1-t) \cos\left(\frac{k\pi}{d}\right) v_1^* + \sqrt{(1-t)^2 \sin^2\left(\frac{k\pi}{d}\right) + t^2 d^2} (v_1^\perp)^*, \end{cases}$$

for  $1 \leq k \leq d - 1$ . Taking the limit at  $t = 1$ , and using the cases 1 and  $w_0$  solved in Proposition 4.8, we get (also thanks to Lemma 5.14):

**Proposition 5.17.** — *The Calogero-Moser left 1-cells are*

$$\{1\}, \quad \{w_0\}, \quad C_s \quad \text{and} \quad C_{s'}.$$

Moreover:

(a) *If  $d$  is odd, then*

$$\chi_{C_s}^{1,\text{CM}} = \chi_{C_{s'}}^{1,\text{CM}} = \chi_1 + \cdots + \chi_{(d-1)/2}.$$

(b) *If  $d$  is even, then*

$$\chi_{C_s}^{1,\text{CM}} = \varepsilon_{s'} + \chi_1 + \cdots + \chi_{(d-2)/2} \quad \text{and} \quad \chi_{C_{s'}}^{1,\text{CM}} = \varepsilon_s + \chi_1 + \cdots + \chi_{(d-2)/2}.$$

The proof of Conjecture 4.11 in this case is complete.

**5.H. Two-sided cells.** — The Kazhdan-Lusztig two-sided 1-cells are

$$\{1\}, \quad \Gamma = C_s \cup C_{s'} \quad \text{and} \quad \{w_0\}.$$

The Kazhdan-Lusztig 1-families are given by

$$\text{Irr}_{\{1\}}^{1,\text{KL}}(W) = \{\varepsilon\}, \quad \text{Irr}_{\Gamma}^{1,\text{KL}}(W) = \text{Irr}(W) \setminus \{\mathbf{1}_W, \varepsilon\} \quad \text{and} \quad \text{Irr}_{\{w_0\}}^{1,\text{KL}}(W) = \{\mathbf{1}_W\}.$$

The Calogero-Moser 1-families have been computed by Bellamy [Bel] (see also [Bon1, Table 5.1]) and coincide with the Kazhdan-Lusztig 1-families. Therefore, the fact that Conjecture 4.11 holds in this case follows immediately from Proposition 4.10.

The proof of Theorem 5.5 is complete.

## 6. Complements

**6.A. Other eigenvalues.** — Keep the notation of the proof of Theorem 5.5. In this proof, it was unnecessary to determine the explicit value of the largest eigenvalue  $\rho_{a,b}$  of  $aA + bB$  and the action of  $aA^\perp + bB^\perp$  on  $E_{\min}(a, b)$  and  $E_{\max}(a, b)$ . This can easily be done, as shown in this subsection.

First, note that  $\text{Tr}(AB) = 0$  because the diagonal coefficients are equal to zero. Also,

$$\text{Tr}(A^2) = \sum_{k=0}^{2d-1} \cos^2\left(\frac{k\pi}{d}\right) \quad \text{and} \quad \text{Tr}(B^2) = 2d^2.$$

The last equality comes from the fact that the coefficient of 1 in  $(\sum_{s \in \text{Ref}(W)} s)^2$  is equal to  $d$ . Therefore,

$$\text{Tr}((aA + bB)^2) = 2a^2 \sum_{k=0}^{d-1} \cos^2\left(\frac{k\pi}{d}\right) + 2b^2 d^2$$

Since the eigenvalue of  $aA + bB$  on  $E_{\min}(a, b)$  is  $-\rho_{a,b}$ , the decomposition (5.11) implies that

$$\text{Tr}((aA + bB)^2) = 2\rho_{a,b}^2 + 2a^2 \sum_{k=1}^{d-1} \cos^2\left(\frac{k\pi}{d}\right).$$

This shows that

$$(6.1) \quad \rho_{a,b} = \sqrt{a^2 + d^2 b^2}$$

It remains to determine the action of  $aA^\perp + bB^\perp$  on  $E_{\min}(a, b)$  and  $E_{\max}(a, b)$ . Both spaces have dimension 1, so this action is by a scalar: if  $\rho$  denotes the scalar by which  $aA^\perp + bB^\perp$  acts on  $E_{\max}(a, b)$ , then  $-\rho$  is the scalar by which  $aA^\perp + bB^\perp$  acts on  $E_{\min}(a, b)$  (again by the proof of Proposition 4.8). Now, note that  $\text{Tr}(A^\perp B^\perp) = 0$  because the diagonal coefficients are equal to zero. Also,

$$\text{Tr}((A^\perp)^2) = \sum_{k=0}^{2d-1} \sin^2\left(\frac{k\pi}{d}\right) \quad \text{and} \quad \text{Tr}((B^\perp)^2) = 2d^2(d-1).$$

Therefore,

$$\text{Tr}((aA^\perp + bB^\perp)^2) = 2a^2 \sum_{k=1}^{d-1} \sin^2\left(\frac{k\pi}{d}\right) + 2b^2 d^2 (d-1).$$



The last equality follows from the formula for  $(b^\perp)^2$  given in the proof of Lemma 5.13. But the decomposition (5.11) implies that

$$\mathrm{Tr}((aA^\perp + bB^\perp)^2) = 2\rho^2 + 2 \sum_{k=1}^{d-1} \left( a^2 \sin^2\left(\frac{k\pi}{d}\right) + b^2 d^2 \right).$$

This shows that  $\rho^2 = 0$  and so

$$(6.2) \quad \rho = 0.$$

Keeping the notation of the proof of Theorem 5.5, this gives the following formula for the paths  $\lambda_1(t)$  and  $\lambda_{w_0}(t)$ :

$$(6.3) \quad \lambda_1(t) = \sqrt{(1-t)^2 + d^2 t^2} v_1^* \quad \text{and} \quad \lambda_{w_0}(t) = -\sqrt{(1-t)^2 + d^2 t^2} v_1^*.$$

**6.B. Some pictures.** — We provide in Figure 1 some pictures of the paths described in (5.16) and (6.3), whenever  $d \in \{3, 4, 5, 6, 7, 8\}$ . In these pictures, we have identified  $V$  and  $V^*$  as all along the paper. The gray points represent the roots, the gray lines represent the reflecting hyperplanes, the blue dots are the points  $w(3dv_1/5)$  (the reason for choosing  $3dv_1/5$  is to have a better view of what happens), the black dots represent the spectrum of the family  $\mathcal{D}^{1, v_1, 0}$  (i.e. they are in bijection with Calogero-Moser left cells), the blue thick curves are the paths  $\lambda_w(t)$  (renormalized as above, i.e. for the family  $\mathcal{D}^{1, tv_1, 3d(1-t)v_1/5}$ ) for  $t \in [0, 1]$  and the blue thin lines are the extensions of these paths for arbitrary values of  $t$ .

## Appendix I Trigonometric identities

We aim to prove that

$$(I.1) \quad \sum_{k=0}^{d-1} \cot\left(\frac{(2k+1)\pi}{2d}\right) \cot\left(\frac{(2k+2j+1)\pi}{2d}\right) = \begin{cases} d^2 - d & \text{if } j = 0, \\ -d & \text{if } 1 \leq j \leq d-1. \end{cases}$$

*Proof.* — For  $n \geq 1$ , let  $f(n) = \sum_{k=1}^{n-1} \cot^2\left(\frac{k\pi}{n}\right)$ . The fact that  $f(n) = \frac{(n-1)(n-2)}{3}$  goes back (at least) to Cauchy. Now,

$$\sum_{k=0}^{d-1} \cot^2\left(\frac{(2k+1)\pi}{2d}\right) = f(2d) - f(d),$$

so the first equality follows easily.

Assume now that  $1 \leq j \leq d-1$ . Since  $\cot(x)\cot(y) = \cot(y-x)(\cot(x) - \cot(y)) - 1$  whenever  $y \not\equiv x \pmod{\pi}$ , we get

$$\begin{aligned} \sum_{k=0}^{d-1} \cot\left(\frac{(2k+1)\pi}{2d}\right) \cot\left(\frac{(2k+2j+1)\pi}{2d}\right) = \\ -d + \cot\left(\frac{j\pi}{d}\right) \sum_{k=0}^{d-1} \left( \cot\left(\frac{(2k+1)\pi}{2d}\right) - \cot\left(\frac{(2k+2j+1)\pi}{2d}\right) \right). \end{aligned}$$

But the sequences  $(k+j)_{0 \leq k \leq d-1}$  and  $(k)_{0 \leq k \leq d-1}$  both cover all the integers modulo  $d$ . So the sum of the terms in the last summand vanishes, as desired.  $\square$

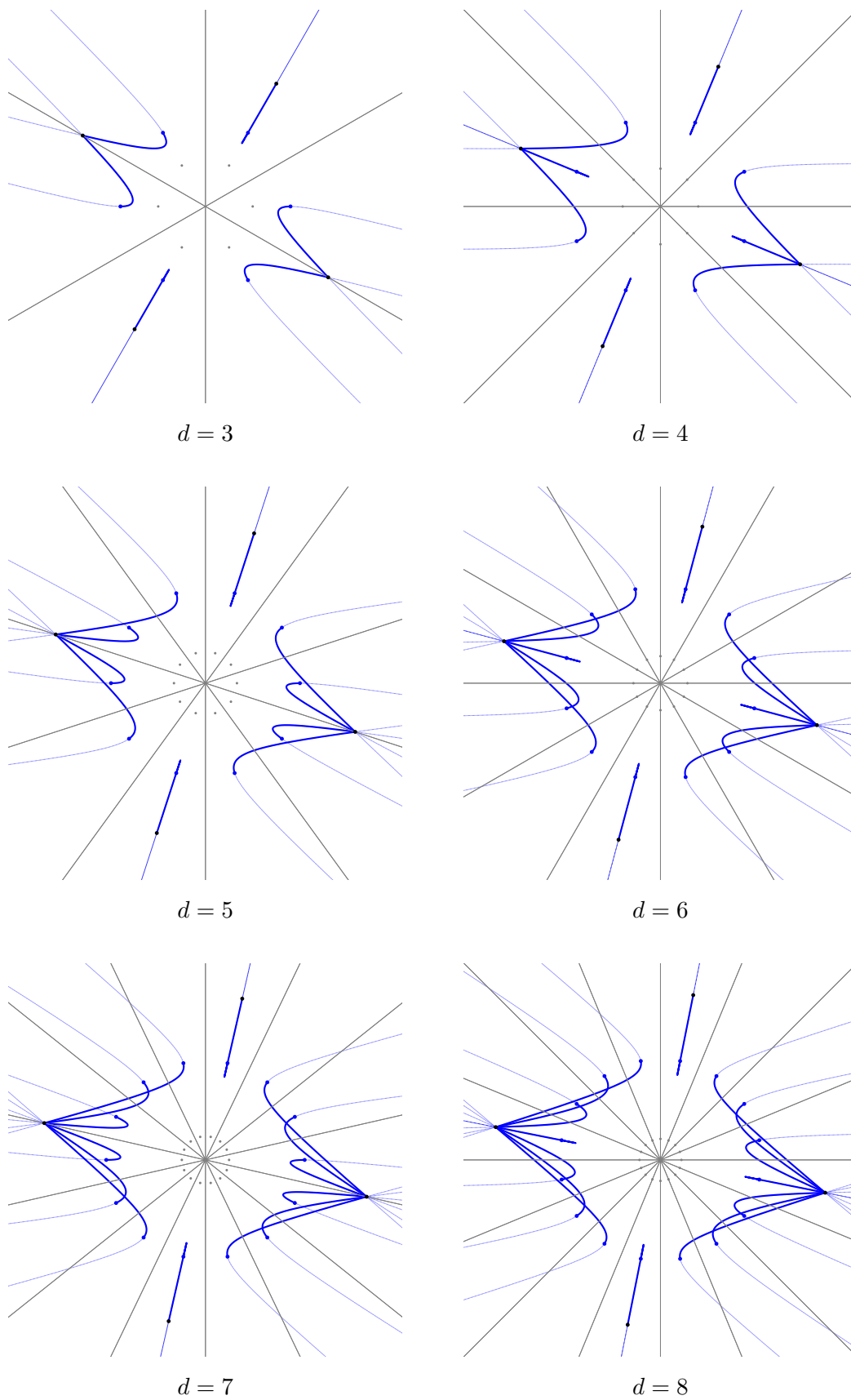


FIGURE 1. Paths  $(\lambda_w)_{w \in W}$  for  $3 \leq d \leq 8$

Note also the following trivial identity:

$$(I.2) \quad \sum_{k=0}^{d-1} \cot\left(\frac{(2k+1)\pi}{2d}\right) = 0$$

(the terms indexed by  $k$  and  $d-1-k$  are opposite to each other). Also, if  $d$  is even, then

$$(I.3) \quad \sum_{k=0}^{d-1} (-1)^k \cot\left(\frac{(2k+1)\pi}{2d}\right) = d.$$

*Proof.* — Let  $\xi = \exp(i\pi/(2d))$  and  $\zeta = \xi^2$ . Then  $\zeta^d = -1$  and

$$\cot\left(\frac{(2k+1)\pi}{2d}\right) = \xi^d \frac{\xi^{2k+1} + \xi^{-2k-1}}{\xi^{2k+1} - \xi^{-2k-1}} = \xi^d \frac{\zeta^{2k+1} + 1}{\zeta^{2k+1} - 1}.$$

Since  $d$  is even, we can write  $d = 2m$ . Then  $\xi^d = \zeta^m$  and

$$(-1)^k = \zeta^{2mk} = \zeta^{m(2k+1)-m} \quad \text{and} \quad (-1)^k \zeta^{2k+1} = \zeta^{(m+1)(2k+1)-m}.$$

Therefore,

$$(-1)^k \cot\left(\frac{(2k+1)\pi}{2d}\right) = \frac{\zeta^{m(2k+1)} + \zeta^{(m+1)(2k+1)}}{\zeta^{2k+1} - 1}.$$

The result follows from [Bon1, (1.10)], specialized at  $X = Y = 1$ .  $\square$

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